Answers

Chapter 1

1. a. $\frac{3}{20} - \frac{1}{20} i$.   b. $\frac{-3}{2} + \frac{11}{2} i$.

   c. $\frac{-1}{2} + \frac{\sqrt{3}}{2} i$.   d. $-1, -i, 1, i, \ldots$.

2. $\pm(1 + 3i)$.

3. $1 + (3 - \sqrt{8})i$; $-1 - (3 + \sqrt{8})i$.

4. a. b., d.: Let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, etc.

   c. Use properties (a), (b) and the fact that a real number is its own conjugate.

5. Use Exercise (4c): $P(\bar{z}) = \overline{P(z)}$.

6. If $z = x + iy = re^{i\theta}$, then $z^2 = x^2 - y^2 + 2ixy = r^2 e^{2i\theta}$, etc.

7. (c) Equality occurs if, and only if, Re z or Im $z = 0$.

8. (b) $|z_1 + z_2| = |z_1| + |z_2|$ if, and only if, $\text{Re}(z_1\bar{z_2}) = |z_1||z_2|$, i.e., if, and only if, $z_1\bar{z_2}$ is a real number, or, equivalently, $\text{Arg } z_1 = \text{Arg } z_2$.

11. Note that $\tan^{-1}$ is always between $\frac{-\pi}{2}$ and $\frac{\pi}{2}$ whereas $\text{Arg } z$ takes values (*modulo* 2π) in the interval 0 to 2π.

12. a. 1, cis $\frac{\pi}{3}$, cis $\frac{2\pi}{3}$, -1, cis $\frac{4\pi}{3}$, cis $\frac{5\pi}{3}$.

   b. cis θ; $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$.

   c. $2^{1/4}$ cis θ; $\theta = \frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6}, \frac{5\pi}{3}$.

13. Note that the nth roots of unity are all zeroes of $z^n - 1 = (z - 1)(z^{n-1} + z^{n-2} \cdots +1)$.

14. Using (13) and the fact that $z^n - 1 = (z - z_1)(z - z_2) \cdots (z - z_n)$ where $z_1, z_2, \ldots, z_n$ are the nth roots of unity, we have (assuming $z_1 = 1$)

$$z^{n-1} + z^{n-2} + \cdots + 1 = (z - z_2) \cdots (z - z_n)$$

   Let $z = 1$, and consider the absolute value of the two sides of the above equation.

15. a. Closed disc centered at i with radius 1; not a region.

   b. The imaginary axis; not a region.

   c. The open half-plane: Re $z > 5/2$; a region.

   d. The upper half of the unit disc, centered at the origin; a region.

   e. The unit circle centered at the origin; not a region.

   f. The circle centered at $\frac{1}{2} i$ with radius $\frac{1}{2}$; not a region.
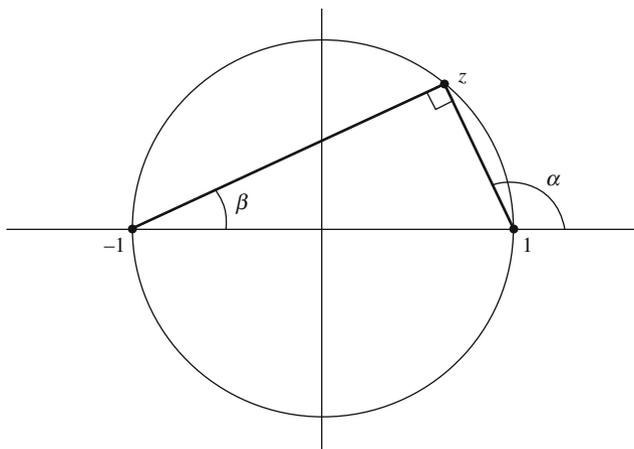
g. The set of points satisfying $r^2 < 2\cos 2\theta$, $r \neq 0$; not connected and, therefore, not a region.

17. $\text{Arg}\left(\frac{z-1}{z+1}\right) = \text{Arg}(z-1) - \text{Arg}(z+1)$. Consider the diagram below for $\text{Re } z > 0$ and an analogous argument for $\text{Re } z < 0$.

21. a. For $|z| \leq r < 1$, $\sum |kz^k| < \sum kr^k < \infty$. Apply Theorem 1.9.

b. For $\text{Re } z \geq 0$, $\sum \left|\frac{1}{k^2+z}\right| \leq \sum \frac{1}{k^2}$. Apply Theorem 1.9.

22. Assume S is polygonally connected, and S is "disconnected" by open sets A and B with $a \in A$, $b \in B$. Consider the polygonal line $L(t)$, $0 \leq t \leq 1$, connecting a and b, and let $c = L(t_0)$, $t_0 = \sup\{t_1 : L(t) \in A, \ 0 \leq t \leq t_1\}$. Then, $c \in A$, but, because A is open, so is $L(t_0 + \delta)$, unless $c = b$.

23. Note that no curve in S (of finite length) can connect a point of S, not on the y axis, with a point on the y axis. Nevertheless, S cannot be "disconnected" because any open set which contains a point of the form $(0, \ t)$, $-1 < t < 1$, would also have to contain points of the form $y = \sin\frac{1}{x}$, $x > 0$.

24. $\zeta \geq \zeta_0 \Rightarrow x^2 + y^2 \geq \frac{\zeta_0}{1-\zeta_0}$

25. $A(x^2 + y^2) + Bx + Cy + D = 0$ is equivalent to $A\left(\frac{\zeta}{1-\zeta}\right) + B\xi\left(\frac{1}{1-\zeta}\right) + C\eta\left(\frac{1}{1-\zeta}\right) + D = 0$ or $A\zeta + B\xi + C\eta + D(1-\zeta) = 0$, $\zeta \neq 1$. Consider $A = 0$ and $A \neq 0$.

26. If $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$, $a_n \neq 0$, then $P(z) = z^n \left(a_n + \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n}\right)$, etc.

27. a. Use formulas (3) for $(\xi, \ \eta, \ \zeta)$ and the fact that $\frac{1}{z} = \frac{1}{x+iy} = \frac{x}{x^2+y^2} - \frac{iy}{x^2+y^2}$ to show that $(\xi', \ \eta', \ \zeta') = (\xi, \ -\eta, \ 1-\zeta)$.

b. Consider the corresponding points on $\sum$ and apply 27 (a).

28. Note that according to (27), $A\xi + B\eta + C\zeta = D$ is equivalent to $A\xi' - B\eta' - C\zeta' = D - C$.

Chapter 2

1. For $P(x, y) = (x + iy)^n$, $P_x = n(x + iy)^{n-1}$ and $P_y = in(x + iy)^{n-1}$.

3. a. Analytic: $P(z) = z^3 - z$.
   b. Not analytic.
   c. Analytic: $P(z) = -iz^2$.

4. $P_y = i P_x$ would imply $P_y \equiv P_x \equiv 0$ because both P_y and P_x would be imaginary.

5. In (3a), $P'(z) = 3z^2 - 1 = 3x^2 - 3y^2 - 1 + i6xy$. In (3c), $P'(z) = -2iz = 2y - 2ixy$.

6. Use the usual properties of limits, etc.

7. $\lim_{h \to 0} \frac{(z+h)^n - z^n}{h} = \lim_{h \to 0} [(z + h)^{n-1} + (z + h)^{n-2}z + \cdots + z^{n-1}] = nz^{n-1}$

8. $\log S_n = \frac{\log n}{n} \to 0$ as $n \to \infty$, so $S_n = e^{\log S_n} \to 1$.

9. a. 1.   b. 1/2.

10. a. R.   b. R.   c. R^2.

11. The radius of convergence must be greater than or equal to $\min(R_1, R_2)$. It may exceed both if, e.g., $b_n = -a_n$ for all n and $R_1, R_2 < \infty$.

12. Use the facts that $\sum \frac{\sin n\theta}{n}$ converges for all θ and $\sum \frac{\cos n\theta}{n}$ converges for $\theta : 0 < \theta < 2\pi$ (both can be proven by Dirichlet's Test).

13. a. Write $a_n = a_1 \left(\frac{a_2}{a_1}\right) \cdots \left(\frac{a_n}{a_{n-1}}\right)$ and note that, according to the hypothesis, for any $\epsilon > 0$, there is some k such that $j > k \Rightarrow \left|\frac{a_j}{a_{j-1}} - L\right| < \epsilon$. Thus, $M_k(L - \epsilon)^{n-k} < a_n < M_k(L + \epsilon)^{n-k}$, etc.
    b. Note that, for $a_n = \frac{1}{n!}$, $\frac{a_{n+1}}{a_n} = \frac{1}{n+1}$ and apply (13a).

14. a. ∞.   b. ∞.   c. e.   d. ∞.

17. If $\sum |a_k|$ and $\sum |b_k|$ converge and if $d_k = \sum_{j=0}^{k} |a_j||b_{k-j}|$, then, clearly, $\sum_{k=0}^{n} d_k \leq \sum_{k=0}^{n} |a_k| \cdot \sum_{k=0}^{\infty} |b_k| \leq \sum_{k=0}^{\infty} |a_k| \sum_{k=0}^{\infty} |b_k|$ so that $\sum d_k$ converges. Moreover $A_n B_n = C_n + R_n$ where $|R_n| = \sum_{\substack{k,j=0 \\ n < k+j \leq 2n}}^{n} a_k b_j \leq d_{n+1} + d_{n+2} + \cdots + d_{2n}$, etc.

18. Use the fact that $\sum a_n z^n$ and $\sum b_n z^n$ converge absolutely within their circles of convergence and apply Exercise (17).

19. a. Let $N \to \infty$ in the identity

$$(1 - z)(1 + z + z^2 + \cdots + z^N) = 1 - z^{N+1}$$

b. $\left(\sum_{n=0}^{\infty} z^n\right) \left(\sum_{n=0}^{\infty} z^n\right) = \sum_{n=0}^{\infty} (n + 1)z^n = \frac{1}{(1-z)^2}$ so that

$$\sum_{n=0}^{\infty} nz^n = \frac{1}{(1 - z)^2} - \frac{1}{(1 - z)} = \frac{z}{(1 - z)^2}.$$

20. If S has an accumulation point at 0, we can find $z_1 \in S$ with $|z_1| < 1$, $z_2 \in S$ with $|z_2| \leq \frac{1}{2}|z_1|$, etc.

21. If $f(z) = 1$ for $z = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$, by (2.13), $f \equiv 1$ and $f' \equiv 0$.

22. Let $g(z) = f(z + a) = \sum C_n z^n$. Then, $C_n = \frac{g^{(n)}(0)}{n!} = \frac{f^{(n)}(a)}{n!}$.

23. a. $|z - 1| < 1.$
    b. all z.
    c. $\left|z - \frac{1}{2}\right| < \frac{1}{2}.$

Chapter 3

1. $f_x = \lim_{\substack{h \to 0 \\ h \text{ real}}} \frac{f(x+h,y)-f(x,y)}{h} = \lim_{\substack{h \to 0 \\ h \text{ real}}} \frac{f(z+h)-f(z)}{h}$

   $f_y = \lim_{\substack{h \to 0 \\ h \text{ real}}} \frac{f(x,y+h)-f(x,y)}{h} = \lim_{\substack{h \to 0 \\ h \text{ real}}} \frac{f(z+ih)-f(z)}{h}$

2. $f_y = 2iy = if_x = 2ix$ if, and only if, $y = x$.

3. The equality in the hint follows from the differentiability of g at $f(z)$. Divide both sides by h to complete the argument.

4. Because $\frac{g^2(z)-g^2(z_0)}{z-z_0} = 1$, $\frac{g(z)-g(z_0)}{z-z_0} = \frac{1}{g(z)+g(z_0)}$, so that $\lim_{z \to z_0} \frac{g(z)-g(z_0)}{z-z_0} = \frac{1}{2g(z_0)}$ by the continuity of g at z_0.

5. Note that $f_x \equiv f_y \equiv 0$, and apply Theorem 1.10.

6. Note that $[f^2(z)]' = 2f(z)f'(z) \equiv 0$.

7. If f maps a region into a straight line or into a circular arc, then there are constants A, B so that $g(z) = Af(z) + B$ maps the region into the imaginary axis or into a circle centered at the origin, respectively. But, then, according to Propositions 3.6 and 3.7, g is constant, and so is f.

8. $f(x, y) = x^2 - y^2 + 2ixy + iC = z^2 + iC$, where C is a real constant.

9. Note that the Cauchy-Riemann equations cannot be satisfied.

10. The Cauchy-Riemann equation $u_x = v_y$ implies, in this case, that $u'(x) = v'(y)$. Because $u'(x)$ is a function of x alone and $v'(y)$ is a function of y alone, both $u'(x)$ and $v'(y)$ are constants; in fact, $u'(x) = v'(y) = a$ and $f(z) = u + iv = az + b$.

11. a. Because $u = e^x \cos y$ and $v = e^x \sin y$, $u_x = v_y = e^x \cos y$ and $u_y = -v_x = -e^x \sin y$.

    b. With $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$
    $e^{z_1+z_2} = e^{x_1+x_2}[\cos(y_1 + y_2) + i \sin(y_1 + y_2)] = e^{x_1}(\cos y_1 + i \sin y_1) \cdot$

    $e^{x_2}(\cos y_2 + i \sin y_2) = e^{z_1}e^{z_2}.$

12. $|e^z| = \sqrt{e^{2x} \cos^2 y + e^{2x} \sin^2 y} = e^x$

13. $e^z = e^x e^{iy} \to \infty$ if $x \to +\infty$ (i.e., along rays in the right half-plane). $e^z \to 0$ if $x \to -\infty$ (i.e., along rays in the left half-plane). On the imaginary axis, $e^z = e^{iy}$ traverses the unit circle infinitely often.

