Appendix A
Supplements

A.1 The Laplace Transform is Injective

In this section, we prove Theorem 1 of Sect. 2.5 that states that the Laplace transform is injective on the set of Laplace transformable continuous functions.[1] Specifically, the statement is

Theorem 1. *Suppose f_1 and f_2 are continuous functions on $[0, \infty)$ and have Laplace transforms. Suppose*

$$\mathcal{L}\{f_1\} = \mathcal{L}\{f_2\}.$$

Then $f_1 = f_2$.

The proof of this statement is nontrivial. It requires a well-known result from advanced calculus which we will assume: the Weierstrass approximation theorem.

Theorem 2 (The Weierstrass Approximation Theorem). *Suppose h is a continuous function on $[0, 1]$. Then for any $\epsilon > 0$, there is a polynomial p such that*

$$|h(t) - p(t)| < \epsilon,$$

for all t in $[0, 1]$.

In essence, the Weierstrass approximation theorem states that a continuous function can be approximated by a polynomial to any degree of accuracy.

[1]The presentation here closely follows that found in *Advanced Calculus* by David Widder, published by Prentice Hall, 1961.

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Lemma 3. *Suppose $h(t)$ is a continuous function so that*

$$\int_0^1 t^n h(t)\,\mathrm{d}t = 0,$$

for each nonnegative integer n. Then $h(t) = 0$ for all $t \in [0,1]$.

Proof. Let $\epsilon > 0$. By the Weierstrass approximation theorem, there is a polynomial p so that

$$|h(t) - p(t)| < \epsilon,$$

for all t in $[0,1]$. Since a polynomial is a linear combination of powers of t, it follows by the linearity of the integral that $\int_0^1 p(t)h(t)\,\mathrm{d}t = 0$. Now observe,

$$\begin{aligned}\int_0^1 (h(t))^2\,\mathrm{d}t &= \int_0^1 h(t)(h(t) - p(t))\,\mathrm{d}t \\ &\le \int_0^1 |h(t)|\,|h(t) - p(t)|\,\mathrm{d}t \\ &\le \epsilon \int_0^1 |h(t)|\,\mathrm{d}t.\end{aligned}$$

Since ϵ is arbitrary, it follows that $\int_0^1 (h(t))^2\,\mathrm{d}t$ can be made as small as we like. This forces $\int_0^1 (h(t))^2\,\mathrm{d}t = 0$. Since $(h(t))^2 \ge 0$, it follows that $(h(t))^2 = 0$, for all $t \in [0,1]$. Therefore, $h(t) = 0$ for all $t \in [0,1]$. □

Theorem 4. *Suppose f is a continuous function on the interval $[0,\infty)$, $F(s) = \mathcal{L}\{f(t)\}(s)$ for $s \ge a$ and $F(a + nl) = 0$ for all $n = 0, 1, \ldots$, for some $l > 0$. Then $f \equiv 0$.*

Proof. Let $g(t) = \int_0^t \mathrm{e}^{-au} f(u)\,\mathrm{d}u$. Since $F(a) = 0$, it follows that $\lim_{t\to\infty} g(t) = 0$. Write

$$F(a + nl) = \int_0^\infty \mathrm{e}^{-(a+nl)t} f(t)\,\mathrm{d}t = \int_0^\infty \mathrm{e}^{-nlt}\mathrm{e}^{-at} f(t)\,\mathrm{d}t$$

and compute using integration by parts with $u = \mathrm{e}^{-nlt}$ and $\mathrm{d}v = \mathrm{e}^{-at} f(t)$. Since $\mathrm{d}u = -nl\mathrm{e}^{-nlt}$ and $v = \int_0^t \mathrm{e}^{-au} f(u)\,\mathrm{d}u = g(t)$, we have

$$\begin{aligned}F(a + nl) &= \mathrm{e}^{-nlt} g(t)\Big|_0^\infty + nl \int_0^\infty \mathrm{e}^{-nlt} g(t)\,\mathrm{d}t. \\ &= nl \int_0^\infty \mathrm{e}^{-nlt} g(t)\,\mathrm{d}t.\end{aligned}$$

Since $F(a + nl) = 0$ we have

$$\int_0^\infty e^{-nlt} g(t)\, dt = 0,$$

for all $n = 1, 2, \ldots$. Now let $x = e^{-lt}$. Then $dx = -l e^{-lt}\, dt = -lx\, dt$ and $t = -\frac{1}{l} \ln x = \frac{1}{l} \ln \frac{1}{x}$. Substituting and simplifying, we get

$$\int_0^1 x^{n-1} g\left(\frac{1}{l} \ln \frac{1}{x}\right) dx = 0,$$

for all $n = 1, 2, \ldots$. By Lemma 3, it follows that $g\left(\frac{1}{l} \ln \frac{1}{x}\right) = 0$ for all $x \in [0, 1]$, and hence, $g(t) = 0$ on $[0, \infty)$. Since $0 = g'(t) = e^{-at} f(t)$ it follows now that $f(t) = 0$ for all $t \in [0, \infty)$.

Proof (Proof of Theorem 1). Suppose $f(t) = f_1(t) - f_2(t)$. Then $\mathcal{L}\{f\}(s) = 0$ for all s. By Theorem, 4 it follows that f is zero and so $f_1 = f_2$. □

A.2 Polynomials and Rational Functions

A ***polynomial of degree*** n is a function of the form

$$p(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0,$$

where $a_n \neq 0$. The ***coefficients*** $a_0, \ldots, a_n$ may be real or complex. We refer to a_n as the ***leading coefficient***. If the coefficients are all real, we say $p(s)$ is a ***real polynomial***. The variable s may also be real or complex. A ***root*** of $p(s)$ is a scalar r such that $p(r) = 0$. Again r may be real or complex. If r is a root of $p(s)$, then there is another polynomial $p_1(s)$ of degree $n - 1$ such that

$$p(s) = (s - r) p_1(s).$$

The polynomial $p_1(s)$ may be obtained by the standard procedure of division of polynomials.

Even though the coefficients may be real, the polynomial $p(s)$ may only have nonreal complex roots. For example, $s^2 + 1$ only has i and –i as roots. Notice in this example that the roots are complex conjugates. This always happens with real polynomials.

Proposition 1. *Suppose $p(s)$ is a real polynomial and $r \in \mathbb{C}$ is a root. Then $\bar{r}$ is also a root.*

Proof. Suppose $p(s) = a_n s^n + \cdots + a_1 s + a_0$, with each coefficient in $\mathbb{R}$. We are given that $p(r) = 0$ from which follows

$$\begin{aligned} p(\bar{r}) &= a_n\bar{r}^n + \cdots + a_1\bar{r} + a_0 \\ &= \overline{a_n r^n} + \cdots + \overline{a_1 r} + \overline{a_0} \\ &= \overline{a_n r^n + \cdots + a_1 r + a_0} \\ &= \overline{p(r)} = \overline{0} = 0. \end{aligned}$$

Thus $\bar{r}$ is also a root. □

The fundamental theorem of algebra addresses the question of whether a polynomial has a root.

Theorem 2 (Fundamental Theorem of Algebra). *Let $p(s)$ be a polynomial of degree greater than 0. Then $p(s)$ has a root $r \in \mathbb{C}$.*

The following corollary follows immediately from the fundamental theorem of algebra.

Corollary 3. *Let $p(s)$ be a polynomial of degree n and $n \geq 1$. Then there are roots $r_1, \ldots, r_n \in \mathbb{C}$ such that*

$$p(s) = a_n(s - r_1)\cdots(s - r_n),$$

where a_n is the leading coefficient of $p(s)$.

Each term of the form $s - r$ is called a ***linear term***: if $r \in \mathbb{R}$ it is a real linear term, and if $r \in \mathbb{C}$, it is a complex linear term. An ***irreducible quadratic*** is a real polynomial $p(s)$ of degree 2 that has no *real* roots. In this case, we may write $p(s) = as^2 + bs + c$ as a sum of squares by a procedure called ***completing the square***:

$$\begin{aligned} p(s) &= as^2 + bs + c \\ &= a\left(s^2 + \frac{b}{a}s + \frac{c}{a}\right) \\ &= a\left(s^2 + \frac{b}{a}s + \frac{b^2}{4a^2} + \frac{4ca - b^2}{4a^2}\right) \\ &= a\left(\left(s + \frac{b}{2a}\right) + \left(\frac{\sqrt{4ca - b^2}}{2a}\right)^2\right) \\ &= a\left((s - \alpha)^2 + \beta^2\right), \end{aligned}$$

where we set $\alpha = -\frac{b}{2a}$ and $\beta = \frac{\sqrt{4ca-b^2}}{2a}$. From this form, we may read off the complex roots $r = \alpha + \mathrm{i}\beta$ and $\bar{r} = \alpha - \mathrm{i}\beta$. We further observe that $as^2 + bs + c = a(s - (\alpha + \mathrm{i}\beta))(s - (\alpha - \mathrm{i}\beta))$.

Corollary 4. *If $p(s)$ is a real polynomial, then $p(s)$ is a product of real linear terms or irreducible quadratics.*

Proof. By fundamental theorem of algebra, $p(s)$ is a product of linear factors of the form $s - r$. By Proposition 1, we have for each nonreal linear factor $s - r$ a corresponding nonreal factor $s - \bar{r}$ of $p(s)$. As observed above, $(s - r)(s - \bar{r})$ is an irreducible quadratic. It follows then that $p(s)$ is a product of real linear terms and irreducible quadratics.

