

Appendix

A.1. Matrix Inversion

This brief section is included only to help you understand Eq. (1.8.5) in the main text and is by no means comprehensive.

Consider the inversion of a 3×3 matrix

$$M = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \quad (\text{A.1.1})$$

The elements of M have been named in this way rather than as M_{ij} , for in the following discussion we will treat the rows as components of the vectors \mathbf{A} , \mathbf{B} , and \mathbf{C} , i.e., in the notation of vector analysis (which we will follow in this section),

$$\mathbf{A} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \text{ and so on}$$

Consider next a triplet of vectors

$$\begin{aligned} \mathbf{A}_R &= \mathbf{B} \times \mathbf{C} \\ \mathbf{B}_R &= \mathbf{C} \times \mathbf{A} \\ \mathbf{C}_R &= \mathbf{A} \times \mathbf{B} \end{aligned} \quad (\text{A.1.2})$$

which are said to be *reciprocal* to \mathbf{A} , \mathbf{B} , and \mathbf{C} . In general,

$$\mathbf{A} \cdot \mathbf{A}_R \neq 0, \quad \mathbf{A} \cdot \mathbf{B}_R = \mathbf{A} \cdot \mathbf{C}_R = 0 \quad \text{and cyclic permutations} \quad (\text{A.1.3})$$

If we construct now a matrix $\bar{\mathbf{M}}$ (called the *cofactor transpose* of \mathbf{M}) whose *columns* are the reciprocal vectors,

$$\bar{\mathbf{M}} = \begin{bmatrix} (a_R)_1 & (b_R)_1 & (c_R)_1 \\ (a_R)_2 & (b_R)_2 & (c_R)_2 \\ (a_R)_3 & (b_R)_3 & (c_R)_3 \end{bmatrix}$$

then

$$\mathbf{M} \cdot \bar{\mathbf{M}} = \begin{bmatrix} \mathbf{A} \cdot \mathbf{A}_R & \mathbf{A} \cdot \mathbf{B}_R & \mathbf{A} \cdot \mathbf{C}_R \\ \mathbf{B} \cdot \mathbf{A}_R & \mathbf{B} \cdot \mathbf{B}_R & \mathbf{B} \cdot \mathbf{C}_R \\ \mathbf{C} \cdot \mathbf{A}_R & \mathbf{C} \cdot \mathbf{B}_R & \mathbf{C} \cdot \mathbf{C}_R \end{bmatrix} = \begin{bmatrix} \mathbf{A} \cdot \mathbf{A}_R & 0 & 0 \\ 0 & \mathbf{B} \cdot \mathbf{B}_R & 0 \\ 0 & 0 & \mathbf{C} \cdot \mathbf{C}_R \end{bmatrix} \quad (\text{A.1.4})$$

Now all three diagonal elements are equal:

$$\begin{aligned} \mathbf{A} \cdot \mathbf{A}_R &= \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{B} \cdot \mathbf{B}_R = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{C} \cdot \mathbf{C}_R \\ &= \det M \end{aligned} \quad (\text{A.1.5})$$

where the last equality follows from the fact that the cross product may be written as a determinant:

$$\mathbf{B} \times \mathbf{C} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \quad (\text{A.1.6})$$

(We shall follow the convention of using two vertical lines to denote a determinant.)
Hence the inverse of the matrix \mathbf{M} is given by

$$\mathbf{M}^{-1} = \frac{\bar{\mathbf{M}}}{\det M} \quad (\text{A.1.7})$$

When does $\det M$ vanish? If one of the vectors, say \mathbf{C} , is a linear combination of the other two; for if

$$\mathbf{C} = \alpha \mathbf{A} + \beta \mathbf{B}$$

then

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{A} \cdot (\mathbf{B} \times \alpha \mathbf{A}) + \mathbf{A} \cdot (\mathbf{B} \times \beta \mathbf{B}) = \mathbf{B} \cdot (\alpha \mathbf{A} \times \mathbf{A}) = 0$$

Thus the determinant vanishes if the rows of the matrix are not linearly independent (LI) and vice versa. If the matrix is used to represent three simultaneous equations, it means not all three equations are independent. The method can be generalized for inverting $n \times n$ matrices, with real or complex elements. One defines a cross product of $n-1$ vectors as

$$\mathbf{A}_1 \times \mathbf{A}_2 \times \cdots \times \mathbf{A}_{n-1} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} & \cdots \\ (a_1)_1 & (a_1)_2 & \cdots & \\ \vdots & \vdots & \vdots & \\ (a_{n-1})_1 & (a_{n-1})_2 & \cdots & (a_{n-1})_n \end{vmatrix} \quad (\text{A.1.8})$$

The resulting vector is orthogonal to the ones in the product, changes sign when we interchange any two of the adjacent ones, and so on, just like its three-dimensional counterpart. If we have a matrix M , whose n rows may be identified with n vectors, $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_n$, then the cofactor transpose has as its columns the reciprocal vectors $\mathbf{A}_{1R}, \dots, \mathbf{A}_{nR}$, where

$$\mathbf{A}_{jR} = \mathbf{A}_{j+1} \times \mathbf{A}_{j+2} \times \cdots \times \mathbf{A}_n \times \mathbf{A}_1 \times \cdots \times \mathbf{A}_{j-1} \quad (\text{A.1.9})$$

One tricky point: the cross product is defined to be orthogonal to the vectors in the product with respect to an inner product

$$\mathbf{A} \cdot \mathbf{B} = \sum A_i B_i$$

and *not*

$$\mathbf{A} \cdot \mathbf{B} = \sum A_i^* B_i$$

even when the components of \mathbf{A} are complex. There is no contradiction here, for the vectors $\mathbf{A}_1, \dots, \mathbf{A}_n$ are fictitious objects that enter a mnemonic and not the elements of the space $\mathbb{V}^n(C)$ on which the operator acts.

