

# Structural Analysis

# SOLID MECHANICS AND ITS APPLICATIONS

Volume 163

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## *Aims and Scope of the Series*

The fundamental questions arising in mechanics are: *Why?*, *How?*, and *How much?* The aim of this series is to provide lucid accounts written by authoritative researchers giving vision and insight in answering these questions on the subject of mechanics as it relates to solids.

The scope of the series covers the entire spectrum of solid mechanics. Thus it includes the foundation of mechanics; variational formulations; computational mechanics; statics, kinematics and dynamics of rigid and elastic bodies; vibrations of solids and structures; dynamical systems and chaos; the theories of elasticity, plasticity and viscoelasticity; composite materials; rods, beams, shells and membranes; structural control and stability; soils, rocks and geomechanics; fracture; tribology; experimental mechanics; biomechanics and machine design.

The median level of presentation is the first year graduate student. Some texts are monographs defining the current state of the field; others are accessible to final year undergraduates; but essentially the emphasis is on readability and clarity.

O.A. Bauchau • J.I. Craig

# Structural Analysis

With Applications to Aerospace Structures

 Springer

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To our wives, Yi-Ling and Nancy, and our families

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## Preface

Engineered structures are almost as old as human civilization and undoubtedly began with rudimentary tools and the first dwellings outside caves. Great progress has been made over thousands of years, and our world is now filled with engineered structures from nano-scale machines to soaring buildings. Aerospace structures ranging from fragile human-powered aircraft to sleek jets and thundering rockets are, in our opinion, among the most challenging and creative examples of these efforts.

The study of mechanics and structural analysis has been an important area of engineering over the past 300 years, and some of the greatest minds have contributed to its development. Newton formulated the most basic principles of equilibrium in the 17<sup>th</sup> century, but fundamental contributions have continued well into the 20<sup>th</sup> century. Today, structural analysis is generally considered to be a mature field with well-established principles and practical tools for analysis and design. A key reason for this is, without doubt, the emergence of the finite element method and its widespread application in all areas of structural engineering. As a result, much of today's emphasis in the field is no longer on structural analysis, but instead is on the use of new materials and design synthesis.

The field of aerospace structural analysis began with the first attempts to build flying machines, but even today, it is a much smaller and narrower field treated in far fewer textbooks as compared to the fields of structural analysis in civil and mechanical engineering. Engineering students have access to several excellent texts such as those by Donaldson [1] and Megson [2], but many other notable textbooks are now out of print.

This textbook has emerged over the past two decades from our efforts to teach core courses in advanced structural analysis to undergraduate and graduate students in aerospace engineering. By the time students enroll in the undergraduate course, they have studied statics and covered introductory mechanics of deformable bodies dealing primarily with beam bending. These introductory courses are taught using texts devoted largely to applications in civil and mechanical engineering, leaving our students with little appreciation for some of the unique and challenging features of aerospace structures, which often involve thin-walled structures made of fiber-reinforced composite materials. In addition, while in widespread use in industry and

the subject of numerous specialized textbooks, the finite element method is only slowly finding its way into general structural analysis texts as older applied methods and special analysis techniques are phased out.

The book is divided into four parts. The first part deals with basic tools and concepts that provide the foundation for the other three parts. It begins with an introduction to the equations of linear elasticity, which underlie all of structural analysis. A second chapter presents the constitutive laws for homogeneous, isotropic and linearly elastic material but also includes an introduction to anisotropic materials and particularly to transversely isotropic materials that are typical of layered composites. The first part concludes with chapter 4, which defines isostatic and hyperstatic problems and introduces the fundamental solution procedures of structural analysis: the displacement method and the force method.

Part 2 develops Euler-Bernoulli beam theory with emphasis on the treatment of beams presenting general cross-sectional configurations. Torsion of circular cross-sections is discussed next, along with Saint-Venant torsion theory for bars of arbitrary shape. A lengthy chapter is devoted to thin-walled beams typical of those used in aerospace structures. Coupled bending-twisting and nonuniform torsion problems are also addressed.

Part 3 introduces the two fundamental principles of virtual work that are the basis for the powerful and versatile energy methods. They provide tools to treat more realistic and complex problems in an efficient manner. A key topic in Part 3 is the development of methods to obtain approximate solution for complex problems. First, the Rayleigh-Ritz method is introduced in a cursory manner; next, applications of the weak statement of equilibrium and of energy principles are presented in a more formal manner; finally, the finite element method applied to trusses and beams is presented. Part 3 concludes with a formal introduction of variational methods and general statements of the energy principles introduced earlier in more applied contexts.