14. a. $2\pi ki$; k any integer.
    b. $(\frac{\pi}{2} + 2\pi k)i$.
    c. $\ln 3 + (2k + 1)\pi i$.
    d. $\frac{1}{2} \ln 2 + (\pi/4 + 2\pi k)i$.

15. a. $2 \sin z \cos z = 2 \left[\frac{1}{2i} (e^{iz} - e^{-iz}) \frac{1}{2} (e^{iz} + e^{-iz}) \right] = \frac{1}{2i} (e^{2iz} - e^{-2iz}) =$
    $\sin 2z$.
  b. $\cos^2 z + \sin^2 z = \frac{1}{4} (e^{iz} + e^{-iz})^2 - \frac{1}{4} (e^{iz} - e^{-iz})^2 = 1$
  c. $(\sin z)' = \frac{1}{2i} (ie^{iz} + ie^{-iz}) = \cos z$.

17. $- \sin z$.

18. $z = (\frac{\pi}{2} + 2\pi k) - i \ln(2 \pm \sqrt{3})$, k any integer.

20. $\sin(x + iy) = \frac{1}{2i} (e^{-y+ix} - e^{y-ix}) = \frac{1}{2i} [e^{-y}(\cos x + i \sin x) - e^{y}(\cos x - i \sin x)] = \frac{1}{2} (e^{-y} \sin x + e^{y} \sin x) + \frac{i}{2} (e^{y} \cos x - e^{-y} \cos x) = \sin x \cosh y + i \cos x \sinh y$.

21. $f(z)f(w) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{n=0}^{\infty} \frac{w^n}{n!} = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{z^k w^{n-k}}{k!(n-k)!} \right) = \sum_{n=0}^{\infty} \frac{(z+w)^n}{n!} = f(z+w)$.

22. Because $e^{iz} = 1 + iz - \frac{z^2}{z!} - \frac{iz^3}{3!} + \cdots$ and $e^{-iz} = 1 - iz - \frac{z^2}{z!} + \frac{iz^3}{3!} + \cdots$,
    $\frac{e^{iz} - e^{-iz}}{2i} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - + \cdots$.

23. $\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$

Chapter 4

1. Use the facts that a 1-1 C^1 mapping $\lambda(t) : [c, d] \to [a, b]$ with $\lambda' > 0$ has a 1-1 C^1 inverse $\lambda^{-1} : [a, b] \to [c, d]$ with $(\lambda^{-1})' > 0$ and, if $\lambda : [c, d] \to [a, b]$ and $\lambda_2 : [e, f] \to [c, d]$, then $\lambda_1 \circ \lambda_2 : [e, f] \to [a, b]$ with all the desired properties of λ_1 and λ_2.

2. $\int_C f(z)dz = \int_0^1 (t^4 + it^4)(2t + i2t)dt = \frac{2}{3} i$. The result is the same as in Example 1, because the curves are equivalent.

3. $\int_C f(z)dz = \int_0^{2\pi} \frac{1}{(\sin t + i \cos t)} (\cos t - i \sin t)dt = \int_0^{2\pi} -idt = -2\pi i$. Note that the curve in this case is in the opposite direction of the curve in Example 2.

4. Let $f = u_1 + iv_1$, $g = u_2 + iv_2$, $C : z(t) = x(t) + iy(t)$ and $\alpha = a + bi$, etc.

5. $F(b) - F(a) = \int_\gamma F'(z)dz$ where γ is any curve with initial point a and terminal point b.

6. As in Lemma 4.9, let $\int_{|z|=1} f(z)dz = Re^{i\theta}$ so that $R = \left| \int_{|z|=1} f(z)dz \right| = \int_{|z|=1} e^{-i\theta} f(z)dz$. Note, then, that

$$R = \int_{|z|=1} e^{-i\theta} f(z)dz = \int_0^{2\pi} f(e^{it})ie^{i(t-\theta)}dt = \int_0^{2\pi} g(t)ie^{i(t-\theta)}dt$$

where $g(t) = f(e^{it})$ is a real-valued function of t

$$= \text{Re} \int_0^{2\pi} g(t)ie^{i(t-\theta)}dt = \int_0^{2\pi} g(t) \sin(\theta - t)dt$$

and because $|g(t)| \leq 1$, $R \leq \int_0^{2\pi} |\sin(\theta - t)|dt = \int_0^{2\pi} |\sin t|dt = 4$.

7. Note that, on any line segment from z_0 to z_1, i.e., if $\gamma(t) = z_0 + t(z_1 - z_0)$, $0 \le t \le 1$, $\int_\gamma 1 dz = z_1 - z_0$ and $\int_\gamma z dz = \frac{z_1^2}{2} - \frac{z_0^2}{2}$, etc.

8. a. Because $z^k = \left(\frac{z^{k+1}}{k+1}\right)'$ and because $\frac{z^{k+1}}{k+1}$ is analytic on C (as long as k is an integer other than -1), by Proposition 4.12, $\int_C z^k dz = 0$.

   b. $\int_C z^k dz = \int_0^{2\pi} R^k e^{ik\theta} i R e^{i\theta} d\theta = i R^{k+1} \int_0^{2\pi} e^{(k+1)i\theta} d\theta = \frac{R^{k+1}}{k+1} e^{(k+1)i\theta} \Big|_{\theta=0}^{2\pi}$
$= 0$.

9. a. $\int_C (z - i) dz = \frac{z^2}{2} - iz \Big|_{-1+i}^{1+i} = 0$.

   b. Let $z(t) = t + i$, $-1 \le t \le 1$. $\int_C (z - i) dz = \int_{-1}^1 t dt = 0$.

Chapter 5

1. $z^2 = 4 + 4(z - 2) + (z - 2)^2$

2. Because all the derivatives of e^z are e^z, $e^z = \sum_{k=0}^\infty e^a \frac{(z-a)^k}{k!} = e^a e^{z-a}$

3. a. $f(z) = -f(-z) \Rightarrow f'(z) = f'(-z)$. Hence, the derivative of an odd function is even. Similarly, the derivative of an even function is odd. Furthermore, if f is odd, i.e., if $f(z) = -f(-z)$, it follows that $f(0) = 0$. So, if f is odd,

$$f(z) = \sum_{\substack{k=1 \\ k \text{ odd}}}^\infty \frac{f^{(k)}(0)}{k!} z^k.$$

   b. By analogous reasoning, an even function has only even powers in its power series expansion about 0.

4. According to Theorem 5.5, $C_k = \frac{1}{2\pi i} \int_C \frac{f(w)}{w^{k+1}} dw$ while, according to Corollary 2.11, $C_k = \frac{f^{(k)}(0)}{k!}$.

5. Let $g(z) = f(z + a)$, so that $g^{(k)}(0) = f^{(k)}(a)$, etc., and apply the result of Exercise 4.

6. a. Using Exercise 4 and the $M - L$ formula, $|C_k| \le \frac{M}{R^k}$.
   b. Apply (6a) with $R = 1$ and $M = 1$ to conclude that $|C_k| \le 1$ for all k.

7. According to Exercise (6a) $|C_j| \le \frac{A+BR^k}{R^j}$. Because $j > k$, we conclude that $C_j = 0$ by letting $R \to \infty$.

8. As in the previous exercise, note that $C_k = 0$ for $k \ge 2$.

9. By Liouville's Theorem, $f'(z)$ is a linear function. Moreover, $|f'(z)| \le |z| \Longrightarrow$ $f'(0) = 0$ so that $f(z) = f(0) + f'(0)z + \frac{f''(0)}{2} z^2 = a + bz^2$ with $|b| = \left|\frac{f''(0)}{2}\right| \le \frac{1}{2}$ because $f''(0) = \frac{1}{2\pi i} \int_{|w|=1} \frac{f'(w)}{w^2} dw \ll 1$.

10. f is bounded in the compact domain: $0 \le x \le 1, 0 \le y \le 1$. Moreover, by the two equations of periodicity, for any $z = x + iy$, $f(z) = f(x, y) = $

$f(x - \text{Int} x, y - \text{Int} y)$, where Int w, the integer part of w, is the greatest integer less than or equal to w. Hence, f is bounded throughout the complex plane and must be constant.

11. By the remarks following the Fundamental Theorem of Algebra, a polynomial of odd degree must have an odd number of zeroes (counting multiplicity). According to Exercise 5 of Chapter 1, a *real* polynomial has a zero at the conjugate of any nonreal zero, so that the complex zeroes come in conjugate pairs. Thus, there must be at least one *real* zero.

12. Let $P(z) = a_n z^n + \cdots a_0 = a_n(z - z_1)(z - z_2) \cdots (z - z_n)$ and note that, if $z_k = a + bi$ and $z_j = a - bi$ are complex conjugates, $(z - z_k)(z - z_j) = z^2 - 2az + (a^2 + b^2)$ is a *real* quadratic polynomial.

13. a. If $v > 0$ for $y > 0$, then we must have $v < 0$ for $y < 0$. (By the Fundamental Theorem of Algebra, $v \geq 0$ for all z is impossible!) Hence $v_y \geq 0$ throughout the real axis. Similarly, if $v < 0$ for $y > 0$, $v_y \leq 0$ throughout the real axis.

    b. Follows from the Cauchy-Riemann equations.

    c. Because $u_x(x, 0)$ is a polynomial in x which is either consistently ≥ 0 or consistently ≤ 0 for all x, $u(x, 0) = \alpha$ cannot have more than one solution. Hence $P(z) = \alpha$ has, at most, one solution for real α and, by the Fundamental Theorem of Algebra, P is a linear polynomial.

14. If $P(z) = (z - \alpha)^k Q(z)$ with $Q(\alpha) \neq 0$, $P'(z) = (z - \alpha)^{k-1}[(z - \alpha)Q'(z) + kQ(z)] = (z - \alpha)^{k-1} R(z)$ with $R(\alpha) \neq 0$. Proceed by induction.

15. Let $f(z) = f(z_0) + \int_{z_0}^{z} f'(w)dw$ where the path of integration is along the ray from 0 to z, beginning at z_0, with $z_0 = t_0 z$, where $t_0 = \sup\{t_1 : |f(tz)| \leq 1, 0 \leq t \leq t_1\}$. Clearly, then $|f(z_0)| \leq \max\{1, |f(0)|\}$ and the integral is bounded by $|z|$.

Chapter 6

1. For any complex α,

$$\frac{1}{z} = \frac{1}{\alpha + (z - \alpha)} = \frac{1}{\alpha\left[1 + \frac{z-\alpha}{\alpha}\right]}$$

$$= \frac{1}{\alpha}\left[1 - \frac{(z - \alpha)}{\alpha} + \frac{(z - \alpha)^2}{\alpha^2} - + \cdots\right]$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k (z - \alpha)^k}{\alpha^{k+1}}.$$

Take $\alpha = 1 + i$.

3. Because $1 + z + z^2 + \cdots = \frac{1}{1-z}$, $1 + 2z + 3z^2 + \cdots = \sum_{n=1}^{\infty} nz^{n-1} = \frac{1}{(1-z)^2}$ and $\sum_{n=1}^{\infty} nz^n = \frac{z}{(1-z)^2}$. Similarly, $\sum_{n=1}^{\infty} n^2 z^{n-1} = \frac{1+z}{(1-z)^3}$ and $\sum_{n=1}^{\infty} n^2 z^n = \frac{z(1+z)}{(1-z)^3}$.

4. If $f(\frac{1}{n}) = \frac{1}{n+1}$, then for all points $z_n = \frac{1}{n}$, $f(z_n) = \frac{1}{\frac{1}{z_n}+1}$ or $f(z_n) = \frac{z_n}{1+z_n}$. Be-
cause $\{z_n\}$ has an accumulation point at 0, this implies that $f(z) = \frac{z}{1+z}$ throughout
its domain of analyticity which yields a contradiction since f was assumed ana-
lytic at $z = -1$.

5. Let z_2 be a fixed real number. Then, $f(z) = \sin(z + z_2)$ and $g(z) = \sin z \cos z_2 + \cos z \sin z_2$ are two entire functions (of z) which agree for all
real values $z = z_1$ and, hence, for all complex values $z = z_1$, as well.
Let $z = z_1$ be *any* such complex number. Then, $f(z) = \sin(z_1 + z)$ and
$g(z) = \sin z_1 \cos z + \cos z_1 \sin z$ agree for all real values $z = z_2$ and, hence,
for all complex values $z = z_2$ as well.

6. If $f(x) = \tan x$, $0 \le x \le 1$, $f(z)$ must equal the analytic function $\frac{\sin z}{\cos z}$ through-
out the domain of analyticity of $f(z)$. Thus, $f(z) = i \Rightarrow \frac{\sin z}{\cos z} = i \Rightarrow \frac{e^{iz}-e^{-iz}}{2i} = \frac{i(e^{iz}+e^{-iz})}{2} \Rightarrow e^{iz} = 0$, which is impossible.

7. Because $|f(z)| > |z|^N$ for large z, $f(z) \to \infty$ as $z \to \infty$ and, hence, f is a
polynomial (Theorem 6.11). Moreover, $|f(z)| \ge |z|^N$ for large z implies that the
degree of $f(z)$ must be at least N.

8. $g(z) = f(z)f(-z)$ is bounded in modulus by 6 throughout $|z| = 1$. Thus $|g(0)| \le 6$
and $|f(0)| \le \sqrt{6}$.

9. $|e^z| = e^x$ so that max $|e^z|$ occurs at a point in the domain with maximal x and
min $|e^z|$ occurs at a point in the domain with minimal x (i.e., at points farthest to
the right and to the left, respectively).

10. Because $z^2 - z = z(z-1)$, the minimum modulus occurs at $z = 0$ and the
maximum modulus (which occurs on the boundary) is assumed at $z = -1$, i.e.,
$\max_{|z|\le 1} |z^2 - z| = 2$; $\min_{|z|\le 1} |z^2 - z| = 0$.