Corollary 5. *Suppose $p(s)$ is a polynomial of degree n and has $m > n$ roots. Then $p(s) = 0$ for all $s \in \mathbb{R}$.*

Proof. This is an immediate consequence of Corollary 3. □

Corollary 6. *Suppose $p_1(s)$ and $p_2(s)$ are polynomials and equal for all $s > A$, for some real number A. Then $p_1(s) = p_2(s)$, for all $s \in \mathbb{R}$.*

Proof. The polynomial $p_1(s) - p_2(s)$ has infinitely many roots so must be zero, identically. Hence, $p_1(s) = p_2(s)$ for all $s \in \mathbb{R}$. □

A ***rational function*** is a quotient of two polynomials, that is, it takes the form $\frac{p(s)}{q(s)}$. A rational function is ***proper*** if the degree of the numerator is less than the degree of the denominator.

Corollary 7. *Suppose $\frac{p_1(s)}{q_1(s)}$ and $\frac{p_2(s)}{q_2(s)}$ are rational functions that are equal for all $s > A$ for some real number A. Then they are equal for all s such that $q_1(s)q_2(s) \neq 0$.*

Proof. Suppose

$$\frac{p_1(s)}{q_1(s)} = \frac{p_2(s)}{q_2(s)}$$

for all $s > A$. Then

$$p_1(s)q_2(s) = p_2(s)q_1(s),$$

for all $s > A$. Since both sides are polynomials, this implies that $p_1(s)q_2(s) = p_2(s)q_1(s)$ for all $s \in \mathbb{R}$. Dividing by $q_1(s)q_2(s)$ gives the result. □

A.3 $\mathcal{B}_q$ Is Linearly Independent and Spans $\mathcal{E}_q$

$\mathcal{B}_q$ Spans $\mathcal{E}_q$

This subsection is devoted to a detailed proof of Theorem 2 of Sect. 2.7. To begin, we will need a few helpful lemmas.

Lemma 1. *Suppose $q(s)$ is a polynomial which factors in the following way: $q(s) = q_1(s)q_2(s)$. Then*

$$\mathcal{B}_{q_1} \subset \mathcal{B}_q,$$
$$\mathcal{R}_{q_1} \subset \mathcal{R}_q,$$
$$\mathcal{E}_{q_1} \subset \mathcal{E}_q.$$

(Of course, the same inclusions for q_2 are valid.)

Proof. Since any irreducible factor (linear or quadratic) of q_1 is a factor of q, it follows by the way $\mathcal{B}_q$ is defined using linear and irreducible quadratic factors of $q(s)$ that $\mathcal{B}_{q_1} \subset \mathcal{B}_q$. Suppose $p_1(s)/q_1(s) \in \mathcal{R}_{q_1}$. Then $p_1(s)/q_1(s) = p_1(s)q_2(s)/q_1(s)q_2(s) = p_1(s)q_2(s)/q(s) \in \mathcal{R}_q$. It follows that $\mathcal{R}_{q_1} \subset \mathcal{R}_q$. Finally, if $f \in \mathcal{E}_{q_1}$, then $\mathcal{L}\{f\} \in \mathcal{R}_{q_1} \subset \mathcal{R}_q$. Hence, $f \in \mathcal{E}_q$ and therefore $\mathcal{E}_{q_1} \subset \mathcal{E}_q$. ◻

Lemma 2. *Let $q(s)$ be a polynomial of degree $n \geq 1$. Then*

$$\mathcal{B}_q \subset \mathcal{E}_q.$$

Proof. We proceed by induction on the degree of $q(s)$. If the degree of $q(s) = 1$, then we can write $q(s) = a(s - \lambda)$, and in this case, $\mathcal{B}_q = \left\{e^{\lambda t}\right\}$. Since

$$\mathcal{L}\left\{e^{\lambda t}\right\} = \frac{1}{s-\lambda} = \frac{a}{a(s-\lambda)} = \frac{a}{q(s)} \in \mathcal{R}_q$$

it follows that $e^{\lambda t} \in \mathcal{E}_q$. Hence, $\mathcal{B}_q \subset \mathcal{E}_q$. Now suppose $\deg q(s) > 1$. According to the fundamental theorem of algebra $q(s)$ must have a linear or irreducible quadratic factor. Thus, $q(s)$ factors in one of the following ways:

1. $q(s) = (s - \lambda)^k q_1(s)$, where $k \geq 1$ and $q_1(s)$ does not contain $s - \lambda$ as a factor.
2. $q(s) = ((s - \alpha)^2 + \beta^2)^k q_1(s)$, where $k \geq 1$ and $q_1(s)$ does not contain $(s - \alpha)^2 + \beta^2$ as a factor.

Since the degree of q_1 is less than the degree of q, we have in both cases by induction that $\mathcal{B}_{q_1} \subset \mathcal{E}_{q_1}$. Lemma 1 implies that $\mathcal{B}_{q_1} \subset \mathcal{E}_q$.

Case 1: $q(s) = (s - \lambda)^k q_1(s)$. Let $f \in \mathcal{B}_q$ be a simple exponential polynomial. Then either $f(t) = t^r e^{\lambda t}$, for some nonnegative integer r less than k, or $f \in \mathcal{B}_{q_1} \subset \mathcal{E}_q$. If $f(t) = t^r e^{\lambda t}$ then, $\mathcal{L}\{f\} = r!/(s - \lambda)^{r+1} \in \mathcal{R}_{(s-\lambda)^{r+1}} \subset \mathcal{R}_q$ by Lemma 1. Thus, $f \in \mathcal{E}_q$, and hence $\mathcal{B}_q \subset \mathcal{E}_q$.

Case 2: $q(s) = ((s - \alpha)^2 + \beta^2)^k q_1(s)$. Let $f \in \mathcal{B}_q$. Then either $f(t) = t^r e^{\alpha t} \operatorname{trig} \beta t$, where trig is sin or cos and r is a nonnegative integer less than k, or $f \in \mathcal{B}_{q_1} \subset \mathcal{E}_q$. If $f(t) = t^r e^{\alpha t} \operatorname{trig} \beta t$, then by Lemma 10 of Sect. 2.6, $\mathcal{L}\{f(t)\} \in \mathcal{R}_{((s-\alpha)^2+\beta^2)^k}$ and hence, by Lemma 1, $\mathcal{L}\{f(t)\} \in \mathcal{R}_q$. It follows that $f \in \mathcal{E}_q$. Hence $\mathcal{B}_q \subset \mathcal{E}_q$. ◻

Proof (of Theorem 2 of Sect. 2.7). Since $\mathcal{E}_q$ is a linear space by Proposition 2 of Sect. 2.6 and $\mathcal{B}_q \subset \mathcal{E}_q$ by Lemma 2, we have

$$\text{Span } \mathcal{B}_q \subset \mathcal{E}_q.$$

To show $\mathcal{E}_q \subset \text{Span } \mathcal{B}_q$, we proceed by induction on the degree of q. Suppose $\deg q(s) = 1$. Then $q(s)$ may be written $q(s) = a(s - \lambda)$, for some constants $a \neq 0$ and $\lambda \in \mathbb{R}$ and $\mathcal{B}_q = \left\{e^{\lambda t}\right\}$. If $f \in \mathcal{E}_q$, then $\mathcal{L}\{f\} = c/(a(s - \lambda))$, for some constant c, and hence, $f = (c/a)e^{\lambda t}$. So $f \in \text{Span } \mathcal{B}_q$ and it follows that $\mathcal{E}_q = \text{Span } \mathcal{B}_q$.

Now suppose $\deg q > 1$. Then $q(s)$ factors in one of the following ways:

1. $q(s) = (s - \lambda)^k q_1(s)$, where $k \geq 1$ and $q_1(s)$ does not contain $s - \lambda$ as a factor.
2. $q(s) = ((s - \alpha)^2 + \beta^2)^k q_1(s)$, where $k \geq 1$ and $q_1(s)$ does not contain $(s - \alpha)^2 + \beta^2$ as a factor.

Since the degree of q_1, is less than the degree of q we have in both cases by induction that $\mathcal{E}_{q_1} = \text{Span } \mathcal{B}_{q_1}$.

Consider the first case: $q(s) = (s - \lambda)^k q_1(s)$. If $f \in \mathcal{E}_q$, then $\mathcal{L}\{f\}$ has a partial fraction decomposition of the form

$$\mathcal{L}\{f\}(s) = \frac{A_1}{s - \lambda} + \cdots + \frac{A_k}{(s - \lambda)^k} + \frac{p_1(s)}{q_1(s)},$$

for some constants $A_1, \ldots, A_k$ and polynomial $p_1(s)$. Taking the inverse Laplace transform, it follows that f is a linear combination of terms in $\left\{e^{\lambda t}, \ldots, t^{k-1}e^{\lambda t}\right\}$ and a function in $\mathcal{E}_{q_1}$. But since $\mathcal{E}_{q_1} = \text{Span } \mathcal{B}_{q_1}$ and $\mathcal{B}_q = \left\{e^{\lambda t}, \ldots, t^{k-1}e^{\lambda t}\right\} \cup \mathcal{B}_{q_1}$, it follows that $f \in \text{Span } \mathcal{B}_q$ and hence $\mathcal{E}_q = \text{Span } \mathcal{B}_q$.