Part 4 covers a selection of advanced topics of particular relevance to aerospace structural analysis. These include introductions to plasticity and thermal stresses, buckling of beams, shear deformations in beams and Kirchhoff plate theory.

In our experience, engineering students generally grasp concepts more quickly when presented first with practical examples, which then lead to broader generalizations. Consequently, most concepts are first introduced by means of simple examples; more formal and abstract statements are presented later, when the student has a better grasp of the significance of the concepts. Furthermore, each chapter provides numerous examples to demonstrate the application of the theory to practical problems. Some of the examples are re-examined in successive chapters to illustrate alternative or more versatile solution methods. Step-by-step descriptions of important solution procedures are provided.

As often as possible, the analysis of structural problems is approached in a unified manner. First, kinematic assumptions are presented that describe the structure's displacement field in an approximate manner; next, the strain field is evaluated based on the strain-displacement relationships; finally, the constitutive laws lead to the stress field for which equilibrium equations are then established. In our experience, this ap-

proach reduces the confusion that students often face when presented with developments that don't seem to follow any obvious direction or strategy but yet, inevitably lead to the expected solution.

The topics covered in parts 1 and 2 along with chapters 9 and 10 from part 3 form the basis for a four semester-hour course in advanced aerospace structural analysis taught to junior and senior undergraduate students. An introductory graduate level course covers part 2 and selected chapters in parts 3 and 4, but only after a brief review of the material in part 1. A second graduate level course focusing on variational end energy methods covers part 3 and selected chapters in part 4. A number of homework problems are included throughout these chapters. Some are straightforward applications of simple concepts, others are small projects that require the use of computers and mathematical software, and others involve conceptual questions that are more appropriate for quizzes and exams.

A thorough study of differential calculus including a basic treatment of ordinary and partial differential equations is a prerequisite. Additional topics from linear algebra and differential geometry are needed, and these are reviewed in an appendix.

Notation is a challenging issue in structural analysis. Given the limitations of the Latin and Greek alphabets, the same symbols are sometimes used for different purposes, but mostly in different contexts. Consequently, no attempt has been made to provide a comprehensive list of symbols, which would lead to even more confusion. Also, in mechanics and structural analysis, sign conventions present a major hurdle for all students. To ease this problem, easy to remember sign conventions are used systematically. Stresses and force resultants are positive on positive faces when acting along positive coordinate directions. Moments and torques are positive on positive faces when acting about positive coordinate directions using the right-hand rule.

In a few instances, new or less familiar terms have been chosen because of their importance in aerospace structural analysis. For instance, the terms “isostatic” and “hyperstatic” structures are used to describe statically determinate and indeterminate structures, respectively, because these terms concisely define concepts that often puzzle and confuse students. Beam bending stiffnesses are indicated with the symbol “ $H$ ” rather than the more common “ $EI$ .” When dealing exclusively with homogeneous material, notation “ $EI$ ” is easy to understand, but in presence of heterogeneous composite materials, encapsulating the spatially varying elasticity modulus in the definition of the bending stiffness is a more rational approach.

It is traditional to use a bold typeface to represent vectors, arrays, and matrices, but this is very difficult to reproduce in handwriting, whether in a lecture or in personal notes. Instead, we have adopted a notation that is more suitable for handwritten notes. Vectors and arrays are denoted using an underline, such as  $\underline{u}$  or  $\underline{F}$ . Unit vectors are used frequently and are assigned a special notation using a single overbar, such as  $\bar{v}_1$ , which denotes the first Cartesian coordinate axis. We also use the overbar to denote non-dimensional scalar quantities, *i.e.*,  $\bar{k}$  is a non-dimensional stiffness coefficient. This is inconsistent, but the two uses are in such different contexts that it should not lead to confusion. Matrices are indicated using a double-underline, *i.e.*,  $\underline{\underline{C}}$  indicates a matrix of  $M$  rows and  $N$  columns.