12. Suppose $|f(z)| + |g(z)|$ assumed its maximum at the interior point z_0 (and not on
the boundary). Let $f(z_0) = Ae^{-i\alpha}$ and $g(z_0) = Be^{-i\beta}$. Then $h(z) = f(z)e^{i\alpha} + g(z)e^{i\beta}$ would satisfy $h(z_0) = |f(z_0)| + |g(z_0)|$ while $|h(z)| \le |f(z)| + |g(z)| < |h(z_0)|$ throughout the boundary. Thus, the analytic function $h(z)$ would assume
its maximum at the interior point z_0 (and not on the boundary) which is impossible.

13. If $P(z) \ne 0$, then, the minimum modulus of $P(z)$ in $|z| \le R$ would have to occur
on the boundary. However, because $P(z) \to \infty$ as $z \to \infty$, we could choose R
so that $|P(z)| > |P(0)|$ for all $|z| = R$, yielding a contradiction.

14. According to Exercise (6b) of Chapter 5 $P(z) = a_0 + a_1 z + \cdots + a_n z^n$
with $|a_k| \le 1$ for $k = 0, 1, \ldots, n$. Consider $Q(z) = \frac{P(z)}{z^n}$ in the annulus
$1 \le |z| \le R$. Throughout $|z| = 1$, $|Q(z)| = |P(z)| \le 1$ and, if $|z| = R$,
$|Q(z)| = |a_n + \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n}| \le |a_n| + \epsilon_R \le 1 + \epsilon_R$ where $\epsilon_R \to 0$ as $R \to \infty$.
Hence, $|Q(z)| = |\frac{P(z)}{z^n}| \le 1$ for all $z : |z| \ge 1 \Rightarrow |P(z)| \le |z|^n$.

[*NOTE:* The related question: What is the maximum value of $P_n(x)$ for real x,
given $|P_n(x)| \le 1$ for $-1 \le x \le 1$, can be answered in terms of Chebychev
polynomials and is considerably more complicated.]

Chapter 7

1. According to the Open Mapping Theorem, the image under f of any open set D containing z_0 in its interior is an open set containing $f(z_0)$ in its interior. Hence, both Re f and Im f assume larger and smaller values in D than the values Re $f(z_0)$ and Im $f(z_0)$.

2. Note that continuous functions map connected sets into connected sets and apply the Open Mapping Theorem to complete the argument.

3. a. By the Open Mapping Theorem, if $f(z)$ is a boundary point of T, z cannot be an interior point of S.

   b. Note that $T = f(S)$ is $\overline{D(0; 4)}$ so that f maps the boundary points: $\{z:|z| = 1,$ Re $z \geq 0\}$ and $\{iy:1 < y < 2$ or $-2 < y < -1\}$ into the interior of T.

4. Because $|f| = 1$ on $C(0; 1)$, by the Maximum Modulus Theorem $|f| \leq 1$ throughout $D(0; 1)$. Because $D(0; 1)$ is compact so is its image under the continuous function f. To show that f maps $D(0; 1)$ onto $D(0; 1)$, consider $f(0)$ and let T be the points on any chord of $C(0; 1)$ passing through $f(0)$. If T were not contained in the range of f, there would have to be some point $w \in T$, $|w| < 1$ which is a boundary point of the range of f. By the previous exercise, however, this is impossible.

5. Let $f(z)$ have zeroes at $\alpha_1, \ldots, \alpha_N$. Then $g(z) = f(z)/(\prod_{i=1}^{N} \frac{z-\alpha_i}{1-\bar{\alpha}_i z})$ would have modulus 1 throughout the circle: $|z| = 1$, and $g(z) \neq 0$ at any points $z:|z| < 1$. Thus, both the maximum and the minimum modulus theorems can be applied to conclude that $|g| \equiv 1$ throughout $D(0; 1)$ and, hence, g is constant i.e., $f(z) = C \prod_{i=1}^{N} \frac{z-\alpha_i}{1-\bar{\alpha}_i z}$. Because f is *entire*, it follows that $\alpha_i = 0$ for all i and, thus, $f(z) = Cz^N$.

8. Note that $\frac{f(z)}{z^2}$ has modulus 1 throughout the boundary of the annulus and apply the Maximum Modulus Theorem.

9. Let $g(z) = \frac{1}{10} f(2z)$. Then $|g| < 1$ for $|z| < 1$ and $g(1/2) = 0$ so that $|g(z)| \leq \left| \frac{z-\frac{1}{2}}{1-\frac{1}{2}z} \right|$. In particular, $|g(1/4)| \leq 2/7$ and $|f(1/2)| \leq 20/7$.

10. Let $g(z) = \frac{f(z)-f(\alpha)}{1-f(\alpha)f(z)}$. Then g is also analytic and bounded by 1 in the unit disc and a direct calculation shows that $g'(\alpha) = \frac{f'(\alpha)}{1-|f(\alpha)|^2}$. Thus, $g'(\alpha) \gg f'(\alpha)$.

11. Because $f(\alpha) = 0$, $f'(\alpha) = \lim_{z \to \alpha} \frac{f(z)}{z-\alpha} \ll \lim_{z \to \alpha} \frac{B_\alpha(z)}{z-\alpha}$ (because $|f(z)| \leq |B_\alpha(z)|$ for all $|z| \leq 1) \Rightarrow f'(\alpha) \ll B'_\alpha(\alpha)$. [NOTE: $B'_\alpha(\alpha) = \frac{1}{1-|\alpha|^2}$.]

12. Consider $g(z) = (z-iR)^2(z+iR)^2 f(z) = (z^2+R^2)^2 f(z)$. As in Proposition 7.3, it can be shown that $|g(z)| \leq 8R^2$ throughout $|z| = R$, and, hence, $|g(z)| = |(z^2 + R^2)^2 f(z)| \leq 8R^2$ as long as $|z| < R$. Thus $|f(z)| \leq \left| \frac{8R^2}{(z^2+R^2)^2} \right|$, and letting $R \to \infty$, we conclude that $f \equiv 0$.

13. a. $\int_\Gamma f(z)dt = \int_\Gamma \int_0^1 \frac{\sin zt}{t} dt\, dz = \int_0^1 \int_\Gamma \frac{\sin zt}{t} dz\, dt = \int_0^1 0\, dt = 0$.

    b. $f(z) = \int_0^1 \frac{\sin zt}{t} dt = \int_0^1 \left(z - \frac{z^3 t^2}{3!} + \frac{z^5 t^4}{5!} - \cdots \right) dt = z - \frac{z^3}{3(3!)} + \frac{z^5}{5(5!)} - + \cdots$.

14. a. $f(z) = \int_0^1 \frac{\sin zt}{t} dt = \int_0^1 \int_0^z \cos zt\, dz\, dt = \int_0^z \int_0^1 \cos zt\, dt\, dz$ so that $f'(z) = \int_0^1 \cos zt\, dt$.

b. According to (13b) $f'(z) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - + \cdots$ and

$$\int_0^1 \cos zt \, dt = \int_0^1 \left(1 - \frac{z^2 t^2}{2!} + \frac{z^4 t^4}{4!} - + \cdots \right) dt$$

$$= 1 - \frac{z^2}{3(2!)} + \frac{z^4}{5(4!)} - + \cdots = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - + \cdots$$

15. Note that g is a linear function of z with $g(0) = z_0$; $g(|z_1 - z_0|) = z_0 + e^{i\theta}|z_1 - z_0| = z_0 + (z_1 - z_0) = z_1$.

16. By the Schwarz Reflection Principle, f can be extended to the entire plane and would then be a bounded entire function. Hence, f is constant.

17. By the Schwarz Reflection Principle, if $f(z) = f(x+iy) = u(x+iy)+iv(x+iy)$, then $f(\bar{z}) = f(x - iy) = u(x - iy) + iv(x - iy) = u(x + iy) - iv(x + iy)$ and $f(-z) = f(-x - iy) = -u(x - iy) + iv(x - iy) = -u(x+iy) - iv(x+iy) = -f(z)$.

19. Note that g is differentiable for $|z| < 1$, and for $|z| > 1$ because $g'(z) = \overline{h'(\bar{z})}$ where $h(z) = f(1/z)$ for $|z| > 1$. Moreover, g is continuous on the upper semicircle so that g is analytic throughout the upper half-plane.

20. Arguing as in Exercise (19), such an analytic function could be extended to a bounded entire function.

21. If $f(x) = |x|$, by the Schwarz Reflection Principle, f can be extended to be analytic in the unit disc. However, $f'(0)$ cannot exist because $\lim_{h\to 0} \frac{f(h)-f(0)}{h}$ yields different values as $h \to 0$ along the positive real axis and along the negative real axis.

Chapter 8

1. γ represents the portion of the ray from α through z to ∞, starting at z. Thus, if z is in the complement of S, so is all of γ. For, if any $z_1 \in \gamma$ belonged to S, so would the entire segment connecting α and z_1, including z.

3. As suggested in the hint, let $C = \{z : |z - \alpha| = r\}$ and consider $\delta_z = \max\{t:D(z; t) \subseteq S\}$. δ_z is a continuous function of $z \in C$ and $\delta = \min_{z \in C} \delta_z$ exists. Hence, the annulus $B = \{z : r - \delta \le |z - \alpha| \le r + \delta\}$ is contained in S. It follows that any $z_0 \in D(\alpha; r)$ must belong to S. For any path γ connecting z_0 to ∞ must intersect C, and, at that point, $d(\gamma, \tilde{S}) \ge \delta$.

4. $\tilde{S}$ is closed because it contains all its accumulation points. Moreover, any point $(x_0, \sin\frac{1}{x_0})$ can be connected by the curve $y = \sin\frac{1}{x}$ to $\left(\frac{1}{k\pi}, 0\right)$ which is within ϵ of $(0, 0)$ as long as $k > \frac{1}{\pi\epsilon}$. The positive y axis, contained in S, then, connects the origin to ∞.

6. As in Lemma 8.3, we can view a simple closed curve Γ with k levels as a union of rectangles and one or more closed curves Γ' with $k - 1$ levels. We can then

define the inside of Γ as the points in the rectangles together with the inside of the closed curves Γ'. As in the proof of Lemma 8.3, it follows by induction that points inside Γ belong to any simply connected domain containing Γ.

7. Suppose $\gamma(t): a \le t \le b$ has $\gamma(t_2) = \gamma(t_1)$. Then γ can be written as a union of γ_1 and γ_2 where $\gamma_1 = \gamma(t); t \in [a, t_1] \cup [t_2, b]$ and $\gamma_2 = \gamma(t); t \in [t_1, t_2]$.

8. Note that, for points z on the negative axis, $\pi i + \int_{-1}^{z} \frac{d\zeta}{\zeta} = \pi i + \int_{-1}^{-|z|} \frac{d\zeta}{\zeta} = \pi i + \ln|z|$. Hence, $\mathrm{Im}(\log z) = \pi i$ for all points on the negative axis. For any z, we can, then, choose the path of integration from -1 to $-|z|$, followed by the circular arc from $-|z|$ to z.

Chapter 9

1. Note that z_0 cannot be a removable singularity nor can it be an essential singularity.

2. No. According to Exercise 1, $|f(z)| \sim \exp\left(\frac{1}{|z|}\right) \Rightarrow f$ has a pole at $z = 0$, but then $|f(z)| \sim \frac{A}{|z|^k}$ near $z = 0$.

4. By Riemann's Principle, the singularity is removable; hence f is (can be considered) entire! But then $|f(z)| \le A|z|$ for large z implies that f is a linear polynomial, and $|f(z)| \le A\sqrt{|z|}$ for large z implies that f is constant.

6. $e^{1/z} \ne 0$. To solve $e^{(1/z)} = w$ for any $w \ne 0$, take $(1/z)$ equal to any of the infinitely many values of $\log w$. Note that infinitely many of these values for $1/z$ correspond to values of z in the unit disc.

7. $f + g$ will have a pole of order $\max(m, n)$ if $m \ne n$ and a (possible) pole of order $\le m$ if $m = n$; $f \cdot g$ will have a pole of order $m + n$; f/g will have a pole of order $m - n$ if $m > n$, a zero of order $n - m$ if $n > m$, and a removable singularity if $m = n$.

9. a. Double pole at $z = 0$; simple pole at $\pm i$.
   b. Simple pole at $z = k\pi$, k any integer.
   c. Same as (b).
   d. Essential singularity at $z = 0$; simple pole at $z = 1$.

11. a. $\sum_{k=-1}^{\infty} (-1)^{k+1} z^{2k}$
    b. $\sum_{k=-\infty}^{\infty} a_k z^k$ with

$$
a_k = \begin{cases} -e & \text{if } k \ge 0, \\ -e + 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{(j-1)!} & \text{if } k = -2j \text{ or } k = -2j + 1, \\ & \qquad j = 1, 2 \cdots \end{cases}
$$

   c. $\sum_{k=-1}^{\infty} \frac{(-1)^{k+1}(z-2)^k}{4^{k+2}}$

14. Use $f(z) = \frac{f(z) - f(-z)}{2}$.

15. a. $\frac{1}{z^2} - \frac{1}{2i(z-i)} + \frac{1}{2i(z+i)}$

b. $\frac{1}{2i(z-i)} - \frac{1}{2i(z+i)}$

16. If $|f(z) - w| > \delta$, $g(z) = \frac{1}{f(z)-w}$ would be analytic in D with zeroes at all the points of $\{z_n\}$ which would imply that g is identically zero.

17. a. Because the range of $\sin \frac{1}{z}$ is dense in the plane, so is the range of $\csc \frac{1}{z} = \frac{1}{\sin \frac{1}{z}}$.

    b. Note that $\csc \frac{1}{z}$ has a pole at all the points $z = \frac{1}{k\pi}$, k an integer.

