Appendix

Algebraic Properties of $\mathbb{R}$

We will assume that you are familiar with the following properties of $\mathbb{R}$.

If x and y are real numbers, then both $x+y$ and $x \cdot y$ are real numbers. Furthermore, addition and multiplication satisfy the following axioms:

A1. (The commutative property for addition) $x+y=y+x$ for all real numbers x and y;

A2. (The associative property for addition) $(x+y)+z=x+(y+z)$ for all real numbers x,y, and z;

A3. (Existence of additive identity) There is a unique real number 0 such that $0+x=x$ for all $x \in \mathbb{R}$;

A4. (Existence of additive inverse) If $x \in \mathbb{R}$, then there is a unique element $-x$ such that $x+(-x)=0$;

M1. (The commutative property for multiplication) $x \cdot y = y \cdot x$ for all real numbers x and y;

M2. (The associative property for multiplication) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for all real numbers x,y, and z;

M3. (Existence of multiplicative identity) There is a unique real number 1, with $1 \neq 0$, such that $1 \cdot x = x$ for all real numbers x.

M4. (Existence of multiplicative inverse) For each nonzero real number x, there exists a unique real number x^{-1} such that $x \cdot x^{-1} = 1$;

D1. (The distributive property) $(x+y) \cdot z = x \cdot z + y \cdot z$ for all real numbers x,y, and z.

We note that this list of properties is not minimal; for example, the uniqueness of the additive identity 0 follows from some of the other properties in the list.

U. Daepp and P. Gorkin, *Reading, Writing, and Proving: A Closer Look at Mathematics*,
Undergraduate Texts in Mathematics, DOI 10.1007/978-1-4419-9479-0,

Order Properties of $\mathbb{R}$

A set satisfying all of the properties above is called a **field**. Thus, $\mathbb{R}$ is an example of a field. In addition, $\mathbb{R}$ has an order defined on it. This means the following:

There is a subset $\mathbb{R}^+$ of $\mathbb{R} \setminus \{0\}$ satisfying:

O1. If $x, y \in \mathbb{R}^+$, then $x \cdot y \in \mathbb{R}^+$;
O2. If $x, y \in \mathbb{R}^+$, then $x + y \in \mathbb{R}^+$;
O3. For every real number x, exactly one of the following three things happens: either $x \in \mathbb{R}^+$, $-x \in \mathbb{R}^+$, or $x = 0$.

If x and y are two real numbers and $x - y \in \mathbb{R}^+$, we write $x > y$ (or $y < x$). The set $\mathbb{R}^+$ is called the **positive real numbers**. Thus $\mathbb{R}$ is a field with an order, and we call it an **ordered field**. The third property, O3, is called the **trichotomy principle**. It is not difficult to show that the results below follow from the statements A1–A4, M1–M4, D1, and O1–O3.

Theorem. *Let x, y, and z be real numbers. Then the following hold:*

1. *If $x < y$ and $y < z$, then $x < z$;*
2. *If $x < y$, then $x + z < y + z$;*
3. *If $x < y$ and $z > 0$, then $x \cdot z < y \cdot z$;*
4. *If $x < y$ and $z < 0$, then $x \cdot z > y \cdot z$;*
5. *If $x \neq 0$, then $x^2 > 0$;*
6. *$1 > 0$;*
7. *If $x > 0$, then $x^{-1} > 0$.*

Proof. We'll do the first and the sixth of these; you can prove the others.

For the proof of (1), note that $y - x \in \mathbb{R}^+$ and $z - y \in \mathbb{R}^+$. By O2 and the associative and commutative properties of addition, $(y - x) + (z - y) = z - x \in \mathbb{R}^+$. Therefore $z - x \in \mathbb{R}^+$ and $x < z$.

For the proof of (6), note that 1 is the multiplicative identity, so $1 \cdot x = x$ for all $x \in \mathbb{R}$. Taking $x = 1$, we get $1^2 = 1 \cdot 1 = 1$. Since $1 \neq 0$, the result now follows from (5). □

Axioms of Set Theory

To give set theory and large parts of mathematics a firm foundation, axioms were developed upon which mathematicians could agree. The rest of set theory, then, needs to follow from these axioms using the rules of logic. Currently the generally accepted axiomatic system is that due to Ernst Zermelo and Abraham Fraenkel, together with the axiom of choice. The abbreviation ZFC is commonly used for this system. (This list of axioms follows that of [41], except for the axiom of choice where we preferred a different version.)

ZFC 1 (Axiom of extension) Two sets are equal if and only if they have the same elements.

ZFC 2 (Axiom of specification) For every set A and every condition $S(x)$, there corresponds a set B whose elements are exactly those elements x of A for which $S(x)$ holds.

ZFC 3 (Axiom of pairing) For every two sets there exists a set to which they both belong.

ZFC 4 (Axiom of unions) For every collection of sets there exists a set that contains all the elements that belong to at least one set of the given collection.