Finally, we are indebted to the many students at Georgia Tech who have given us helpful and constructive feedback over the past decade as we developed the course notes that are the predecessor of this book. We have tried to constructively utilize their initial confusion and probing questions to clarify and refine the treatment of important but confusing topics. We are also grateful for the many discussions and valuable feedback from our colleagues, Profs. Erian Armanios, Sathya Hanagud, Dewey Hodges, George Kardomateas, Massimo Ruzzene, and Virgil Smith, several of whom have used our notes for teaching advanced aerospace structural analysis here at Georgia Tech.

Atlanta, Georgia,  
July 2009

*Olivier Bauchau*  
*James Craig*

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# Contents

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## Part I Basic tools and concepts

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<b>1</b>	<b>Basic equations of linear elasticity</b> . . . . .	<b>3</b>
1.1	The concept of stress . . . . .	3
1.1.1	The state of stress at a point . . . . .	3
1.1.2	Volume equilibrium equations . . . . .	7
1.1.3	Surface equilibrium equations . . . . .	10
1.2	Analysis of the state of stress at a point . . . . .	11
1.2.1	Stress components acting on an arbitrary face . . . . .	11
1.2.2	Principal stresses . . . . .	13
1.2.3	Rotation of stresses . . . . .	14
1.2.4	Problems . . . . .	19
1.3	The state of plane stress . . . . .	20
1.3.1	Equilibrium equations . . . . .	20
1.3.2	Stresses acting on an arbitrary face within the sheet . . . . .	21
1.3.3	Principal stresses . . . . .	22
1.3.4	Rotation of stresses . . . . .	24
1.3.5	Special states of stress . . . . .	26
1.3.6	Mohr's circle for plane stress . . . . .	27
1.3.7	Lamé's ellipse . . . . .	30
1.3.8	Problems . . . . .	31
1.4	The concept of strain . . . . .	33
1.4.1	The state of strain at a point . . . . .	34
1.4.2	The volumetric strain . . . . .	37
1.5	Analysis of the state of strain at a point . . . . .	38
1.5.1	Rotation of strains . . . . .	38
1.5.2	Principal strains . . . . .	40
1.6	The state of plane strain . . . . .	41
1.6.1	Strain-displacement relations for plane strain . . . . .	41
1.6.2	Rotation of strains . . . . .	42

1.6.3	Principal strains . . . . .	43
1.6.4	Mohr's circle for plane strain . . . . .	44
1.7	Measurement of strains . . . . .	45
1.7.1	Problems . . . . .	49
1.8	Strain compatibility equations . . . . .	50
<b>2</b>	<b>Constitutive behavior of materials . . . . .</b>	<b>53</b>
2.1	Constitutive laws for isotropic materials . . . . .	55
2.1.1	Homogeneous, isotropic, linearly elastic materials . . . . .	55
2.1.2	Thermal effects . . . . .	59
2.1.3	Problems . . . . .	61
2.1.4	Ductile materials . . . . .	63
2.1.5	Brittle materials . . . . .	65
2.2	Allowable stress . . . . .	66
2.3	Yielding under combined loading . . . . .	68
2.3.1	Tresca's criterion . . . . .	68
2.3.2	Von Mises' criterion . . . . .	70
2.3.3	Comparing Tresca's and von Mises' criteria . . . . .	71
2.3.4	Problems . . . . .	73
2.4	Material selection for structural performance . . . . .	73
2.4.1	Strength design . . . . .	74
2.4.2	Stiffness design . . . . .	74
2.4.3	Buckling design . . . . .	75
2.5	Composite materials . . . . .	76
2.5.1	Basic characteristics . . . . .	76
2.5.2	Stress diffusion in composites . . . . .	78
2.6	Constitutive laws for anisotropic materials . . . . .	82
2.6.1	Constitutive laws for a lamina in the fiber aligned triad . . . . .	85
2.6.2	Constitutive laws for a lamina in an arbitrary triad . . . . .	87
2.7	Strength of a transversely isotropic lamina . . . . .	94
2.7.1	Strength of a lamina under simple loading conditions . . . . .	94
2.7.2	Strength of a lamina under combined loading conditions . . . . .	95
2.7.3	The Tsai-Wu failure criterion . . . . .	96
2.7.4	The reserve factor . . . . .	98
<b>3</b>	<b>Linear elasticity solutions . . . . .</b>	<b>101</b>
3.1	Solution procedures . . . . .	102
3.1.1	Displacement formulation . . . . .	103
3.1.2	Stress formulation . . . . .	103
3.1.3	Solutions to elasticity problems . . . . .	104
3.2	Plane strain problems . . . . .	110
3.3	Plane stress problems . . . . .	111
3.4	Plane strain and plane stress in polar coordinates . . . . .	113
3.5	Problem featuring cylindrical symmetry . . . . .	116
3.5.1	Problems . . . . .	133