18. If f is a polynomial, according to the Fundamental Theorem of Algebra, the range of f is the full complex plane. Otherwise, note that $f(1/z)$ has an essential singularity at 0.

Chapter 10

1. a. $\frac{1}{z^4+z^2} = \frac{1}{z^2(z^2+1)} = \frac{1}{z^2}(1 - z^2 + z^4 - \cdots)$ around $z = 0$. Hence Res $\left(\frac{1}{z^4+z^2}; 0\right) = 0$; 0 is a double pole.

   $\frac{1}{z^4+z^2} = \frac{1}{z^2(z+i)(z-i)}$. Hence, $\frac{1}{z^4+z^2}$ has a simple pole at i, with Res $= \frac{i}{2}$, and a simple pole at $z = -i$, with Res $= \frac{-i}{2}$ (see Chapter 9, Exercise (15a)).

   b. $\cot z = \frac{\cos z}{\sin z}$ has a simple pole at every integral multiple of π with Res$(\cot z; \pi k) = 1$ for all k.

   c. $\csc z = \frac{1}{\sin z}$ has a simple pole at every integral multiple of π with Res$(\csc z; \pi k) = \frac{1}{\cos(\pi k)} = (-1)^k$.

   d. $\frac{\exp \frac{1}{z^2}}{z-1}$ has a simple pole at $z = 1$ with Res $= e$. Around $z = 0$,

   $$\frac{e^{1/z^2}}{z-1} = \left(1 + \frac{1}{z^2} + \frac{1}{2!z^4} + \frac{1}{3!z^6} \cdots\right)(-1 - z - z^2 \cdots).$$

   Hence, Res $\left(\frac{e^{1/z^2}}{z-1}; 0\right) = -e + 1$ (see Chapter 9, Exercise (11b); 0 is an essential singularity!).

   e. $\frac{1}{z^2+3z+2} = \frac{1}{(z+1)(z+2)}$. Hence $\frac{1}{z^2+3z+2}$ has simple poles at -1 and -2 with Res $\left(\frac{1}{z^2+3z+2}; -1\right) = 1$; Res $\left(\frac{1}{z^2+3z+2}; -2\right) = -1$.

   f. Essential singularity at $z = 0$ with Res $= 1$.

   g. Essential singularity at $z = 0$ with Res $= 9/2$.

   h. If $b^2 - 4ac \neq 0$, there are simple poles at $\frac{-b \pm \sqrt{b^2-4ac}}{2a}$ with residues of $\frac{1}{\pm\sqrt{b^2-4ac}}$. If $b^2 - 4ac = 0$, $\frac{1}{az^2+bz+c} = \frac{1}{a\left(z+\frac{b}{2a}\right)^2}$, so that there is a double pole at $z = \frac{-b}{2a}$ with residue zero.

2. a. $2\pi i$ (see (1b)).

   b.

$$2\pi i \sum_{k=1}^{3} \frac{1}{(z_k - 4) \cdot 3z_k^2} = \frac{2\pi i}{3} \sum_{k=1}^{3} \frac{z_k}{z_k - 4} \text{ where } z_k = e^{2k\pi i/3}, \quad k = 1, 2, 3,$$

$$= \frac{-2\pi i}{63}$$

   c. $2\pi i$ (see Exercise (1f)).

   d. $9\pi i$ (see Exercise (1g)).

3. Let C be any regular closed curve surrounding $z = 0$ and *not* surrounding any of the other singularities: $z = 2\pi ki, \ k = \pm 1, \ \pm 2, \ \ldots$. Then $\text{Res}\left(\frac{1}{(1-e^{-z})^n}; 0\right) = \frac{1}{2\pi i} \int_C \frac{dz}{(1-e^{-z})^n}$. Letting

$$w = 1 - e^{-z},$$

$$e^{-z} = 1 - w$$

$$-e^{-z}dz = -dw$$

$$dz = \frac{dw}{e^{-z}} = \frac{dw}{1 - w}$$

   and

$$\frac{1}{2\pi i} \int_C \frac{dz}{(1 - e^{-z})^n} = \frac{1}{2\pi i} \int_{C^*} \frac{dw}{w^n(1 - w)}$$

   where C^* is the image under $w = 1 - e^{-z}$ of C. [To see that C^* surrounds 0 and not 1 in the w-plane, we can consider C to be the boundary of the rectangle $-1 \leq x \leq 1; \ \frac{-\pi}{2} \leq y \leq \frac{\pi}{2}$ in which case C^* can be seen to be the left half of the annulus centered at 1 with inner radius $1/e$ and outer radius e.] Thus, $\text{Res}\left(\frac{1}{(1-e^{-z})^n}; 0\right) = \text{Res}\left(\frac{1}{(w^n(1-w)}; 0\right) = 1$ because $\frac{1}{(w^n(1-w)} = \frac{1}{w^n}(1 + w + w^2 + \cdots)$ has $\text{Res} = 1$ at $w = 0$ for all n.

6. $\frac{f(z+h)-f(z)}{h} = \int_\gamma \varphi(w)\left[\frac{\frac{1}{w-(z+h)} - \frac{1}{w-z}}{h}\right] dw$. Because $z \notin \gamma$, we can take $\lim_{h\to 0}$ inside the integral and $\lim_{h\to 0} \frac{f(z+h)-f(z)}{h} = \int_\gamma \frac{\varphi(w)}{(w-z)^2} dw$. In particular, because $f(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w-z} dw$ where C is a regular curve surrounding z, it follows that $f'(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{(w-z)^2} dw$ and continuing inductively we can prove Theorem 10.11.
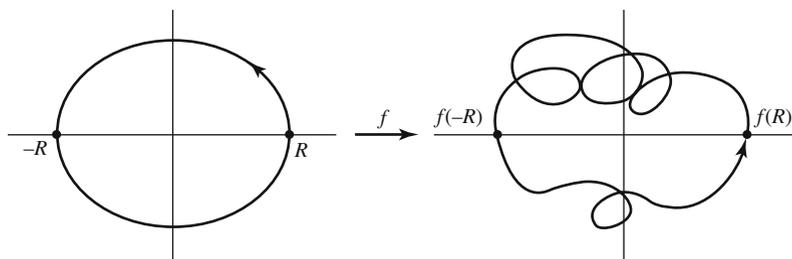
7. Consider the image of the circle $|z| = R$ under the mapping $w = f(z)$. Because $f(z)$ is real if, and only if, z is real, f maps the entire upper semicircle $|z| = R$; $y > 0$ into either the upper half-plane or the lower half-plane, and, likewise, f maps the entire lower semicircle $|z| = R$, $y < 0$ into either the upper or the lower half-plane. Because $\Delta \operatorname{Arg} w$ is at most π in either the upper or lower half-plane, it follows that $\Delta \operatorname{Arg} f(z)$, as z traverses $|z| = R$, is, at most, 2π and $\mathbb{Z}(f)$ in $|z| \leq R, = \frac{1}{2\pi} \Delta \operatorname{Arg} f(z) \leq 1$.

9. a. 0 because $|3e^z| \geq \frac{3}{e} > |z|$ on $|z| = 1$

   b. 1 because $|z| > \left| \frac{1}{3} e^z \right|$

   c. On $|z| = 2$, $|z^4| > |5z - 1|$. On $|z| = 1$, $|5z| > |z^4 + 1|$. Hence, there are 3 zeroes in the annulus.

   d. Note that, on $|z| = 1$, $|5z^4| = 5 \geq |z^6 + 3z^2 - 1|$ with equality possible only at $z = \pm i$. Because $z^6 - 5z^4 + 3z^2 - 1 \neq 0$ at $z = \pm i$, it follows that there are 4 zeroes in $|z| \leq 1$.

11. $\operatorname{Res}\left(\frac{z^m f'(z)}{f(z)}; z_k \right) = p \cdot z_k^m$ where p is the order of the zero at z_k.

12. Note that $1 + z + \frac{z^2}{2!} + \cdots \frac{z^n}{n!} \to e^z$ which has no zeroes anywhere. Because the convergence is *uniform* in $|z| \leq R$, the result follows.

14. Use the fact that $|a_n z^n| > |a_{n-1} z^{n-1} + \cdots + a_0|$ on the circle $|z| = R$ for sufficiently large R.

15. To show that $J(\lambda)$ is defined and continuous, note that $|f| > |g|$ throughout γ implies that $f + \lambda g$ is nonzero throughout γ for all $\lambda : 0 \leq \lambda \leq 1$.

16. Let $f(z) = \sqrt{z^2 - 1} = \exp\left(\frac{1}{2} \int_{\sqrt{2}}^z \frac{2\zeta d\zeta}{\zeta^2 - 1} \right)$. Without loss of generality, we can assume that the path of integration is in the upper half-plane, if $\operatorname{Im} z > 0$, and in the lower half-plane, if $\operatorname{Im} z < 0$. To show that $\lim_{z \to x} f(z)$ exists for $-\infty < x < -1$, we must establish that the same limit exists as we approach x through the upper half-plane or the lower half-plane. The difference between the limits equals $\int_C \frac{2\zeta}{\zeta^2 - 1} d\zeta$ where C is a regular closed curve surrounding $\zeta = \pm 1$. By the argument principle,

$$\int_C \frac{2\zeta}{\zeta^2 - 1} d\zeta = 2\pi i \sum \operatorname{Res}\left(\frac{2\zeta}{\zeta^2 - 1}; \pm 1 \right) = 4\pi i$$

Hence, $\lim_{z \to x} \sqrt{z^2 - 1}$ exists because $e^{w/2} = e^{(w+4\pi i)/2}$. By Theorem 7.7, then f is analytic in the plane minus $[-1, 1]$.

17. Define $f(z) = \sqrt[3]{(z-1)(z-2)(z-3)}$ as $\exp\left(\frac{1}{3}\log\left[(z-1)(z-2)(z-3)\right]\right)$
    where

$$\log\left[(z-1)(z-2)(z-3)\right] = \int_4^z \frac{\left[(z-1)(z-2)(z-3)\right]'}{(z-1)(z-2)(z-3)}\, dz + \log 6$$

for z in the plane minus the interval $(-\infty,\ 3]$. Show then that $f(z)$ defines a function which is continuous at all points x on the real axis with $x < 1$.

Chapter 11

1. a. $\int_{-\infty}^{\infty} \frac{x^2}{(1+x^2)^2}\, dx = 2\pi i\ \text{Res}\left(\frac{z^2}{(z+i)^2(z-i)^2}; i\right) = 2\pi i f'(i)$ (where $f(z) = \frac{z^2}{(z+i)^2}) = \frac{\pi}{2}$.

   b. $\int_0^{\infty} \frac{x^2}{(x^2+4)^2(x^2+9)}\, dx = \frac{1}{2} \cdot 2\pi i \sum \text{Res}\left(\frac{z^2}{(z^2+4)^2(z^2+9)}; 2i, 3i\right)$. Note that
   $\text{Res}\left(\frac{z^2}{(z^2+4)^2(z^2+9)}; 3i\right) = \frac{(3i)^2}{((3i)^2+4)^2 6i} = \frac{3}{50}i$ and $\text{Res}\left(\frac{z^2}{(z^2+4)^2(z^2+9)}; 2i\right) = f'(2i)$, with $f(z) = \frac{z^2}{(z+2i)^2(z^2+9)}$, and equals $\frac{-13}{200}i$. Thus, $\int_0^{\infty} \frac{x^2}{(x^2+4)^2(x^2+9)} = \pi i\left(-\frac{1}{200}i\right) = \frac{\pi}{200}$.

   c. Use the fact that $z^4 + z^2 + 1 = 0$ when $z^2 = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$ or when $z^2 = e^{i\frac{2}{3}\pi}$ or $e^{i\frac{4}{3}\pi}$. Thus, $z^4 + z^2 + 1$ has zeroes in the *upper* half-plane at $z_1 = e^{i\pi/3}$ and at $z_2 = e^{i\frac{2}{3}\pi}$.

$$\text{Res}\left(\frac{1}{z^4+z^2+1}; z_1\right) = \frac{1}{4z^3+2z}\Big|_{z=e^{i\pi/3}} = \frac{1}{-3+\sqrt{3}i},$$

$$\text{Res}\left(\frac{1}{z^4+z^2+1}; z_2\right) = \frac{1}{4z^3+2z}\Big|_{z=e^{i\frac{2}{3}\pi}} = \frac{1}{3+\sqrt{3}i}$$

   so that

$$\int_0^{\infty} \frac{dx}{x^4+x^2+1} = \frac{1}{2}2\pi i\left(\frac{1}{-3+\sqrt{3}i} + \frac{1}{3+\sqrt{3}i}\right) = \frac{\sqrt{3}\pi}{6}.$$

   d. $\int_0^{\infty} \frac{\sin x}{x(1+x^2)}\, dx = \frac{1}{2}\text{Im}\int_{-\infty}^{\infty} \frac{e^{ix}-1}{x(1+x^2)}\, dx = \frac{1}{2}\text{Im}\left(2\pi i\ \text{Res}\left(\frac{e^{iz}-1}{z(1+z^2)}; i\right)\right) = \frac{1}{2}\text{Im}\left(\frac{\pi(e-1)i}{e}\right) = \frac{\pi(e-1)}{2e}$.

   e. $\int_0^{\infty} \frac{\cos x}{1+x^2}\, dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2}dx = \frac{1}{2}\text{Re}\int_{-\infty}^{\infty} \frac{e^{ix}\, dx}{1+x^2} = \frac{1}{2} \cdot 2\pi i\ \text{Res}\left(\frac{e^{iz}}{1+z^2}; i\right) = \pi/2e$.

f. $\int_0^\infty \frac{dx}{x^3+8} = -\sum_{k=1}^3 \text{Res}\left(\frac{\log z}{z^3+8}; z_k\right)$ where $z_1 = 2e^{i\pi/3}$, $z_2 = 2e^{i\pi} = -2$;