Now consider the second case: $q(s) = ((s - \alpha)^2 + \beta^2)^k q_1(s)$. If $f \in \mathcal{E}_q$, then $\mathcal{L}\{f\}$ has a partial fraction decomposition of the form

$$\mathcal{L}\{f\}(s) = \frac{A_1 s + B_1}{(s - \alpha)^2 + \beta^2} + \cdots + \frac{A_k s + B_k}{((s - \alpha)^2 + \beta^2)^k} + \frac{p_1(s)}{q_1(s)},$$

for some constants $A_1, \ldots, A_k, B_1, \ldots, B_k$ and polynomial $p_1(s)$. Taking the inverse Laplace transform, it follows from Corollary 11 of Sect. 2.5 that f is a linear combination of terms in

$$\left\{e^{\alpha t}\cos\beta t, e^{\alpha t}\sin\beta t, \ldots, t^{k-1}e^{\alpha t}\cos\beta t, t^{k-1}e^{\alpha t}\sin\beta t\right\}$$

and a function in $\mathcal{E}_{q_1}$. But since $\mathcal{E}_{q_1} = \text{Span } \mathcal{B}_{q_1}$ and

$$\mathcal{B}_q = \left\{e^{\alpha t}\cos\beta t, e^{\alpha t}\sin\beta t, \ldots, t^{k-1}e^{\alpha t}\cos\beta t, t^{k-1}e^{\alpha t}\sin\beta t\right\} \cup \mathcal{B}_{q_1}$$

it follows that $f \in \text{Span } \mathcal{B}_q$ and hence $\mathcal{E}_q = \text{Span } \mathcal{B}_q$. □

The Linear Independence of $\mathcal{B}_q$

Let $q(s)$ be a nonconstant polynomial. This subsection is devoted to showing that $\mathcal{B}_q$ is linearly independent. To that end, we derive a sequence of lemmas that will help us reach this conclusion.

Lemma 3. *Suppose $q(s) = (s-\lambda)^m q_1(s)$ where $(s-\lambda)$ is not a factor of $q_1(s)$. Suppose $f_1 \in \mathcal{E}_{(s-\lambda)^m}$, $f_2 \in \mathcal{E}_{q_1}$, and*

$$f_1 + f_2 = 0.$$

Then $f_1 = 0$ and $f_2 = 0$.

Proof. Let $F_1(s) = \mathcal{L}\{f_1\}(s)$ and $F_2(s) = \mathcal{L}\{f_2\}(s)$. Then $F_1(s) = \frac{p(s)}{(s-\lambda)^r}$, for some $r \le m$ and some polynomial $p(s)$ with no factor of $s-\lambda$. Similarly, $F_2(s) = \frac{p_1(s)}{q_1(s)}$, for some polynomial $p_1(s)$. Thus,

$$\frac{p(s)}{(s-\lambda)^r} + \frac{p_1(s)}{q_1(s)} = 0. \tag{1}$$

By Corollary 7 of Appendix A.2, equation (1) holds for all s different from λ and the roots of $q_1(s)$. Since $q_1(s)$ contains no factor of $s-\lambda$, it follows that $q_1(\lambda) \neq 0$. Consider the limit as s approaches λ in (1). The second term approaches the finite value $p_1(\lambda)/q_1(\lambda)$. Since the sum is 0, the limit of the left side is 0 and this implies that the limit of the first term is finite as well. But this can only happen if $p(s) = 0$. It follows that $F_1(s) = 0$ and hence $f_1 = 0$. This in turn implies that $f_2 = 0$. □

Lemma 4. *Suppose $q(s) = ((s-\alpha)^2 + \beta^2)^m q_1(s)$ where $(s-\alpha)^2 + \beta^2$ is not a factor of $q_1(s)$. Suppose $f_1 \in \mathcal{E}_{((s-\alpha)+\beta^2)^m}$, $f_2 \in \mathcal{E}_{q_1}$, and*

$$f_1 + f_2 = 0.$$

Then $f_1 = 0$ and $f_2 = 0$.

Proof. Let $F_1(s) = \mathcal{L}\{f_1\}(s)$ and $F_2(s) = \mathcal{L}\{f_2\}(s)$. Then $F_1(s) = \frac{p(s)}{((s-\alpha)^2+\beta^2)^r}$, for some $r \le m$ and some polynomial $p(s)$ with no factor of $(s-\alpha)^2+\beta^2$. Similarly, $F_2(s) = \frac{p_1(s)}{q_1(s)}$, for some polynomial $p_1(s)$. Thus,

$$\frac{p(s)}{((s-\alpha)^2+\beta^2)^m} + \frac{p_1(s)}{q_1(s)} = 0. \tag{2}$$

Again, by Corollary 7 of Appendix A.2, equation (2) holds for all s different from $\alpha \pm \mathrm{i}\beta$ and the roots of $q_1(s)$. Since $q_1(s)$ contains no factor of $(s-\alpha)^2 + \beta^2$ it follows that $q_1(\alpha + \mathrm{i}\beta) \neq 0$. Consider the limit as s approaches the complex number $\alpha + \mathrm{i}\beta$ in (2). The second term approaches a finite value. Since the sum is 0, the limit of the left side is 0 and this implies that the limit of the first term is finite as well. But this can only happen if $p(s) = 0$. It follows that $F_1(s) = 0$ and hence $f_1 = 0$. This in turn implies that $f_2 = 0$. ☐

Lemma 5. *Let $q(s) = (s - \lambda)^m$. Then $\mathcal{B}_q$ is linearly independent.*

Proof. By definition, $\mathcal{B}_q = \left\{e^{\lambda t}, te^{\lambda t}, \ldots, t^{m-1}e^{\lambda t}\right\}$. Suppose

$$c_0 e^{\lambda t} + \cdots + c_{m-1} t^{m-1} e^{\lambda t} = 0.$$

Dividing both sides by $e^{\lambda t}$ gives

$$c_0 + c_1 t + \cdots + c_{m-1} t^{m-1} = 0.$$

Evaluating the at $t = 0$ gives $c_0 = 0$. Taking the derivative of both sides and evaluating at $t = 0$ gives $c_1 = 0$. In general, taking the rth derivative of both sides and evaluating at $t = 0$ gives $c_r = 0$, for $r = 1, 2, \ldots, m = 1$. It follows that all of the coefficients are necessarily 0, and this implies linear independence. ☐

Lemma 6. *Let $q(s) = ((s - \alpha)^2 + \beta^2)^m$. Then $\mathcal{B}_q$ is linearly independent.*

Proof. By definition, $\mathcal{B}_q = \left\{e^{\alpha t}\cos\beta t,\ e^{\alpha t}\sin\beta t, \ldots,\ t^{m-1}e^{\alpha t}\cos\beta t,\ t^{m-1}e^{\alpha t} \sin\beta t\right\}$. A linear combination of $\mathcal{B}_q$ set to 0 takes the form

$$p_1(t)e^{\alpha t}\cos\beta t + p_2(t)e^{\alpha t}\sin\beta t = 0, \tag{3}$$

where p_1 and p_2 are polynomials carrying the coefficients of the linear combination. Thus, it is enough to show p_1 and p_2 are 0. Dividing both sides of Equation (3) by $e^{\alpha t}$ gives

$$p_1(t)\cos\beta t + p_2(t)\sin\beta t = 0. \tag{4}$$

Let m be an integer. Evaluating at $t = \frac{2\pi m}{\beta}$ gives $p_1\left(\frac{2\pi m}{\beta}\right) = 0$, since $\sin 2\pi m = 0$ and $\cos 2\pi m = 1$. It follows that p_1 has infinitely many roots. By Corollary 5, of Sect. A.2 this implies $p_1 = 0$. In a similar way, $p_2 = 0$. ☐

Theorem 7. *Let q be a nonconstant polynomial. View $\mathcal{B}_q$ as a set of functions on $I = [0, \infty)$. Then $\mathcal{B}_q$ is linearly independent.*

Proof. Let k be the number of roots of q. Our proof is by induction on k. If q has but one root, then $q(s) = (s - \lambda)^n$ and this case is taken care of by Lemma 5. Suppose now that $k > 1$. We consider two cases. If $q(s) = (s - \lambda)^m q_1(s)$, where $q_1(s)$ has no factor of $s - \lambda$, then a linear combination of functions in $\mathcal{B}_q$ has the form $f_1(t) + f_2(t)$, where $f_1(t) = c_0 e^{\lambda t} + \cdots + c_{m-1}t^{m-1}e^{\lambda t}$ is a linear combination of functions in $\mathcal{B}_{(s-\lambda)^m}$ and $f_2(t)$ is a linear combination of functions in $\mathcal{B}_{q_1}$. Suppose

$$f_1(t) + f_2(t) = 0.$$

By Lemma 3, f_1 is identically zero and by Lemma 5, the coefficients $c_0, \ldots, c_{m-1}$ are all zero. It follows that $f_2 = 0$. Since $q_1(s)$ has one fewer root than q, it follows by induction the coefficients of the functions in $\mathcal{B}_{q_1}$ that make up f_2 are all zero. If follows that $\mathcal{B}_q$ is linearly independent.