Exercise A.1.1. Using the method described above, show that

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 2 \\ -1 & 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -2 & 1 \\ 2 & -5 & 4 \\ -1 & 3 & -2 \end{bmatrix}$$

and

$$\begin{bmatrix} 2 & 1 & 3 \\ 4 & 1 & 2 \\ 0 & -1 & 2 \end{bmatrix}^{-1} = \frac{1}{12} \begin{bmatrix} -4 & 5 & 1 \\ 8 & -4 & -8 \\ 4 & -2 & 2 \end{bmatrix}$$

Theorem A.1.1. If $\Omega|V\rangle = |0\rangle$ implies $|V\rangle = |0\rangle$ then Ω^{-1} exists.

Proof. Let $|V_1\rangle, \dots, |V_n\rangle$ be a LI basis in \mathbb{V}^n . Then another LI basis is generated by the action of Ω , i.e., $\Omega|V_1\rangle, \dots, \Omega|V_n\rangle$ is also a LI basis. To see this, let us assume the contrary, that there exists a relation of the form

$$\sum_i \alpha_i \Omega|V_i\rangle = 0$$

with not all $\alpha_i = 0$. Upon pulling out Ω , because it is linear, we get

$$\Omega\left(\sum_i \alpha_i |V_i\rangle\right) = 0$$

which, when combined with the assumed property of Ω , implies that

$$\sum_i \alpha_i |V_i\rangle = |0\rangle$$

with not all $\alpha_i = 0$, which is not true. So we can conclude that every vector $|V'\rangle$ in \mathbb{V}^n may be written as a *unique* linear combination in the new basis generated by Ω as

$$|V'\rangle = \sum_i \alpha_i \Omega|V_i\rangle$$

In terms of $|V\rangle = \sum_i \alpha_i |V_i\rangle$, we see that *every* $|V'\rangle$ in \mathbb{V}^n may be written as

$$|V'\rangle = \Omega|V\rangle$$

where $|V\rangle$ is *unique*. In other words, we can think of *every* $|V'\rangle$ in \mathbb{V}^n as arising from a *unique* source $|V\rangle$ in \mathbb{V}^n under the action of Ω . Define an operator Λ whose action on any vector $|V'\rangle$ in \mathbb{V}^n is to take it back to its unique source $|V\rangle$. (If the source of $|V'\rangle$ were not unique—say, because there are two vectors $|V_1\rangle$ and $|V_2\rangle$ that are mapped into $|V'\rangle$ by Ω —then we could not define Λ , for acting on $|V'\rangle$, it would not know whether to give $|V_1\rangle$ or $|V_2\rangle$.) The action of Λ is then

$$\Lambda|V'\rangle = |V\rangle, \quad \text{where } |V'\rangle = \Omega|V\rangle$$

We may identify Λ as the inverse of Ω ,

$$\Lambda = \Omega^{-1} \quad \text{or} \quad \Lambda\Omega = I$$

since for any $|V'\rangle$ in \mathbb{V}^n

$$\Lambda|V'\rangle = \Lambda\Omega|V\rangle = |V\rangle \quad \text{Q.E.D.}$$

A.2. Gaussian Integrals

We discuss here all the Gaussian integrals that we will need. Consider

$$I_0(\alpha) = \int_{-\infty}^{\infty} e^{-\alpha x^2} dx, \quad \alpha > 0 \quad (\text{A.2.1})$$

This integral cannot be evaluated by conventional methods. The trick is to consider

$$I_0^2(\alpha) = \int_{-\infty}^{\infty} e^{-\alpha x^2} dx \int_{-\infty}^{\infty} e^{-\alpha y^2} dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\alpha(x^2+y^2)} dx dy$$

Switching to polar coordinates in the x - y plane,

$$\begin{aligned} I_0^2(\alpha) &= \int_0^{\infty} \int_0^{2\pi} e^{-\alpha \rho^2} \rho d\rho d\phi \\ &= \pi/\alpha \end{aligned}$$

Therefore

$$I_0(\alpha) = (\pi/\alpha)^{1/2} \quad (\text{A.2.2})$$

By differentiating with respect to α we can get all the integrals of the form

$$I_{2n}(\alpha) = \int_{-\infty}^{\infty} x^{2n} e^{-\alpha x^2} dx$$

For example,

$$\begin{aligned} I_2(\alpha) &= \int_{-\infty}^{\infty} x^2 e^{-\alpha x^2} dx = -\frac{\partial}{\partial \alpha} \int_{-\infty}^{\infty} e^{-\alpha x^2} dx \\ &= -\frac{\partial}{\partial \alpha} I_0(\alpha) = \frac{1}{2\alpha} \left(\frac{\pi}{\alpha}\right)^{1/2} \end{aligned} \quad (\text{A.2.3})$$

The integrals $I_{2n+1}(\alpha)$ vanish because these are integrals of odd functions over an even interval $-\infty$ to $+\infty$. Equations (A.2.2) and (A.2.3) are valid even if α is purely imaginary.

