PART III
THE CLASSICAL LIE ALGEBRAS AND THEIR REPRESENTATIONS

As we indicated at the outset, the analysis we have just carried out of the structure of $\mathfrak{sl}_2\mathbb{C}$ and $\mathfrak{sl}_3\mathbb{C}$ and their representations carries over to other semisimple complex Lie algebras. In Lecture 14 we codify this structure, using the pattern of the examples we have worked out so far to give a model for the analysis of arbitrary semisimple Lie algebras and stating some of the most important facts that are true in general. As usual, we postpone proofs of many of these facts until Part IV and the Appendices, the main point here being to introduce a unifying approach and language. The facts themselves will all be seen explicitly on a case-by-case basis for the classical Lie algebras $\mathfrak{sl}_n\mathbb{C}$, $\mathfrak{sp}_{2n}\mathbb{C}$, and $\mathfrak{so}_n\mathbb{C}$, which are studied in some detail in Lectures 15–20.

Most of the development follows the outline we developed in Lectures 11–13, the main goal being to describe the irreducible representations as explicitly as we can, and to see the decomposition of naturally occurring representations, both algebraically and geometrically. While most of the representations are found inside tensor powers of the standard representations, for the orthogonal Lie algebras this only gives half of them, and one needs new methods to construct the other "spin" representations. This is carried out using Clifford algebras in Lecture 20.

We also make the tie with Weyl's construction of representations of $GL_n\mathbb{C}$ from Lecture 6, which arose from the representation theory of the symmetric groups. We show in Lecture 15 that these are the irreducible representations of $\mathfrak{sl}_n\mathbb{C}$; in Lecture 17 we show how to use them to construct the irreducible representations of the symplectic Lie algebras, and in Lecture 19 to give the nonspin representation of the orthogonal Lie algebras. These give useful descriptions of the irreducible representations, and powerful methods for decomposing other representations, but they are not necessary for the logical progression of the book, and many of these decompositions can also be deduced from the Weyl character formula which we will discuss in Part IV.