Now suppose $q(s) = ((s-\alpha)^2 + \beta^2)^m q_1(s)$, where $q_1(s)$ has no factor of $(s-\alpha)^2 + \beta^2$. Then a linear combination of functions in $\mathcal{B}_q$ has the form $f_1(t) + f_2(t)$, where $f_1(t) = c_0 e^{\alpha t} \cos \beta t + \cdots + c_{m-1} t^{m-1} e^{\alpha t} \cos \beta t + d_0 e^{\alpha t} \sin \beta t + \cdots + d_{m-1} t^{m-1} e^{\alpha t} \sin \beta t$ is a linear combination of functions in $\mathcal{B}_{((s-\alpha)^2+\beta)^m}$ and $f_2(t)$ is a linear combination of functions in $\mathcal{B}_{q_1}$. Suppose

$$f_1(t) + f_2(t) = 0.$$

By Lemma 4 f_1 is identically zero, and by Lemma 6 the coefficients $c_0, \ldots, c_{m-1}$ and $d_0, \ldots, d_{m-1}$ are all zero. It follows that $f_2 = 0$. By induction, the coefficients of the functions in $\mathcal{B}_{q_1}$ that make up f_2 are all zero. If follows that $\mathcal{B}_q$ is linearly independent. ∎

A.4 The Matrix Exponential

In this section, we verify some properties of the matrix exponential that have been used in Chap. 9.

First, we argue that the matrix exponential

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots + \frac{A^n}{n!} + \cdots$$

converges absolutely for all $n \times n$ matrices A. By this we mean that each entry of e^A converges absolutely. For convenience, let $E = e^A$. Let $a = \max\left\{\left|A_{i,j}\right| : 1 \le i, j \le n\right\}$. If $a = 0$, then $A = 0$ and the series that defines e^A reduces to I, the $n \times n$ identity. We will thus assume $a > 0$. Consider the (i, j) entry of A^2. We have

$$\begin{aligned}
\left|(A^2)_{i,j}\right| &= \left|\sum_{l=1}^{n} A_{i,l} A_{l,j}\right| \\
&\le \sum_{l=1}^{n} |A_{i,l}| \left|A_{l,j}\right| \\
&\le \sum_{l=1}^{n} a^2 = na^2.
\end{aligned}$$

In a similar way, the (i, j) entry of A^3 satisfies

$$\begin{aligned}\left|(A^3)_{i,j}\right| &= \left|\sum_{l=1}^{n} A_{i,l}(A^2)_{l,j}\right| \\ &\le \sum_{l=1}^{n} |A_{i,l}|\left|(A^2)_{l,j}\right| \\ &\le \sum_{l=1}^{n} ana^2 = n^2a^3.\end{aligned}$$

By induction, we have

$$\left|(A^k)_{i,j}\right| \le n^{k-1}a^k.$$

It follows from this estimate that

$$\begin{aligned}\left|E_{i,j}\right| &= \left|(I)_{i,j} + (A)_{i,j} + \frac{(A^2)_{i,j}}{2!} + \frac{(A^3)_{i,j}}{3!}\cdots\right| \\ &\le \delta_{i,j} + \left|(A)_{i,j}\right| + \frac{\left|(A^2)_{i,j}\right|}{2!} + \frac{\left|(A^3)_{i,j}\right|}{3!}\cdots \\ &\le \delta_{i,j} + a + \frac{na^2}{2!} + \frac{n^2a^3}{3!} + \frac{n^3a^4}{4!} + \cdots \\ &= \delta_{i,j} + \sum_{k=1}^{\infty}\frac{n^{k-1}a^k}{k!} \\ &\le 1 + \frac{1}{n}\sum_{k=0}^{\infty}\frac{(na)^k}{k!} = 1 + \frac{1}{n}\mathrm{e}^{an}.\end{aligned}$$

It follows now that the series for $E_{i,j}$ converges absolutely.

If we replace A by At, we get that each entry of e^{At} is an absolutely convergent power series in t with infinite radius of convergence. Furthermore, the last inequality above shows that the (i, j) entry of e^{At} is bounded by $1 + \frac{1}{n}\mathrm{e}^{\mathrm{ant}}$ and hence of exponential type.

A.5 The Cayley–Hamilton Theorem

Theorem 1 (Cayley–Hamilton). *Let A be an $n \times n$ matrix and $c_A(s)$ its characteristic polynomial. Then*

$$c_A(A) = 0.$$

The following lemma will be helpful for the proof.

Lemma 2. *Let j be a nonnegative integer and let $a \in \mathbb{C}$. Then*

$$\mathbf{D}^l \left(t^j \mathrm{e}^{at}\right)|_{t=0} = \mathbf{D}^j t^l |_{t=a}.$$

Proof. The derivative formula $\mathbf{D}^l \left(y\mathrm{e}^{at}\right) = \left((\mathbf{D}+a)^l\, y\right)\mathrm{e}^{at}$ implies

$$\begin{aligned}
\mathbf{D}^l(t^j \mathrm{e}^{at})|_{t=0} &= ((\mathbf{D}+a)^l t^j)|_{t=0} \\
&= \sum_{k=0}^{l} \binom{l}{k} a^{l-k}(\mathbf{D}^k t^j)|_{t=0} \\
&= \begin{cases} 0 & \text{if } l < j \\ \dfrac{a^{l-j}\, l!}{(l-j)!} & \text{if } l \geq j \end{cases} \\
&= \mathbf{D}^j t^l |_{t=a}.
\end{aligned}$$

□

Proof (of Cayley-Hamilton Theorem). Let $c_A(s)$ be the characteristic polynomial of A. Suppose $\lambda_1, \cdots, \lambda_m$ are the roots of c_A with corresponding multiplicities $r_1, \cdots, r_m$. Then

$$\mathcal{B}_{c_A} = \left\{t^j \mathrm{e}^{\lambda_k t} : j = 0, \ldots, r_k - 1, k = 1, \ldots, m\right\}$$

is the standard basis for $\mathcal{E}_{c_A}$. As in Fulmer's method, Sect. 9.4, we may assume that there are $n \times n$ matrices $M_{j,k},\ j = 0, \ldots, r_k - 1, k = 1, \ldots, m$, so that

$$\mathrm{e}^{At} = \sum_{k=1}^{m} \sum_{j=0}^{r_k-1} t^j \mathrm{e}^{\lambda_k t} M_{j,k}.$$

Differentiating both sides l times and evaluating at $t = 0$ gives

$$A^l = \sum_{k=1}^{m} \sum_{j=0}^{r_k-1} \mathbf{D}^l \left(t^j \mathrm{e}^{\lambda_k t}\right)|_{t=0} M_{j,k} = \sum_{k=1}^{m} \sum_{j=0}^{r_k-1} \mathbf{D}^j t^l |_{t=\lambda_k} M_{j,k},$$

with the second equality coming from Lemma 2. Now let $p(s) = c_0 + c_1 s + \cdots + c_N s^N = \sum_{l=0}^{N} c_l s^l$ be any polynomial. Then

$$\begin{aligned}
p(A) &= \sum_{l=0}^{N} c_l A^l \\
&= \sum_{l=0}^{N} \sum_{k=1}^{m} \sum_{j=0}^{r_k-1} c_l \mathbf{D}^j t^l |_{t=\lambda_k} M_{j,k}
\end{aligned}$$

$$= \sum_{k=1}^{m} \sum_{j=0}^{r_k-1} \mathbf{D}^j \left(\sum_{l=0}^{N} c_l t^l \right) |_{t=\lambda_k} M_{j,k}$$

$$= \sum_{k=1}^{m} \sum_{j=0}^{r_k-1} p^{(j)}(\lambda_k) M_{j,k}. \tag{1}$$

For the characteristic polynomial, we have $c_A(s) = \prod_{k=1}^{m}(s - \lambda_k)^{r_k}$, and hence, $c_A^{(j)}(\lambda_k) = 0$ for all $j = 0, \ldots, r_k - 1$ and $k = 1, \ldots m$. Now let $p(s) = c_A(s)$ in Equation (1) to get

$$c_A(A) = \sum_{k=1}^{m} \sum_{j=0}^{r_k-1} c_A^{(j)}(\lambda_k) M_{j,k} = 0.$$

□

Appendix B
Selected Answers

Section 1.1

1. $P' = kP$

3. $h'(t) = \lambda\sqrt{h(t)}$

5. order: 2; Standard form: $y'' = t^3/y'$.

7. order: 2; Standard form: $y'' = -(3y + ty')/t^2$.

9. order: 4; standard form: $y^{(4)} = \sqrt[3]{(1-(y''')^4)/t}$.

11. order: 3; standard form: $y''' = 2y'' - 3y' + y$.

13. y_1, y_2, and y_3

15. y_1, y_2, and y_4

17. y_1, y_2, and y_4

19.

$$y'(t) = 3c\mathrm{e}^{3t}$$
$$3y + 12 = 3(c\mathrm{e}^{3t} - 4) + 12 = 3c\mathrm{e}^{3t} - 12 + 12 = 3c\mathrm{e}^{3t}.$$

Note that $y(t)$ is defined for all $t \in \mathbb{R}$.

21.

$$y'(t) = \frac{c\mathrm{e}^t}{(1 - c\mathrm{e}^t)^2}$$

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$$y^2(t) - y(t) = \frac{1}{(1-ce^t)^2} - \frac{1}{1-ce^t} = \frac{1-(1-ce^t)}{(1-ce^t)^2} = \frac{ce^t}{(1-ce^t)^2}.$$

If $c \le 0$, then the denominator $1 - ce^t > 0$ and $y(t)$ has domain $\mathbb{R}$. If $c > 0$, then $1-ce^t = 0$ if $t = \ln\frac{1}{c} = -\ln c$. Thus, $y(t)$ is defined either on the interval $(-\infty, -\ln c)$ or $(-\ln c, \infty)$.