Consider next

$$I_0(\alpha, \beta) = \int_{-\infty}^{\infty} e^{-\alpha x^2 + \beta x} dx \quad (\text{A.2.4})$$

By completing the square on the exponent, we get

$$I_0(\alpha, \beta) = e^{\beta^2/4\alpha} \int_{-\infty}^{\infty} e^{-\alpha(x-\beta/2\alpha)^2} dx = e^{\beta^2/4\alpha} \left(\frac{\pi}{\alpha}\right)^{1/2} \quad (\text{A.2.5})$$

These results are valid even if α and β are complex, provided $\text{Re } \alpha > 0$. Finally, by applying to both sides of the equation

$$\int_0^{\infty} e^{-ar} dr = \frac{1}{a}$$

the operator $(-d/da)^n$, we obtain

$$\int_0^{\infty} r^n e^{-ar} dr = \frac{n!}{a^{n+1}}$$

Consider this integral with $\alpha=1$ and n replaced by $z-1$, where z is an arbitrary complex number. This defines the *gamma function* $\Gamma(z)$

$$\Gamma(z) = \int_0^{\infty} r^{z-1} e^{-r} dr$$

For real, positive and integral z ,

$$\Gamma(z) = (z-1)!$$

A.3. Complex Numbers

A complex variable z can be written in terms of two real variables x and y , and $i = (-1)^{1/2}$, as

$$z = x + iy \quad (\text{A.3.1})$$

Its *complex conjugate* z^* is defined to be

$$z^* = x - iy \quad (\text{A.3.2})$$

One may invert these two equations to express the *real and imaginary parts*, x and y , as

$$x = \frac{1}{2}(z + z^*), \quad y = (z - z^*)/2i \quad (\text{A.3.3})$$

The *modulus squared* of z , defined to be zz^* , equals

$$zz^* \equiv |z|^2 = (x + iy)(x - iy) = x^2 + y^2 \quad (\text{A.3.4})$$

You may verify that $z = z'$ implies that $x = x'$ and $y = y'$ by considering the modulus of $z - z'$.

From the power-series expansions

$$\sin x = x - x^3/3! + x^5/5! - \dots$$

$$\cos x = 1 - x^2/2! + x^4/4! - \dots$$

one can deduce that

$$e^{ix} = \cos x + i \sin x \quad (\text{A.3.5})$$

It is clear that e^{ix} has unit modulus (x is real).

The expression $z = x + iy$ gives z in *Cartesian form*. The *polar form* is

$$\begin{aligned} z = x + iy &= (x^2 + y^2)^{1/2} \left[\frac{x}{(x^2 + y^2)^{1/2}} + i \frac{y}{(x^2 + y^2)^{1/2}} \right] \\ &= \rho (\cos \theta + i \sin \theta) \\ &= \rho e^{i\theta} \end{aligned}$$

where

$$\rho = (x^2 + y^2)^{1/2} \quad \text{and} \quad \theta = \tan^{-1}(y/x) \quad (\text{A.3.6})$$

Clearly

$$|z| = \rho \quad (\text{A.3.7})$$

Each complex number $z = x + iy$ may be visualized as a point (x, y) in the x - y plane. This plane is also called the *complex z plane*.

A.4. The $i\epsilon$ Prescription

We will now derive and interpret the formula

$$\frac{1}{x \mp i\epsilon} = \mathcal{P} \frac{1}{x} \pm i\pi \delta(x) \quad (\text{A.4.1})$$

where $\varepsilon \rightarrow 0$ is a positive infinitesimally small quantity. Consider an integral of the form

$$I = \lim_{\varepsilon \rightarrow 0} \int_{-\infty}^{\infty} \frac{f(x) dx}{x - i\varepsilon}. \quad (\text{A.4.2})$$

Viewing this as the integral on the real axis of the complex $z = x + iy$ plane, we see that the integrand has an explicit pole at $z = i\varepsilon$ in addition to any singularities f might have. We assume f has no singularities on or infinitesimally close to the real axis. As long as ε is fixed, there is no problem with the integral. For example, if f has some poles in the upper half-plane and vanishes fast enough to permit our closing the contour in the upper half-plane, the integral equals $2\pi i$ times the sum of the residues of the poles of f and the pole at $z = i\varepsilon$. Likewise, if we change the sign of the ε term, we simply drop the contribution from the explicit pole, which is now in the lower half-plane.

What if $\varepsilon \rightarrow 0$? Now the pole is going to ram (from above) into our contour which runs along the x -axis. So we prepare for this as follows. Since the only singularity near the real axis is the explicit pole at $z = i\varepsilon$, we make the following deformation of the contour without changing the value of I : the contour runs along the real axis from $-\infty$ to $-\varepsilon'$, (ε' is another positive infinitesimal) goes around counterclockwise, below the origin in a semicircle of radius ε' , and resumes along the real axis from $x = \varepsilon'$ to ∞ . The nice thing is that we can now set $\varepsilon = 0$, which brings the pole to the origin. The three parts of the integration contour contribute as follows:

$$\begin{aligned} I &= \lim_{\varepsilon' \rightarrow 0} \left[\int_{-\infty}^{-\varepsilon'} \frac{f(x) dx}{x} + \int_{\varepsilon'}^{\infty} \frac{f(x) dx}{x} + i\pi f(0) \right] \\ &\equiv \mathcal{P} \int_{-\infty}^{\infty} \frac{f(x) dx}{x} + i\pi f(0). \end{aligned} \quad (\text{A.4.3})$$

The sum of the two integrals in the limit $\varepsilon' \rightarrow 0$ is defined as the *principal value integral* denoted by the symbol \mathcal{P} . In the last term, which is restricted to the infinitesimal neighbourhood of the origin, we have set the argument of the smooth function f to zero and done the integral of dz/z counterclockwise around the *semicircle* to get $i\pi$.

