Undergraduate Texts in Mathematics

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Ulrich Daepp • Pamela Gorkin

Reading, Writing, and Proving

A Closer Look at Mathematics

Second Edition

Springer

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For Hannes and Madeleine

Preface

You are probably about to teach or take a "first course in proof techniques," or maybe you just want to learn more about mathematics. No matter what the reason, a student who wishes to learn the material in this book likes mathematics and we hope to keep it that way. At this point, students have an intuitive sense of why things are true, but not the exposure to the detailed and critical thinking necessary to survive in the mathematical world. We have written this book to bridge this gap.

In our experience, students beginning this course have little training in rigorous mathematical reasoning; they need guidance. At the end, they are where they should be; on their own. Our aim is to teach the students to read, write, and do mathematics independently, and to do it with clarity, precision, and care. If we can maintain the enthusiasm they have for the subject, or even create some along the way, our book has done what it was intended to do.

Reading. This book was written for a course we teach to first- and second-year college students. The style is informal. A few problems require calculus, but these are identified as such. Students will also need to participate while reading proofs, prodded by questions (such as, "Why?"). Many detailed examples are provided in each chapter. Since we encourage the students to draw pictures, we include many illustrations as well. Exercises, designed to teach certain concepts, are also included. These can be used as a basis for class discussion, or preparation for the class. Students are expected to solve the exercises before moving on to the problems. Complete solutions to all of the exercises are provided at the end of each chapter. Problems of varying degrees of difficulty appear at the end of each chapter. Some problems are simply proofs of theorems that students are asked to read and summarize; others supply details to statements in the text. Though many of the remaining problems are standard, we hope that students will solve some of the unique problems presented in each chapter.

Writing. The bad news is that it is not easy to write a proof well. The good news is that with proper instruction, students quickly learn the basics of writing. We try to write in a way that we hope is worthy of imitation, but we also provide students with "tips" on writing, ranging from the (what should be) obvious to the insider's preference ("Don't start a sentence with a symbol.").

Proving. How can someone learn to prove mathematical results? There are many theories on this. We believe that learning mathematics is the same as learning to play an instrument or learning to succeed at a particular sport. Someone must provide the background: the tips, information on the basic skills, and the insider's "know-how." Then the student has to practice. Musicians and athletes practice hours a day, and it's not surprising that most mathematicians do, too. We will provide students with the background; the exercises and problems are there for practice. The instructor observes, guides, teaches and, if need be, corrects. As with anything else, the more a student practices, the better she or he will become at solving problems.

Using this book. What should be in a book like this one? Even a quick glance at other texts on this subject will tell you that everyone agrees on certain topics: logic, quantifiers, basic set theoretic concepts, mathematical induction, and the definition and properties of functions. The depth of coverage is open to debate, of course. We try to cover logic and quantifiers fairly quickly, because we believe that students can only fully appreciate the fundamentals of mathematics when they are applied to interesting problems.

What is also apparent is that after these essential concepts, everyone disagrees on what should be included. Even we prefer to vary our approach depending on our students. We have tried to provide enough material for a flexible approach.

- *The Minimal Approach.* If you need only the basics, cover Chapters 1–18. (If you assume the well ordering principle, or decide to accept the principle of mathematical induction without proof, you can also omit Chapters 12 and 13.)
- *The Usual Approach.* This approach includes Chapters 1–18 and Chapters 21—24. (This is easily doable in a standard semester, if the class meets three hours per week.)
- *The Algebra Approach.* For an algebraic slant to the course, cover Chapters 1–18, omitting Chapter 13 and including Chapters 27 and 28.
- *The Analysis Approach.* For a slant toward analysis, cover Chapters 1–23. (Include Chapter 24, if time allows. This is what we usually cover in our course.) Include as much material from Chapters 25 and 26 as time allows. Students usually enjoy an introduction to metric spaces.
- *Projects.* We have included projects intended to let students demonstrate what they can do when they are on their own. We indicate prerequisites for each project, and have tried to vary them enough that they can be assigned throughout the semester. The results in these projects come from different areas that we find particularly interesting. Students can be guided to a project at their level. Since there are open-ended parts in each project, students can take these projects as far as they want. We usually encourage the students to work on these in groups.
- *Notation.* A word about some of our symbols is in order here. In an attempt to make this book user-friendly, we indicate the end of a proof with the well-known symbol □. The end of an example or exercise is designated by ○. If a problem is used later in the text, we designate it by **Problem**#. We also have a fair number

of "nonproofs." These are "proofs" with errors, gaps, or both; the students are asked to find the flaw and to fix it. We conclude such "proofs" with the symbol ⍰. Every other symbol will be defined when we introduce you to it. Definitions are incorporated in the text for ease of reading and the terms defined are given in boldface type.

Presenting. We also hope that students will make the transition to thinking of themselves as members of a mathematical community. We encourage the students we have in this class to attend talks, give talks, go to conferences, read mathematical books, watch mathematical movies, read journal articles, and talk with their colleagues about the things in this course that interest them. Our (incomplete, but lengthy) list of references should serve a student well as a starting point. Each of the projects works well as the basis of a talk for students, and we have included some background material in each section. We begin the chapter on projects with some tips on speaking about mathematics.

What's new in this edition. We have made many changes to the first edition. First, all exercises now have solutions and every chapter, except for the first, has at least twenty problems of varying difficulty. As a result, the text has now roughly twice as many problems than before. As in the first edition, definitions are incorporated in the text. In this edition, all definitions newly introduced in a chapter appear again in a section with formal statements of the new definitions. We have included a detailed description of definitions by recursion and a recursion theorem. We've added axioms of set theory to the appendix. We have included new projects: one on the axiom of choice and one on complex numbers. We have added some interesting pieces to two projects, *Picture Proofs* and *The Best Number of All (and Some Other Pretty Good Ones).*

Some chapters have been changed or added. The first edition's Chapter 12, which required more of students than previous chapters, has been broken into two chapters, now enumerated Chapters 12 and 13. If the instructor wishes, it is possible to simply assume the results in Chapter 13 and omit the chapter. We have also included a new chapter, Chapter 24, on the Cantor–Schröder–Bernstein theorem. We feel that this is the proper culmination to Chapters 21–23 and a wonderful way to end the course, but be forewarned that it is not an easy chapter.

Thanks to many of you who used the text, we were able to pinpoint areas where we could improve many of our explanations, provide more motivation, or present a different perspective. Our goal was to find simpler, more precise explanations, and we hope that we have been successful. One new feature of this text that may interest instructors of the course: We have written solutions to every third problem. These are available on our website (see below).

Of course, we have updated our reference list, made corrections to errors that appeared in the first version, and, most likely, introduced new errors in the second version. We hope you will send us corrections to errors that you find in the text, as well as any suggestions you have for improvement.

We hope that through reading, writing, proving, and presenting mathematics, we can produce students who will make good colleagues in every sense of the word.

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We thank Hannes Daepp for creating a website to accompany the text. This website contains complete solutions to all problems numbered $3n$, where n is a positive integer. It also contains corrections to both editions of the text.

http://www.facstaff.bucknell.edu/udaepp/readwriteprove/

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