

A

Appendix: mathematical tools

A.1 Notation

It is traditional to use a bold typeface to represent vectors, arrays, and matrices. While this typographical convention is elegant in print, it is difficult to reproduce in handwriting or on a white board in a lecture hall. Students are then faced with the confusing dilemma of using a notation in handwriting that does not match that used in textbooks. The notation used in this book is selected to eliminate this problem. The printed notation uses single and double underlines to indicate arrays and matrices, respectively, and these are easily reproduced in handwriting.

Vectors and arrays are denoted using an underline, *i.e.*, \underline{u} or \underline{F} . A vector is first order tensors such as a position, displacement, or force vector. An array is a container used to store a collection of scalars. When defining the components of an array, the scalars it consists of are listed in a column delimited by curly braces, see eq. (A.2).

Unit vectors are vectors of unit magnitude and are denoted with an overbar; for instance, \bar{i}_1 indicates the first unit vector of a triad, or \bar{n} denotes a unit vector in a particular direction in Euclidean space. The overbar is also used to denote non-dimensional scalar quantities; for instance, \bar{k} denotes a non-dimensional stiffness coefficient.

Matrices are indicated using a double underline. For instance, $\underline{\underline{C}}$ indicates a matrix with M rows and N columns, see eq. (A.5). Matrices are used to store the components of second order tensors such as the stress and strain tensors. They are also used to store the coefficients of linear systems of equations.

The indicial notation is used throughout the book. The traditional notation, $\bar{i}, \bar{j}, \bar{k}$, for a Cartesian axis system is replaced by $\mathcal{I} = (\bar{i}_1, \bar{i}_2, \bar{i}_3)$ and the corresponding coordinates $x, y,$ and $z,$ become $x_1, x_2,$ and $x_3,$ respectively. Similarly, force components commonly denoted $F_x, F_y,$ and F_z become $F_1, F_2,$ and $F_3,$ respectively.

A.2 Vectors, arrays, matrices and linear algebra

Vectors are a fundamental part of mechanics and provide a powerful abstraction for manipulating forces, moments, and displacements in statics and mechanics of deformable solids. Arrays and matrices are also very useful constructs in mechanics, especially when dealing with vectors and linear algebra concepts. Vectors will frequently be represented by arrays, and array operations can be used to carry out vector operations. Arrays and matrices provide powerful tools to represent sets of simultaneous linear algebraic equations and express their solutions. Many numerical procedures for approximating the solution of complex mechanics problems will be described in terms of arrays and matrices. Coordinate transformations can also be represented in a compact manner using rotation matrices.

This section provides a summary of some of the key properties of vectors, arrays and matrices that are used in this book. The presentation is by no means complete or rigorous; in most cases, useful results will be presented without proof. Introductory texts such as that of Strang [10] provide in-depth coverage of linear algebra and its applications.

A.2.1 Vectors, arrays and matrices

Vectors

Vectors describe quantities that have both a magnitude and a direction, whereas *scalars* have only a signed magnitude. In this book, the term vector will be used to describe a quantity with a magnitude and a direction in Euclidean space, that is, a quantity with three independent directional components. Typically, these three components are defined in a Cartesian coordinate system, but cylindrical and spherical coordinate systems may also be used. Cartesian coordinates are defined by a triad,

$$\mathcal{I} = (\bar{i}_1, \bar{i}_2, \bar{i}_3), \quad (\text{A.1})$$

where \bar{i}_1 , \bar{i}_2 , and \bar{i}_3 are three mutually orthogonal unit vectors.

Vector quantities in this book include forces, moments, positions, and displacements. An underscore is used to indicate a vector, \underline{v} .

Arrays

An *array* is a container used to store a collection of scalars. An underscore¹ is used to indicate an array, \underline{a} . The N elements of an array are arranged into a column of size N . In this book, an array is always defined as a column,

¹ Many texts use a bold font to indicate arrays or vectors but this is not adopted here because of the difficulty of creating a bold symbol in handwriting.

$$\underline{a} = \left\{ \begin{array}{c} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_N \end{array} \right\}. \tag{A.2}$$

The i^{th} element of the array, a_i , is identified by a subscript that indicates its position in the array. Curly braces will be used to denote an array.

The *transpose* of an array of size N is a row of N elements, and a superscript $(\cdot)^T$ is used to indicate a transpose. Thus, the transpose of array \underline{a} defined in eq. (A.2) is written as

$$\underline{a}^T = \{a_1, a_2, a_3, \dots, a_N\}. \tag{A.3}$$

Frequently, the following notation will also be used

$$\underline{a} = \{a_1, a_2, a_3, \dots, a_N\}^T. \tag{A.4}$$

Equations (A.2), (A.3), and (A.4), all define the same column array, \underline{a} .

The components of a vector can be stored in an array with three elements. For instance, array $\underline{f} = \{f_1, f_2, f_3\}^T$ could represent a force vector with components f_1 , f_2 , and f_3 in a Cartesian system.

Matrices

A matrix is a container used to store a collection of N arrays all of the same size. Each array is of size M and forms a column of the matrix, which is of size $M \times N$. When specifying the size of a matrix, the notation $M \times N$ will be used: the matrix consists of M rows and N columns. A matrix of size 2×3 consists of 2 rows and 3 columns. A double underscore is used to indicate a matrix,

$$\underline{\underline{A}} = [\underline{a}_1 \ \underline{a}_2 \ \dots \ \underline{a}_N] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1N} \\ a_{21} & a_{22} & \dots & a_{2N} \\ \dots & \dots & \dots & \dots \\ a_{M1} & a_{M2} & \dots & a_{MN} \end{bmatrix}. \tag{A.5}$$

The elements of a matrix with the same indices (or subscripts) define the *diagonal of the matrix*. When the number of rows is equal to the number of columns, the matrix is said to be a *square matrix*.

The *transpose of a matrix* is represented as $\underline{\underline{A}}^T$ and is defined by switching the rows and columns in the original matrix,

$$\underline{\underline{A}}^T = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{M1} \\ a_{12} & a_{22} & \dots & a_{M2} \\ \dots & \dots & \dots & \dots \\ a_{1N} & a_{2N} & \dots & a_{NM} \end{bmatrix}. \tag{A.6}$$

The matrix transpose can also be defined by simply reversing the subscripts of all the individual elements. The transpose of a square matrix is also a square matrix, but the transpose of a matrix of size $M \times N$ is a matrix of size $N \times M$.

A *symmetric matrix* is a square matrix that is identical to its transpose, that is, a matrix for which $\underline{\underline{A}} = \underline{\underline{A}}^T$. A *skew-symmetric matrix* is a square matrix whose transpose is also its negative, that is, a matrix for which $\underline{\underline{A}}^T = -\underline{\underline{A}}$. Any square matrix can be expressed as the sum of its symmetric and skew-symmetric parts,

$$\underline{\underline{A}} = \frac{1}{2} (\underline{\underline{A}} + \underline{\underline{A}}^T) + \frac{1}{2} (\underline{\underline{A}} - \underline{\underline{A}}^T) = \underline{\underline{A}}_s + \underline{\underline{A}}_a, \quad (\text{A.7})$$

where $\underline{\underline{A}}_s$ is symmetric because $\underline{\underline{A}}_s^T = (\underline{\underline{A}} + \underline{\underline{A}}^T)^T = \underline{\underline{A}}_s$ and $\underline{\underline{A}}_a$ is skew-symmetric because $\underline{\underline{A}}_a^T = (\underline{\underline{A}} - \underline{\underline{A}}^T)^T = -\underline{\underline{A}}_a$.

A *diagonal matrix* is a matrix whose only non-zero elements lie along its diagonal. The *identity matrix* is a square diagonal matrix whose diagonal elements are all unity.

A.2.2 Vector, array and matrix operations

Basic operations

Vectors, arrays and matrices can be added or subtracted only if all quantities are of the same dimensions. Consequently, only vectors can be added to vectors, arrays to arrays and matrices to matrices. The resulting vector, array or matrix is a new quantity of the same type and dimension as the those being added or subtracted.

Vectors, arrays, or matrices can also be multiplied by a signed constant, resulting in a new vector, array, or matrix whose elements are each multiplied by the same signed constant. These operations follow the associative, distributive and commutative rules of scalar algebra.

Scalar product

Let a_1 , a_2 , and a_3 be the component of vector \underline{a} in a given triad, and b_1 , b_2 , and b_3 those of vector \underline{b} in the same triad. The *scalar product* of the vectors, denoted $\underline{a} \cdot \underline{b}$, is defined as

$$\underline{a} \cdot \underline{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 = \|\underline{a}\| \|\underline{b}\| \cos(\widehat{ab}), \quad (\text{A.8})$$

where $\|\underline{a}\|$ and $\|\underline{b}\|$ are the magnitudes of vectors \underline{a} and \underline{b} , respectively, and \widehat{ab} denotes the angle between vectors \underline{a} and \underline{b} . The scalar product is so named because this operation involving two vectors results in a scalar quantity. The scalar product is also referred to as the *dot product*, because of the notation used to represent this operation.