Eq. (A.4.1) is a compact way to say all this. It is understood that Eq. (A.4.1) is to be used inside an integral only and that inside an integral the factor $1/(x - i\varepsilon)$ leads to two terms: the first, $\mathcal{P}(1/x)$, leads to the principal value integral, and the second, $i\pi\delta(x)$, leads to $i\pi f(0)$.

It is clear that if we reverse the sign of the ε term, we change the sign of the delta function since the semicircle now goes around the pole in the clockwise direction. The principal part is not sensitive to this change of direction and is unaffected.

It is clear that if we replace x by $x - a$ the pole moves from the origin to $x = a$ and $f(0)$ gets replaced by $f(a)$ so that we may write

$$\frac{1}{(x - a) \mp i\varepsilon} = \mathcal{P} \frac{1}{(x - a)} \pm i\pi\delta(x - a) \quad (\text{A.4.4})$$

It is clear that the limits on x need not be $\pm\infty$ for the formula to work.

Finally, note that according to Eq. (A.4.4) the difference between the integrals with two signs of ε is just $2\pi if(a)$. This too agrees with the present analysis in terms of the integral I in Eq. (A.4.2) since in the difference of the two integrals the contribution along the real axis cancels due to opposite directions of travel except for the part near the pole where the difference of the two semicircles (one going above and going below the pole) is a circle around the pole.

Answers to Selected Exercises

Chapter 1

$$1.8.1. (1) |\omega=1\rangle \rightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad |\omega=2\rangle \rightarrow \frac{1}{(30)^{1/2}} \begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix}, \quad |\omega=4\rangle \rightarrow \frac{1}{(10)^{1/2}} \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

(2) No, no.

1.8.2. (1) Yes

$$(2) |\omega=0\rangle \rightarrow \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad |\omega=1\rangle \rightarrow \frac{1}{2^{1/2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad |\omega=-1\rangle \rightarrow \frac{1}{2^{1/2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

1.8.10. $\omega=0, 0, 2; \lambda=2, 3, -1.$

Chapter 4

4.2.1. (1) 1, 0, -1

$$(2) \langle L_x \rangle = 0, \langle L_x^2 \rangle = 1/2, \Delta L_x = 1/2^{1/2}$$

$$(3) |L_x=1\rangle \rightarrow \begin{bmatrix} 1/2 \\ 1/2^{1/2} \\ 1/2 \end{bmatrix}, \quad |L_x=0\rangle \rightarrow \begin{bmatrix} -1/2^{1/2} \\ 0 \\ 1/2^{1/2} \end{bmatrix},$$

$$|L_x=-1\rangle \rightarrow \begin{bmatrix} 1/2 \\ -1/2^{1/2} \\ 1/2 \end{bmatrix}$$

$$(4) \quad P(L_x=1)=1/4, \quad P(L_x=0)=1/2, \quad P(L_x=-1)=1/4$$

$$(5) \quad |\psi\rangle \rightarrow \frac{1}{(1/4+1/2)^{1/2}} \begin{bmatrix} 1/2 \\ 0 \\ 1/2^{1/2} \end{bmatrix} = \text{projection of } |\psi\rangle \text{ on the } L_z^2=1 \text{ eigen-}$$

space. $P(L_z^2=1)=3/4$. If L_z is measured $P(L_z=1)=1/3$, $P(L_z=-1)=2/3$. Yes, the state changes.

- (6) No. To see this right away note that if $\delta_1=\delta_2=\delta_3=0$, $|\psi\rangle=1|L_x=1\rangle$ and if $\delta_1=\delta_3=0$ and $\delta_2=\pi$, $|\psi\rangle=|L_x=-1\rangle$. [See answer to part (3).] The vectors $|\psi\rangle$ and $e^{i\theta}|\psi\rangle$ are physically equivalent only in the sense that they generate the same probability distribution for any observable. This does not mean that when the vector $|\psi\rangle$ appears as a part of a linear combination it can be multiplied by an arbitrary phase factor. In our example one can only say, for instance, that

$$|\psi\rangle' = e^{-i\delta_1}|\psi\rangle \\ = \frac{1}{2}|L_z=1\rangle + \frac{e^{i(\delta_2-\delta_1)}}{2^{1/2}}|L_z=0\rangle + \frac{e^{i(\delta_3-\delta_1)}}{2}|L_z=-1\rangle$$

is physically equivalent to $|\psi\rangle$. Although $|\psi\rangle'$ has different coefficients from $|\psi\rangle$ in the linear expansion, it has the same "direction" as $|\psi\rangle$. In summary, then, the relative phases $\delta_2-\delta_1$ and $\delta_3-\delta_1$ are physically relevant but the overall phase is not, as you will have seen in the calculation of $P(L_x=0)$.