<b>4</b>	<b>Engineering structural analysis</b> .....	137
4.1	Solution approaches .....	137
4.2	Bar under constant axial force .....	138
4.3	Hyperstatic systems .....	144
4.3.1	Solution procedures .....	145
4.3.2	The displacement or stiffness method .....	146
4.3.3	The force or flexibility method .....	151
4.3.4	Problems .....	156
4.3.5	Thermal effects in hyperstatic system .....	157
4.3.6	Manufacturing imperfection effects in hyperstatic system ...	161
4.3.7	Problems .....	164
4.4	Pressure vessels .....	165
4.4.1	Rings under internal pressure .....	165
4.4.2	Cylindrical pressure vessels .....	166
4.4.3	Spherical pressure vessels .....	167
4.4.4	Problems .....	168
4.5	Saint-Venant's principle .....	169

---

**Part II Beams and thin-wall structures**

---

<b>5</b>	<b>Euler-Bernoulli beam theory</b> .....	173
5.1	The Euler-Bernoulli assumptions .....	174
5.2	Implications of the Euler-Bernoulli assumptions .....	175
5.3	Stress resultants .....	177
5.4	Beams subjected to axial loads .....	178
5.4.1	Kinematic description .....	179
5.4.2	Sectional constitutive law .....	179
5.4.3	Equilibrium equations .....	180
5.4.4	Governing equations .....	181
5.4.5	The sectional axial stiffness .....	182
5.4.6	The axial stress distribution .....	182
5.4.7	Problems .....	185
5.5	Beams subjected to transverse loads .....	186
5.5.1	Kinematic description .....	186
5.5.2	Sectional constitutive law .....	187
5.5.3	Equilibrium equations .....	188
5.5.4	Governing equations .....	189
5.5.5	The sectional bending stiffness .....	191
5.5.6	The axial stress distribution .....	193
5.5.7	Rational design of beams under bending .....	194
5.5.8	Problems .....	211
5.6	Beams subjected to combined axial and transverse loads .....	217
5.6.1	Kinematic description .....	217
5.6.2	Sectional constitutive law .....	217

5.6.3	Equilibrium equations	218
5.6.4	Governing equations	219
<b>6</b>	<b>Three-dimensional beam theory</b>	<b>223</b>
6.1	Kinematic description	224
6.2	Sectional constitutive law	225
6.3	Sectional equilibrium equations	227
6.4	Governing equations	229
6.5	Decoupling the three-dimensional problem	230
6.5.1	Definition of the principal axes of bending	231
6.5.2	Decoupled governing equations	232
6.6	The principal centroidal axes of bending	233
6.6.1	The bending stiffness ellipse	235
6.7	The neutral axis	236
6.8	Evaluation of sectional stiffnesses	240
6.8.1	The parallel axis theorem	240
6.8.2	Thin-walled sections	242
6.8.3	Triangular area equivalence method	243
6.8.4	Useful results	244
6.8.5	Problems	247
6.9	Summary of three-dimensional beam theory	248
6.9.1	Discussion of the results	255
6.10	Problems	255
<b>7</b>	<b>Torsion</b>	<b>261</b>
7.1	Torsion of circular cylinders	261
7.1.1	Kinematic description	262
7.1.2	The stress field	264
7.1.3	Sectional constitutive law	265
7.1.4	Equilibrium equations	266
7.1.5	Governing equations	266
7.1.6	The torsional stiffness	267
7.1.7	Measuring the torsional stiffness	267
7.1.8	The shear stress distribution	268
7.1.9	Rational design of cylinders under torsion	270
7.1.10	Problems	271
7.2	Torsion combined with axial force and bending moments	271
7.2.1	Problems	274
7.3	Torsion of bars with arbitrary cross-sections	275
7.3.1	Introduction	275
7.3.2	Saint-Venant's solution	276
7.3.3	Saint-Venant's solution for a rectangular cross-section	284
7.3.4	Problems	289
7.4	Torsion of a thin rectangular cross-section	290
7.5	Torsion of thin-walled open sections	292