If a_i and b_i are the components of arrays \underline{a} and \underline{b} , both of size N , the scalar product of the two arrays, denoted $\underline{a} \cdot \underline{b}$, is defined as

$$\underline{a} \cdot \underline{b} = a_1 b_1 + a_2 b_2 + \dots + a_N b_N = \sum_{i=1}^N a_i b_i. \tag{A.9}$$

The scalar product is also expressed with the following notation

$$\underline{a} \cdot \underline{b} = \underline{a}^T \underline{b} = \underline{b}^T \underline{a} = \sum_{i=1}^N a_i b_i. \tag{A.10}$$

Norm of an array

The *norm* of array, \underline{a} , denoted $\|\underline{a}\|$, is defined as

$$\|\underline{a}\| = \sqrt{\underline{a} \cdot \underline{a}}. \tag{A.11}$$

This norm is also called the magnitude or length of the array. The norm is always a non-negative scalar quantity.

A vector whose norm is unity is called a *unit vector* and is denoted with an overscore. Thus, \bar{a} is a vector with unit magnitude or a unit vector. The definition of a triad, see eq. (A.1), involves three unit vectors, denoted \bar{v}_1 , \bar{v}_2 , and \bar{v}_3 . The scalar product of two unit vectors is the cosine of the angle between them as can be seen from eq. (A.8). Also, the scalar product of a unit vector and another vector is the projection of that vector in the unit vector direction.

Matrix determinant

The determinant of a matrix is a scalar quantity, denoted $\det(\underline{A})$, that plays an important role in linear algebra. The determinant of a matrix is defined as the sum of the entries of any row (or column) times its co-factors: $\det(\underline{A}) = a_{i1}C_{i1} + a_{i2}C_{i2} + \dots + a_{iN}C_{iN}$, where the co-factor is defined as $C_{ij} = (-1)^{i+j} \det(\underline{M}_{ij})$; \underline{M}_{ij} is the sub-matrix obtained by deleting the i^{th} row and j^{th} column of matrix \underline{A} .

This formal recursive definition is not necessarily the most computationally efficient manner to compute the determinant of a matrix. Many efficient numerical algorithms to perform this task are available in most numerical analysis software packages.

The determinant of a 2×2 matrix is easy to evaluate: $\det(\underline{A}) = a_{11}a_{22} - a_{21}a_{12}$. The determinant of a 3×3 matrix is: $\det(\underline{A}) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$. For matrices of larger size, it is preferable to rely on computer software.

Several properties of the determinant are important to note:

1. The determinant of a product of matrices is the product of the determinants: $\det(\underline{A} \underline{B}) = \det(\underline{A}) \det(\underline{B})$.
2. The determinant of the transpose is the same as the determinant of the matrix: $\det(\underline{A}^T) = \det(\underline{A})$.
3. The determinant of a diagonal matrix is the product of the diagonal elements.

4. Interchanging two rows or columns changes the sign of the determinant.
5. Adding or subtracting a multiple of one row (or column) to another row (or column) does not change the determinant.
6. If two rows (or columns) are the same or multiples of each other, the determinant is zero.
7. If a row (or column) is zero, the determinant is zero.
8. A matrix whose determinant is zero is called a *singular matrix*.

Vector product

Let $a_1, a_2,$ and a_3 be the component of vector \underline{a} , and $b_1, b_2,$ and b_3 those of vector \underline{b} , both resolved in the same triad $\mathcal{I} = (\bar{i}_1, \bar{i}_2, \bar{i}_3)$. The *vector product* of two vectors, denoted $\underline{a} \times \underline{b}$ yields a vector quantity defined as

$$\underline{a} \times \underline{b} = (a_2b_3 - a_3b_2)\bar{i}_1 + (a_3b_1 - a_1b_3)\bar{i}_2 + (a_1b_2 - a_2b_1)\bar{i}_3. \quad (\text{A.12})$$

The vector product can also be defined in a more geometric fashion as $\underline{a} \times \underline{b} = \|\underline{a}\| \|\underline{b}\| \sin(\widehat{ab}) \bar{n}$, where \bar{n} is a unit vector perpendicular to both \underline{a} and \underline{b} and whose direction is determined by the right-hand rule. It follows that $\underline{a} \times \underline{a} = 0, \bar{i}_1 \times \bar{i}_2 = \bar{i}_3,$ and $\underline{a} \times \underline{b} = -\underline{b} \times \underline{a}$.

Matrix multiplication

Let matrix \underline{A} be of size $M \times K$ and matrix \underline{B} of size $K \times N$. The product of these two matrices, simply denoted $\underline{A}\underline{B}$, results in a third matrix, \underline{C} , of size $M \times N$, whose components are

$$c_{ij} = \sum_{k=1}^K a_{ik}b_{kj}, \quad i = 1, 2, \dots, M, \quad j = 1, 2, \dots, N, \quad (\text{A.13})$$

Multiplication of two matrices is only possible if the number of columns of the first matrix matches the number of rows of the second.

Let matrix \underline{A} be of size $M \times K$ and array \underline{b} of size K . The product of the matrix by the array, simply denoted $\underline{A}\underline{b}$, results in an array, \underline{c} , of size M , whose components are

$$c_i = \sum_{k=1}^K a_{ik}b_k, \quad i = 1, 2, \dots, M. \quad (\text{A.14})$$

From these definitions, the following properties can easily be proved.

1. $\underline{A}\underline{B} \neq \underline{B}\underline{A}$: matrix multiplication is not a commutative operation.
2. $(\underline{A}\underline{B})^T = \underline{B}^T \underline{A}^T$.
3. Operation $\underline{b}\underline{A}$ is not defined because of dimension mismatch between the array and matrix.
4. Operation $\underline{b}^T \underline{A}$ is defined if array \underline{b} is of size equal to the number of rows in \underline{A} .
5. $(\underline{A}\underline{B})\underline{C} = \underline{A}(\underline{B}\underline{C})$ (associative rule).
6. $\underline{A}(\underline{B} + \underline{C}) = \underline{A}\underline{B} + \underline{A}\underline{C}$ (distributive rule).
7. Product of a matrix by the identity matrix gives the matrix itself: $\underline{A}\underline{I} = \underline{I}\underline{A} = \underline{A}$.

Matrix inverse

Matrix division is not defined, but the *inverse of a square matrix* is defined. By definition, the multiplication of a square matrix, $\underline{\underline{A}}$, of size $N \times N$ by its inverse, denoted $\underline{\underline{A}}^{-1}$, produces the identity matrix of the same size,

$$\underline{\underline{A}} \underline{\underline{A}}^{-1} = \underline{\underline{A}}^{-1} \underline{\underline{A}} = \underline{\underline{I}}. \tag{A.15}$$

If the determinant of a matrix vanishes, its inverse does not exist. Note that the product of a matrix by its inverse is commutative.

Calculation of the inverse of a matrix is difficult when its dimensions exceed 3. For a matrix of size 2×2 , the inverse is

$$\underline{\underline{A}}^{-1} = \frac{1}{\det(\underline{\underline{A}})} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}. \tag{A.16}$$

For square matrices of size larger than 3, numerical software packages should be used to compute the inverse. Detailed descriptions of the numerical procedures can be found in Strang [10].

A.2.3 Solutions of simultaneous linear algebraic equations

Matrix and array notations provide powerful abstractions for dealing with sets of simultaneous linear algebraic equations. A set of N linear equations in N unknowns, q_i , can be written explicitly as

$$\begin{aligned} a_{11}q_1 + a_{12}q_2 + \dots + a_{1N}q_N &= b_1 \\ a_{21}q_1 + a_{22}q_2 + \dots + a_{2N}q_N &= b_2 \\ &\vdots \\ a_{N1}q_1 + a_{N2}q_2 + \dots + a_{NN}q_N &= b_N. \end{aligned}$$

When expressed in matrix notation, this problem takes a much more compact form,

$$\underline{\underline{A}} \underline{\underline{q}} = \underline{\underline{b}}, \tag{A.17}$$

where $\underline{\underline{A}}$ is the square matrix of size $N \times N$ storing the coefficients of the system, $\underline{\underline{q}}$ the array of size N storing the unknowns of the problem, and $\underline{\underline{b}}$ the array of size N storing the right-hand side coefficients. Expanding the matrix-array product using eq. (A.14) will show that the compact notation of eq. (A.17) is equivalent to the more explicit form given above.

To find the solution of the system of equation, the inverse of matrix $\underline{\underline{A}}$, denoted $\underline{\underline{A}}^{-1}$, is computed first. Equation (A.17) is then pre-multiplied by this inverse to find $\underline{\underline{A}}^{-1} \underline{\underline{A}} \underline{\underline{q}} = \underline{\underline{A}}^{-1} \underline{\underline{b}}$. In view of the definition of the inverse, eq. (A.15), this implies $\underline{\underline{I}} \underline{\underline{q}} = \underline{\underline{A}}^{-1} \underline{\underline{b}}$, and finally

$$\underline{q} = \underline{A}^{-1}\underline{b}. \quad (\text{A.18})$$

If matrix \underline{A} is singular, its inverse does not exist and the linear system cannot be solved.

When the right-hand side coefficients vanish, the system of linear equations of called a *homogeneous system*,

$$\underline{A}\underline{q} = 0. \quad (\text{A.19})$$

The *trivial solution*, $\underline{q} = 0$, is clearly a solution of the system of equations. If matrix \underline{A} is not singular, it is the only possible solution. When matrix \underline{A} is singular, however, an infinite number of solutions exist.

A.2.4 Eigenvalues and eigenvectors

The following matrix equation

$$\underline{A}\underline{q} = \lambda\underline{q}, \quad (\text{A.20})$$

is called an *eigenvalue problem*. Matrix \underline{A} is a known square matrix of size $N \times N$, \underline{q} an unknown array of size N , and λ an unknown scalar.