Chapter 5

$$5.4.2. (a) \quad R = (maV_0)^2 / (\hbar^4 k^2 + m^2 a^2 V_0^2); \quad T = 1 - R$$

$$(b) \quad T = (\cosh^2 2\kappa a + \alpha^2 \sinh^2 2\kappa a)^{-1} \text{ where } i\kappa \text{ is the complex wave number for } |x| \leq a \text{ and } \alpha = (V_0 - 2E) / [4E(V_0 - E)]^{1/2}.$$

Chapter 7

$$7.4.2. \quad 0, \quad 0, \quad (n+1/2)\hbar/m\omega, \quad (n+1/2)m\omega\hbar, \quad (n+1/2)\hbar. \text{ Note that the recipe } m\omega \rightarrow (m\omega)^{-1} \text{ is at work here.}$$

$$7.4.5. (1) \quad (1/2^{1/2})(|0\rangle e^{-i\omega t/2} + |1\rangle e^{-3i\omega t/2})$$

$$(2) \quad \langle X(t) \rangle = (\hbar/2m\omega)^{1/2} \cos \omega t, \quad \langle P(t) \rangle = -(m\omega\hbar/2)^{1/2} \sin \omega t$$

$$(3) \quad \langle \dot{X}(t) \rangle = (i\hbar)^{-1} \langle [X, H] \rangle = \langle P(t) \rangle / m, \quad \langle \dot{P}(t) \rangle = -m\omega^2 \langle X(t) \rangle. \text{ By eliminating } \langle \dot{P} \rangle \text{ we can get an equation for } \langle X(t) \rangle \text{ and vice versa and solve it using the initial values } \langle X(0) \rangle \text{ and } \langle P(0) \rangle, \text{ e.g., } \langle X(t) \rangle = \langle X(0) \rangle \cos \omega t + [\langle P(0) \rangle / m\omega] \sin \omega t.$$

Chapter 10

10.3.2. $3^{-1/2}[|334\rangle + |343\rangle + |433\rangle]$

Chapter 12

12.6.1. $E = -\hbar^2/2\mu a_0^2, \quad V = -\hbar^2/\mu a_0 r$

Chapter 13

13.3.1. Roughly 200 MeV.

13.3.2. Roughly 1 Å.

Chapter 14

14.3.5. $M = \left(\frac{\alpha + \delta}{2}\right)I + \left(\frac{\beta + \gamma}{2}\right)\sigma_x + i\left(\frac{\beta - \gamma}{s}\right)\sigma_y + \left(\frac{\alpha - \delta}{2}\right)\sigma_z$

14.3.7. (1) $2^{1/4}(\cos \pi/8 + i(\sin \pi/8)\sigma_x)$.

(2) $2/3I - 1/3\sigma_x$.

(3) σ_x

14.4.4. Roughly 2×10^{-9} second.

14.4.6. $(e\hbar/2mc) \tanh(e\hbar B/2mckT)\mathbf{k}$

14.5.2. (1) Roughly one part in a million.

(2) 10^{10} G.

14.5.3. $1/2, 1/4, 0$.

14.5.4. $\left(\frac{1 + \cos \theta}{2}\right)^2$

Chapter 15

15.2.2. (1) $\langle 1\ 1, 1/2(-1/2)|3/2\ 1/2\rangle = (1/3)^{1/2}$
 $\langle 1\ 0, 1/2\ 1/2|3/2\ 1/2\rangle = (2/3)^{1/2}$
 $\langle 1\ 1, 1/2(-1/2)|1/2\ 1/2\rangle = (2/3)^{1/2}$
 $\langle 1\ 0, 1/2\ 1/2|1/2\ 1/2\rangle = -(1/3)^{1/2}$

$$\begin{aligned}
 (2) \quad |jm\rangle &= |2, 1\rangle = 2^{-1/2}|m_1=1, m_2=0\rangle + 2^{-1/2}|m_1=0, m_2=1\rangle \\
 |2, 0\rangle &= 6^{-1/2}|1, -1\rangle + \left(\frac{2}{3}\right)^{1/2}|0, 0\rangle + \left(\frac{1}{6}\right)^{1/2}| -1, 1\rangle \\
 |1, 1\rangle &= 2^{-1/2}|1, 0\rangle - 2^{-1/2}|0, 1\rangle \\
 |1, 0\rangle &= 2^{-1/2}|1, -1\rangle - 2^{-1/2}| -1, 1\rangle \\
 |0, 0\rangle &= 3^{-1/2}|1, -1\rangle - 3^{-1/2}|0, 0\rangle + 3^{-1/2}| -1, 1\rangle
 \end{aligned}$$

The others are either zero, obvious, or follow from Eq. (15.2.11).

$$15.2.6. \quad \mathbb{P}_+ = \frac{(2\mathbf{L}\cdot\mathbf{S})/\hbar^2 + l + 1}{2l + 1}, \quad \mathbb{P}_- = \frac{l - (2\mathbf{L}\cdot\mathbf{S})/\hbar^2}{2l + 1}$$

Chapter 16

$$16.1.2. \quad E(a_0) = 10E_0/\pi^2$$

$$16.1.3. \quad -ma_0^2V_0^2/\pi\hbar^2$$

$$16.1.4. \quad E(a_0) = \frac{1}{2}\hbar\omega\left(\frac{12}{11}\right)^{1/2}$$

$$16.2.4. \quad \text{Roughly } 1.5 \times 10^{17} \text{ seconds or } 10^{10} \text{ years.}$$

