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Algebra

Revised Third Edition

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FOREWORD

The present book is meant as a basic text for a one-year course in algebra, at the graduate level.

A perspective on algebra

As I see it, the graduate course in algebra must primarily prepare students to handle the algebra which they will meet in all of mathematics: topology, partial differential equations, differential geometry, algebraic geometry, analysis, and representation theory, not to speak of algebra itself and algebraic number theory with all its ramifications. Hence I have inserted throughout references to papers and books which have appeared during the last decades, to indicate some of the directions in which the algebraic foundations provided by this book are used; I have accompanied these references with some motivating comments, to explain how the topics of the present book fit into the mathematics that is to come subsequently in various fields; and I have also mentioned some unsolved problems of mathematics in algebra and number theory. The *abc* conjecture is perhaps the most spectacular of these.

Often when such comments and examples occur out of the logical order, especially with examples from other branches of mathematics, of necessity some terms may not be defined, or may be defined only later in the book. I have tried to help the reader not only by making cross-references within the book, but also by referring to other books or papers which I mention explicitly.

I have also added a number of exercises. On the whole, I have tried to make the exercises complement the examples, and to give them aesthetic appeal. I have tried to use the exercises also to drive readers toward variations and applications of the main text, as well as toward working out special cases, and as openings toward applications beyond this book.

Organization

Unfortunately, a book must be projected in a totally ordered way on the page axis, but that's not the way mathematics "is", so readers have to make choices how to reset certain topics in parallel for themselves, rather than in succession.

I have inserted cross-references to help them do this, but different people will make different choices at different times depending on different circumstances.

The book splits naturally into several parts. The first part introduces the basic notions of algebra. After these basic notions, the book splits in two major directions: the direction of algebraic equations including the Galois theory in Part II; and the direction of linear and multilinear algebra in Parts III and IV. There is some sporadic feedback between them, but their unification takes place at the next level of mathematics, which is suggested, for instance, in §15 of Chapter VI. Indeed, the study of algebraic extensions of the rationals can be carried out from two points of view which are complementary and interrelated: representing the Galois group of the algebraic closure in groups of matrices (the linear approach), and giving an explicit determination of the irrationalities generating algebraic extensions (the equations approach). At the moment, representations in GL_2 are at the center of attention from various quarters, and readers will see GL_2 appear several times throughout the book. For instance, I have found it appropriate to add a section describing all irreducible characters of $GL_2(F)$ when F is a finite field. Ultimately, GL_2 will appear as the simplest but typical case of groups of Lie types, occurring both in a differential context and over finite fields or more general arithmetic rings for arithmetic applications.

After almost a decade since the second edition, I find that the basic topics of algebra have become stable, with one exception. I have added two sections on elimination theory, complementing the existing section on the resultant. Algebraic geometry having progressed in many ways, it is now sometimes returning to older and harder problems, such as searching for the effective construction of polynomials vanishing on certain algebraic sets, and the older elimination procedures of last century serve as an introduction to those problems.

Except for this addition, the main topics of the book are unchanged from the second edition, but I have tried to improve the book in several ways.

First, some topics have been reordered. I was informed by readers and reviewers of the tension existing between having a textbook usable for relatively inexperienced students, and a reference book where results could easily be found in a systematic arrangement. I have tried to reduce this tension by moving all the homological algebra to a fourth part, and by integrating the commutative algebra with the chapter on algebraic sets and elimination theory, thus giving an introduction to different points of view leading toward algebraic geometry.

The book as a text and a reference

In teaching the course, one might wish to push into the study of algebraic equations through Part II, or one may choose to go first into the linear algebra of Parts III and IV. One semester could be devoted to each, for instance. The chapters have been so written as to allow maximal flexibility in this respect, and I have frequently committed the crime of lèse-Bourbaki by repeating short arguments or definitions to make certain sections or chapters logically independent of each other.

Granting the material which under no circumstances can be omitted from a basic course, there exist several options for leading the course in various directions. It is impossible to treat all of them with the same degree of thoroughness. The precise point at which one is willing to stop in any given direction will depend on time, place, and mood. However, any book with the aims of the present one must include a choice of topics, pushing ahead·in deeper waters, while stopping short of full involvement.