7.5.1	Problems .....	294
<b>8</b>	<b>Thin-walled beams .....</b>	<b>297</b>
8.1	Basic equations for thin-walled beams.....	297
8.1.1	The thin wall assumption .....	297
8.1.2	Stress flows .....	298
8.1.3	Stress resultants .....	299
8.1.4	Sign conventions .....	301
8.1.5	Local equilibrium equation .....	302
8.2	Bending of thin-walled beams .....	303
8.2.1	Problems .....	304
8.3	Shearing of thin-walled beams .....	307
8.3.1	Shearing of open sections .....	308
8.3.2	Evaluation of stiffness static moments .....	308
8.3.3	Shear flow distributions in open sections .....	309
8.3.4	Problems .....	317
8.3.5	Shear center for open sections .....	318
8.3.6	Problems .....	324
8.3.7	Shearing of closed sections .....	325
8.3.8	Shearing of multi-cellular sections .....	329
8.3.9	Problems .....	332
8.4	The shear center .....	334
8.4.1	Calculation of the shear center location .....	334
8.4.2	Problems .....	341
8.5	Torsion of thin-walled beams .....	343
8.5.1	Torsion of open sections .....	343
8.5.2	Torsion of closed section .....	343
8.5.3	Comparison of open and closed sections .....	345
8.5.4	Torsion of combined open and closed sections .....	346
8.5.5	Torsion of multi-cellular sections .....	347
8.5.6	Problems .....	351
8.6	Coupled bending-torsion problems .....	354
8.6.1	Problems .....	361
8.7	Warping of thin-walled beams under torsion .....	362
8.7.1	Kinematic description .....	362
8.7.2	Stress-strain relations .....	363
8.7.3	Warping of open sections .....	364
8.7.4	Problems .....	369
8.7.5	Warping of closed sections .....	369
8.7.6	Warping of multi-cellular sections .....	371
8.8	Equivalence of the shear and twist centers .....	371
8.9	Non-uniform torsion .....	372
8.9.1	Problems .....	376
8.10	Structural idealization .....	377
8.10.1	Sheet-stringer approximation of a thin-walled section .....	378

8.10.2	Axial stress in the stringers . . . . .	381
8.10.3	Shear flow in the sheet components . . . . .	381
8.10.4	Torsion of sheet-stringer sections . . . . .	384
8.10.5	Problems . . . . .	390

---

**Part III Energy and variational methods**

---

<b>9</b>	<b>Virtual work principles . . . . .</b>	<b>395</b>
9.1	Introduction . . . . .	395
9.2	Equilibrium and work fundamentals . . . . .	396
9.2.1	Static equilibrium conditions . . . . .	396
9.2.2	Concept of mechanical work . . . . .	399
9.3	Principle of virtual work . . . . .	400
9.3.1	Principle of virtual work for a single particle . . . . .	400
9.3.2	Kinematically admissible virtual displacements . . . . .	405
9.3.3	Use of infinitesimal displacements as virtual displacements . . . . .	410
9.3.4	Principle of virtual work for a system of particles . . . . .	412
9.4	Principle of virtual work applied to mechanical systems . . . . .	415
9.4.1	Generalized coordinates and forces . . . . .	420
9.4.2	Problems . . . . .	425
9.5	Principle of virtual work applied to truss structures . . . . .	428
9.5.1	Truss structures . . . . .	428
9.5.2	Solution using Newton's law . . . . .	431
9.5.3	Solution using kinematically admissible virtual displacements . . . . .	432
9.5.4	Solution using arbitrary virtual displacements . . . . .	433
9.6	Principle of complementary virtual work . . . . .	437
9.6.1	Compatibility equations for a planar truss . . . . .	438
9.6.2	Principle of complementary virtual work for trusses . . . . .	441
9.6.3	Complementary virtual work . . . . .	445
9.6.4	Applications to trusses . . . . .	446
9.6.5	Problems . . . . .	449
9.6.6	Unit load method for trusses . . . . .	449
9.6.7	Problems . . . . .	455
9.7	Internal virtual work in beams and solids . . . . .	456
9.7.1	Beam bending . . . . .	457
9.7.2	Beam twisting . . . . .	458
9.7.3	Three-dimensional solid . . . . .	459
9.7.4	Euler-Bernoulli beam . . . . .	461
9.7.5	Problems . . . . .	462
9.7.6	Unit load method for beams . . . . .	462
9.7.7	Problems . . . . .	470
9.8	Application of the unit load method to hyperstatic problems . . . . .	472
9.8.1	Force method for trusses . . . . .	473
9.8.2	Force method for beams . . . . .	480