$z_3 = 2e^{i\frac{5}{3}\pi}$. Note then that $\text{Res}\left(\frac{\log z}{z^3+8}; z_k\right) = \frac{-z_k \log z_k}{24}$ and $\sum_{k=1}^3 \text{Res}\left(\frac{\log z}{z^3+8}; z_k\right)$

$= -\frac{\sqrt{3}\pi}{18}$.

g. $(1-e^{(a-1)2\pi i})\int_0^\infty \frac{x^{a-1}}{1+x}dx = 2\pi i \text{Res}\left(\frac{z^{a-1}}{1+z}; -1\right) = 2\pi i e^{(a-1)\pi i}$. Since

$e^{\pi i} = -1;\ e^{2\pi i} = 1, \int_0^\infty \frac{x^{a-1}}{1+x}dx = \frac{-2\pi i e^{a\pi i}}{1-e^{2a\pi i}} = \frac{-2\pi i}{e^{-a\pi i} - e^{a\pi i}} = $

$\frac{\pi}{\sin(\pi a)}$.

h.

$$\int_0^{2\pi} \frac{dx}{(2+\cos x)^2} = \frac{4}{i}\int_{|z|=1} \frac{z}{(z^2+4z+1)^2}dz$$

$$= 8\pi \text{Res}\left(\frac{z}{(z^2+4z+1)^2}; -2+\sqrt{3}\right)$$

$$= 8\pi f'(-2+\sqrt{3}), \quad \text{with } f(z) = \frac{z}{(z+2+\sqrt{3})^2}$$

$$= 8\pi \frac{\sqrt{3}}{18} = \frac{4}{9}\sqrt{3}\pi$$

i.

$$\int_0^{2\pi} \frac{\sin^2 x}{5+3\cos x}dx = \frac{i}{2}\int_{|z|=1} \frac{(z^2-1)^2}{z^2(3z^2+10z+3)}dz$$

$$= -\pi \sum \text{Res}\left(\frac{(z^2-1)^2}{z^2(3z^2+10z+3)}; -\frac{1}{3}, 0\right).$$

The result follows by noting that the Res at $-1/3$ is $8/9$ and the Res at 0, which

equals $f'(0)$ with $f(z) = \frac{(z^2-1)^2}{3z^2+10z+3}$, is equal to $-\frac{10}{9}$.

j. $\int_0^{2\pi} \frac{dx}{a+\cos x} = \frac{2}{i}\int_{|z|=1} \frac{dz}{z^2+2az+1} = 4\pi \text{Res}\left(\frac{1}{z^2+2az+1}; z_0\right)$ where z_0 is the zero

of $z^2+2az+1$ with $|z_0| < 1$. Thus $z_0 = \begin{cases} -a+\sqrt{a^2-1} & \text{if } a > 1 \\ -a-\sqrt{a^2-1} & \text{if } a < -1 \end{cases}$

$$\int_0^{2\pi} \frac{dx}{a+\cos x} = \frac{4\pi}{2z_0+2a} = \frac{2\pi}{z_0+a} = \begin{cases} 2\pi/\sqrt{a^2-1} & \text{if } a > 1 \\ -2\pi/\sqrt{a^2-1} & \text{if } a < -1 \end{cases}$$

2. Let C_R, Γ_R be as in 11.1 (I). $\int_{C_R} \frac{e^{2iz}-1-2iz}{z^2}dz = 0$ because the integrand is entire.

Thus, $-2\int_{-R}^R \frac{\sin^2 x}{x^2}dx - 2i\int_{\Gamma_R} \frac{dz}{z} + \int_{\Gamma_R} \frac{e^{2iz}-1}{z^2}dz = 0$, and letting $R \to \infty$, we

find $\int_{-\infty}^\infty \frac{\sin^2 x}{x^2}dx = \pi$.

3. Let C_n be the indicated contour; Γ_n the circular segment. Then, $\int_{C_n} \frac{1}{1+z^n} dz = 2\pi i \,\text{Res}\left(\frac{1}{1+z^n}; e^{i\pi/n}\right) = \frac{-2\pi i}{n} e^{i\pi/n}$. Note that $\int_{\Gamma_n} \frac{dz}{1+z^n} \to 0$ as the radius of $\Gamma_n \to \infty$. Hence, letting $R \to \infty$, we find

$$\lim_{R \to \infty} \int_{C_n} \frac{1}{1+z^n} dz = \int_0^\infty \frac{1}{1+x^n} dx - e^{2\pi i/n} \int_0^\infty \frac{1}{1+x^n} dx,$$

and

$$(1 - e^{2\pi i/n}) \int_0^\infty \frac{1}{1+x^n} dx = -2\pi i e^{i\pi/n}.$$

Thus,

$$\int_0^\infty \frac{1}{1+x^n} dx = \frac{2\pi i e^{\pi i/n}}{e^{2\pi i/n} - 1} = \frac{2\pi i}{e^{\pi i/n} - e^{-\pi i/n}} = \frac{\pi/n}{\sin(\pi/n)}.$$

7. a. Note that on Γ_R, $|e^{iz^2}| = e^{-2xy}$ where $z = x + iy$. Because $x \geq \frac{R}{\sqrt{2}}$, $|e^{iz^2}| \leq e^{-Ry}$. Dividing Γ_R into the lower part $L = \{z \in \Gamma_R : y \leq h\}$ and an upper part $U = \{z \in \Gamma_R : y \geq h\}$, $\int_{\Gamma_R} e^{iz^2} dz = \int_L e^{iz^2} dz + \int_U e^{iz^2} dz \ll 2h + e^{-Rh}\left(\frac{\pi}{4}\right) R$ by the usual M-L formula. Choosing $h = \frac{1}{\sqrt{R}}$, e.g., we see that $\int_{\Gamma_R} e^{iz^2} dz \to 0$ as $R \to \infty$.

   b. $\int_{C_R} e^{iz^2} dz = 0$ where C_R is the boundary of the indicated sector. Parametrizing and letting $R \to \infty$, we see by (a) that $\int_0^\infty e^{ix^2} dx - e^{i\pi/4} \int_0^\infty e^{-x^2} dx = 0 \Rightarrow \int_0^\infty \cos x^2 dx + i \int_0^\infty \sin x^2 dx = e^{i\pi/4} \int_0^\infty e^{-x^2} dx$. Using the fact that $\int_0^\infty e^{-x^2} dx = \sqrt{\pi}/2$ and equating real and imaginary parts shows that $\int_0^\infty \cos x^2 dx = \int_0^\infty \sin x^2 dx = \frac{\sqrt{2\pi}}{4}$.

8. [Note that $\int_{|z|=R} \frac{P(z)}{Q(z)} dz = 2\pi i \sum_k \text{Res}\left(\frac{P}{Q}; z_k\right)$ where $\{z_k\}$ represent the zeroes of Q in $|z| < R$ (assuming that $Q \neq 0$ on $|z| = R$.)] If we choose R large enough to encompass all the zeroes of Q, $\int_{|z|=R} \frac{P(z)}{Q(z)} dz = 2\pi i \sum \text{Res}\left(\frac{P}{Q}\right)$. On the other hand, letting $R \to \infty$ and applying the usual $M - L$ estimates, $\int_{|z|=R} \frac{P(z)}{Q(z)} dz \to 0$. Hence, $\sum \text{Res}\left(\frac{P}{Q}\right) = 0$.

9. a. $2\sum_{n=1}^\infty \frac{1}{n^2+1} = -\sum_{k=1}^3 \text{Res}\left(\frac{\pi \cot(\pi z)}{1+z^2}; z_k\right)$ where $z_1 = 0; z_2 = i; z_3 = -i = -\left(1 + \frac{\pi \cot(\pi i)}{i}\right) = -1 + \pi\left(\frac{e^{2\pi}+1}{e^{2\pi}-1}\right)$.

   $\sum_{n=1}^\infty \frac{1}{n^2+1} = -\frac{1}{2} + \frac{\pi}{2}\left(\frac{e^{2\pi}+1}{e^{2\pi}-1}\right)$.

   b. $\sum_{n=1}^\infty \frac{1}{n^4} = -\frac{1}{2}\text{Res}\left(\frac{\pi \cot(\pi z)}{z^4}; 0\right) = \frac{\pi^4}{90}$

   c. $1 + 2\sum_{n=1}^\infty \frac{(-1)^n}{n^2+1} = -\sum_{z_k=\pm i} \text{Res}\left(\frac{\pi}{(\sin \pi z)(z^2+1)}; z_k\right) = -\frac{\pi}{i\sin(\pi i)} = \frac{2\pi}{e^\pi - e^{-\pi}}$.

   $\sum_{n=1}^\infty \frac{(-1)^n}{n^2+1} = \frac{\pi e^\pi}{e^{2\pi}-1} - \frac{1}{2}$.

13. Because $\binom{3n}{n} = \frac{1}{2\pi i} \int_{|z|=R} \frac{(1+z)^{3n}}{z^{n+1}} dz$ for any $R > 0$, $\sum_{n=0}^{\infty} \binom{3n}{n} \frac{1}{8^n} = -\frac{8}{2\pi i} \times$

$\int_{|z|=1/2} \frac{dz}{(z+1)^3 - 8z} = -\frac{8}{2\pi i} \int_{|z|=1/2} \frac{dz}{z^3+3z^2-5z+1}$ (since $\left| \frac{(1+z)^3}{8z} \right| < \frac{27}{32}$ for $|z| =$

$1/2$). Because $z^3 + 3z^2 - 5z + 1 = (z-1)(z^2 + 4z - 1)$, the only zero of

$z^3 + 3z^2 - 5z + 1$ inside $|z| = \frac{1}{2}$ is at $z = -2 + \sqrt{5}$, and $\sum_{n=0}^{\infty} \binom{3n}{n} \frac{1}{8^n} =$

$-8 \operatorname{Res} \left(\frac{1}{z^3+3z^2-5z+1}; -2+\sqrt{5} \right) = \frac{5+3\sqrt{5}}{5}$.

14. Because $\binom{2n}{n} x^n = \frac{1}{2\pi i} \int_{|z|=R} \left(\frac{(1+z)^2 x}{z} \right)^n \frac{dz}{z}$ and because $\left| \frac{(1+z)^2 x}{z} \right| \le |4x| < 1$

throughout $|z| = 1$, $\sum_{n=0}^{\infty} \binom{2n}{n} x^n = \frac{1}{2\pi i} \int_{|z|=1} \frac{-1}{(1+z)^2 x - z} dz$. Note that $(1 +$

$z)^2 x - z = xz^2 + (2x - 1)z + x$ has zeroes at $(1 - 2x \pm \sqrt{1 - 4x})/2x$

and $(1 - 2x - \sqrt{1 - 4x})/2x$ is inside the unit circle. Thus, $\sum_{n=0}^{\infty} \binom{2n}{n} x^n =$

$-\operatorname{Res} \left(-\frac{1}{(1+z)^2 x - z}; \frac{1-2x-\sqrt{1-4x}}{2x} \right) = \frac{1}{\sqrt{1-4x}}$.

15. Note that $\max_{a^2+b^2=4} a^2 b = \max_{0 \le b \le 2} (4b - b^3) = \frac{16}{9}\sqrt{3}$.

16. a. To maximize $\frac{a^2 b}{R^2}$ as in the diagram below, note that $(2R)^2 + 2^2 = 2(a^2 + b^2)$,

   so that $a^2 + b^2 = 2(R^2 + 1)$ and $\frac{a^2 b}{R^2} = \frac{b(6-b^2)}{2}$ because $R^2 = 2$.

   Hence, $\max \left| \frac{(z-1)^2(z+1)}{z^2} \right| = \frac{1}{2} \max(6b - b^3)$ occurs when $b = \sqrt{2}$ and

   $\max \left| \frac{(z-1)^2(z+1)}{z^2} \right| = 2\sqrt{2}$.
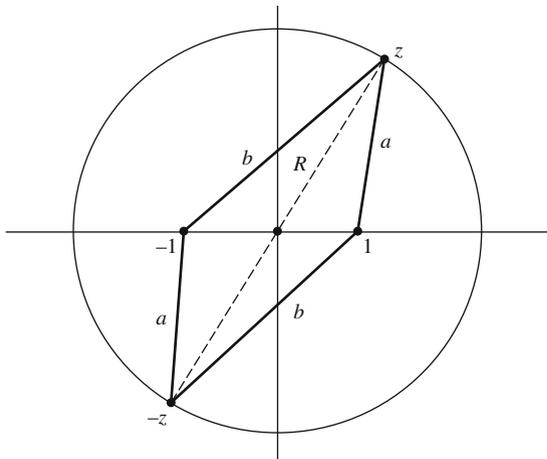
   b. As in the diagram above

$$\max_{|z|=R} \left| \frac{(z-1)^2(z+1)}{z^2} \right| = \max \frac{a^2 b}{R^2} = \max \frac{b(2R^2 + 2 - b^2)}{R^2},$$

which occurs when $b^2 = \frac{1}{3}(2R^2 + 2)$, and is equal to $\frac{4\sqrt{6}}{9}\frac{(R^2+1)^{3/2}}{R^2}$. But $\min_{0<R<\infty}\frac{4\sqrt{6}}{9}\frac{(R^2+1)^{3/2}}{R^2}$ occurs when $R = \sqrt{2}$ and equals $2\sqrt{2}$ as we saw in Exercise (16a).