23.

$$y'(t) = \frac{-ce^t}{ce^t - 1}$$
$$-e^y - 1 = -e^{-\ln(ce^t-1)} - 1 = \frac{-1}{ce^t-1} - 1 = \frac{-ce^t}{ce^t-1}.$$

25.

$$y'(t) = -(c-t)^{-2}(-1) = \frac{1}{(c-t)^2}$$
$$y^2(t) = \frac{1}{(c-t)^2}.$$

The denominator of $y(t)$ is 0 when $t = c$. Thus, the two intervals where $y(t)$ is defined are $(-\infty, c)$ and (c, ∞).

27. $y(t) = \frac{e^{2t}}{2} - t + c.$

29. $y(t) = t + \ln|t| + c$

31. $y(t) = -\frac{2}{3}\sin 3t + c_1 t + c_2$

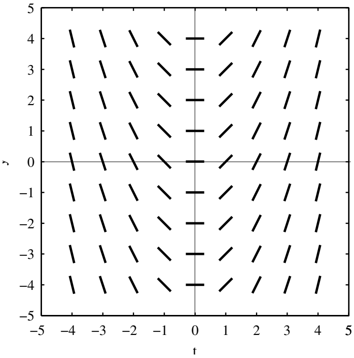
33. $y(t) = 3e^{-t} + 3t - 3$

35. $y(t) = -18(t+1)^{-1}$

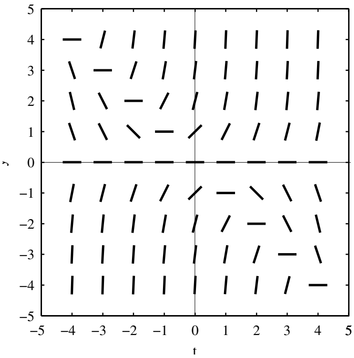
37. $y(t) = -te^{-t} - e^{-t}.$

Section 1.2

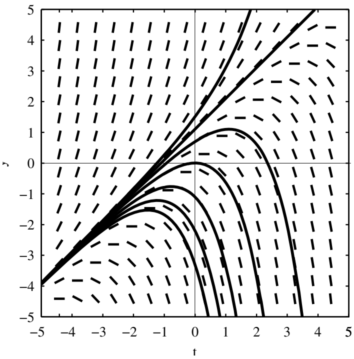
1. $$y' = t$$

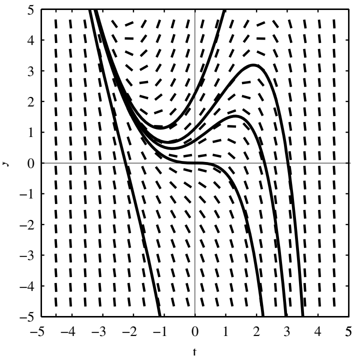


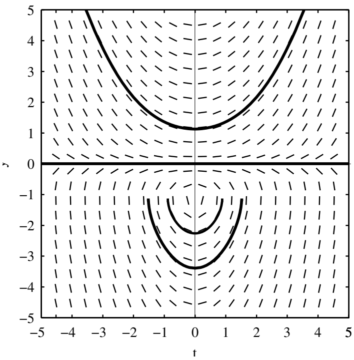
3. $$y' = y(y + t)$$



5.



7. 

9. 

11. $y = 0$

13. $y = \pm 1$

15. $y = -t + (2n + 1)\pi, n \in \mathbb{Z}$

17. $2yy' - 2t - 3t^2 = 0$

19. $y' = \frac{3y}{t} - t.$

Section 1.3

1. Separable

3. Not separable

5. Separable

7. Not separable

9. Separable
11. $t^2(1-y^2)=c$, c a real constant
13. $y^5=\frac{5}{2}t^2+10t+c$, c a real constant
15. $y=1-c\cos t$, c any real constant
17. $y=\frac{4ce^{4t}}{1+ce^{4t}}$, c a real number, and $y=4$
19. $y=\tan(t+c)$, c a real constant
21. $\ln(y+1)^2-y=\tan^{-1}t+c$, c a real constant, and $y=-1$
23. $y=\frac{1}{\ln|1-t|+c}$, c real constant
25. $y=0$
27. $y=4e^{-t^2}$
29. $y=2\sqrt{u^2+1}$
31. $y(t)=\tan\left(-\frac{1}{t}+1+\frac{\pi}{3}\right)$, $(a, b)=(6/(6+5\pi), 6/(6-\pi))$.
33. $\approx$ 212 million years old
35. $\approx$ 64 min
37. $t\approx 8.2$ min and $T(20)\approx 67.7°$
39. $\approx 205.5°$
41. 602
43. $\approx$ 3.15 years
45. 3857
47. 1,400

Section 1.4

1. $y=\frac{1}{4}e^{t}-\frac{9}{4}e^{-3t}$.
3. $y=te^{2t}+4e^{2t}$.
5. $y=\frac{e^t}{t}-\frac{e}{t}$

7. $y = \frac{\sin(t^2)}{2t} + \frac{c}{t}$.

9. $y = 4\sin 4t - 3\cos 4t + ce^{3t}$

11. $z = t^2 + 1 + ce^{t^2}$

13. $y = 1 - e^{-\sin t}$

15. $y = -t - \frac{1}{2} - \frac{3}{2}t^{-2}$.

17. $y = \frac{1}{a+b}e^{bt} + ce^{-at}$

19. $y = (t + c)\sec t$

21. $y = t^n e^t + ct^n$

23. $y = \frac{1}{5}t^2 - \frac{9}{5}t^{-3}$

25. $y(t) = \frac{1}{t} + (4a - 2)t^{-2}$

27. $y = (10 - t) - \frac{8}{10^4}(10 - t)^4$, for $0 \le t \le 10$. After 10 min, there is no salt in the tank.

29. (a) 10 min, (b) ≈ 533.33 g

31. (a) Differential equation: $P'(t) + (r/V)P(t) = rc$. If P_0 denotes the initial amount of pollutant in the lake, then $P(t) = Vc + (P_0 - Vc)e^{-(r/V)t}$. The limiting concentration is c.
   (b) $t_{1/2} = (V/r)\ln 2$; $t_{1/10} = (V/r)\ln 10$
   (c) Lake Erie: $t_{1/2} = 1.82$ years, $t_{1/10} = 6.05$ years, Lake Ontario: $t_{1/2} = 5.43$ years, $t_{1/10} = 18.06$ years

33. $y_2(t) = 10(5 + t) - \frac{500\ln(5+t)}{5+t} + \frac{500\ln 5 - 250}{5+t}$

Section 1.5

1. $y = t\tan(\ln|t| + \pi/4)$

3. $y = 2t$

5. $y = \pm t\sqrt{1 + kt}$, $k \in \mathbb{R}$

7. $y = t\sin(\ln|t| + c)$ and $y = \pm t$

9. $$y = \frac{1}{1 - t}$$

11. $y = \pm\dfrac{1}{\sqrt{1 + ce^{t^2}}}$ and $y = 0$

13. $y = \dfrac{1}{-5 + c\sqrt{1 - t^2}}$ and $y = 0$.

15. $y = -\sqrt{1 + 3e^{-t^2}}$

17. $y = \pm\dfrac{1}{\sqrt{t + \frac{1}{2} + ce^{2t}}}$ and $y = 0$

19. $-2y + \ln|2t - 2y| = c, c \in \mathbb{R}$ and $y = t$

21. $y - \tan^{-1}(t + y) = c, c \in \mathbb{R}$

23. $y = \pm\sqrt{ke^t - t}, k \in \mathbb{R}$

25. $y = e^{t-1+ce^{-t}}, c \in \mathbb{R}$

Section 1.6

1. $t^2 + ty^2 = c$

3. Not Exact

5. Not Exact

7. $(y - t^2)^2 - 2t^4 = c$

9. $y^4 = 4ty + c$

Section 1.7

1. $y(t) = 1 + \int_1^t uy(u)\,du$

3. $y(t) = 1 + \int_0^t \dfrac{u - y(u)}{u + y(u)}\,du$

5.

$$\begin{aligned} y_0(t) &= 1 \\ y_1(t) &= \frac{1+t^2}{2} \\ y_2(t) &= \frac{5}{8} + \frac{t^2}{4} + \frac{t^4}{8} \\ y_3(t) &= \frac{29}{48} + \frac{5t^2}{16} + \frac{t^4}{16} + \frac{t^6}{48}. \end{aligned}$$

7. $y_1(t) = \frac{t^2}{2}$; $y_2(t) = \frac{t^2}{2} + \frac{t^5}{20}$; $y_3(t) = \frac{t^2}{2} + \frac{t^5}{20} + \frac{t^8}{160} + \frac{t^{11}}{4400}$.

9.

$$\begin{aligned} y_0(t) &= 0 \\ y_1(t) &= t + \frac{t^3}{3} \\ y_2(t) &= t + \frac{t^7}{7 \cdot 3^2} \\ y_3(t) &= t + \frac{t^{15}}{15 \cdot 7^2 \cdot 3^4} \\ y_4(t) &= t + \frac{t^{31}}{31 \cdot 15^2 \cdot 7^4 \cdot 3^8} \\ y_5(t) &= t + \frac{t^{63}}{63 \cdot 31^2 \cdot 15^4 \cdot 7^8 \cdot 3^{16}} \end{aligned}$$

11. Not guaranteed unique

13. Unique solution

15.

$$\begin{aligned} y_0(t) &= 1 \\ y_1(t) &= 1 + at \\ y_2(t) &= 1 + at + \frac{a^2t^2}{2} \\ y_3(t) &= 1 + at + \frac{a^2t^2}{2} + \frac{a^3t^3}{3!} \end{aligned}$$

$$\vdots$$

$$y_n(t) = 1 + at + \frac{a^2t^2}{2} + \cdots + \frac{a^n t^n}{n!}.$$

$y(t) = \lim_{n\to\infty} y_n(t) = e^{at}$; it is a solution; there are no other solutions.