To solve the problem, it is recast in the homogeneous form as

$$(\underline{A} - \lambda\underline{I})\underline{q} = 0. \quad (\text{A.21})$$

If matrix $(\underline{A} - \lambda\underline{I})$ is non singular, *i.e.*, if $\det(\underline{A} - \lambda\underline{I}) \neq 0$, the only possible solution is the trivial solution, $\underline{q} = 0$.

If matrix $(\underline{A} - \lambda\underline{I})$ is singular, *i.e.*, if $\det(\underline{A} - \lambda\underline{I}) = 0$, non-trivial solutions becomes possible. For the determinant of matrix $(\underline{A} - \lambda\underline{I})$ to vanish, scalar λ must satisfy the following equation

$$\det \begin{bmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} - \lambda \end{bmatrix} = 0.$$

When the determinant is expanded, the following N^{th} order polynomial equation for λ results

$$\lambda^N + c_1\lambda^{N-1} + c_{N-2}\lambda^{N-2} + \cdots + c_N = 0, \quad (\text{A.22})$$

where the coefficients, c_i , are determined for the evaluation of the determinant. This equation is called the *characteristic equation*. The N values λ which satisfy this characteristic equation are called *eigenvalues*, and for each eigenvalue, a non-trivial solution can be found, called an *eigenvector*.

If all the entries of matrix \underline{A} are real, the N eigenvalues could be real or complex number. It can be shown, however, that for symmetric matrices, the eigenvalues and associated eigenvectors are always real [10]. For each distinct eigenvalue, λ_i , eq. (A.21) can be solved for the corresponding eigenvector, \underline{q}_i . Repeated eigenvalues require different treatment, see [10]. If one or more eigenvalues are zero, matrix \underline{A} is

singular. Because eq. (A.21) is a homogeneous equation, the solutions for the eigenvectors are not unique. Each eigenvector is defined within an arbitrary constant. A unique definition of the eigenvectors is obtained by requiring their norm to be unity.

The eigenvectors can be used to diagonalize matrix \underline{A} . First, eq. (A.21) can be written for each eigenvectors and the results are collected in a matrix form as

$$\underline{A} [q_1, q_2, \dots, q_N] = [\lambda_1 q_1, \lambda_2 q_2, \dots, \lambda_N q_N] = [q_1, q_2, \dots, q_N] \begin{bmatrix} \lambda_1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \lambda_N \end{bmatrix}.$$

Next, the *eigenvector matrix*, \underline{S} , is constructed; each column of this matrix stores one of the eigenvectors of \underline{A} , i.e., $\underline{S} = [q_1 \ q_2 \ \dots \ q_N]$. With this notation, the above equation can be written in a compact form as

$$\underline{A} \underline{S} = \underline{S} \underline{A}, \tag{A.23}$$

where \underline{A} is the diagonal matrix whose diagonal elements contain the eigenvalues. Pre-multiplying eq. (A.23) by the inverse of the eigenvector matrix then leads to

$$\underline{S}^{-1} \underline{A} \underline{S} = \underline{A}, \tag{A.24}$$

This transformation is called the diagonalization of matrix \underline{A} .

A.2.5 Positive-definite and quadratic forms

Matrix-array products of the form $\Phi = q^T \underline{A} q$, where Φ is a scalar, are frequently encountered in structural analysis. Expanding the array and matrix product leads to

$$\Phi = \{q_1, q_2, \dots, q_N\} \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1N} \\ a_{21} & a_{22} & \dots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & \dots & a_{NN} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ \vdots \\ q_N \end{Bmatrix} = \sum_{i=1}^N \sum_{j=1}^N a_{ij} q_i q_j.$$

Scalar Φ is clearly a quadratic expression in q_i . Consequently, Φ is referred to as a *quadratic form*.

A symmetric matrix, \underline{A} , is a *positive-definite matrix* if the quadratic form

$$q^T \underline{A} q \geq 0, \tag{A.25}$$

for any non-zero, real valued q and it is equal to zero only when $q = 0$. It is possible to show that the eigenvalues of a positive-definite matrix are all only positive, and non-zero. It is also possible to show that a symmetric matrix with positive, non-zero eigenvalues is positive-definite.

A.2.6 Partial derivatives of a linear form

Consider a scalar, Φ , that is a linear function of three variables, q_1 , q_2 , and q_3 ,

$$\Phi = 13q_1 + 8q_2 - 19q_3.$$

The derivatives of this scalar with respect to the three variables are obtained by using elementary rules of calculus to find

$$\frac{\partial \Phi}{\partial q_1} = 13, \quad \frac{\partial \Phi}{\partial q_2} = 8, \quad \frac{\partial \Phi}{\partial q_3} = -19.$$

It is convenient to express this scalar in a more compact form using the following matrix notation

$$\Phi = \{q_1, q_2, q_3\} \begin{Bmatrix} 13 \\ 8 \\ -19 \end{Bmatrix} = \underline{q}^T \underline{Q}. \quad (\text{A.26})$$

The derivatives of scalar Φ also can be represented using the compact notation. First, the array of partial derivatives is defined as

$$\frac{\partial \Phi}{\partial \underline{q}} = \left\{ \frac{\partial \Phi}{\partial q_1}, \frac{\partial \Phi}{\partial q_2}, \frac{\partial \Phi}{\partial q_3} \right\}^T,$$

from which it follows that $\partial \Phi / \partial \underline{q} = \{13, 8, -19\}^T = \underline{Q}$. The desired partial derivatives are then readily obtained as

$$\frac{\partial \Phi}{\partial \underline{q}} = \frac{\partial}{\partial \underline{q}} (\underline{q}^T \underline{Q}) = \underline{Q}. \quad (\text{A.27})$$

These derivatives are identical to those obtained from elementary rules of calculus.

A.2.7 Partial derivatives of a quadratic form

Consider a quadratic form defined by symmetric matrix, \underline{A} , and array \underline{q} ,

$$\begin{aligned} \Phi &= \frac{1}{2} \underline{q}^T \underline{A} \underline{q} = \frac{1}{2} \{q_1, q_2, q_3\}^T \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} \\ &= \frac{a_{11}}{2} q_1^2 + \frac{a_{22}}{2} q_2^2 + \frac{a_{33}}{2} q_3^2 + a_{12} q_1 q_2 + a_{23} q_2 q_3 + a_{13} q_1 q_3. \end{aligned}$$

The derivatives of this scalar with respect to the three variables are obtained by using elementary rules of calculus to find

$$\begin{aligned} \frac{\partial \Phi}{\partial q_1} &= a_{11} q_1 + a_{12} q_2 + a_{13} q_3, \\ \frac{\partial \Phi}{\partial q_2} &= a_{12} q_1 + a_{22} q_2 + a_{23} q_3, \\ \frac{\partial \Phi}{\partial q_3} &= a_{13} q_1 + a_{23} q_2 + a_{33} q_3. \end{aligned} \quad (\text{A.28})$$

First, the array of partial derivatives is defined as

$$\frac{\partial \Phi}{\partial \underline{q}} = \left\{ \frac{\partial \Phi}{\partial q_1}, \frac{\partial \Phi}{\partial q_2}, \frac{\partial \Phi}{\partial q_3} \right\}^T,$$

and using this with eq. (A.28) results in the desired partial derivative

$$\frac{\partial \Phi}{\partial \underline{q}} = \frac{\partial}{\partial \underline{q}} \left(\frac{1}{2} \underline{q}^T \underline{A} \underline{q} \right) = \underline{A} \underline{q}. \tag{A.29}$$

These derivatives are identical to those obtained from elementary rules of calculus.

A.2.8 Stationarity and quadratic forms

Consider a scalar, Π defined as

$$\Pi(\underline{q}) = \frac{1}{2} \underline{q}^T \underline{A} \underline{q} - \underline{q}^T \underline{b}, \tag{A.30}$$

where \underline{q} is an array of size N that stores N independent variables, \underline{A} is a known symmetric, positive-definite matrix of size $N \times N$, and \underline{b} an array of size N storing known coefficients. Determine the value of the independent variables that make Π stationary, *i.e.*, for which the partial derivatives of Π all vanish $\partial \Pi / \partial \underline{q} = \underline{0}$.

The derivatives are found using eqs. (A.27) and (A.29) as

$$\frac{\partial \Pi}{\partial \underline{q}} = \frac{\partial}{\partial \underline{q}} \left(\frac{1}{2} \underline{q}^T \underline{A} \underline{q} - \underline{q}^T \underline{b} \right) = \underline{A} \underline{q} - \underline{b} = \underline{0}, \tag{A.31}$$

The vanishing of the partial derivatives of Π leads to a system of linear equations, $\underline{A} \underline{q} = \underline{b}$, which can be solved with the help of eq. (A.18) to find $\underline{q} = \underline{A}^{-1} \underline{b}$.

A.2.9 Minimization and quadratic forms

Consider once again the scalar Π defined by eq. (A.30). Determine the value of the independent variables that make Π minimum. An elegant solution of this problem is given by Strang [10].