Chapter 12

3. Note that $\int_{\Gamma_R} e^z \ln z\, dz = 0$ where Γ_R is the closed curve indicated below,
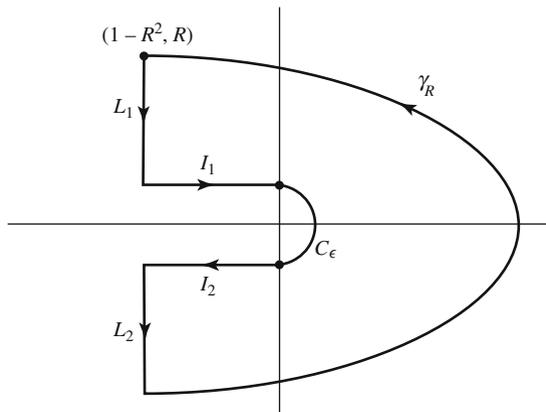
i.e., $\Gamma_R = \gamma_R \cup L_1 \cup I_1 \cup C_\epsilon \cup I_2 \cup L_2$. Because $|e^z \ln z| \le e^{1-R^2}(\ln R + \pi)$ for $\operatorname{Re} z = 1 - R^2$, $\int_{L_1 \cup L_2} e^z \ln z\, dz \to 0$ as $R \to \infty$. Similarly, $\int_{C_\epsilon} e^z \ln z\, dz \to 0$ as $\epsilon \to 0$. Hence $\int_\gamma e^z \ln z\, dz = \int_0^\infty e^{-x}(\ln x + \pi i)dx - \int_0^\infty e^{-x}(\ln x - \pi i)dx = 2\pi i$.

4.

$$\sum(-1)^k \binom{n}{k}^{1/3} = \frac{1}{2\pi i}\int_C [f(z)]^{1/3}\frac{\pi}{\sin(\pi z)}dz$$

$$\ll \frac{1}{\pi}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty}\frac{\pi^{2/3}}{|\sin(\pi z)|^{2/3}\sqrt[3]{|z(1-z)\cdots(1-\frac{z}{n})|}}dz$$

$$\ll \frac{A}{\sqrt[3]{\sqrt{n+1}}} = \frac{A}{(n+1)^{1/6}}$$

5. a. On $\operatorname{Re} z = -\frac{3}{4}$,

$$\frac{1}{\sqrt{\left|(1-z)\cdots\left(1-\frac{z}{n}\right)\right|}} \le \frac{1}{\sqrt{\left(1+\frac{3}{4}\right)\left(1+\frac{3/4}{2}\right)\cdots\left(1+\frac{3/4}{n}\right)}}$$

$$= \frac{1}{\sqrt{\prod_{k=1}^{n} \left(1 + \frac{3}{4k}\right)}}$$

$$\leq \frac{1}{\sqrt{\left[\prod_{k=1}^{n} \left(1 + \frac{1}{k}\right)\right]^{3/4}}}$$

$$= \frac{1}{(n + 1)^{3/8}}.$$

Hence $\sum_{k=0}^{n}(-1)^k \sqrt{\binom{n}{k}} \ll \frac{A}{n^{3/8}}$.

b. Let Re $z = -1 + \delta = -t$. Then $\left|(1 - z)\cdots(1 - \frac{z}{n})\right| \geq$ $\prod_{k=1}^{n}(1 + \frac{t}{k}) \geq \left[\prod_{k=1}^{n}\left(1 + \frac{1}{k}\right)\right]^t = (n + 1)^{1-\delta}$. Also, because

$|\sin(x+iy)| \geq |\sin x|$, $|\sin(\pi z)| \geq |\sin(\pi \delta)| \geq 2\delta$. Thus, $\left|\sum_{k=1}^{n}(-1)^k \sqrt{\binom{n}{k}}\right|$

$\leq 2 \int_{-1+\delta-i\infty}^{-1+\delta+i\infty} \frac{dz}{\sqrt{\sin(\pi z)(1-z)\cdots(1-\frac{z}{n})}} \leq \frac{A}{\sqrt{\delta(n+1)^{1-\delta}}}$ and, taking $\delta = \frac{1}{\log n}$,

$\left|\sum_{k=1}^{n}(-1)^k \sqrt{\binom{n}{k}}\right| \leq \frac{B\sqrt{\log n}}{\sqrt{n}}$.

Chapter 13

1. Note that the solutions of $z^k = \alpha$ are uniformly distributed around the circle of radius $|\alpha|^{1/k}$. Hence z^k is 1-1 in any set of the form $S_{\alpha,\beta} = \{z : \alpha < \text{Arg } z < \beta; \beta - \alpha < \frac{2\pi}{k}\}$.

2. $x = x_0$ is mapped into a circle centered at 0 with radius e^{x_0}. $y = y_0$ is mapped into the ray $re^{iy_0}; r > 0$.

3. i. $f_3 \circ f_2 \circ f_1$ where $f_1(z) = \frac{\pi}{3}(z + 2)$; $f_2(z) = e^{iz}$; $f_3(z) = \frac{z-i}{z+i}$.

   ii. $f(z) = \frac{4z}{6-z}$ (see Theorem 13.23).

   iii. $f(z) = \frac{4}{\pi} \log z$

   iv. $f_3 \circ f_2 \circ f_1$ with $f_1(z) = \sqrt{z}$, $f_2(z) = -\frac{(z-1)^2}{4(z+1)^2}$ and $f_3(z) = \frac{z-i}{z+i}$.

8. a. Note that lines are mapped into lines.

   b. By considering $g(z) = \alpha f(\beta z + \gamma) + \delta$, we can assume without loss of generality that f maps a rectangle R of the form: $0 \leq x \leq a$, $0 \leq y \leq b$ onto a rectangle S of the form: $0 \leq x \leq c$, $0 \leq y \leq d$. *If we assume, in addition that f maps the boundary curves of R onto the boundary curves of S,* it will follow from the Schwarz Reflection Principle that Re $f(z)$ and Im $f(z)$ both grow (at most) linearly. Thus $|f(z)| \leq A|z| + B$ and, by the Extended Liouville Theorem, f is linear. See Example 2 in Section 7.2 and exercise 23 of Chapter 7. The fact that the boundary lines are, in fact, mapped onto the boundary lines, is proven in 14.9.

9. If $f_1(z) = \frac{a_1 z + b_1}{c_1 z + d_1}$, $f_2(z) = \frac{a_2 z + b_2}{c_2 z + d_2}$, then $f_2 \circ f_1(z) = \frac{Az + B}{Cz + D}$ with $A = a_2 a_1 + b_2 c_1$, $B = a_2 b_1 + b_2 d_1$, $C = a_1 c_2 + c_1 d_2$, $D = b_1 c_2 + d_1 d_2$ and $AD - BC = (a_1 d_1 - b_1 c_1)(a_2 d_2 - b_2 c_2)$. Also, as in the text, if $f(z) = \frac{az + b}{cz + d}$, $f^{-1}(z) = \frac{dz - b}{-cz + a}$. The other group properties are easily established.

10. a. $|z| = 1$.
    b. The line $x = -\frac{1}{2}$.
    c. $C(-\frac{2}{3}; \frac{1}{3})$. Note that $\frac{1}{z-2} = f_2 \circ f_1$ with $f_1(z) = z - 2$, $f_2(z) = \frac{1}{z}$ and use Lemma 13.10.

11. By Theorem 13.15, $f(z) = e^{i\theta} z$ and $e^{i\theta} = f'(0) > 0$.

12. Let $f(z) = f_1 \circ f_2^{-1}(z)$. Then $f(0) = 0$, and $f'(0) = f_1'(f_2^{-1}(0)) \cdot \frac{1}{f_2'(f_2^{-1}(0))} = \frac{f_1'(z_0)}{f_2'(z_0)} > 0$ so that, by Exercise 11, $f(z) \equiv z$ and $f_2 \equiv f_1$.

13. Use Theorems 13.15–13.17 and the fact that any disc or half-plane can be mapped onto the unit disc or upper half-plane, respectively, by a linear mapping.

14. The lower half-plane.

15. $\sqrt{\frac{az^2 + b}{cz^2 + d}}$; a, b, c, d real; $ad - bc > 0$.

16. Use the hint given with the exercise and the fact that $h_1 \circ h_2$ is of the form $\frac{az+b}{cz+d}$; a, b, c, d real as in the proof of Exercise 9.

17. (a) $i, -i$. (b) 0.

18. $T(z) = \frac{(z-z_2)(z_3-z_1)}{(z-z_1)(z_3-z_2)}$ maps $z_1, z_2, z_3 \to \infty, 0, 1$, respectively, and because it is bilinear, it maps the circle (or line) containing z_1, z_2, z_3 onto the real line. Moreover, T is 1-1 so that $(z_1, z_2, z_3, z_4) = T(z_4)$ is real-valued if, and only if, z_4 lies on the circle (or line) containing z_1, z_2, z_3.

19. (a) $w = -\frac{1}{z}$. (b) $w = z + i$. (c) $w = \frac{z-i}{z+i}$.

20. Note that $\frac{z-\alpha}{1-\bar{\alpha}z}$ maps $|z| < 1$ onto $|z| < 1$ and $g(z) = a\left(\frac{4z-1-\beta}{1-\beta(4z-1)}\right)$ maps $|z - \frac{1}{4}| < \frac{1}{4}$ onto $|z| < a$. Equating coefficients leads to $\alpha = 2 - \sqrt{3}$.

Chapter 14

1. Since g is locally irrotational and source-free, F is well defined and analytic. As z moves along a curve C from z_1 to z_2

$$F(z_2) - F(z_1) = \int_{t_1}^{t_2} (u - iv)(dx + i dy)$$

$$= \int_{t_1}^{t_2} u\,dx + v\,dy + i(u\,dy - v\,dx)$$

where $C : z(t)$; $z_k = z(t_k)$. If Re $F(z)$ is constant throughout C, $\int_{t_1}^{t_2} u\,dx + v\,dy \equiv 0$, which implies that $g = u + iv$ is orthogonal to the (tangent) direction vector $\left(\frac{dx}{dt}, \frac{dy}{dt}\right)$.

2. Note that curves satisfying Im $F(z) = $ constant are orthogonal to curves satis-
fying Re $F(z) = $ constant since the two families of curves are the preimages
under F of the orthogonal curves: $x = $ constant and $y = $ constant. Alternatively,
argue as in Exercise 1, that $\int_C u\,dy - v\,dx = 0$ implies that the vector (u, v) is
orthogonal to the direction $\left(\frac{dy}{dt}, -\frac{dx}{dt}\right)$.

3. a. The hyperbolas $xy = c$.
   b. Rays from the origin.

4. Note that, if $F(z) = z + A_0 + \frac{A_1}{z} + \frac{A_2}{z^2} + \cdots$, with $A_k = a_k + ib_k$; a_k, b_k real,
then Im $F(e^{i\theta}) = \sin\theta + b_0 - \sum_{k=1}^{\infty} a_k \sin(k\theta) + \sum_{k=1}^{\infty} b_k \cos(k\theta)$. Hence, by
the uniqueness of the Fourier series, $a_1 = 1$; $a_k = 0$ for $k > 1$ and $b_k = 0$ for
$k \geq 1$, i.e., $F(z) = z + A_0 + \frac{1}{z}$.

5. a. For $f(z) = 2z + \frac{1}{z}$, $f(e^{i\theta}) = 3\cos\theta + i\sin\theta = u + iv$ with $\frac{u^2}{9} + v^2 = 1$.
   b. Take $f(z) = f_2 \circ f_1$ where f_1 is the inverse of $2z + \frac{1}{z}$, i.e., $f_1(z) = \frac{z+\sqrt{z^2-8}}{4}$
   and $f_2(z) = z + \frac{1}{z}$.

6. Let $g(z) = e^{i\theta}\left[\frac{f(z)-f(z_0)}{1-\overline{f(z_0)}f(z)}\right]$. Because $g'(z_0) = \frac{e^{i\theta}f'(z_0)}{1-|f(z_0)|^2}$, choose
$\theta = -\text{Arg}\,f'(z_0)$.

9. If $R \neq \mathbb{C}$, let $f_1 : R \to U$ and $f_2 : R \to U$ be such that $f_1(z_1) = f_2(z_2) = 0$.
Then, $f = f_2^{-1} \circ f_1$ is the desired mapping. If $R = \mathbb{C}$, let $f(z) = z - z_1 + z_2$.

10. If $f_1: \mathbb{C} \to R$ were conformal, then $f = f_2 \circ f_1$, where $f_2: R \to U$
would be a conformal mapping of $\mathbb{C}$ onto U which is impossible by Liouville's
Theorem.

11. a. Note, as in the proof of the Riemann Mapping Theorem, part B, that
   $|g'(z_0)| < \frac{1}{\delta}$ where δ is such that $\bar{D}(z_0; \delta) \subset R$.
   b. Let $\varphi_1(z)$ be the Riemann mapping function $\varphi_1 : R \to U$ with $\varphi_1(z_0) = 0$,
   $\varphi_1'(z_0) = M > 0$ and let $\varphi_2 : R \to U$ with $\varphi_2(z_0) = 0$, $\varphi_2'(z_0) = M^* \geq M$.
   Let $f(z) = \varphi_2 \circ \varphi_1^{-1}(z)$. Then f is analytic in the unit disc; $|f| < 1$ there,
   $f(0) = 0$ and $f'(0) = \frac{\varphi_2'(z_0)}{\varphi_1'(z_0)} = \frac{M^*}{M} \geq 1$. Hence, by Schwarz' lemma,
   $M^* = M$ and $f(z) = z$, i.e., $\varphi_2 = \varphi_1$.

Chapter 15

2. $e^{\epsilon i z^2}$

3. e^z maps the lines $y = \pm\frac{\pi}{2}$ onto the imaginary axis; hence, $e^{e^z} \ll 1$ on the
boundary of D. If $f(z) \ll A_\epsilon e^{\epsilon e^z}$ in D, then $g(z) = f(\log z) \ll A_\epsilon e^{\epsilon|z|}$ in the
right half-plane.

Chapter 16

1. $u + v = \text{Re}[(1 - i)f]$; $uv = \text{Re}\left(\frac{-if^2}{2}\right)$.