17. Yes

19. 1. $y(t) = t + ct^2$.
    2. Every solution satisfies $y(0) = 0$. There is no contradiction to Theorem 5 since, in standard form, the equation is $y' = \frac{2}{t}y - 1 = F(t, y)$ and $F(t, y)$ is not continuous for $t = 0$.

21. No

Section 2.1

1. $Y(s) = \frac{2}{s-4}$ and $y(t) = 2e^{4t}$
3. $Y(s) = \frac{1}{(s-4)^2}$ and $y(t) = te^{4t}$
5. $Y(s) = \frac{1}{s+2} + \frac{1}{s-1}$ and $y(t) = e^{-2t} + e^t$
7. $Y(s) = \frac{3s+3}{s^2+3s+2} = \frac{3(s+1)}{(s+1)(s+2)} = \frac{3}{s+2}$ and $y(t) = 3e^{-2t}$
9. $Y(s) = \frac{s-1}{s^2+25}$ and $y(t) = \frac{-1}{5}\sin 5t + \cos 5t$
11. $Y(s) = \frac{1}{s+4}$ and $y(t) = e^{-4t}$
13. $Y(s) = \frac{1}{(s+2)^2} + \frac{1}{(s+2)^3} = \frac{1}{(s+2)^2} + \frac{1}{2}\frac{2}{(s+2)^3}$ and $y(t) = te^{-2t} + \frac{1}{2}t^2e^{-2t}$

Section 2.2

Compute the Laplace transform of each function given below directly from the integral definition given in (1).

1. $\dfrac{3}{s^2} + \dfrac{1}{s}$
3. $\dfrac{1}{s-2} - \dfrac{3}{s+1}$

5. $\dfrac{5}{s-2}$

7. $\dfrac{2}{s^3} - \dfrac{5}{s^2} + \dfrac{4}{s}$

9. $\dfrac{1}{s+3} + \dfrac{7}{(s+4)^2}$

11. $\dfrac{s+2}{s^2+4}$

13. $\dfrac{2}{(s+4)^3}$

15. $\dfrac{2}{s^3} + \dfrac{2}{(s-2)^2} + \dfrac{1}{s-4}$

17. $\dfrac{24}{(s+4)^5}$

19. $\dfrac{1}{(s-3)^2}$

21. $\dfrac{12s^2-16}{(s^2+4)^3}$

23. $\dfrac{2s}{s^2+1} - \dfrac{2}{s}$

25. $\dfrac{\ln(s+6) - \ln 6}{s}$

27. $\dfrac{2b^2}{s(s^2+4b^2)}$

29. $\dfrac{1}{2}\left(\dfrac{a-b}{s^2+(a-b)^2} + \dfrac{a+b}{s^2+(a+b)^2}\right)$

31. $\dfrac{b}{s^2-b^2}$

33. $\dfrac{b}{s^2-b^2}$

(a) Show that $\Gamma(1) = 1$.
(b) Show that Γ satisfies the recursion formula $\Gamma(\beta+1) = \beta\Gamma(\beta)$. (*Hint*: Integrate by parts.)
(c) Show that $\Gamma(n+1) = n!$ when n is a nonnegative integer.

Section 2.3

1.

The $s-1$ *-chain*	
$\frac{5s+10}{(s-1)(s+4)}$	$\frac{3}{s-1}$
$\frac{2}{s+4}$	

3.

The $s-5$ *-chain*	
$\frac{1}{(s+2)(s-5)}$	$\frac{1/7}{(s-5)}$
$\frac{-1/7}{(s+2)}$	

5.

The $s-1$ *-chain*	
$\frac{3s+1}{(s-1)(s^2+1)}$	$\frac{2}{s-1}$
$\frac{-2s+1}{s^2+1}$	

7.

The $s+3$ *-chain*	
$\frac{s^2+s-3}{(s+3)^3}$	$\frac{3}{(s+3)^3}$
$\frac{s-2}{(s+3)^2}$	$\frac{-5}{(s+3)^2}$
$\frac{1}{s+3}$	$\frac{1}{s+3}$
0	

9.

The s + 1 -chain	
$\frac{s}{(s+2)^2(s+1)^2}$	$\frac{-1}{(s+1)^2}$
$\frac{s+4}{(s+2)^2(s+1)}$	$\frac{3}{s+1}$
$\frac{-3s-8}{(s+2)^2}$	

11.

The s − 5 -chain	
$\frac{1}{(s-5)^5(s-6)}$	$\frac{-1}{(s-5)^5}$
$\frac{1}{(s-5)^4(s-6)}$	$\frac{-1}{(s-5)^4}$
$\frac{1}{(s-5)^3(s-6)}$	$\frac{-1}{(s-5)^3}$
$\frac{1}{(s-5)^2(s-6)}$	$\frac{-1}{(s-5)^2}$
$\frac{1}{(s-5)(s-6)}$	$\frac{-1}{s-5}$
$\frac{1}{s-6}$	

13. $\frac{13/8}{s-5} - \frac{5/8}{s+3}$

15. $\frac{23}{12(s-5)} + \frac{37}{12(s+7)}$

17. $\frac{25}{8(s-7)} - \frac{9}{8(s+1)}$

19. $\frac{1}{2(s+5)} - \frac{1}{2(s-1)} + \frac{1}{s-2}$

21. $\frac{7}{(s+4)^4}$

23. $\dfrac{3}{(s+3)^3} - \dfrac{5}{(s+3)^2} + \dfrac{1}{s+3}$

25. $\dfrac{1}{54}\left(\dfrac{5}{s-5} + \dfrac{21}{(s+1)^2} + \dfrac{3}{(s-5)^2} - \dfrac{5}{s+1}\right)$

27. $\dfrac{-2}{(s+2)^2} - \dfrac{3}{s+2} - \dfrac{1}{(s+1)^2} + \dfrac{3}{s+1}$

29. $\dfrac{12}{(s-3)^3} + \dfrac{-14}{(s-3)^2} + \dfrac{15}{s-3} + \dfrac{-16}{s-2} + \dfrac{1}{s-1}$

31. $\dfrac{2}{(s-2)^2} + \dfrac{5}{s-2} + \dfrac{3}{(s-3)^2} - \dfrac{5}{s-3}$

33. $Y(s) = \frac{-3}{(s+1)^2} - \frac{1}{s+1} + \frac{1}{s-2}$ and $y(t) = -3te^{-t} - e^{-t} + e^{2t}$

35. $Y(s) = \frac{-30}{s^2} + \frac{24}{s} - \frac{26}{s+1} + \frac{1}{s-5}$ and $y(t) = -30t + 24 - 26e^{-t} + e^{5t}$

37. $Y(s) = \frac{2s-3}{(s-1)(s-2)} + \frac{4}{s(s-1)(s-2)} = \frac{2}{s} + \frac{3}{s-2} - \frac{3}{s-1}$ and $y(t) = 2 + 3e^{2t} - 3e^{t}$

Section 2.4

1.

The s^2+1-chain	
$\dfrac{1}{(s^2+1)^2(s^2+2)}$	$\dfrac{1}{(s^2+1)^2}$
$\dfrac{-1}{(s^2+1)(s^2+2)}$	$\dfrac{-1}{(s^2+1)}$
$\dfrac{1}{s^2+2}$	

3.

The s^2+3 -chain	
$\dfrac{8s+8s^2}{(s+3)^3(s^2+1)}$	$\dfrac{12-4s}{(s^2+3)^3}$
$\dfrac{4(s-1)}{(s^2+3)^2(s^2+1)}$	$\dfrac{2-2s}{(s^2+3)^2}$
$\dfrac{2(s-1)}{(s^2+3)(s^2+1)}$	$\dfrac{1-s}{s^2+3}$
$\dfrac{s-1}{s^2+1}$	

5.