Define a second array, \underline{p} , that is the same size as \underline{q} but otherwise arbitrary and construct the following scalar function

$$\Phi(\underline{p}) - \Phi(\underline{q}) = \frac{1}{2} \underline{p}^T \underline{A} \underline{p} - \underline{p}^T \underline{b} - \frac{1}{2} \underline{q}^T \underline{A} \underline{q} + \underline{q}^T \underline{b}.$$

Next, let $\underline{b} = \underline{A} \underline{q}$ to find

$$\Phi(\underline{p}) - \Phi(\underline{q}) = \frac{1}{2} \underline{p}^T \underline{A} \underline{p} - \underline{p}^T \underline{A} \underline{q} + \frac{1}{2} \underline{q}^T \underline{A} \underline{q} = \frac{1}{2} (\underline{p} - \underline{q})^T \underline{A} (\underline{p} - \underline{q}). \tag{A.32}$$

Because matrix \underline{A} is symmetric, $\underline{p}^T \underline{A} \underline{q} = (\underline{p}^T \underline{A} \underline{q})^T = \underline{q}^T \underline{A} \underline{p}$. This identity is used to obtain the last equality. By definition of a positive-definite matrix, eq. (A.25), the

last expression of eq. (A.32) must be positive, resulting in $\Phi(\underline{p}) > \Phi(\underline{q})$. Because \underline{p} is arbitrary, this means that the minimum value of Π is achieved for $\underline{b} = \underline{A} \underline{q}$, which can be solved to find $\underline{q} = \underline{A}^{-1} \underline{b}$

Comparing the results obtained here with those of the previous section, it is concluded that if \underline{A} is a symmetric, positive-definite matrix, the stationary point of Π is a minimum.

A.2.10 Least-square solution of linear systems with redundant equations

Consider a system of N linear equations, $\underline{A} \underline{x} = \underline{b}$, where \underline{A} is an $N \times N$ square matrix, \underline{x} the array storing the N unknowns of the problem, and \underline{b} the known right-hand side array. If \underline{A} is not singular, the solution of this system is simply $\underline{x} = \underline{A}^{-1} \underline{b}$, see eq. (A.18). Consider next a system of N linear equations, $\underline{A} \underline{x} = \underline{b}$, where \underline{A} is a rectangular matrix of size $N \times M$, $N > M$, \underline{x} the array of size M storing the unknowns of the problem, and \underline{b} the known right-hand side array. This problem features more equations than unknowns. Such a system is known as an *over-determined system of equations*, and in general, no solution exists.

To obtain an approximate solution of the problem, it is assumed that each of the N equations is not exactly satisfied, but rather, presents an error, hopefully small. This is written as $\underline{A} \underline{x} - \underline{b} = \underline{e}$, where \underline{e} is the array of errors. A solution is now sought that minimizes the square of the norm of the error array, and this can be stated as

$$\min_{\underline{x}} \|\underline{e}\|^2 = \min_{\underline{x}} \|(\underline{A} \underline{x} - \underline{b})\|^2 = \min_{\underline{x}} [(\underline{A} \underline{x} - \underline{b})^T (\underline{A} \underline{x} - \underline{b})].$$

The minimum of this quadratic expression is obtained by requiring the vanishing of its derivatives with respect to \underline{x} , *i.e.* $\partial \|\underline{e}\|^2 / \partial \underline{x} = 0$. This results in the following equation

$$\frac{\partial \|\underline{e}\|^2}{\partial \underline{x}} = 2 \underline{x}^T \underline{A}^T \underline{A} - 2 \underline{b}^T \underline{A} = 0,$$

which, after taking the transpose, can be recast as $(\underline{A}^T \underline{A}) \underline{x} = \underline{A}^T \underline{b}$. Note that $(\underline{A}^T \underline{A})$ is now a square matrix of size $M \times M$, and hence, this linear system of equations is readily solved as

$$\underline{x} = (\underline{A}^T \underline{A})^{-1} \underline{A}^T \underline{b}, \tag{A.33}$$

provided that matrix $(\underline{A}^T \underline{A})$ is not singular. Equation (A.33) provides a *least-squares solution* of the over-determined system of equations.

A.2.11 Problems

Problem A.1. Evaluation of a quadratic form

Evaluate the quadratic form $\Phi = \underline{q}^T \underline{A}_a \underline{q}$, where \underline{A}_a is a skew symmetric form.

A.3 Coordinate systems and transformations

A.3.1 The rotation matrix

Consider two orthonormal bases, $\mathcal{I} = (\bar{i}_1, \bar{i}_2, \bar{i}_3)$ and $\mathcal{I}^* = (\bar{i}_1^*, \bar{i}_2^*, \bar{i}_3^*)$, as shown in fig. A.1. The relative orientation of these two bases is arbitrary.

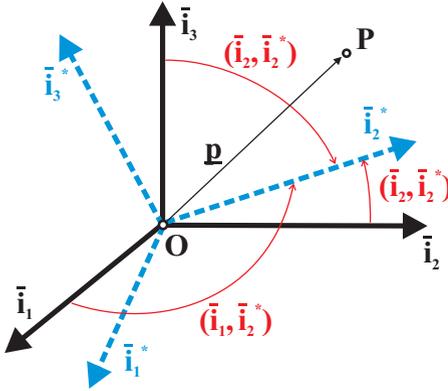


Fig. A.1. Position of a point in two coordinate systems.

Let ℓ_1, ℓ_2 and ℓ_3 be the direction cosines of unit vector \bar{i}_1^* with respect to axes \bar{i}_1, \bar{i}_2 , and \bar{i}_3 , respectively, i.e., $\ell_1 = \bar{i}_1^* \cdot \bar{i}_1 = \cos(\bar{i}_1^*, \bar{i}_1)$, $\ell_2 = \bar{i}_1^* \cdot \bar{i}_2 = \cos(\bar{i}_1^*, \bar{i}_2)$, and $\ell_3 = \bar{i}_1^* \cdot \bar{i}_3 = \cos(\bar{i}_1^*, \bar{i}_3)$. The direction cosines of unit vector \bar{i}_2^* are defined in a similar manner as $m_1 = \bar{i}_2^* \cdot \bar{i}_1 = \cos(\bar{i}_2^*, \bar{i}_1)$, $m_2 = \bar{i}_2^* \cdot \bar{i}_2 = \cos(\bar{i}_2^*, \bar{i}_2)$, and $m_3 = \bar{i}_2^* \cdot \bar{i}_3 = \cos(\bar{i}_2^*, \bar{i}_3)$; these quantities are highlighted in fig. A.1. Finally, the direction cosines of unit vector \bar{i}_3^* are $n_1 = \bar{i}_3^* \cdot \bar{i}_1 = \cos(\bar{i}_3^*, \bar{i}_1)$, $n_2 = \bar{i}_3^* \cdot \bar{i}_2 = \cos(\bar{i}_3^*, \bar{i}_2)$, and $n_3 = \bar{i}_3^* \cdot \bar{i}_3 = \cos(\bar{i}_3^*, \bar{i}_3)$. With these definitions, it becomes possible to express the unit

vectors of basis \mathcal{I}^* as linear combinations of those of basis \mathcal{I} ,

$$\bar{i}_1^* = \ell_1 \bar{i}_1 + \ell_2 \bar{i}_2 + \ell_3 \bar{i}_3, \tag{A.34a}$$

$$\bar{i}_2^* = m_1 \bar{i}_1 + m_2 \bar{i}_2 + m_3 \bar{i}_3, \tag{A.34b}$$

$$\bar{i}_3^* = n_1 \bar{i}_1 + n_2 \bar{i}_2 + n_3 \bar{i}_3. \tag{A.34c}$$

Similarly, the unit vectors of basis \mathcal{I} can be expressed as linear combinations of those of basis \mathcal{I}^* ,

$$\bar{i}_1 = \ell_1 \bar{i}_1^* + m_1 \bar{i}_2^* + n_1 \bar{i}_3^*, \tag{A.35a}$$

$$\bar{i}_2 = \ell_2 \bar{i}_1^* + m_2 \bar{i}_2^* + n_2 \bar{i}_3^*, \tag{A.35b}$$

$$\bar{i}_3 = \ell_3 \bar{i}_1^* + m_3 \bar{i}_2^* + n_3 \bar{i}_3^*. \tag{A.35c}$$

It is convenient to defined the *direction cosine matrix* or *rotation matrix*,

$$\underline{R} = \begin{bmatrix} \cos(\bar{i}_1^*, \bar{i}_1) & \cos(\bar{i}_2^*, \bar{i}_1) & \cos(\bar{i}_3^*, \bar{i}_1) \\ \cos(\bar{i}_1^*, \bar{i}_2) & \cos(\bar{i}_2^*, \bar{i}_2) & \cos(\bar{i}_3^*, \bar{i}_2) \\ \cos(\bar{i}_1^*, \bar{i}_3) & \cos(\bar{i}_2^*, \bar{i}_3) & \cos(\bar{i}_3^*, \bar{i}_3) \end{bmatrix} = \begin{bmatrix} \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \\ \ell_3 & m_3 & n_3 \end{bmatrix}. \tag{A.36}$$

This matrix fully defines the orientation of basis \mathcal{I}^* with respect to basis \mathcal{I} because it stores the direction cosines of each of the three unit vectors defining \mathcal{I}^* with respect to \mathcal{I} . The direction cosines must satisfy the following relationships: $\ell_1^2 + \ell_2^2 + \ell_3^2 = 1$, $m_1^2 + m_2^2 + m_3^2 = 1$, and $n_1^2 + n_2^2 + n_3^2 = 1$. It follows that the rotation matrix has the following property

$$\underline{\underline{R}}^T \underline{\underline{R}} = \begin{bmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix} \begin{bmatrix} \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \\ \ell_3 & m_3 & n_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \underline{\underline{I}}, \quad (\text{A.37})$$

where the vanishing of the off-diagonal terms arises from of the orthogonality among the unit vectors themselves: $\bar{i}_1 \cdot \bar{i}_2^* = (\ell_1 \bar{i}_1 + \ell_2 \bar{i}_2 + \ell_3 \bar{i}_3) \cdot (m_1 \bar{i}_1 + m_2 \bar{i}_2 + m_3 \bar{i}_3) = \ell_1 m_1 + \ell_2 m_2 + \ell_3 m_3 = 0$. Since the product $\underline{\underline{R}}^T \underline{\underline{R}}$ is equal to the identity matrix, $\underline{\underline{R}}^T$ must be the inverse of the rotation matrix $\underline{\underline{R}}$, or $\underline{\underline{R}}^{-1} = \underline{\underline{R}}^T$. Such matrices are said to be *orthogonal matrices*, and therefore, the *matrix of direction cosines is an orthogonal matrix*.