2. If $g = f_x$, then $g_{xx} + g_{yy} = (f_{xx} + f_{yy})_x = 0$. Similarly, f_y is harmonic.

3. If $g = u^2$, $g_{xx} + g_{yy} = 2u(u_{xx} + u_{yy}) + 2(u_x^2 + u_y^2) = 2(u_x^2 + u_y^2)$ which cannot be identically zero unless u is constant.

4. A direct calculation shows that, if $u = \log(x^2 + y^2)$, $u_{xx} + u_{yy} = 0$. If u were the real part of an analytic $f(z)$, $z \neq 0$, f would have to agree up to an additive constant with the analytic function $\log z$ in the simply connected domain $0 < \text{Arg } z < 2\pi$. But then, $f(z)$, like $\log z$, would not be continuous on the positive real axis.

5. Note that, if $v(r, \theta) = u(r \cos\theta, r \sin\theta)$, then, $r^2 v_{rr} + r v_r + v_{\theta\theta} = r^2(u_{xx} + u_{yy})$. Hence, Laplace's equation becomes

$$r^2 v_{rr} + r v_r + v_{\theta\theta} = 0 \text{ or } r\frac{\partial}{\partial r}(r v_r) + v_{\theta\theta} = 0.$$

i.e., $\frac{\partial}{\partial r}(r v_r) + \frac{1}{r} v_{\theta\theta} = 0$. If v depends on r alone, $v_{\theta\theta} = 0$, and the above differential equation implies

$$v = a \ln r + b.$$

Note also that, if $v_r = 0$ and v is harmonic, then $v_{\theta\theta} = 0$, or $v(\theta) = a\theta + b$.

6. $2\pi i f(z) = \int_{C_R} \frac{f(\zeta)}{\zeta - z} d\zeta - \int_{C_R} \frac{f(\zeta)}{\zeta - \bar{z}} d\zeta = \int_{C_R} \frac{(z - \bar{z})f(\zeta)}{(\zeta - z)(\zeta - \bar{z})} d\zeta$. Let Γ_R be the semicircular arc $|z| = R$, $\text{Im } z \geq 0$. Then, as $R \to \infty$, $\int_{\Gamma_R} \frac{(z - \bar{z})f(\zeta)}{(\zeta - z)(\zeta - \bar{z})} d\zeta \to 0$ by the M-L formula, so that

$$2\pi i f(z) = 2i \int_{-\infty}^{\infty} \frac{y f(t)}{(t - x)^2 + y^2} dt,$$

and, dividing by $2\pi i$ and equating real parts,

$$u(x + iy) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y u(t)}{(t - x)^2 + y^2} dt$$

7. Note that $\text{Re}(z^3) = x^3 - 3xy^2 = 4x^3 - 3x$ on the unit circle. Hence

$$u(x, y) = \frac{1}{4}(x^3 - 3xy^2 + 3x).$$

8. $u(z) = \frac{3}{2} - \frac{1}{\pi}\text{Arg}\left(\frac{z-1}{z+1}\right)$ where the Arg takes values between $\frac{\pi}{2}$ and $\frac{3\pi}{2}$.

$$u(z) = k \Rightarrow \text{Arg}\left(\frac{z - 1}{z + 1}\right) = \theta = \pi\left(\frac{3}{2} - k\right).$$

Thus, for $\frac{1}{2} < k < 1$, z lies on the upper arc of a circle for which $[-1, 1]$ is a chord and for which the lower arc has 2θ degrees. For $0 < k < \frac{1}{2}$, z lies on congruent arcs in the lower half-plane. The chord $[-1, 1]$ is the level curve for $k = \frac{1}{2}$.

9. $1 - \frac{1}{\pi} \operatorname{Arg} z = 1 - \frac{1}{\pi} \theta$ (See the note following the solution to exercise 5.)

10. $\sin z$ maps the strip onto the upper half-plane with the boundary of the strip being mapped onto the real line segments $(-\infty, -1)$, $[-1, 1]$, $(1, \infty)$.

   In the upper half-plane: $w > 0$, $\operatorname{Arg}(w^2 - 1)$ has the values 2π, π, 0 on the intervals $(-\infty, -1)$, $(-1, 1)$ and $(1, \infty)$, respectively, and $\operatorname{Arg}(w^2 - 1) = \operatorname{Im}[\log(w^2 - 1)]$ is the imaginary part of a function analytic in the upper half-plane. Thus, the desired solution is

$$u(x, y) = \frac{1}{\pi} \operatorname{Arg}(\sin^2 z - 1) = \frac{1}{\pi} \operatorname{Arg}(-\cos^2 z)$$

Note, for example, that on the y axis, $u(0, y) = \frac{1}{\pi} \operatorname{Arg}(-\cosh^2 y) = 1$.

11. By Theorem 16.3, if $e^z - P(z)$ does not have infinitely many zeroes,

$$e^z - P(z) = Q(z)e^{R(z)}$$

where Q, R are polynomials. Considering the growth at infinity, it follows that $R(z) = z$, $Q(z) = 1$, and $P(z) = 0$. Similarly, for $\sin z - P(z)$.

12. If a function f of order j does not have infinitely many zeroes,

$$f(z) = Q(z)e^{P(z)}.$$

But if $f \neq 0$, Q is a constant and f can be written in the form $f(z) = e^{P(z)}$. Finally, because f is of order j, P is a polynomial of degree j.

Chapter 17

1. $\prod_{k=2}^{N} \left(1 - \frac{1}{k^2}\right) = \prod_{k=2}^{N} \frac{(k-1)(k+1)}{k^2} = \frac{N+1}{2N}$. Hence $P_N \to \frac{1}{2}$ as $N \to \infty$.

2. $\prod_{k=2}^{N} \left[1 + \frac{(-1)^k}{k}\right] = \left(\frac{3}{2}\right) \left(\frac{2}{3}\right) \left(\frac{5}{4}\right) \left(\frac{4}{5}\right) \cdots \left[1 + \frac{(-1)^N}{N}\right]$. Hence,

$$P_N = \begin{cases} 1 & \text{if } N \text{ is odd} \\ 1 + \dfrac{1}{N} & \text{if } N \text{ is even} \end{cases}$$

and $P_N \to 1$ as $N \to \infty$.

4. $\log(1 + z_k) - z_k = -\frac{z_k^2}{2} + \frac{z_k^3}{3} - + \cdots \ll z_k^2$ if $|z_k| \leq \frac{1}{2}$. Hence, if $\sum |z_k|^2$ converges, so does $\sum [\log(1 + z_k) - z_k]$ and, because $\sum z_k$ converges, it follows that $\sum \log(1 + z_k)$ converges. By Proposition 17.2, then, $\Pi(1 + z_k)$ converges.

5. Because $\sum z_k = \sum \frac{(-1)^k}{\sqrt{k}}$ converges, the convergence of $\sum \log(1+z_k)$ is equiva-

 lent to the convergence of $\sum [\log(1+z_k) - z_k]$. But the latter is $\sum_k \left(-\frac{z_k^2}{2} + \frac{z_k^3}{3} - \right.$

 $\left. +\cdots \right)$ and, for $k \geq 4$, $\log(1+z_k) - z_k \leq \frac{-1}{6k}$ so that $\sum \log(1+z_k)$ diverges.

6. $(1+z)(1+z^2)\cdots(1+z^{2^{N-1}}) = 1+z+z^2+\cdots+z^{2^N-1} \to \sum_{k=0}^{\infty} z^k = \frac{1}{1-z}$
 uniformly for $|z| \leq r < 1$.

7. $\prod_{k=1}^{\infty} \left(1 - \frac{z}{k^2}\right)$

8. Using the power series expansion for $\sin z$, it can be seen that $\frac{\sin \pi \sqrt{z}}{\pi \sqrt{z}}$ is entire
 and equal to zero if $z = k^2$; $k = 1, 2, \cdots$. Note, also, that, according to
 Proposition 17.8, the solutions in (7) and (8) are identical.

9. $f(z) = \prod_{k=1}^{\infty} \left(1 - \frac{4z^2}{(2k+1)^2}\right) \Big/ \cos \pi z$ is entire and zero-free. As in
 Proposition 17.8, it can be shown (considering the magnitude of f on a square of
 side $2N$ centered at the origin) that $|f(z)| \leq A \exp(|z|^{3/2})$ and that f is, in fact,
 constant, so that $f(z) = f(0) = 1$. The product form also be derived from the
 identity:

$$\cos \pi z = \frac{\sin 2\pi z}{2 \sin \pi z}.$$

10. a. $\prod_{k=1}^{\infty} \left[1 - \frac{1}{k(1-z)}\right] \exp\left(\frac{1}{k(1-z)}\right)$.

 b. Le $\{z_k\}$ be a sequence of distinct points with $\lim_{k\to\infty} z_k = z_0$. Then, an
 entire function can be defined with zeroes at the points $\lambda_k = \frac{1}{z_0 - z_k}$. Setting
 $g(z) = f\left(\frac{1}{z_0 - z}\right)$, g will be analytic for $z \neq z_0$ and equal to zero at the points
 of the original sequence $\{z_k\}$.

11. $F'(z) = \frac{1}{2\pi i} \int_C \int_a^b \frac{\varphi(\zeta,t)}{(\zeta-z)^2} dt \, d\zeta = \int_a^b \frac{1}{2\pi i} \int_C \frac{\varphi(z,t)}{(\zeta-z)^2} d\zeta \, dt = \int_a^b \frac{\partial}{\partial z} (\varphi(z,t)) dt$.

12. Because h is continuous, $|h| \leq M$ on $[\alpha, \beta]$. For any $\epsilon > 0$, $\int_\alpha^{x-\epsilon} \frac{h(u)y}{(u-x)^2+y^2} du$
 and $\int_{x+\epsilon}^\beta \frac{h(u)y}{(u-x)^2+y^2} du$ are each bounded in absolute value by $\frac{M y(\beta-\alpha)}{\epsilon^2}$ whereas
 $\int_{x-\epsilon}^{x+\epsilon} \frac{h(u)y}{(u-x)^2+y^2} du = h(\overline{x}) \cdot 2 \tan^{-1}\left(\frac{\epsilon}{y}\right)$ where $x - \epsilon < \overline{x} < x + \epsilon$. Hence, as
 $y \to 0$, $\int_\alpha^\beta \frac{h(u)y}{(u-x)^2+y^2} du$ can be made arbitrarily close to $\pi h(x)$.

13. $f(z) = \int_0^1 \frac{dt}{1-zt} = -\frac{\log(1-z)}{z}$ which is analytic in $\mathbb{C} - [1, \infty)$. By the argument
 principle, $\Delta \text{Arg}(1 - z) = 2\pi i$ as z circles the point $z = 1$; hence, f has a jump
 discontinuity of $\frac{2\pi i}{x}$ as z crosses from the upper half-plane to the lower half-
 plane at $z = x > 1$. Note also that, if we consider $g(z) = \int_0^\infty \frac{dt}{e^t - z}$ (Example 2
 following 17.9), setting $u = e^{-t}$, it can be shown that $g(z) = \int_0^1 \frac{dt}{1-zt}$.

Chapter 18

1. Assuming $\log z$ is the principle branch, i.e., $\operatorname{Im} \log z = 0$ on the positive axis, it follows that $\operatorname{Im} g_1(z)$ will be between $-\pi$ and $-\frac{\pi}{2}$ in the third quadrant, whereas $\operatorname{Im} g_2(z)$ will be between π and $\frac{3\pi}{2}$.

3. Note that $f(-z) = \sum a_n z^n$; $a_n \geq 0$, and apply Theorem 18.3.

4. a. Because $\frac{1}{n^{1/3}} = \frac{1}{\Gamma(1/3)} \int_0^\infty e^{-nt} t^{-2/3} dt$,

$$\sum \frac{z^n}{n^{1/3}} = \frac{1}{\Gamma(1/3)} \int_0^\infty \sum (ze^{-t})^n t^{-2/3} dt$$
$$= \frac{1}{\Gamma(1/3)} \int_0^\infty \frac{z}{t^{2/3}(e^t - z)} dt$$

which is analytic outside of the interval $[1, \infty)$

   b. Since $\frac{1}{n^2+1} = \int_0^\infty e^{-nt} \sin t \, dt$,

$$\sum \frac{z^n}{n^2 + 1} = \sum \int_0^\infty (ze^{-t})^n \sin t \, dt$$
$$= \int_0^\infty \sum (ze^{-t})^n \sin t \, dt = \int_0^\infty \frac{e^t \sin t}{e^t - z} dt$$

which is analytic outside of the interval $[1, \infty)$

5. Make the change-of-variables $u = nt$.

6. Setting $u = t^2$ yields

$$\int_0^\infty e^{-t^2} dt = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}.$$

7. Because $e^{-t/n} = 1 - \frac{t}{n} + \frac{t^2}{2n^2} - + \cdots$, $0 \leq e^{-t/n} - \left(1 - \frac{t}{n}\right) \leq \frac{t^2}{2n^2}$, if $t \leq n$, so that $e^{-t} - \left(1 - \frac{t}{n}\right)^n \leq e \cdot \frac{e^{-t} t^2}{2n}$ and

$$\int_0^n t^{z-1}\left(1 - \frac{t}{n}\right)^n dt - \int_0^n t^{z-1} e^{-t} dt \ll \frac{e}{2n} \int_0^n t^{z-1} e^{-t} t^2 \, dt$$
$$\leq \frac{e}{2n}\Gamma(\operatorname{Re} z + 2)$$

which approaches 0 as $n \to \infty$.