The s^2+2s+2 -chain	
$\dfrac{1}{(s^2+2s+2)^2(s^2+2s+3)^2}$	$\dfrac{1}{(s^2+2s+2)^2}$
$\dfrac{-(s^2+2s+4)}{(s^2+2s+2)(s^2+2s+3)^2}$	$\dfrac{-2}{s^2+2s+2}$
$\dfrac{2s^2+4s+7}{(s^2+2s+3)^2}$	

7. $\dfrac{1}{10}\left(\dfrac{3}{s-3}+\dfrac{1-3s}{s^2+1}\right)$

9. $\dfrac{9s^2}{(s^2+4)^2(s^2+1)} = \dfrac{12}{(s^2+4)^2}+\dfrac{1}{s^2+4}-\dfrac{1}{s+1}$

11. $\dfrac{2}{s-3}+\dfrac{6-2s}{(s-3)^2+1}$

13. $\dfrac{-5s+15}{(s^2-4s+8)^2}+\dfrac{-s+3}{s^2-4s+8}+\dfrac{1}{s-1}$

15. $\dfrac{s+1}{(s^2+4s+6)^2}+\dfrac{2s+2}{s^2+4s+6}+\dfrac{s+1}{(s^2+4s+5)^2}-\dfrac{2s+2}{s^2+4s+5}$

17. $Y(s)=\frac{1}{(s+2)^2}+\frac{4s}{(s^2+4)(s+2)^2}=\frac{1}{s^2+4}$ and $y(t)=\frac{1}{2}\sin 2t$

19. $Y(s)=\frac{1}{s^2+4}+\frac{3}{(s^2+9)(s^2+4)}=\frac{8}{5}\frac{1}{s^2+4}-\frac{3}{5}\frac{1}{s^2+9}$ and $y(t)=\frac{4}{5}\sin 2t-\frac{1}{5}\sin 3t$

Section 2.5

1. -5
3. $3t - 2t^2$
5. $3\cos 2t$
7. $-11te^{-3t} + 2e^{-3t}$
9. $e^{2t} - e^{-4t}$
11. $\frac{-1}{6}t^3e^{2t} + \frac{3}{2}t^2e^{2t} + 2te^{2t}$
13. $te^t + e^t + te^{-t} - e^{-t}$
15. $4te^{2t} - e^{2t} + \cos 2t - \sin 2t$
17. $3 - 6t + e^{2t} - 4e^{-t}$
19. $2e^{-t}\cos 2t - e^{-t}\sin 2t$
21. $e^{4t}\cos t + 3e^{4t}\sin t$
23. $e^t\cos 3t$
25. $2t\sin 2t$
27. $2te^{-2t}\cos t + (t-2)e^{-2t}\sin t$
29. $4te^{-4t}\cos t + (t-4)e^{-4t}\sin t$
31. $\frac{1}{256}\big((3-4t^2)e^t\sin 2t - 6te^t\cos 2t\big)$
33. $\frac{1}{48}\big((-t^3+3t)e^{4t}\sin t - 3t^2e^{4t}\cos t\big)$
35. $y(t) = \cos t - \sin t + 2(\sin t - t\cos t) = \cos t + \sin t - 2t\cos t$
37. $Y(s) = \frac{8s}{(s^2+1)^3}$ and $y(t) = t\sin t - t^2\cos t$

Section 2.6

1. $\mathcal{B}_q = \left\{e^{4t}\right\}$

3. $\mathcal{B}_q = \{1,\, e^{-5t}\}$
5. $\mathcal{B}_q = \{e^{3t},\, te^{3t}\}$
7. $\mathcal{B}_q = \{e^{3t},\, e^{-2t}\}$
9. $\mathcal{B}_q = \{e^{t/2},\, e^{4t/3}\}$
11. $\mathcal{B}_q = \left\{e^{(2+\sqrt{3})t},\, e^{(2-\sqrt{3})t}\right\}$
13. $\mathcal{B}_q = \{e^{-3t/2},\, te^{-3t/2}\}$
15. $\mathcal{B}_q = \{\cos(5t/2),\, \sin(5t/2)\}$
17. $\mathcal{B}_q = \{e^t \cos 2t,\, e^t \sin 2t\}$
19. $\mathcal{B}_q = \{e^{-3t}, te^{-3t}, t^2e^{-3t}, t^3e^{-3t}\}$
21. $\mathcal{B}_q = \{e^t, te^t, t^2e^t\}$
23. $\mathcal{B}_q = \{e^{-2t}\cos t, e^{-2t}\sin t, te^{-2t}\cos t, te^{-2t}\sin t\}$
25. $\mathcal{B}_q = \{\cos t,\, \sin t,\, t\cos t,\, t\sin t,\, t^2\cos t,\, t^2\sin t,\, t^3\cos t,\, t^3\sin t\}$

Section 2.7

1. Yes
3. Yes
5. Yes
7. No
9. No
11. No
13. $\mathcal{B}_q = \{e^t,\, e^{-t},\, \cos t,\, \sin t\}$
15. $\mathcal{B}_q = \{e^t,\, te^t,\, t^2e^t,\, e^{-7t},\, te^{-7t}\}$
17. $\mathcal{B}_q = \{e^{-2t},\, te^{-2t},\, t^2e^{-2t},\, \cos 2t,\, \sin 2t,\, t\cos 2t,\, t\sin 2t\}$
19. $\mathcal{B}_q = \{e^{2t},\, te^{2t},\, e^{-3t},\, te^{-3t},\, t^2e^{-3t}\}$.
21. $\mathcal{B}_q = \{e^{-4t},\, te^{-4t},\, e^{-3t}\cos 2t,\, e^{-3t}\sin 2t,\, te^{-3t}\cos 2t,\, te^{-3t}\sin 2t\}$

23. $\mathcal{B}_q = \{e^{3t},\ te^{3t},\ t^2e^{3t},\ e^{-t}\cos 3t,\ e^{-t}\sin 3t,\ te^{-t}\cos 3t,\ te^{-t}\sin 3t\}$

25. $\mathcal{B}_q = \{e^{t/2},\ e^t,\ te^t\}$

27. $\mathcal{B}_q = \left\{\cos\sqrt{3}t,\ \sin\sqrt{3}t, \cos\sqrt{2}t,\ \sin\sqrt{2}t\right\}$

Section 2.8

1. $\dfrac{t^3}{6}$

3. $3(1-\cos t)$

5. $\frac{1}{13}(2e^{3t} - 2\cos 2t - 3\sin 2t)$

7. $\frac{1}{108}(18t^2 - 6t - 6 - e^{-6t})$

9. $\frac{1}{6}(e^{2t} - e^{-4t})$

11. $\frac{1}{a^2+b^2}(be^{at} - b\cos bt - a\sin bt)$

13. $\begin{cases} \frac{b\sin at - a\sin bt}{b^2-a^2} & \text{if } b \neq a \\ \frac{\sin at - at\cos at}{2a} & \text{if } b = a \end{cases}$

15. $\begin{cases} \frac{a\sin at - b\sin bt}{a^2-b^2} & \text{if } b \neq a \\ \frac{1}{2a}(at\cos at + \sin at) & \text{if } b = a \end{cases}$

17. $F(s) = \dfrac{4}{s^3(s^2+4)}$

19. $F(s) = \dfrac{6}{s^4(s+3)}$

21. $F(s) = \dfrac{4}{(s^2+4)^2}$

23. $\dfrac{1}{4}(-e^t + e^{5t})$

25. $\dfrac{1}{2}t\sin t$

27. $\frac{1}{13}(2e^{3t} - 2\cos 2t - 3\sin 2t)$

29. $\frac{e^{at} - e^{bt}}{a - b}$

31. $\int_0^t g(x)\cos\sqrt{2}(t - x)\,dx$

33. $t - \sin t$

35. $\frac{1}{54}(2 - 6t + 9t^2 - 2e^{-3t})$

37. $\frac{1}{162}(2 - 3t\sin 3t - 2\cos 3t)$

Section 3.1

1. No

3. No

5. No

7. Yes; $(\boldsymbol{D}^2 - 7\boldsymbol{D} + 10)(y) = 0$, $q(s) = s^2 - 7s + 10$, homogeneous

9. Yes; $\boldsymbol{D}^2(y) = -2 + \cos t$, $q(s) = s^2$, nonhomogeneous

11. (a) $6e^t$
    (b) 0
    (c) $\sin t - 3\cos t$

13. (a) 0
    (b) 0
    (c) 1

15. $y(t) = \cos 2t + c_1 e^t + c_2 e^{4t}$ where c_1, c_2 are arbitrary constants.

17. $y(t) = \cos 2t + e^t - e^{4t}$

19. $\boldsymbol{L}(e^{rt}) = (ar^2 + br + c)e^{rt}$

Section 3.2

1. Linearly independent
3. Linearly dependent
5. Linearly dependent
7. Linearly independent
9. Linearly independent
11. Linearly independent
13. Linearly independent
15. t
17. $10t^{29}$
19. $e^{(r_1+r_2+r_3)t}(r_3 - r_1)(r_3 - r_2)(r_2 - r_1)$
21. 12
23. $c_1 = -2/5, c_2 = 1$
25. $a_1 = 3, a_2 = 3$

Section 3.3

1. $y(t) = c_1 e^{2t} + c_2 e^{-t}$
3. $y(t) = c_1 e^{-4t} + c_2 e^{-6t}$
5. $y(t) = c_1 e^{-4t} + c_2 t e^{-4t}$
7. $y(t) = c_1 e^{-t} \cos 2t + c_2 e^{-t} \sin 2t$
9. $y(t) = c_1 e^{-9t} + c_2 e^{-4t}$
11. $y(t) = c_1 e^{-5t} + c_2 t e^{-5t}$
13. $y = \frac{e^t - e^{-t}}{2}$
15. $y = t e^{5t}$

17. $q(s) = s^2 + 4s - 21$, $w(e^{3t}, e^{-7t}) = -10e^{-4t}$, $K = -10$.

19. $q(s) = s^2 - 6s + 9$, $w(e^{3t}, te^{3t}) = e^{6t}$, $K = 1$.

21. $q(s) = s^2 - 2s + 5$, $w(e^t \cos 2t, e^t \sin 2t) = 2e^{2t}$, $K = 2$.

Section 3.4

1. $y_p(t) = a_1 e^{3t}$

3. $y_p(t) = a_1 t e^{2t}$

5. $y_p(t) = a_1 \cos 5t + a_2 \sin 5t$

7. $y_p(t) = a_1 t \cos 2t + a_2 t \sin 2t$

9. $y_p(t) = a_1 e^{-2t} \cos t + a_2 e^{-2t} \sin t$

11. $y = -te^{-2t} + c_1 e^{-2t} + c_2 e^{5t}$

13. $y = \frac{1}{2} t^2 e^{-t} + c_1 e^{-t} + c_2 t e^{-t}$

15. $y = \frac{1}{2} e^{-3t} + c_1 e^{-2t} \cos t + c_2 e^{-2t} \sin t$

17. $y = -t^2 - 2 + c_1 e^t + c_2 e^{-t}$

19. $y = \frac{1}{2} t^2 e^{2t} + c_1 e^{2t} + c_2 t e^{2t}$.