A.3.2 Rotation of vector components

Consider now the position vector of point \mathbf{P} , denoted \underline{p} in fig. A.1, and its components expressed in the two bases, \mathcal{I} and \mathcal{I}^* ,

$$\underline{p} = p_1 \bar{i}_1 + p_2 \bar{i}_2 + p_3 \bar{i}_3 = p_1^* \bar{i}_1^* + p_2^* \bar{i}_2^* + p_3^* \bar{i}_3^*. \quad (\text{A.38})$$

The components of vector \underline{p} in basis \mathcal{I} are denoted p_1 , p_2 , and p_3 , whereas those in basis \mathcal{I}^* are denoted p_1^* , p_2^* , and p_3^* . On the right-hand side of this equation, the unit vectors of basis \mathcal{I}^* will now be expressed in term of their counterparts in basis \mathcal{I} using eq. (A.34) to find

$$\begin{aligned} p_1 \bar{i}_1 + p_2 \bar{i}_2 + p_3 \bar{i}_3 &= p_1^* (\ell_1 \bar{i}_1 + \ell_2 \bar{i}_2 + \ell_3 \bar{i}_3) + p_2^* (m_1 \bar{i}_1 + m_2 \bar{i}_2 + m_3 \bar{i}_3) \\ &\quad + p_3^* (n_1 \bar{i}_1 + n_2 \bar{i}_2 + n_3 \bar{i}_3). \end{aligned}$$

A scalar product of this result by \bar{i}_1 , \bar{i}_2 , and \bar{i}_3 yields three equations, $p_1 = p_1^* \ell_1 + p_2^* m_1 + p_3^* n_1$, $p_2 = p_1^* \ell_2 + p_2^* m_2 + p_3^* n_2$, and $p_3 = p_1^* \ell_3 + p_2^* m_3 + p_3^* n_3$, respectively. These equations relate the components of vector \underline{p} in bases \mathcal{I} and \mathcal{I}^* , and can be summarized in a compact matrix form as

$$\begin{Bmatrix} p_1 \\ p_2 \\ p_3 \end{Bmatrix} = \begin{bmatrix} \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \\ \ell_3 & m_3 & n_3 \end{bmatrix} \begin{Bmatrix} p_1^* \\ p_2^* \\ p_3^* \end{Bmatrix} = \underline{\underline{R}} \begin{Bmatrix} p_1^* \\ p_2^* \\ p_3^* \end{Bmatrix},$$

where the rotation matrix, $\underline{\underline{R}}$, is defined in eq. (A.36). The rotation matrix expresses the linear relationship between the components of vector \underline{p} in bases \mathcal{I} and \mathcal{I}^* . Of course, the inverse relationship is easy to find

$$\begin{Bmatrix} p_1 \\ p_2 \\ p_3 \end{Bmatrix} = \underline{\underline{R}} \begin{Bmatrix} p_1^* \\ p_2^* \\ p_3^* \end{Bmatrix} \iff \begin{Bmatrix} p_1^* \\ p_2^* \\ p_3^* \end{Bmatrix} = \underline{\underline{R}}^T \begin{Bmatrix} p_1 \\ p_2 \\ p_3 \end{Bmatrix}, \quad (\text{A.39})$$

because the rotation matrix is orthogonal, see eq. (A.37). The rotation matrix “rotates the components of vector \underline{p} ” from one coordinate system to the other. A vector is a mathematical entity characterized by a magnitude and orientation in space. For

practical reasons, however, it is often easier to represent a vector by its components in a specific basis. For instance, the three components, $p_1, p_2,$ and $p_3,$ represent vector \underline{p} in basis \mathcal{I} . Had a different basis been selected, say \mathcal{I}^* , the same vector \underline{p} would have been represented by a different set of components, $p_1^*, p_2^*,$ and p_3^* . When vector \underline{p} is represented in two different bases, \mathcal{I} and \mathcal{I}^* , the corresponding components, $p_1, p_2, p_3,$ and $p_1^*, p_2^*, p_3^*,$ respectively, must be related by eqs. (A.39).

While the above development has focused on the position vector of an arbitrary point \mathbf{P} , similar arguments could have been used for other vectors, such as displacement vectors or force vectors. Equations (A.39) are very general and express the relationship between the components of any vector in two different bases. In fact, eqs. (A.39) can be taken as the definition of a vector quantity: *a vector is a mathematical entity whose components in two different bases are related by eqs. (A.39).*

A.3.3 The rotation matrix in two dimensions

The previous section has focused on coordinate transformations in three dimensions. In many cases, however, a simpler, two-dimensional transformation is sufficient. Consider, for instance, the plane stress or plane strain problems investigated in sections 1.3 or 1.6, respectively. Two unit vectors, say \bar{i}_1 and $\bar{i}_2,$ define the plane of the problem, whereas \bar{i}_3 is normal to this plane, as depicted in fig. A.2. A second set of unit vectors, \bar{i}_1^* and $\bar{i}_2^*,$ is now selected such that the angle between axes \bar{i}_1 and \bar{i}_1^* is θ . Note that since (\bar{i}_1, \bar{i}_2) and $(\bar{i}_1^*, \bar{i}_2^*)$ define the same plane, $\bar{i}_3 = \bar{i}_3^*$ are both normal to this plane.

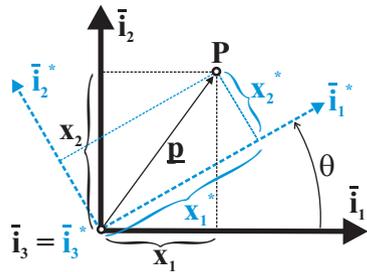


Fig. A.2. Position of a point in two coordinate systems.

Two orthonormal bases, $\mathcal{I} = (\bar{i}_1, \bar{i}_2, \bar{i}_3)$ and $\mathcal{I}^* = (\bar{i}_1^*, \bar{i}_2^*, \bar{i}_3^*),$ have now been defined, a situation similar to that of section A.3.1. Many of the direction cosines of the present problem, however, have special values, because $\bar{i}_3 = \bar{i}_3^*$. For instance, $\ell_1 = \bar{i}_1^* \cdot \bar{i}_1 = \cos \theta, \ell_2 = \bar{i}_1^* \cdot \bar{i}_2 = \sin \theta,$ and $\ell_3 = \bar{i}_1^* \cdot \bar{i}_3 = 0.$ Similarly, $m_1 = \bar{i}_2^* \cdot \bar{i}_1 = -\sin \theta, m_2 = \bar{i}_2^* \cdot \bar{i}_2 = \cos \theta,$ and $m_3 = \bar{i}_2^* \cdot \bar{i}_3 = 0;$ finally, $n_1 = \bar{i}_3^* \cdot \bar{i}_1 = 0, n_2 = \bar{i}_3^* \cdot \bar{i}_2 = 0,$ and $n_3 = \bar{i}_3^* \cdot \bar{i}_3 = 1.$ The rotation matrix defined by eq. (A.36) now simplifies to

$$\underline{\underline{R}} = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The entries in the last line and column simply imply that the component of a vector along axis $\bar{i}_3 = \bar{i}_3^*$ is unaffected by the change of basis. In many cases, it is not necessary to use a 3×3 rotation matrix; the use of the smaller size, 2×2 rotation matrix is then preferable,

$$\underline{\underline{R}} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}. \tag{A.40}$$

The fact that the direction cosine matrix is an orthogonal matrix, as show in eq. (A.37), is now a straightforward consequence of trigonometric identities,

$$\underline{\underline{R}}\underline{\underline{R}}^T = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \underline{\underline{I}}. \quad (\text{A.41})$$

A.3.4 Rotation of vector components in two dimensions

Consider now the position vector of point \mathbf{P} , denoted \underline{p} in fig. A.2, and its components expressed in the two bases, \mathcal{I} and \mathcal{I}^* ,

$$\underline{p} = p_1 \bar{v}_1 + p_2 \bar{v}_2 = p_1^* \bar{v}_1^* + p_2^* \bar{v}_2^*. \quad (\text{A.42})$$

Vector \underline{p} is assumed to be in plane (\bar{v}_1, \bar{v}_2) , and hence its component along axis $\bar{v}_3 = \bar{v}_3^*$ vanishes. The components of vector \underline{p} in basis \mathcal{I} are denoted p_1 and p_2 , whereas those in basis \mathcal{I}^* are denoted p_1^* and p_2^* . Following the procedure developed in section A.3.2, the following relationship is found between these two sets of components $p_1 \bar{v}_1 + p_2 \bar{v}_2 = p_1^* (\cos \theta \bar{v}_1 + \sin \theta \bar{v}_2) + p_2^* (-\sin \theta \bar{v}_1 + \cos \theta \bar{v}_2)$. A scalar product of this result by \bar{v}_1 and \bar{v}_2 the following equations

$$\begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{Bmatrix} p_1^* \\ p_2^* \end{Bmatrix} = \underline{\underline{R}} \begin{Bmatrix} p_1^* \\ p_2^* \end{Bmatrix}.$$

The rotation matrix expresses the linear relationship between the components of vector \underline{p} in bases \mathcal{I} and \mathcal{I}^* . Of course, the inverse relationship is easy to find

$$\begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix} = \underline{\underline{R}} \begin{Bmatrix} p_1^* \\ p_2^* \end{Bmatrix} \iff \begin{Bmatrix} p_1^* \\ p_2^* \end{Bmatrix} = \underline{\underline{R}}^T \begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix}, \quad (\text{A.43})$$

because the rotation matrix is orthogonal, see eq. (A.41). These equations should be compared to their three-dimensional counterparts, eq. (A.39).