9.8.3	Combined truss and beam problems . . . . .	485
9.8.4	Multiple redundancies . . . . .	487
9.8.5	Problems . . . . .	489
<b>10</b>	<b>Energy methods . . . . .</b>	<b>493</b>
10.1	Conservative forces . . . . .	494
10.1.1	Potential for internal and external forces . . . . .	497
10.1.2	Calculation of the potential functions . . . . .	498
10.2	Principle of minimum total potential energy . . . . .	500
10.2.1	Non-conservative external forces . . . . .	503
10.3	Strain energy in springs . . . . .	504
10.3.1	Rectilinear springs . . . . .	504
10.3.2	Torsional springs . . . . .	508
10.3.3	Bars . . . . .	509
10.3.4	Problems . . . . .	513
10.4	Strain energy in beams . . . . .	514
10.4.1	Beam under axial loads . . . . .	514
10.4.2	Beam under transverse loads . . . . .	515
10.4.3	Beam under torsional loads . . . . .	516
10.4.4	Relationship with virtual work . . . . .	517
10.5	Strain energy in solids . . . . .	519
10.5.1	Three-dimensional solid . . . . .	519
10.5.2	Three-dimensional beams . . . . .	520
10.6	Applications to trusses and beams . . . . .	521
10.6.1	Applications to trusses . . . . .	521
10.6.2	Problems . . . . .	525
10.6.3	Applications to beams . . . . .	527
10.7	Development of a finite element formulation for trusses . . . . .	529
10.7.1	General description of the problem . . . . .	530
10.7.2	Kinematics of an element . . . . .	531
10.7.3	Element elongation and force . . . . .	533
10.7.4	Element strain energy and stiffness matrix . . . . .	533
10.7.5	Element external potential and load array . . . . .	535
10.7.6	Assembly procedure . . . . .	535
10.7.7	Alternative description of the assembly procedure . . . . .	538
10.7.8	Derivation of the governing equations . . . . .	539
10.7.9	Solution procedure . . . . .	540
10.7.10	Solution procedure using partitioning . . . . .	542
10.7.11	Post-processing . . . . .	544
10.7.12	Problems . . . . .	547
10.8	Principle of minimum complementary energy . . . . .	548
10.8.1	The potential of the prescribed displacements . . . . .	548
10.8.2	Constitutive laws for elastic materials . . . . .	550
10.8.3	The principle of minimum complementary energy . . . . .	551
10.8.4	The principle of least work . . . . .	553

10.8.5	Problems .....	559
10.9	Energy theorems .....	560
10.9.1	Clapeyron's theorem .....	561
10.9.2	Castigliano's first theorem .....	563
10.9.3	Crotti-Engesser theorem .....	564
10.9.4	Castigliano's second theorem .....	565
10.9.5	Applications of energy theorems .....	565
10.9.6	The dummy load method .....	570
10.9.7	Unit load method revisited .....	572
10.9.8	Problems .....	576
10.10	Reciprocity theorems .....	577
10.10.1	Betti's theorem .....	577
10.10.2	Maxwell's theorem .....	579
10.10.3	Problems .....	581
<b>11</b>	<b>Variational and approximate solutions .....</b>	<b>583</b>
11.1	Approach .....	583
11.2	Rayleigh-Ritz method for beam bending .....	585
11.2.1	Statement of the problem .....	585
11.2.2	Description of the Rayleigh-Ritz method .....	586
11.2.3	Discussion of the Rayleigh-Ritz method .....	587
11.2.4	Problems .....	602
11.3	The strong and weak statements of equilibrium .....	603
11.3.1	The weak form for beams under axial loads .....	604
11.3.2	Approximate solutions for beams under axial loads .....	608
11.3.3	Problems .....	616
11.3.4	The weak form for beams under transverse loads .....	617
11.3.5	Approximate solutions for beams under transverse loads .....	621
11.3.6	Problems .....	624
11.3.7	Equivalence with energy principles .....	625
11.3.8	The principle of minimum total potential energy .....	627
11.3.9	Treatment of the boundary conditions .....	629
11.3.10	Summary .....	638
11.4	Formal procedures for the derivation of approximate solutions .....	638
11.4.1	Basic approximations .....	639
11.4.2	Principle of virtual work .....	640
11.4.3	The principle of minimum total potential energy .....	643
11.4.4	Problems .....	650
11.5	A finite element formulation for beams .....	654
11.5.1	General description of the problem .....	655
11.5.2	Kinematics of an element .....	657
11.5.3	Element displacement field .....	658
11.5.4	Element curvature field .....	660
11.5.5	Element strain energy and stiffness matrix .....	660
11.5.6	Element external potential and load array .....	661