Table of Constants

$$\hbar c = 1973.3 \text{ eV } \text{\AA}$$

$$\alpha = e^2/\hbar c = 1/137.04$$

$$mc^2 = 0.511 \text{ MeV} \quad (m \text{ is the electron mass})$$

$$Mc^2 = 938.28 \text{ MeV} \quad (M \text{ is the proton mass})$$

$$a_0 = \hbar^2/me^2 = 0.511 \text{ \AA}$$

$$e\hbar/2mc = 0.58 \times 10^{-8} \text{ eV/G} \quad (\text{Bohr magneton})$$

$$k = 8.62 \times 10^{-5} \text{ eV/K}$$

$$kT \simeq 1/40 \text{ eV at } T = 300 \text{ K} \quad (\text{room temperature})$$

$$1 \text{ eV} = 1.6 \times 10^{-12} \text{ erg}$$

Mnemonics for Hydrogen

In the ground state,

$$v/c \equiv \beta = \alpha$$

$$E_1 = -T = -\frac{1}{2}mv^2 = -\frac{1}{2}mc^2\alpha^2$$

$$mva_0 = \hbar$$

In higher states, $E_n = E_1/n^2$.

Index

- Absorption spectrum, 368
- Accidental degeneracy
 - free-particle case, 426
 - harmonic oscillator case, 352, 423
 - hydrogen atom case, 359, 422
- Actinides, 371
- Active transformations, 29, 280
- Adjoint, 13, 25, 26
- Aharonov–Bohm effect, 497
- Angular momentum
 - addition of
 - $\mathbf{J} + \mathbf{J}$, 408
 - $\mathbf{L} + \mathbf{S}$, 414
 - $\mathbf{S} + \mathbf{S}$, 403
 - commutation rules, 319
 - eigenfunctions, 324, 333
 - eigenvalue problem of, 321
 - spin, 373
 - in three dimensions, 318
 - in two dimensions, 308
- Anticommutation relations, 640
- Anti-Hermitian operators, 27
- Antisymmetric states, 261
- Anyons, 607

- Balmer series, 367
- Basis, 6
- Berry phase, 592
- Berry potential, 603
- Bohr magneton, 389
- Bohr model, 364
- Bohr radius, 244, 357
- Bohr-Sommerfeld quantization rule, 448
- Born approximation
 - time-dependent, 529
 - time-independent, 534
 - validity of, 543

- Bose-Einstein statistics, 271
- Bosons, 263
- Bound states, 160, 445
 - energy quantization in, 160
- Bra, 11
- Breit-Wigner form, 551
- de Broglie waves, 112, 366
- Double well, 616
 - tunneling in, 616

- Canonical commutation rule, 131
- Canonically conjugate operators, 69
- Canonical momentum, 80
 - electromagnetic case, 84
- Canonical transformations
 - active, 97
 - introduction to, 92
 - point transformations, 94
 - regular, 97
- Center of mass (CM), 85
- Centrifugal barrier, 340
- Characteristic equation, 33
- Characteristic polynomial, 33
- Chemical potential, 641
- Classical limit, 179
- Classical radius of electron, 364
- Clebsch–Gordan coefficients, 412
- Cofactor matrix, 656
- Coherent states
 - fermionic, 642
 - oscillator, 607
 - spin, 636
- Collapse of state vector, 122, 139
- Commutator, 20
- Compatible variables, 129
- Completeness relation, 23, 59
- Complete set of observables, 133

- Complex numbers, 660
- Compton scattering, 123
- Compton wavelength, 246
 - electronic, 363
- Condon-Shortley convention, 410
- Configuration space, 76
- Consistency test
 - for three-dimensional rotations, 318
 - for translations, 306, 312
 - for translations and rotations, 310
- Coordinates
 - canonical, 94
 - center-of-mass, 85
 - cyclic, 81
 - relative, 85
- Correlation function, 628
 - connected, 634
- Correlation length, 629
- Correspondence principle, 197
- Coulomb scattering, 531
- Coupled mass problem, 46
- Creation operator, 205
- Cross section
 - in CM frame, 557
 - differential, 526, 529
 - for Gaussian potential, 533
 - for hard sphere, 549
 - in lab frame, 559
 - partial, 548
 - photoelectric, 506
 - Rutherford, 531
 - for Yukawa potential, 531
- Cyclotron frequency, 588

- Dalgarno and Lewis method, 462
- Darwin term, 572
- Degeneracy, 38, 44, 120
- Density matrix, 133
- Derivative operator, 63
 - eigenvalue problem for, 66
 - matrix elements of, 64
- Destruction operator, 205
- Determinant, 29
- Diagonalization
 - of Hermitian operator, 40
 - simultaneous, 43
- Differential cross section, 526, 529
- Dipole approximation, 502
- Dipole moment, 463
- Dipole selection rule, 465
- Dirac delta function, 60
 - definition of, 60
 - derivatives of, 61
 - Gaussian approximation for, 61
 - integral representation of, 63
 - three-dimensional, 342
- Dirac equation
 - electromagnetic, 566
 - free particle, 565
- Dirac monopole, 605
- Dirac notation, 3
- Dirac string, 605
- Direct product
 - of operators, 250
 - spaces, 249
- Double-slit experiment, 108
 - quantum explanation of, 175
- Dual spaces, 11