A.4 Orthogonality properties of trigonometric functions

Trigonometric functions enjoy remarkable orthogonality properties, which are often used to obtain series solution of various problems. The Kronecker delta symbol will be used to express these properties in a compact manner and is defined as

$$\delta_{ij} = \begin{cases} 1 & i = j, \\ 0 & i \neq j. \end{cases} \quad (\text{A.44})$$

Consider now the product of sine or cosine functions with different wave numbers, m an n . The integration of these products leads to the following results

$$\int_0^1 \sin m\pi\eta \sin n\pi\eta \, d\eta = \frac{\delta_{mn}}{2}, \quad (\text{A.45a})$$

$$\int_0^1 \cos m\pi\eta \cos n\pi\eta \, d\eta = \frac{\delta_{mn}}{2}. \quad (\text{A.45b})$$

The integration of the product of two sine functions with different wave numbers vanishes, except when the two wave numbers are identical; the same is true for the cosine function. Equation (A.45a) expresses the orthogonality of the sine functions: the set of functions, $\sin m\pi\eta$, $m = 1, 2, \dots, \infty$, is said to be orthogonal over the range $\eta \in [0, 1]$ because eq. (A.45a) holds. The cosine functions, $\cos m\pi\eta$, $m = 1, 2, \dots, \infty$, are also orthogonal over the same range. Similarly, the following results can be verified,

$$\int_{-1/2}^{+1/2} \sin m\pi\eta \sin n\pi\eta \, d\eta = \frac{\delta_{mn}}{2}, \tag{A.46a}$$

$$\int_{-1/2}^{+1/2} \cos m\pi\eta \cos n\pi\eta \, d\eta = \frac{\delta_{mn}}{2}. \tag{A.46b}$$

The sine functions, $\sin m\pi\eta$, $m = 1, 2, \dots, \infty$, are also orthogonal over the range $\eta \in [-1/2, +1/2]$.

The following definite integral are also useful

$$\int_{-1/2}^{+1/2} \cos m\pi\eta \, d\eta = \frac{2}{m\pi} \begin{cases} 0, & m \text{ even,} \\ (-1)^{(m-1)/2}, & m \text{ odd.} \end{cases} \tag{A.47}$$

$$\int_0^1 \sin m\pi\eta \, d\eta = \frac{2}{m\pi} \begin{cases} 0, & m \text{ even,} \\ 1, & m \text{ odd.} \end{cases} \tag{A.48}$$

A.5 Gauss-Legendre quadrature

When applying energy methods, the computation of the stiffness matrix and load array involves integrations of the product of the shape functions by the stiffness properties of the structure. As the number of assumed shape function increases, it becomes increasingly cumbersome to perform all these integration in closed form, especially when the expression for the shape functions becomes complex.

To circumvent this problem, numerical integration can be used. A very powerful tool for numerical integration is the Gauss-Legendre quadrature scheme. In its simplest form [11], this scheme approximately evaluates an integral by the following sum

$$\int_{-1}^{+1} f(\eta) \, d\eta \approx \sum_{i=1}^N w_i f(\eta_i), \tag{A.49}$$

where η_i , $i = 1, 2, \dots, N$ are the *Gauss-Legendre quadrature points*, and w_i the associated *weights*. The Gauss-Legendre quadrature points are often called *sampling points*, because the integral is evaluated by sampling the value of the integrand at these points. Table A.1 lists the Gauss-Legendre quadrature points and associated weights for $N = 2, 3$, and 4. The fundamental property of the N point Gauss-Legendre quadrature scheme is that it exactly integrates a polynomial of degree $2N - 1$.

Table A.1. Gauss points and associated weights for $N = 2, 3,$ and 4 .

N	η_i	w_i
2	$\pm\sqrt{1/3}$	1
3	0 $\pm\sqrt{3/5}$	8/9 5/9
4	$\pm\sqrt{(3 - 2\sqrt{6/5})/7}$ $\pm\sqrt{(3 + 2\sqrt{6/5})/7}$	$(18 + \sqrt{30})/36$ $(18 - \sqrt{30})/36$

To illustrate the application of the Gauss-Legendre quadrature scheme, consider the following integral

$$I = \int_{-1}^{+1} [x^4 - 5x^3 + 3x^2 + 5x] dx = 2.4.$$

At first, the 2-point quadrature formula is used to find

$$I \approx \left[\left(\frac{1}{3}\right)^2 + 5 \left(\frac{1}{3}\right)^{3/2} + 3\frac{1}{3} - 5 \left(\frac{1}{3}\right)^{1/2} \right] + \left[\left(\frac{1}{3}\right)^2 - 5 \left(\frac{1}{3}\right)^{3/2} + 3\frac{1}{3} + 5 \left(\frac{1}{3}\right)^{1/2} \right] = \frac{20}{9} = 2.22.$$

This 2-point formula exactly integrates a polynomial of degree $2 \times 2 - 1 = 3$; hence, an approximate answer is expected for this integral involving a polynomial of degree four. The approximate answer only incurs a 7.4% error. Next, the 3-point quadrature formula is used, leading to

$$I \approx \frac{5}{9} \left[\left(\frac{3}{5}\right)^2 + 5 \left(\frac{3}{5}\right)^{3/2} + 3\frac{3}{5} - 5 \left(\frac{3}{5}\right)^{1/2} \right] + \frac{5}{9} \left[\left(\frac{3}{5}\right)^2 - 5 \left(\frac{3}{5}\right)^{3/2} + 3\frac{3}{5} + 5 \left(\frac{3}{5}\right)^{1/2} \right] = \frac{60}{25} = 2.4.$$

This 3-point formula exactly integrates a polynomial of degree $3 \times 2 - 1 = 5$; hence, the exact solution is recovered.

Next, consider the following integral involving transcendental function

$$I = \int_1^5 \frac{1}{x} dx = [\ln x]_1^5 = \ln 5 = 1.609.$$

To recast the problem in the standard form, a change of variable, $x = 2\eta + 3$, is first performed. The Jacobian of the coordinate transformation is readily evaluated, $dx/d\eta = 2$. The 2-point quadrature formula then yields a first approximation of the integral

$$I = \int_{-1}^{+1} \frac{1}{2\eta + 3} \frac{dx}{d\eta} d\eta \approx 2 \left[\frac{1}{-2\sqrt{1/3} + 3} + \frac{1}{2\sqrt{1/3} + 3} \right] = \frac{36}{23} = 1.565,$$

which only involves a 2.75% error. To improve the approximation, the 3-point quadrature formula is used, leading to

$$I \approx \frac{2}{9} \left[\frac{5}{-2\sqrt{3/5} + 3} + \frac{8}{3} + \frac{5}{2\sqrt{3/5} + 3} \right] = \frac{476}{297} = 1.603.$$

The error is now reduced to about 0.42%. Higher order Gauss-Legendre quadrature scheme can be derived that involve an increasing number of sampling points and associated weights. This data have been tabulated, see Abramowitz and Stegun [12], or can be readily calculated [11].

For integration over a rectangular domain, the basic Gauss-Legendre quadrature scheme of eq. (A.49) is generalized as

$$\int_{-1}^{+1} \int_{-1}^{+1} f(\eta, \zeta) d\eta d\zeta \approx \sum_{i=1}^N \sum_{j=1}^M w_i w_j f(\eta_i, \zeta_j), \quad (\text{A.50})$$

where the sampling points, η_i and ζ_j , and associated weights, w_i and w_j , respectively, are those listed in table A.1.

References

1. B.K. Donaldson. *Analysis of Aircraft Structures*. Cambridge University Press, Cambridge, New York, second edition, 2008.
2. T.H.G. Megson. *Aircraft Structures for Engineering Students*. Elsevier Aerospace Engineering Series, Oxford, fourth edition, 2007.
3. D.J. McGill and W.W. King. *An Introduction to Dynamics*. PWS-KENT Publishing Company, Englewood Cliffs, New Jersey, third edition, 1995.
4. J.H. Ginsberg. *Advanced Engineering Dynamics*. Cambridge University Press, Cambridge, second edition, 1998.
5. R. Ewing. *Calculus of Variations with Applications*. Dover Publications, Inc., New York, 1985.
6. R. Weinstock. *Calculus of Variations with Applications to Physics and Engineering*. Dover Publications, Inc., New York, 1974.
7. F.B. Hildebrand. *Advanced Calculus for Applications*. Prentice Hall, Inc., Englewood Cliffs, New Jersey, second edition, 1976.
8. T.J.R. Hughes. *The Finite Element Method*. Prentice Hall, Inc., Englewood Cliffs, New Jersey, 1992.
9. K.J. Bathe. *Finite Element Procedures*. Prentice Hall, Inc., Englewood Cliffs, New Jersey, 1996.
10. G. Strang. *Linear Algebra and its Applications*. Thomson-Brooks/Cole, Toronto, 2006.
11. W.H. Press, B.P. Flannery, S.A. Teutolsky, and W.T. Vetterling. *Numerical Recipes. The Art of Scientific Computing*. Cambridge University Press, Cambridge, 1990.
12. M. Abramowitz and I.A. Stegun. *Handbook of Mathematical Functions*. Dover Publications, Inc., New York, 1964.