11.5.7	Assembly procedure . . . . .	662
11.5.8	Alternative description of the assembly procedure . . . . .	664
11.5.9	Derivation of the governing equations . . . . .	666
11.5.10	Solution procedure . . . . .	666
11.5.11	Summary . . . . .	670
11.5.12	Problems . . . . .	671
<b>12</b>	<b>Variational and energy principles . . . . .</b>	<b>673</b>
12.1	Mathematical preliminaries . . . . .	673
12.1.1	Stationary point of a function . . . . .	674
12.1.2	Lagrange multiplier method . . . . .	675
12.1.3	Stationary point of a definite integral . . . . .	677
12.2	Variational and energy principles . . . . .	679
12.2.1	Review of the equations of linear elasticity . . . . .	680
12.2.2	The principle of virtual work . . . . .	682
12.2.3	The principle of complementary virtual work . . . . .	683
12.2.4	Strain and complementary strain energy density functions . . . . .	685
12.2.5	The principle of minimum total potential energy . . . . .	686
12.2.6	The principle of minimum complementary energy . . . . .	688
12.2.7	Energy theorems . . . . .	690
12.2.8	Hu-Washizu's principle . . . . .	690
12.2.9	Hellinger-Reissner's principle . . . . .	694
12.3	Applications of variational and energy principles . . . . .	695
12.3.1	The shear lag problem . . . . .	697
12.3.2	The Saint-Venant torsion problem . . . . .	701
12.3.3	The Saint-Venant torsion problem using the Prandtl stress function . . . . .	703
12.3.4	The non-uniform torsion problem . . . . .	707
12.3.5	The non-uniform torsion problem (closed sections) . . . . .	709
12.3.6	The non-uniform torsion problem (open sections) . . . . .	712
12.3.7	Problems . . . . .	713

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**Part IV Advanced topics**

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<b>13</b>	<b>Introduction to plasticity and thermal stresses . . . . .</b>	<b>721</b>
13.1	Yielding under combined loading . . . . .	721
13.1.1	Introduction to yield criteria . . . . .	722
13.1.2	Tresca's criterion . . . . .	724
13.1.3	Von Mises' criterion . . . . .	724
13.1.4	Problems . . . . .	725
13.2	Applications of yield criteria to structural problems . . . . .	725
13.2.1	Problems . . . . .	731
13.2.2	Plastic bending . . . . .	732
13.2.3	Problems . . . . .	737

13.2.4	Plastic torsion . . . . .	737
13.3	Thermal stresses in structures . . . . .	741
13.3.1	The direct method . . . . .	742
13.3.2	Problems . . . . .	746
13.3.3	The constraint method . . . . .	746
13.4	Application to bars, trusses and beams . . . . .	748
13.4.1	Applications to bars and trusses . . . . .	748
13.4.2	Problems . . . . .	753
13.4.3	Application to beams . . . . .	753
13.4.4	Problems . . . . .	760
<b>14</b>	<b>Buckling of beams . . . . .</b>	<b>763</b>
14.1	Rigid bar with root torsional spring . . . . .	763
14.1.1	Analysis of a perfect system . . . . .	763
14.1.2	Analysis of an imperfect system . . . . .	765
14.2	Buckling of beams . . . . .	767
14.2.1	Equilibrium equations . . . . .	767
14.2.2	Buckling of a simply-supported beam (equilibrium approach) . . . . .	769
14.2.3	Buckling of a simply-supported beam (imperfection approach) . . . . .	771
14.2.4	Work done by the axial force . . . . .	774
14.2.5	Buckling of a simply-supported beam (energy approach) . . . . .	776
14.2.6	Applications to beam buckling . . . . .	780
14.2.7	Buckling of beams with various end conditions . . . . .	784
14.2.8	Problems . . . . .	784
14.3	Buckling of sandwich beams . . . . .	789
<b>15</b>	<b>Shearing deformations in beams . . . . .</b>	<b>793</b>
15.1	Introduction . . . . .	793
15.1.1	A simplified approach . . . . .	794
15.1.2	An equilibrium approach . . . . .	795
15.1.3	Problems . . . . .	801
15.2	Shear deformable beams: an energy approach . . . . .	801
15.2.1	Shearing effects on beam deflections . . . . .	805
15.2.2	Shearing effects on buckling . . . . .	812
15.2.3	Problems . . . . .	814
<b>16</b>	<b>Kirchhoff plate theory . . . . .</b>	<b>819</b>
16.1	Governing equations of Kirchhoff plate theory . . . . .	820
16.1.1	Kirchhoff assumptions . . . . .	820
16.1.2	Stress resultants . . . . .	824
16.1.3	Equilibrium equations . . . . .	826
16.1.4	Constitutive laws . . . . .	828
16.1.5	Stresses due to in-plane forces and bending moments . . . . .	830
16.1.6	Summary of Kirchhoff plate theory . . . . .	830