- Ehrenfest's theorem, 180
- Eigenket, 30
- Eigenspace, 37
- Eigenvalue problem, 30
- Eigenvector, 30
- Einstein temperature, 220
- Electromagnetic field
 - interactions with matter, 83, 90, 499
 - quantization of, 506
 - review of, 492
- Ensemble
 - classical, 125
 - mixed, 133
 - quantum, 125
- Euclidean Lagrangian, 614
- Euler angles, 333
- Euler-Lagrange equations, 79
- Exchange operator, 278
- Exclusion principle, 264
- Expectation value, 127

- Fermi-Dirac statistics, 270
- Fermionic oscillator, 640
 - thermodynamics of, 642
- Fermi's golden rule, 483
- Fermions, 263
- Field, 2
- Filling factor, 591
- Fine-structure constant, 362
- Fine-structure correction, 367, 466
- Fourier transform, 62
- Free-particle problem
 - cartesian coordinates, 151
 - spherical coordinates, 426
- Functional, 77
- Functions of operators, 54

- Gauge
 - Coulomb, 494
 - invariance, 493, 496
 - transformations, 493, 496
- Gaussian integrals, 659

- Gaussian potential, 533
- Generalized force, 80
- Generalized potential, 84
- Geometric phase, 593
- Gram-Schmidt theorem, 14
- Grassmann numbers, 642
- Green's function, 534
- Gyromagnetic ratio, 386

- Hamiltonian formulation, 86
- Hamilton's equations, 88
- Harmonic oscillator
 - classical, 83
 - fermionic, 640
 - isotropic, 260, 351
 - quantum, 185
 - in the coordinate basis, 189
 - in the energy basis, 202
 - energy levels of, 194
 - propagator for, 196
 - wave functions of, 195, 202
 - thermodynamics of, 219
 - three-dimensional, 260, 351
 - two-dimensional, 316
- Heisenberg picture, 147, 490
- Hermite polynomials, 490
- Hermitian operators, 27
 - diagonalization of, 40
 - simultaneous, 43
 - eigenbases of, 36
 - eigenvalues of, 35
 - eigenvectors of, 36
 - infinite-dimensional, 65
- Hilbert space, 67
 - bosonic, 265
 - fermionic, 265
 - normal mode problem in, 70
 - for two particles, 265
- 't Hooft, 619
- Hydrogen atom
 - degeneracy of, 359
 - energy levels of, 356
 - 21-cm line, 408
 - wave functions of, 356, 357
- Hyperfine interaction, 407

- Ideal measurement, 122
- Identical particles
 - bosons, 263
 - definition of, 260
 - fermions, 263
- Identity operator, 19
- Impact parameter, 523
- Improper vectors, 67
- Incompatible variables, 128
- Induced emission, 521
- Inelasticity, 554
- Infinite-dimensional spaces, 57
- Inner product, 8
- Inner product space, 7
- Inverse of operator, 20, 655
- Ionic bond, 370
- Irreducible space, 330
- Irreducible tensor operator, 418
- Ising model, 627

- Ket, 3
- Klein-Gordon equation, 564
- Kronecker's delta, 10

- Lagrangian, 76
 - for electromagnetic interactions, 83
- Laguerre polynomial, 356
- Lamb shift, 574
- Landau Level, 587, 588
- Laughlin wave function, 592
- Laughlin quasihole, 607
- Least action principle, 77
- Legendre transform, 87
- Linear independence, 4
- Linear operators, 18
- Lorentz spinor, 566
- Lowering operator
 - angular momentum, 322
 - for harmonic oscillator, 205
 - see also* Destruction operator
- Lowest Landau Level, 588
- Lyman series, 367

- Magnetic moment, 385
- Magnetic quantum number, 314
- Matrix elements, 20
- Matrix inversion, 655
- Mendelev, 370
- Metastable states, 553
- Minimum uncertainty packet, 241
- Multielectron atoms, 369

- Negative absolute temperature, 394
- Norm, 9
- Normal modes, 52
- Number operator, 207
- Numerical estimates, 361

- Operators, 18
 - adjoint of, 25
 - anti-Hermitian, 27
 - conjugate, 69
 - derivatives of, 55
 - functions of, 54