Index

- Aeroelasticity, 357
- Affine transformation, 859, 899
- Airy's stress function, 111
- Analytical mechanics, 395
- Angular distortion, 34
- Anisotropic plate, 851, 883
- Anticlastic bending, 837
- Area static moment, 308
- Axial force, 177

- Balanced laminate, 844, 847, 852
- Beltrami-Michell's equations, 104
- Bending moment, 178, 826
- Bending stiffness ellipse, 235
- Bending stiffness matrix, 830, 842
- Bifurcation point, 511
- Boolean matrix, 537, 663
- Boundary conditions
 - essential, 606
 - geometric, 606, 681, 697
 - natural, 680, 697
- Bredt-Batho formula, 343
- Buckling
 - equation, 770, 779, 781
 - load, 512, 764
 - mode shape, 771
 - of a rigid bar, 510, 763–766
 - of beams, 767–784
- Bungee cord, 507

- Calculus of variations, 412, 527, 673–679
- Cauchy's tetrahedron, 11, 21
- Center of twist, 362
- Central solution, 170

- Centroid, 188
- Centroidal bending stiffness, 188
- Characteristic equation, 922
- Circular plates, 865
- Closing shear flow, 326
- Co-location method, 285
- Coefficient of thermal expansion, 59, 742
- Compatibility equations, 50
 - Truss, 438
- Compatible
 - strain field, 681
 - virtual strain field, 682
- Complementary strain energy, 505, 550, 685
- Complementary strain energy density
 - for beam bending, 516
 - for beam extension, 515
 - for beam torsion, 517
 - for three-dimensional beam, 521
 - for three-dimensional solid, 520
- Complementary strain energy density function, 685
- Complementary virtual work, 441, 445
- Configuration
 - deformed, 768
 - reference, 33, 768
- Connectivity matrix, 536, 663
- Conservative forces, 494–499
- Conservative system, 498
- Constitutive laws, 102, 681
- Constraint method
 - for bars and trusses, 748
 - for beams, 753
 - for thermoelasticity, 742, 746–748

- Continuous problems, 527
- Convergence plot, 595
- Corner force, 836
- Corner force bending, 836, 838
- Coupled bending-torsion problems, 354
- Coupling stiffness matrix, 842
- Critical load, 512, 764
- Cross bending stiffness, 226
- Curvature interpolation matrix, 885

- Dead loads, 499
- Deformation energy, 498
- Deformed configuration, 33, 768, 897
- Degree of freedom, 413, 586, 609, 639, 884
- Degree of redundancy, 144
- Design
 - buckling, 75
 - stiffness, 74
 - strength, 74
- Deviatoric stress tensor, 723
- Direct method for thermoelasticity, 742–746
- Direction cosine, 10, 11
- Direction cosine matrix, 927
- Discrete problems, 527
- Displacement gradient interpolation
 - array, 777, 778
 - matrix, 905
- Displacement gradients array, 904
- Displacement interpolation array, 640, 885
- Displacement method, 146
- Displacements
 - infinitesimal, 410
 - virtual, 410
- Dot product, 918
- Driving force, 443
- Dummy load method, 570–572

- Eigenvalue problem, 922
- Eigenvalues, 922
- Eigenvectors, 922
- Elastic axis, 356
- Elastic limit, 726
- Elastic material, 685
- Elastic stability, 763
- Element load array, 535
- Element stiffness matrix, 534, 661
- Energy
 - complementary strain, 505, 685
 - deformation, 498
 - internal, 498
 - strain, 498, 685
 - stress, 505, 685
 - total potential, 498
- Energy theorems, 560, 690
 - Castigliano's first theorem, 563
 - Castigliano's second theorem, 565
 - Clapeyron's theorem, 561
 - Crotti-Engesser theorem, 564
- Engineering constant, 85
- Engineering structural analysis, 137
- Equilibrium equations, 7–11, 101, 680
 - force, 8, 680
 - moment, 9
 - traction, 11, 680
- Equivalent stress, 70, 725
- Essential boundary conditions, 606, 619
- Euler buckling load, 771
- Euler's
 - first law, 397, 415
 - second law, 398, 415
- Euler-Bernoulli beams
 - assumptions, 174–177
 - axial stiffness, 182
 - axial stress distribution, 182, 193
 - bending stiffness, 191
 - equilibrium equations, 180, 188, 218
 - governing equations, 181, 189, 219
 - kinematic description, 179, 186, 217
 - stress resultants, 177
 - under axial loads, 178–186
 - under combined loads, 217–221
 - under transverse loads, 186–217
- Euler-Cauchy differential equation, 116
- Euler-Lagrange equation, 678, 696
- External complementary virtual work, 444
- External force, 407, 413, 421

- Factor of safety, 67
- Failure load, 67
- Finite dimensional problems, 527
- Finite element method, 584
 - beam element assembly procedure, 662
 - beam element stiffness matrix, 661
 - Beam governing equations, 666
 - for beams, 654
 - for trusses, 529
 - Kinematics of beam element, 657
 - Strain energy in beam element, 660

- Flexibility matrix, 203, 204
- Flexibility method, 146, 151
- Flow
 - axial stress, 299
 - shear stress, 299
- Force
 - axial, 177
 - conservative, 494–499
 - external, 407, 413, 421
 - in-plane, 824
 - internal, 407, 413, 421
 - natural, 407
 - non-conservative, 503
 - residual, 161, 728
 - shear, 177
 - thermal, 161
 - transverse shear, 825
- Force equilibrium equations, 8, 680
- Force method, 146, 151, 473
- Four-point bending test, 192, 815
- Functional, 527, 585, 677

- Galerkin's method, 609, 622, 639, 640
- Gauss-Legendre quadrature, 931–933
 - points, 931
 - weights, 931
- Generalized
 - coordinates, 420
 - forces, 420
- Generalized Hooke's law, 55
- Geometric boundary conditions, 606, 619, 681, 697
- Geometric nonlinearities, 510, 513
- Geometric stiffness matrix, 778, 905
- Global displacement array, 657
- Global load array, 537, 664
- Global stiffness matrix, 537, 664

- Hellinger-Reissner's principle, 694–695
- Homogeneous system, 922
- Hooke's law, 55
- Hu-Washizu's principle, 690–694
- Hydrostatic pressure, 26, 722

- Imperfect system, 765
- In-plane force, 824
- In-plane loads matrix, 904
- In-plane stiffness matrix, 829, 842
- Incompressible material, 59

- Incremental work, 399
- Ineffective length, 81
- Infeasible direction, 406
- Infinite dimensional problems, 527
- Infinitesimal displacements, 410
- Influence coefficients, 203, 204, 579
- Internal energy, 498
- Internal force, 407, 413, 421
- Internal virtual work, 456–461
 - for beam bending, 457
 - for beam twisting, 458
 - for Euler-Bernoulli beams, 461
 - for solid, 459
- Interpolation matrix
 - curvature, 777
 - strain, 777
- Invariants
 - strain, 41
 - stress, 14
- Isotropic plate, 883

- Kinematically admissible
 - displacement field, 681
- Kinematically admissible direction, 406
- Kinematically admissible virtual displacements, 405
- Kinematically inadmissible direction, 406
- Kirchhoff plate theory, 819

- Lagrange multiplier, 676
- Lagrange multiplier method, 675
- Lamé's ellipse, 30
- Laminate
 - balanced, 844, 847, 852
 - special orthotropic, 847, 899
- Laminated composite plate, 840
- Laplace's equation, 279
- Lay-up parameters, 843, 846, 849
- Least-square solution, 926
- Limit of proportionality, 63
- Linear elasticity, 137
- Linearly elastic spring, 504
- Load array, 613, 623, 641, 886
- Load factor, 67
- Local crippling, 908

- Manufacturing imperfection, 161
- Material
 - anisotropic, 53, 83

- brittle, 65
- compliance matrix, 57, 520
- ductile, 63
- elastic, 685
- elastic-perfectly plastic, 64
- engineering constants, 85
- homogeneous, 53
- isotropic, 53, 84
- linearly elastic, 55
- monoclinic, 84
- orthotropic, 84
- perfectly plastic, 64
- plastic flow, 63
- stiffness matrix, 57, 58, 520
- strain hardening, 64
- transversely isotropic, 84
- triclinic, 83
- Material line, 34
- Material nonlinearities, 513
- Matrix
 - definition, 917
 - diagonal, 918
 - identity, 918
 - inverse, 921
 - multiplication, 920
 - of direction cosine, 927
 - orthogonal, 927
 - rotation, 927
 - skew-symmetric, 918
 - square, 917
 - symmetric, 918
 - transpose, 917
- Mechanical work, 399
- Modulus
 - bulk, 59
 - elasticity, 55
 - shear, 56
 - Young's, 55
- Modulus-weighted centroid, 188
- Mohr's circle
 - for plane strain, 44
 - for plane stress, 27
- Moment equilibrium equations, 9
- Moment-curvature relationship, 188