16.2	The bending problem	831
16.2.1	Typical boundary conditions	832
16.2.2	Simple plate bending solutions	836
16.2.3	Problems	839
16.3	Anisotropic plates	840
16.3.1	Laminated composite plates	840
16.3.2	Constitutive laws for laminated composite plates	841
16.3.3	The in-plane stiffness matrix	842
16.3.4	Problems	844
16.3.5	The bending stiffness matrix	845
16.3.6	The coupling stiffness matrix	847
16.3.7	Problems	849
16.3.8	Directionally stiffened plates	849
16.3.9	Governing equations for anisotropic plates	851
16.4	Solution techniques for rectangular plates	853
16.4.1	Navier's solution for simply supported plates	853
16.4.2	Problems	858
16.4.3	Lévy's solution	859
16.4.4	Problems	864
16.5	Circular plates	865
16.5.1	Governing equations for the bending of circular plates	865
16.5.2	Circular plates subjected to loading presenting circular symmetry	868
16.5.3	Problems	871
16.5.4	Circular plates subjected to arbitrary loading	872
16.5.5	Problems	874
16.6	Energy formulation of Kirchhoff plate theory	875
16.6.1	The virtual work done by the internal stresses	876
16.6.2	The virtual work done by the applied loads	880
16.6.3	The principle of virtual work for Kirchhoff plates	881
16.6.4	The principle of minimum total potential energy for Kirchhoff plates	882
16.6.5	Approximate solutions for Kirchhoff plates	884
16.6.6	Solutions based on partial approximation	891
16.6.7	Problems	895
16.7	Buckling of plates	896
16.7.1	Equilibrium formulation	897
16.7.2	Energy formulation	904
16.7.3	Problems	913
<b>A</b>	<b>Appendix: mathematical tools</b>	<b>915</b>
A.1	Notation	915
A.2	Vectors, arrays, matrices and linear algebra	916
A.2.1	Vectors, arrays and matrices	916
A.2.2	Vector, array and matrix operations	918

A.2.3	Solutions of simultaneous linear algebraic equations . . . . .	921
A.2.4	Eigenvalues and eigenvectors . . . . .	922
A.2.5	Positive-definite and quadratic forms . . . . .	923
A.2.6	Partial derivatives of a linear form . . . . .	924
A.2.7	Partial derivatives of a quadratic form . . . . .	924
A.2.8	Stationarity and quadratic forms . . . . .	925
A.2.9	Minimization and quadratic forms . . . . .	925
A.2.10	Least-square solution of linear systems with redundant equations . . . . .	926
A.2.11	Problems . . . . .	926
A.3	Coordinate systems and transformations . . . . .	927
A.3.1	The rotation matrix . . . . .	927
A.3.2	Rotation of vector components . . . . .	928
A.3.3	The rotation matrix in two dimensions . . . . .	929
A.3.4	Rotation of vector components in two dimensions . . . . .	930
A.4	Orthogonality properties of trigonometric functions . . . . .	930
A.5	Gauss-Legendre quadrature . . . . .	931
<b>References</b> . . . . .		935
<b>Index</b> . . . . .		937