9. $f(z) = 1 - \frac{1}{2^z} + \frac{1}{3^z} - + \cdots = \left(1 - \frac{2}{2^z}\right)\zeta(z)$ so that f is certainly analytic, like $\zeta(z)$, for $z \neq 1$. Moreover, $\lim_{z \to 1} f(z) = \lim_{z \to 1} \frac{2^z - 2}{2^z(z-1)} = \ln 2$ so that f is analytic at $z = 1$ as well.

10. Because $\zeta(z) \to \infty$ as $z \to 1$, $\Pi\left(1 - \frac{1}{p}\right)$ diverges to 0. Because $\sum \frac{1}{p^2}$ converges, this implies that $\sum \frac{1}{p}$ diverges (see Chapter 17, Exercise 4).

Chapter 19

1. Consider $f(z) = \tan z - z$ inside the square centered at the origin and with sides of length $2\pi N$, whose boundary is denoted C_N. Then, the number of poles of $f(z)$ inside C_N = the number of zeros of $\cos z$ inside C_N = $2N$. The number of *real* zeros of f inside C_N is $2N+1$ since f has a triple zero at the origin and $\tan x = x$ has exactly one solution in each of the intervals $[(2k-1)\frac{\pi}{2}, (2k+1)\frac{\pi}{2}]$; $k = \pm 1, \pm 2, \ldots, \pm(N-1)$.

   Let c = the number of complex, *nonreal* zeros of $\tan z - z$ inside C_N. Then

$$\frac{1}{2\pi i} \int_{C_N} \frac{\tan^2 z}{\tan z - z} dz = \mathbb{Z} - \mathbb{P} = 1 + c$$

   by the calculations above.

   By the usual $M - L$ estimates and the fact that $|\tan z| < 1 + \epsilon$ throughout C_N (where $\epsilon \to 0$ as $N \to \infty$), it follows that

$$\mathbb{Z} - \mathbb{P} = 1 + c < 2.$$

   Hence $c = 0$.

2. With $f_2(z) = \frac{z^2}{(1+z^2)(\tan z - z)}$, note that $\int_{C_N} f_2(z)dz \to -2\pi i$ whereas

$$\int_{C_N} f_2(z)dz \to 2\pi i \left[\sum_{\substack{k=1 \\ x_k \neq 0}}^{\infty} \frac{\sin^2 x_k}{x_k^2} + \text{Res}(f_2; i) + \text{Res}(f_2; -i) + \text{Res}(f_2; 0) \right].$$

   Note, then, that $\text{Res}(f_2; i) = \text{Res}(f_2; -i) = -\left(\frac{e^2+1}{4}\right)$ whereas $\text{Res}(f_2; 0) = 3$ because $\frac{z^2}{\tan z - z}$ has a simple pole at $z = 0$ and $\lim_{z\to 0} \frac{z^3}{\tan z - z} = 3$. Hence, $\sum_{\substack{k=0 \\ x_k \neq 0}}^{\infty} \frac{\sin^2 x_k}{x_k^2} = \frac{e^2-7}{2}$ and $\text{Var}\left(\frac{\sin^2 x}{x}\right) = 2\sum_{\substack{k=1 \\ x_k \neq 0}}^{\infty} \frac{\sin^2 x_k}{x_k^2} + 2 = e^2 - 5$.

3. Let $f(z) = \frac{e^z-1}{z^2(e^z-z)}$. Then $\int_{C_N} f(z)dz \to 0$ as $N \to \infty$ if C_N is a square centered at the origin with sides of length $2\pi N$. At the same time, $\int_{C_N} f(z)dz \to 2\pi i \left(\sum \frac{1}{z_k^2} + \text{Res}(f; 0)\right)$ where the sum is taken over the zeros of $e^z - z$. Because $\text{Res}(f; 0) = +1$ it follows that $\sum \frac{1}{z_k^2} = -1$.

4. As in Section 19.3, a solution $\{a_k\}$, $\{b_k\}$, would imply

$$1 + a_1 z + \frac{a_2 z^2}{2!} + \cdots = e^{\alpha z} = 1 + \alpha z + \frac{\alpha^2 z^2}{2!} + \cdots$$

$$1 + b_1 z + \frac{b_2 z^2}{2!} + \cdots = e^{\beta z} = 1 + \beta z + \frac{\beta^2 z^2}{2!} + \cdots$$

so that $\alpha = a_1$ and $a_k = a_1^k$, $k = 2, 3, \ldots$, and $\beta = b_1$, $b_k = b_1^k$; $k = 2, 3, \ldots$. Thus, there would be infinitely many solutions of the form $\{a_k\}$, $\{b_k\}$ with $a_1, b_1 \geq 0$; $a_1 + b_1 = 2$ and $a_k = a_1^k$; $b_k = b_1^k$ for $k = 2, 3, \ldots$.

5. Suppose d_1 is relatively prime to all d_j, $j \neq 1$, and assume that the desired partition is possible. Then, as in Section 19.5,

$$\frac{z}{1-z} = \frac{z^{a_1}}{1-z^{d_1}} + \frac{z^{a_2}}{1-z^{d_2}} + \cdots + \frac{z^{a_k}}{1-z^{d_k}}$$

for $|z| < 1$. Then, if we let $z \to e^{2\pi i/d_1}$, the first term on the right side of the equality would approach infinity whereas all the others would approach a finite limit. Thus, the partition is impossible. (In fact, according to this argument, the partition would be impossible as long as one of the differences is not a divisor of any of the others.)

9. $\sum_{n=2}^{\infty} \left| \frac{1}{np^{nz}} \right| = \sum_{n=2}^{\infty} \frac{1}{np^{nx}} \leq \frac{1}{2} \sum_{n=2}^{\infty} \frac{1}{p^{nx}} = \frac{1}{2p^{2x}} \frac{p^x}{(p^x-1)} \leq \frac{2}{p^{2x}}$ for $x > \frac{1}{2}$.

In fact, for $x \geq \frac{1}{2} + \delta$, $\sum_{\substack{n=2 \\ p \text{ prime}}}^{\infty} \left| \frac{1}{np^{nz}} \right| \leq \sum_{p \text{ prime}} \frac{2}{p^{1+2\delta}} \leq \sum_{n=1}^{\infty} \frac{2}{n^{1+2\delta}} < \infty$.

Thus, $\sum_{\substack{n \geq 2 \\ p \text{ prime}}} \frac{1}{np^{nz}}$ is uniformly convergent on compacta and is analytic in the half-plane $\operatorname{Re} z > \frac{1}{2}$.

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Appendices

I A Note on Simply Connected Regions

The definition of simple connectedness (8.1) led to a relatively easy proof of the General Closed Curve Theorem (8.6). At the same time, while Definition 8.1 was somewhat complicated, we were able to establish the very intuitive result that a simply connected region contains, along with any closed polygonal path, all of the points which are "inside" the path. (See Lemma 8.3 and Exercise 6 of Chapter 8). This property of a simply connected region can be generalized. That is, a simply connected region contains, along with *any* closed curve, all the points inside the curve. The difficulty in proving the general result lies in defining the "inside" of a general closed curve. If we limit ourselves to smooth closed curves, however, we can use complex integrals to define the "inside" of the curve and we can prove the above property of simply connected regions.

Definition

If Γ is a smooth closed curve, we say that a point $z_0 \notin \Gamma$ is *inside* Γ if $\int_\Gamma \frac{dz}{z-z_0} \neq 0$. The totality of such points is called the *inside of* Γ. Note that a similar definition (10.4), under more limited circumstances, is given in Chapter 10.

Lemma

If D is a simply connected region, Γ is a closed curve contained in D and $z_0 \in \tilde{D}$, then there exists a differentiable curve $\gamma(t)$ which connects z_0 to ∞ and which does not intersect Γ.

Proof

According to Definition 8.1, there exists a continuous curve γ, connecting z_0 to ∞ with $d(\gamma, \tilde{D}) < \epsilon$. If we take $\epsilon = \frac{1}{2} d(\Gamma, \tilde{D})$, γ will not intersect Γ. Moreover, since $\gamma \to \infty$, for some N, $t \geq N \Rightarrow |\gamma(t)| \geq \max\{|z| : z \in \Gamma\}$. We can, then, redefine $\gamma(t) = \frac{t}{N}\gamma(N)$ for $t \geq N$ so that γ will be differentiable (γ' will actually be constant) for $t \geq N$. Finally, because $\gamma(t)$, $0 \leq t \leq N$, can be uniformly approximated by a differentiable curve, there exists a curve γ with all of the desired properties. $\square$

Theorem

If D is a simply connected region and Γ is a smooth closed curve contained in D, then the inside of Γ is contained in D.

Proof

If not, there would be $z_0 \in \tilde{D}$ for which $\int_\Gamma \frac{dz}{z-z_0} \neq 0$. Let γ be a differentiable curve, connecting z_0 to ∞ and not intersecting Γ (as in the above lemma), and define

$$I(t) = \int_\Gamma \frac{dz}{z - \gamma(t)}, \quad t \geq 0.$$

$I(t)$ can be differentiated with respect to t and

$$I'(t) = \gamma'(t) \int_\Gamma \frac{dz}{[z - \gamma(t)]^2}.$$

The above integral is clearly 0 (for all t) since the integrand has, as a primitive, the function $\frac{-1}{z-\gamma(t)}$. Thus we can conclude that $I(t)$ is constant. On the other hand, $I(0) \neq 0$ (since $\gamma(0) = z_0$) and $I(t) \to 0$ as $t \to \infty$ since the integrand approaches 0 uniformly, which yields the desired contradiction. $\qquad\square$

II   Circulation and Flux as Contour Integrals

Let C be a closed curve given by $z(t) = x(t) + iy(t)$. Then a vector tangent to C is given by

$$\dot{z}(t) = \frac{dx}{dt} + i\frac{dy}{dt}$$

and a normal vector to C is given by

$$\frac{dy}{dt} - i\frac{dx}{dt}.$$

(If C is parametrized so that the tangent points in the counter-clockwise direction, the above normal vector points "outward.") Suppose $g = u + iv$ represents a flow function throughout C. Then the circulation around C is found by integrating the tangential component of g against the arclength, and the flux across C is given by the corresponding integral of the normal component of g. Let σ, τ represent the circulation and flux, respectively, and recall that the component of a vector $\vec{\alpha}$ in the direction of $\vec{\beta}$ is given by $(\vec{\alpha} \circ \vec{\beta}/|\vec{\beta}|)$. Then

$$\sigma = \int_C \left(u\frac{dx}{dt} + v\frac{dy}{dt} \right) dt = \int_C u\,dx + v\,dy$$

and

$$\tau = \int_C \left(u\frac{dy}{dt} - v\frac{dx}{dt} \right) dt = \int_C u\,dy - v\,dx.$$

Note, finally, that if $f = \bar{g} = u - iv$,

$$\int_C f(z)dz = \int_C (u - iv)(dx + i\,dy) = \sigma + i\tau.$$

III Steady-State Temperatures; The Heat Equation

Let the function u denote the temperature at the points of a region D and assume that u is independent of time. Then $u = u(x, y)$ is a real-valued function of the position (x, y), and we wish to show that it is harmonic. To this end, we note two basic facts:

1. Heat flows in the direction of cooler temperatures, and the amount of heat crossing a curve per unit of time is proportional to the length of the curve and the difference in temperature across the two sides. Thus the amount of heat crossing a horizontal line of length Δx is equal to $Ku_y \Delta x$, while the amount of heat crossing a vertical line of length Δy is given by $Ku_x \Delta y$.
2. The total increase in heat (the amount of heat entering minus the amount of heat leaving) in any square $S \subset D$ must be zero. Otherwise, the temperature at points of S would change, contrary to our assumption that u is independent of time.

Using these two facts, we can obtain the following proof that u is harmonic, assuming $u \in C^2$.

Suppose that S is any square in D with horizontal and vertical sides of length h and assume without loss of generality that the lower left vertex is $(0,0)$. Note that for any function $f(x, y)$ with continuous partial derivatives at the origin,

$$f(x, y) - f(0, 0) = f(x, y) - f(x, 0) + f(x, 0) - f(0, 0)$$
$$= yf_y(x, \xi) + xf_x(\eta,\ 0)$$

so that

(3) $\qquad f(x, y) - f(0, 0) = y(f_y(0, 0) + \epsilon_1) + x(f_x(0, 0) + \epsilon_2)$

where ϵ_1 and $\epsilon_2 \to 0$ as $(x, y) \to (0,\ 0)$. To obtain a formula for the change in the amount of heat in S per unit time, we first calculate the loss of heat through the top side minus the increase through the bottom side. According to (1), over any subinterval Δx, this is given by

$$[Ku_y(x, h) - Ku_y(x, 0)]\Delta x.$$

But according to (3),

$$u_y(x,\ h) = u_y(0, 0) + xu_{yx}(0, 0) + hu_{yy}(0, 0) + \epsilon_1 x + \epsilon_2 h$$
$$u_y(x,\ 0) = u_y(0,\ 0) + xu_{yx}(0, 0) + \epsilon_3 x$$

so that

$$u_y(x, h) - u_y(x, 0) = hu_{yy}(0, 0) + \epsilon_4 h$$

where $\epsilon_4 \to 0$ as $h \to 0$. The net decrease in heat from the (two) subintervals thus equals $K[hu_{yy}(0,0) + \epsilon_4 h]\Delta x$, and the net loss through the top and bottom sides is given by

$$K[h^2 u_{yy}(0,0) + \epsilon_4 h^2].$$

Similarly, the net loss through the vertical sides is given by

$$K[h^2 u_{xx}(0,0) + \epsilon_5 h^2].$$

and since the overall decrease must be zero,

$$u_{xx}(0,0) + u_{yy}(0,0) + \epsilon_4 + \epsilon_5 = 0.$$

Since, finally, h could have been chosen as small as possible, we conclude

$$u_{xx}(0,0) + u_{yy}(0,0) = 0$$

and since the origin is in no way special, it follows that u is harmonic throughout D.

Index