- Hermitian, 27
- identity, 22
- infinite-dimensional, 63
- inverse of, 20
- linear, 18
- matrix elements of, 21
- product of, 20
- projection, 22
- unitary, 28
- Optical theorem, 548, 555
- Orthogonality, 9
- Orthogonal matrix, 28
- Orthonormality, 9
- Outer product, 23
- Paramagnetic resonance, 392
- Parity invariance, 297
- Partial wave
 - amplitude, 545
 - expansion, 545
- Particle in a box, 157, 259
- Paschen series, 367
- Passive transformation, 29, 280
- Path integral
 - coherent state, 607, 610
 - configuration space, 582
 - definition, 223
 - fermionic, 646
 - free particle, 225, 582
 - imaginary time, 614
 - phase space, 586
 - recipe, 223
 - and Schrödinger's equation, 229
 - statistical mechanics, 624
- Pauli equation, 568
- Pauli exclusion principle, 264
- Pauli matrices, 381
- Periodic table, 370
- Perturbations
 - adiabatic, 478
 - periodic, 482
 - sudden, 477
 - time-independent, 451
- Phase shift, 546
- Phase space, 88
- Phonons, 198
- Photoelectric effect, 111, 499
- Photons, 110, 198
 - quantum theory of, 516
- Physical Hilbert space, 67
- Pictures
 - Heisenberg, 147, 490
 - interaction (Dirac), 485
 - Schrödinger, 147, 484
- Planck's constant, 111
- Poisson brackets, 92
 - invariance of, 96
- Polarizability, 464
- P operator, 116
- Population inversion, 395
- Postulates, 115, 211
- Probability amplitude, 111, 121
- Probability current density, 166
- Probability density, 121
- Product basis, 403
- Projection operator, 23
- Propagator
 - for coupled masses, 51
 - Feynman's, 578
 - for free particle, 153
 - for Gaussian packet, 154
 - for harmonic oscillator, 615
 - for (classical) string, 72
- Proper vectors, 67
- Pseudospin, 639
- Quadrupole tensor, 425
- Quanta, 197
- Quantization of energy, 160
- Quantum Hall Effect (QHE), 589
- Radial equation
 - in three dimensions, 339
 - in two dimensions, 316
- Radial part of wave function
 - in three dimensions, 339
 - in two dimensions, 316
- Raising operator
 - for angular momentum, 222
 - for harmonic oscillator, 205
- Range of potential, 525
- Rare earth elements, 371
- Ray, 118
- Recursion relation, 193
- Reduced mass, 86
- Reduced matrix element, 420
- Reflection coefficient, 168
- Resonances, 550
- Rotations
 - generators of (classical), 100
 - generators of (quantum), 308
 - invariance under (classical), 100
 - invariance under (quantum), 310
- Runge-Lenz vector, 360, 422
- Rutherford cross section, 531
- Rydberg, 355
- Scattering
 - general theory, 523
 - of identical particles, 560
 - from step potential, 167
 - of two particles, 555
- Scattering amplitude, 527

- Schrödinger equation
 - equivalence to path integral, 229
 - time-dependent, 116, 143
 - time-independent, 145
- Schrödinger picture, 147, 484
- Schwartz inequality, 16
- Selection rule
 - angular momentum, 458, 459
 - dipole, 459
 - general, 458
- Shell, 370
- Singlet, 405
- S matrix
 - definition of, 529
 - partial wave, 547
- Spectroscopic notation, 350
 - modified, 415
- Spherical Bessel functions, 348
- Spherical Hankel functions, 348
- Spherical harmonics, 335, 336
- Spherical Neumann functions, 348
- Spin, 325, 373
- Spinor, 375
- Spin-orbit interaction, 468
- Spin statistics theorem, 264
- Spontaneous decay, 517
- Spontaneous emission, 521
- Square-well potential, 164
- Stark effect, 459, 465
- Stationary states, 146
- Statistics, 264
 - determination of, 269
- Stern-Gerlach experiment, 399
- Subspaces, 17
- Superposition principle, 117
- Symmetric states, 263
- Symmetries
 - classical, 98
 - quantum, 279
 - spontaneous breakdown of, 620
- Tensor
 - antisymmetric (ϵ_{ijk}), 319
 - cartesian, 417
 - irreducible, 418
 - operator, 417
 - quadrupole, 425
 - second rank, 418
 - spherical, 417
- Thermal wavelength, 625
- Thomas factor, 468, 571
- Thomas-Reiche-Kuhn rule, 457
- Time-ordered integral, 148
- Time-ordering symbol, 633, 651
- Time-reversal symmetry, 301
- Time translation invariance, 294
- Top state, 410
- Total S basis, 405
- Trace, 30
- Transformations, 29
 - active, 29, 97, 280
 - canonical, 92
 - generator of, 99, 283
 - identity, 98
 - passive, 29, 280, 284
 - point, 94
 - regular, 97
 - unitary, 27
- Translated state, 280
- Translation
 - finite, 289
 - generator of, 100, 283
 - operator, 280
- Translational invariance
 - implications of, 98, 292
 - in quantum theory, 279
- Transmission coefficient, 168
- Transverse relaxation time, 395
- Triangle inequality, 116, 412
- Triplets, 405
- Tunneling, 175, 616
- Two-particle Hilbert space, 247
- Uncertainty, 128
- Uncertainty principle
 - applications of, 198
 - derivation of, 237
 - energy-time, 245
 - physical basis of, 140
- Uncertainty relation, 138
- Unitarity bound, 548
- Unitary operator, 27
 - eigenvalues of, 39
 - eigenvectors of, 39
- Variational method, 429
- Vector addition coefficients, 412
- Vectors
 - components of, 6
 - improper, 67
 - inner product of, 8
 - norm of, 9
 - orthogonality of, 9
 - outer product of, 25
 - proper, 67
- Vector operator, 313
- Vector space
 - axioms for, 2
 - basis for, 6
 - dimensionality of, 5
 - field of, 2
 - of Hilbert, 67
 - infinite dimensional, 57
 - subspace of, 17

- Virial theorem, 212
 - for hydrogen, 359, 471
- Wave functions, 121
- Wave-particle duality, 113
- Waves
 - interference of, 108
 - matter, 112
 - plane, 108
- Wick's theorem, 645
- Wigner-Eckart theorem, 420
- WKB approximation
 - and bound states, 445
 - introduction to, 435
 - and path integrals, 438
 - three-dimensional, 449
 - and tunneling, 444
- X operator, 68
 - matrix elements of, 68
- Yukawa potential, 531
- Zeeman effect, 398
- Zero point energy, 198