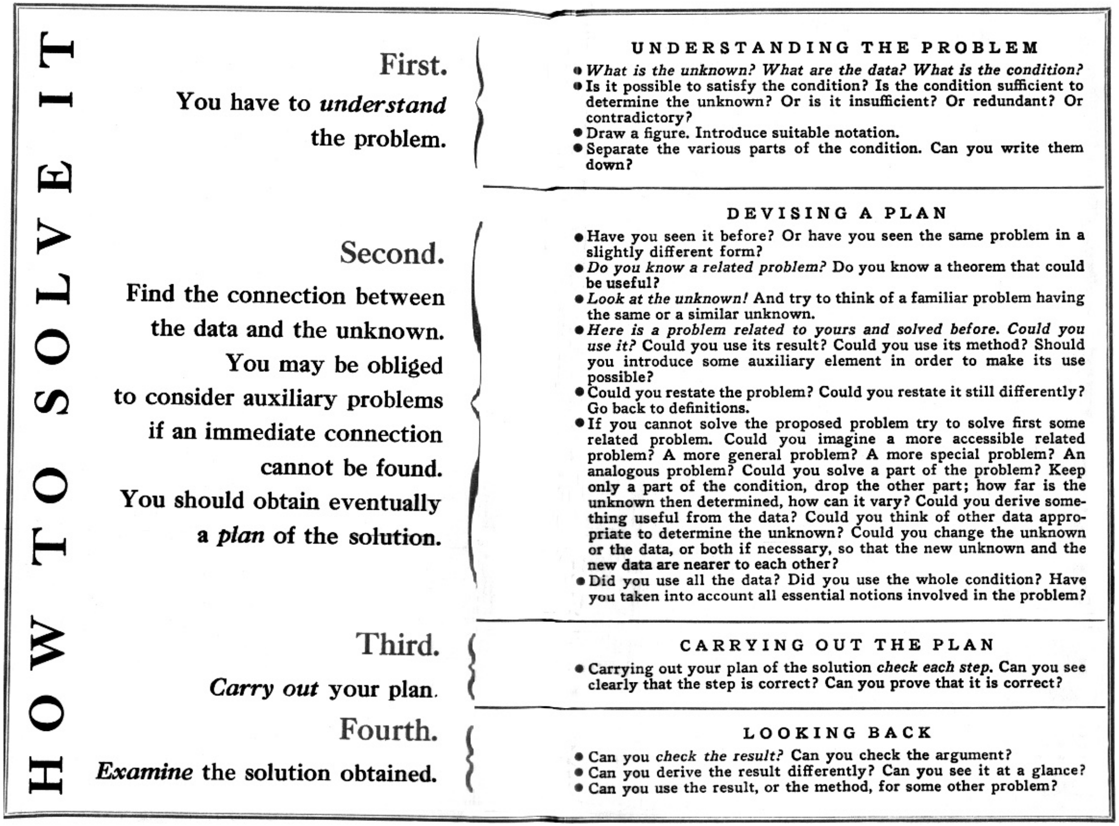
ZFC 5 (Axiom of powers) For each set there exists a collection of sets that contains, among its elements, all the subsets of the given set.

ZFC 6 (Axiom of infinity) There exists a set containing 0 and containing the successor of each of its elements.
(Recall that $0 = \emptyset$ and the successor of x is $x^+ = x \cup \{x\}$.)

ZFC 7 (Axiom of substitution) If A is a set and $S(a,b)$ is a sentence such that for each a in A the set $\{b : S(a,b)\}$ can be formed, then there exists a function $F : A \to \{\{b : S(a,b)\} : a \in A\}$ such that $F(a) = \{b : S(a,b)\}$.

ZFC 8 (Axiom of choice) Given a nonempty collection $\mathscr{F}$ of nonempty sets, there is a function $f : \mathscr{F} \to \bigcup_{A \in \mathscr{F}} A$ such that $f(A) \in A$.

Pólya's List



HOW TO SOLVE IT

First.

You have to *understand* the problem.

UNDERSTANDING THE PROBLEM

- *What is the unknown? What are the data? What is the condition?*
- Is it possible to satisfy the condition? Is the condition sufficient to determine the unknown? Or is it insufficient? Or redundant? Or contradictory?
- Draw a figure. Introduce suitable notation.
- Separate the various parts of the condition. Can you write them down?

Second.

Find the connection between the data and the unknown. You may be obliged to consider auxiliary problems if an immediate connection cannot be found. You should obtain eventually a *plan* of the solution.

DEVISING A PLAN

- Have you seen it before? Or have you seen the same problem in a slightly different form?
- *Do you know a related problem?* Do you know a theorem that could be useful?
- *Look at the unknown!* And try to think of a familiar problem having the same or a similar unknown.
- *Here is a problem related to yours and solved before. Could you use it?* Could you use its result? Could you use its method? Should you introduce some auxiliary element in order to make its use possible?
- Could you restate the problem? Could you restate it still differently? Go back to definitions.
- If you cannot solve the proposed problem try to solve first some related problem. Could you imagine a more accessible related problem? A more general problem? A more special problem? An analogous problem? Could you solve a part of the problem? Keep only a part of the condition, drop the other part; how far is the unknown then determined, how can it vary? Could you derive something useful from the data? Could you think of other data appropriate to determine the unknown? Could you change the unknown or the data, or both if necessary, so that the new unknown and the new data are nearer to each other?
- Did you use all the data? Did you use the whole condition? Have you taken into account all essential notions involved in the problem?

Third.

Carry out your plan.

CARRYING OUT THE PLAN

- Carrying out your plan of the solution *check each step*. Can you see clearly that the step is correct? Can you prove that it is correct?

Fourth.

Examine the solution obtained.

LOOKING BACK

- Can you *check the result?* Can you check the argument?
- Can you derive the result differently? Can you see it at a glance?
- Can you use the result, or the method, for some other problem?

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