There can be no universal agreement on these matters, not even between the author and himself. Thus the concrete decisions as to what to include and what not to include are finally taken on grounds of general coherence and aesthetic balance. Anyone teaching the course will want to impress their own personality on the material, and may push certain topics with more vigor than I have, at the expense of others. Nothing in the present book is meant to inhibit this.

Unfortunately, the goal to present a fairly comprehensive perspective on algebra required a substantial increase in size from the first to the second edition, and a moderate increase in this third edition. These increases require some decisions as to what to omit in a given course.

Many shortcuts can be taken in the presentation of the topics, which admits many variations. For instance, one can proceed into field theory and Galois theory immediately after giving the basic definitions for groups, rings, fields, polynomials in one variable, and vector spaces. Since the Galois theory gives very quickly an impression of depth, this is very satisfactory in many respects.

It is appropriate here to recall my original indebtedness to Artin, who first taught me algebra. The treatment of the basics of Galois theory is much influenced by the presentation in his own monograph.

Audience and background

As I already stated in the forewords of previous editions, the present book is meant for the graduate level, and I expect most of those coming to it to have had suitable exposure to some algebra in an undergraduate course, or to have appropriate mathematical maturity. I expect students taking a graduate course to have had some exposure to vector spaces, linear maps, matrices, and they will no doubt have seen polynomials at the very least in calculus courses.

My books *Undergraduate Algebra* and *Linear Algebra* provide more than enough background for a graduate course. Such elementary texts bring out in parallel the two basic aspects of algebra, and are organized differently from the present book, where both aspects are deepened. Of course, some aspects of the linear algebra in Part III of the present book are more "elementary" than some aspects of Part II, which deals with Galois theory and the theory of polynomial equations in several variables. Because Part II has gone deeper into the study of algebraic equations, of necessity the parallel linear algebra occurs only later in the total ordering of the book. Readers should view both parts as running simultaneously.

Unfortunately, the amount of algebra which one should ideally absorb during this first year in order to have a proper background (irrespective of the subject in which one eventually specializes) exceeds the amount which can be covered physically by a lecturer during a one-year course. Hence more material must be included than can actually be handled in class. I find it essential to bring this material to the attention of graduate students.

I hope that the various additions and changes make the book easier to use as a text. By these additions, I have tried to expand the general mathematical perspective of the reader, insofar as algebra relates to other parts of mathematics.

Acknowledgements

I am indebted to many people who have contributed comments and criticisms for the previous editions, but especially to Daniel Bump, Steven Krantz, and Diane Meuser, who provided extensive comments as editorial reviewers for Addison-Wesley. I found their comments very stimulating and valuable in preparing this third edition. I am much indebted to Barbara Holland for obtaining these reviews when she was editor. I am also indebted to Karl Matsumoto who supervised production under very trying circumstances. I thank the many people who have made suggestions and corrections, especially George Bergman and students in his class, Chee-Whye Chin, Ki-Bong Nam, David Wasserman, Randy Scott, Thomas Shiple, Paul Vojta, Bjorn Poonen and his class, in particular Michael Manapat.

For the 2002 and beyond Springer printings

From now on, *Algebra* appears with Springer-Verlag, like the rest of my books. With this change, I considered the possibility of a new edition, but decided against it. I view the book as very stable. The only addition which I would make, if starting from scratch, would be some of the algebraic properties of SL_n and GL_n (over $\mathbf{R}$ or $\mathbf{C}$), beyond the proof of simplicity in Chapter XIII. As things stood, I just inserted some exercises concerning some aspects which everybody should know. The material actually is now inserted in a new edition of *Undergraduate Algebra*, where it properly belongs. The algebra appears as a supporting tool for doing analysis on Lie groups, cf. for instance Jorgenson/Lang *Spherical Inversion on $SL_n(\mathbf{R})$*, Springer Verlag 2001.

I thank specifically Tom von Foerster, Ina Lindemann and Mark Spencer for their editorial support at Springer, as well as Terry Kornak and Brian Howe who have taken care of production.