- Natural boundary conditions, 680, 697
- Natural force, 407
- Navier's equations, 103
- Necking, 63

- Neutral axis, 195
- Neutral axis, 236–237
- Newton's first law, 396
- Newton's third law, 397
- Non-conservative forces, 503
- Non-uniform torsion, 362, 372–374, 707, 894
- Nonlinearly elastic spring, 505
- Normal material line, 820

- Octahedral face, 18
- Orthogonal matrix, 927

- Parallel axis theorem, 240
- Participation factor, 639
- Perfect system, 765
- Plane strain rotation, 42
- Plastic bending, 732–737
- Plastic limit, 726
- Plate
 - anisotropic, 851, 883
 - bending stiffness, 832
 - boundary conditions, 832
 - circular, 865
 - corner force bending, 836
 - corner forces, 836
 - curvatures, 823
 - energy formulation, 875
 - governing equation, 831
 - isotropic, 883
 - Lévy's solution, 859
 - mid-plane strains, 823
 - mid-plane symmetry, 852
 - Navier's solution, 853
 - specially orthotropic, 852, 883
 - strain energy, 882
- Poisson's equation, 280
- Poisson's ratio, 55
- Potential
 - definition, 496
- Potential of
 - a conservative force, 495
 - dead loads, 499
 - dislocations, 550
 - externally applied loads, 528
 - prescribed displacements, 550, 552
 - the body forces, 687
 - the external loads, 498
 - the prescribed displacements, 688

- the surface tractions, 687
- Prandtl's stress function, 279, 703
- Pressure vessel, 26
- Principal
 - strain, 40, 43
 - stress, 23
- Principal centroidal
 - axes of bending, 231
 - bending stiffness, 234
- Principle of
 - complementary virtual work, 437–444, 683
 - least work, 553
 - minimum complementary energy, 548–560
 - minimum total potential energy, 500–504
 - reciprocity of shear stresses, 9
 - virtual work, 396, 400–415, 433, 434, 682
- Principle of virtual work
 - for a particle, 400
 - for a particle system, 412
 - for trusses, 432
- Quadratic form, 923
- Rational design of
 - beams under bending, 194
 - cylinders under torsion, 270
- Rayleigh-Ritz method, 586–588
- Reciprocity theorems, 577
 - Betti's theorem, 577
 - Maxwell's theorem, 579
- Rectilinear spring, 504
- Reference configuration, 33, 768, 897
- Relative elongation, 34
- Residual
 - force, 161
 - stress, 161
- Residual force, 162, 728
- Rotation matrix, 927
- Rotational spring, 504, 508
- Saint-Venant solution for torsion, 277, 701, 703
- Saint-Venant's principle, 169
- Scalar product, 918
- Second order tensors, 234
- Sectional stiffness matrix, 226
- Shape function, 586, 609, 639, 658, 659, 884
- Shear center, 318–321, 334
 - for open sections, 318
- Shear flexibility parameter, 806
- Shear force, 177
- Shear lag, 82, 697
- Shear stiffness, 794, 797
- Shear stiffness of sandwich beam, 800
- Shear strain components
 - engineering, 39
 - tensor, 39
- Specially orthotropic laminate, 847, 899
- Specially orthotropic plate, 852, 883
- Spring
 - linearly elastic, 504
 - nonlinearly elastic, 505
 - rectilinear, 504
 - rotational, 504, 508
 - torsional, 504, 508
- Stacking sequence, 843, 846, 849
- State of strain at a point, 34–41
- State of stress
 - hydrostatic, 26, 722
 - in thin-walled pressure vessels, 26
 - pure shear, 26
- State of stress at a point, 11–16
- Static moments
 - area, 308
 - stiffness, 308
- Statically
 - determinate, 141, 144
 - indeterminate, 141, 144
- Statically admissible
 - forces, 442
 - stress field, 680
 - virtual forces, 442
 - virtual stress field, 684
- Stiffness first moments, 307
- Stiffness matrix, 613, 623, 641, 886
 - bending, 830, 842
 - coupling, 842
 - in-plane, 829, 842
- Stiffness method, 146
- Stiffness static moments, 307, 308
- Strain
 - axial, 37
 - compatibility equations, 50
 - direct, 37
 - gauges, 46
 - invariants, 41

- measurement of, 45
- plane strain state, 41–45
- principal, 40, 41, 43
- principal directions, 41
- principal plane strains, 43
- shear, 37
- shearing, 37
- tensor, 40
- the concept of, 33–38
- thermal, 59, 741, 742
- volumetric, 37
- Strain energy, 494, 498, 685
 - in bar torsion, 516
 - in beam bending, 515
 - in beam extension, 514
 - in beams, 514–519
 - in plates, 882
 - in solids, 519
 - in springs, 504–509
 - in three-dimensional beams, 520
- Strain energy density
 - for beam bending, 516
 - for beam extension, 515
 - for beam torsion, 517
 - for three-dimensional beam, 520
 - for three-dimensional solid, 520
- Strain energy density function, 685
- Strain gauge rosette, 47
- Strain gauges, 46
- Strain interpolation array, 640
- Strain interpolation matrix, 885
- Strain, plane strain rotation, 42
- Strain-displacement equations, 101
- Strain-displacement relationships, 681
- Stress
 - allowable, 66
 - concentration factor, 124, 130
 - direct, 6
 - engineering, 6
 - equivalent, 70, 725
 - failure, 63
 - invariants, 14
 - normal, 6
 - plane strain state, 58
 - plane stress state, 20–25, 58
 - principal, 14, 722
 - principal directions, 14, 722
 - residual, 161
 - shear, 6
 - state at a point, 3
 - the concept of, 3–6
 - thermal, 59, 741, 743
 - true, 63
 - yield, 63
- Stress energy, 505, 685
- Stress resultants, 177, 824
- Stress vector, 4
- Strong statement of equilibrium, 603
- Structural analysis, 137
- Structural idealization, 377
 - sheets, 378
 - stringers, 378
- Surface traction, 10
- Synclastic bending, 837
- System of particles, 397, 414
- Systems
 - hyperstatic, 141, 144, 431, 440
 - isostatic, 141, 144, 431, 440
- Tensor
 - first order, 6
 - Second order, 40
 - second order, 6
 - strain, 40
- Test array, 613
- Test function, 604
- Thermal
 - force, 161
 - strain, 59, 741, 742
 - stress, 59, 158, 741, 743
- Thermoelasticity, 741
 - constraint method, 742, 746–748
 - direct method, 742–746
 - equivalent thermal body forces, 747
 - equivalent thermal surface tractions, 747
- Thin-walled beams
 - assumptions, 297
 - axial flow, 299
 - bending of, 303
 - non-uniform torsion, 372–374
 - shear center, 371
 - shear center location, 334
 - shear flow, 299
 - shearing of closed sections, 325
 - shearing of open sections, 308
 - stress resultants, 299
 - structural idealization, 377
 - torsion, 343

- torsion of closed sections, 343
- torsion of open sections, 343
- torsional stiffness, 345
- twist center, 371
- warping, 362
- warping of closed sections, 369
- warping of open sections, 364
- Three-point bending test, 815
- Timoshenko beam theory, 794
- Torsion
 - non-uniform, 362, 372–374, 707
 - uniform, 277, 702
- Torsion of
 - a rectangular section, 284–288
 - a thin rectangular strip, 290
 - an elliptical section, 281
 - bars of arbitrary shape, 275–281
 - circular cylinders, 261–270
 - thin-walled open sections, 292
- Torsional spring, 504, 508
- Torsional stiffness, 266
- Total complementary energy, 689
- Total potential energy, 498, 687
- Total stress energy, 688
- Total vertical load, 833
- Traction equilibrium equations, 11, 680
- Transverse shear force, 825
- Trivial solution, 922
- Truss
 - elongation-displacement equations, 429
 - method of joints, 431
 - pinned joints, 428
 - planar, 428
 - spatial, 428
 - structures, 428–431
- Twist rate, 263
- Twisting moment, 826
- Uniform torsion, 277, 702
- Unit load method, 449
 - for beams, 462
 - for combined trusses and beams, 485
 - for hyperstatic beams, 480
 - for hyperstatic problems, 472
 - for hyperstatic trusses, 473
 - for multiple redundancies, 487
 - for trusses, 449
- Unit load method
 - for shear-deformable beams, 811
- Unit load system, 451, 463
- Unit vector, 919
- Vector product, 920
- Virtual
 - displacements, 400, 410
 - forces, 441
 - work, 400
- Virtual displacements
 - displacement dependent forces, 411
 - rigid bodies, 411
- Virtual work
 - external, 413
 - internal, 413
- Warping, 276
- Warping shear flow, 372
- Warping stiffness, 373, 708, 894
- Weak statement of equilibrium, 603
 - for beams under axial loads, 604
 - for beams under transverse loads, 617
- Weighting function, 604
- Work
 - incremental, 399
 - mechanical, 399
- Yield criterion
 - maximum shear stress, 69
 - Tresca's, 68
 - Von Mises', 70
- Yield envelope, 72, 724
- Yield stress, 63
- Yielding under combined loading, 721–725