Serge Lang
New Haven 2004

Logical Prerequisites

We assume that the reader is familiar with sets, and with the symbols $\cap$, $\cup$, $\supset$, $\subset$, $\in$. If A, B are sets, we use the symbol $A \subset B$ to mean that A is contained in B but may be equal to B. Similarly for $A \supset B$.

If $f: A \to B$ is a mapping of one set into another, we write

$$x \mapsto f(x)$$

to denote the effect of f on an element x of A. We distinguish between the arrows $\to$ and $\mapsto$. We denote by $f(A)$ the set of all elements $f(x)$, with $x \in A$.

Let $f: A \to B$ be a mapping (also called a map). We say that f is **injective** if $x \neq y$ implies $f(x) \neq f(y)$. We say f is **surjective** if given $b \in B$ there exists $a \in A$ such that $f(a) = b$. We say that f is **bijective** if it is both surjective and injective.

A subset A of a set B is said to be **proper** if $A \neq B$.

Let $f: A \to B$ be a map, and A' a subset of A. The restriction of f to A' is a map of A' into B denoted by $f \mid A'$.

If $f: A \to B$ and $g: B \to C$ are maps, then we have a composite map $g \circ f$ such that $(g \circ f)(x) = g(f(x))$ for all $x \in A$.

Let $f: A \to B$ be a map, and B' a subset of B. By $f^{-1}(B')$ we mean the subset of A consisting of all $x \in A$ such that $f(x) \in B'$. We call it the **inverse image** of B'. We call $f(A)$ the **image** of f.

A **diagram**

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "h"'] & & B \arrow[dl, "g"] \\ & C & \end{tikzcd}$$

is said to be **commutative** if $g \circ f = h$. Similarly, a **diagram**

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\varphi"'] & B \arrow[d, "g"] \\ C \arrow[r, "\psi"'] & D \end{tikzcd}$$

is said to be **commutative** if $g \circ f = \psi \circ \varphi$. We deal sometimes with more complicated diagrams, consisting of arrows between various objects. Such diagrams are called commutative if, whenever it is possible to go from one object to another by means of two sequences of arrows, say

$$A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} A_n$$

and

$$A_1 \xrightarrow{g_1} B_2 \xrightarrow{g_2} \cdots \xrightarrow{g_{m-1}} B_m = A_n,$$

then

$$f_{n-1} \circ \cdots \circ f_1 = g_{m-1} \circ \cdots \circ g_1,$$

in other words, the composite maps are equal. Most of our diagrams are composed of triangles or squares as above, and to verify that a diagram consisting of triangles or squares is commutative, it suffices to verify that each triangle and square in it is commutative.

We assume that the reader is acquainted with the integers and rational numbers, denoted respectively by **Z** and **Q**. For many of our examples, we also assume that the reader knows the real and complex numbers, denoted by **R** and **C**.

Let A and I be two sets. By a family of elements of A, indexed by I, one means a map $f: I \to A$. Thus for each $i \in I$ we are given an element $f(i) \in A$. Although a family does not differ from a map, we think of it as determining a collection of objects from A, and write it often as

$$\{f(i)\}_{i \in I}$$

or

$$\{a_i\}_{i \in I},$$

writing a_i instead of $f(i)$. We call I the indexing set.

We assume that the reader knows what an equivalence relation is. Let A be a set with an equivalence relation, let E be an equivalence class of elements of A. We sometimes try to define a map of the equivalence classes into some set B. To define such a map f on the class E, we sometimes first give its value on an element $x \in E$ (called a representative of E), and then show that it is independent of the choice of representative $x \in E$. In that case we say that f is **well defined**.

We have products of sets, say finite products $A \times B$, or $A_1 \times \cdots \times A_n$, and products of families of sets.

We shall use Zorn's lemma, which we describe in Appendix 2.

We let $\#(S)$ denote the number of elements of a set S, also called the **cardinality** of S. The notation is usually employed when S is finite. We also write $\#(S) = \text{card}(S)$.

CONTENTS

Part One The Basic Objects of Algebra