21. $y = te^{2t} - \frac{2}{5} e^{2t} + c_1 e^{-3t} + c_2 t e^{-3t}$

23. $y = \frac{1}{4} t e^{-3t} \sin(2t) + c_1 e^{-3t} \cos(2t) + c_2 e^{-3t} \sin(2t)$

25. $y = \frac{-1}{12} e^{3t} + \frac{10}{21} e^{6t} + \frac{135}{84} e^{-t}$

27. $y = 2e^{2t} - 2\cos t - 4 \sin t$

Section 3.5

1. $y = \frac{1}{32} e^{-6t} + c_1 e^{2t} + c_2 e^{-2t}$

3. $y = te^{-2t} + c_1 e^{-2t} + c_2 e^{-3t}$

5. $y = -te^{-4t} + c_1 e^{2t} + c_2 e^{-4t}$

7. $y = te^{2t} - \frac{2}{5}e^{2t} + c_1 e^{-3t} + c_2 t e^{-3t}$

9. $y = -3t^2 e^{4t} \cos 3t + te^{4t} \sin 3t + c_1 e^{4t} \cos 3t + c_2 e^{4t} \sin 3t$

11. $y = \frac{1}{2} \sin t + c_1 e^{-t} + c_2 t e^{-t}$

Section 3.6

1. $k = 32$ lbs/ft

3. $k = 490$ N/m

5. $\mu = 8$ lbs s/ft

7. 400 lbs

9. $6y'' + 20y = 0$, $y(0) = .1$, $y'(0) = 0$; $y = \frac{1}{10} \cos \sqrt{\frac{10}{3}}\, t$, undamped free or simple harmonic motion; $A = 1/10$, $\beta = \sqrt{10/3}$, and $\phi = 0$

11. $\frac{1}{2}y'' + 2y' + 32y = 0$, $y(0) = 1$, $y'(0) = 1$; $y = e^{-2t} \cos \sqrt{60}t + \frac{3}{\sqrt{60}} e^{-2t} \sin \sqrt{60}t = \sqrt{\frac{23}{20}} e^{-2t} \cos(\sqrt{60}t + \phi)$, where $\phi = \arctan \sqrt{60}/20 \approx .3695$; underdamped free motion.

13. $y'' + 96y = 0$, $y(0) = 0$, $y'(0) = 2/3$; $y = \frac{\sqrt{6}}{36} \sin \sqrt{96}t = \frac{\sqrt{6}}{36} \cos\left(\sqrt{96}t - \frac{\pi}{2}\right)$; undamped free or simple harmonic motion and crosses equilibrium.

Section 3.7

1. $q(t) = -\frac{3}{100}e^{-20t} \cos 20t - \frac{3}{100}e^{-3t} \sin 20t + \frac{3}{100}$ and $I(t) = \frac{6}{5}e^{-20t} \cos(20t - \pi/2)$

3. $q(t) = \frac{4}{5}e^{-10t} + 7te^{-10t} - \frac{4}{5} \cos 5t + \frac{3}{5} \sin 5t$ and $I(t) = -e^{-10t} - 70te^{-10t} + 4 \sin 5t + 3 \cos 5t$

5. $q(t) = \frac{1}{75}(\cos 25t - \cos 100t)$; the capacitor will not overcharge.

Section 4.1

1. Yes; $(\boldsymbol{D}^3 - 3\boldsymbol{D})y = e^t$, order 3, $q(s) = s^3 - 3s$, nonhomogeneous

3. No

5. (a) 0
   (b) 0
   (c) 0

7. (a) $10e^{-t}$
   (b) 0
   (c) 0

9. $y(t) = te^{2t} + c_1e^{2t} + c_2e^{-2t} + c_3$

11. $y(t) = te^{2t} + e^{2t} + 2e^{-2t} - 1$

Section 4.2

1. $y(t) = c_1e^{-t} + c_2e^{-\frac{1}{2}t}\cos\frac{\sqrt{3}}{2}t + c_3e^{-\frac{1}{2}t}\sin\frac{\sqrt{3}}{2}t$

3. $y(t) = c_1e^t + c_2e^{-t} + c_3\sin t + c_4\cos t$

5. $y(t) = c_1e^t + c_2e^{-t} + c_3e^{2t} + c_4e^{-2t}$

7. $y(t) = c_1e^{-2t} + c_2\cos 5t + c_3\sin 5t$

9. $y(t) = c_1e^t + c_2e^{-3t} + c_3te^{-3t} + c_4t^2e^{-3t}$

11. $y = e^t - 3e^{-t} + \cos t + 2\sin t$

Section 4.3

1. $y = cte^{-t}$

3. $y = ce^{2t}$

5. $y = \frac{1}{2}te^t + c_1e^{-t} + c_2e^t + c_3$

7. $y = \frac{1}{12}te^{2t} + c_1e^t + c_2e^{-t} + c_3e^{2t} + c_4e^{-2t}$

9. $y = \frac{1}{2}te^t + c_1e^{-t} + c_2e^t + c_3$

11. $y = \frac{t^2}{8} + c_1 + c_2 \cos 2t + c_3 \sin 2t$

13. $y = -t(\sin t + \cos t) + c_1 e^t + c_2 \cos t + c_2 \sin t$

Section 4.4

1. $y_1(t) = -4e^{2t} + 6e^{4t}$ and $y_2(t) = -4e^{2t} + 3e^{4t}$

3. $y_1(t) = \cos 2t - \sin 2t$ and $y_2(t) = -\cos 2t + \sin 2t$

5. $y_1(t) = -20e^{-t} + 40e^t - 20e^{2t}$ and $y_2(t) = -6e^{-t} + 20e^t - 12e^{2t}$

7. $y_1(t) = 3e^{-t} - 3\cos 3t + \sin 3t$ and $y_2(t) = 3e^{-t} + 3\cos 3t + 3\sin 3t$

9. $y_1(t) = \cos t + \sin t + 2\cos 2t + \sin 2t$ and $y_2(t) = 2\cos t + 2\sin t - 2\cos 2t - \sin 2t$

13. $y_1(t) = 2e^t \cos 2t + 2e^t \sin 2t$ and $y_2(t) = -2e^t \cos 2t + 2e^t \sin 2t$

15. $y_1(t) = 20 - 19\cos t - 2\sin t$ and $y_2(t) = 8 - 8\cos t + 3\sin t$

Section 4.5

1. $y(t) = 10e^{-5t}$, asymptotically stable

3. $y(t) = e^t + e^{3t}$, unstable

5. $y(t) = e^{-2t}\sin(t)$, stable

7. $y(t) = e^{-3t} + 4te^{-3t}$, stable

9. $y(t) = e^t(\sin(t) + \cos(t))$, unstable

11. $y(t) = e^{-t}$, marginally stable

13. $y(t) = e^{-t}$

15. $y(t) = \frac{1}{4}(e^{2t} - e^{-2t})$

17. $y(t) = 1 - \cos(t)$

Section 5.1

1. Not linear

3. Yes, nonhomogeneous, yes

5. Yes, nonhomogeneous, no

7. Yes, nonhomogeneous, no

9. Not linear

11. Yes, homogeneous, no

13. 1. $\boldsymbol{L}(\frac{1}{t}) = 0$
    2. $\boldsymbol{L}(1) = -1$
    3. $\boldsymbol{L}(t) = 0$
    4. $\boldsymbol{L}(t^r) = (r^2 - 1)t^r$

15. $C = \frac{-1}{2}$

17. (3) a. $y(t) = \mathrm{e}^{-t} - \mathrm{e}^{t} + 2t$
    (3) b. $y(t) = \mathrm{e}^{-t} + (0)\mathrm{e}^{t} + (1)t = \mathrm{e}^{-t} + t$
    (3) c. $y(t) = \mathrm{e}^{-t} + -\mathrm{e}^{t} + 3t$
    (3) d. $y(t) = \mathrm{e}^{-t} + (a-1)\mathrm{e}^{t} + (b-a+2)t$

19. $(-\infty, 0)$

21. $(0, \pi)$

23. $(3,\ \infty)$

25. The initial condition occurs at $t = 0$ which is precisely where $a_2(t) = t^2$ has a zero. Theorem 6 does not apply.

27. The assumptions say that $y_1(t_0) = y_2(t_0)$ and $y_1'(t_0) = y_2'(t_0)$. Both y_1 and y_2 therefore satisfies the same initial conditions. By the uniqueness part of Theorem 6 $y_1 = y_2$.

Section 5.2

1. Dependent; $2t$ and $5t$ are multiples of each other.

3. Independent

5. Independent

11. 1. Suppose $at^3 + b\left|t^3\right| = 0$ on $(-\infty, \infty)$. Then for $t = 1$ and $t = -1$ we get

$$a + b = 0$$
$$-a + b = 0.$$

These equations imply $a = b = 0$. So y_1 and y_2 are linearly independent.

2. Observe that $y_1'(t) = 3t^2$ and $y_2'(t) = \begin{cases} -3t^2 & \text{if } t < 0 \\ 3t^2 & \text{if } t \geq 0. \end{cases}$ If $t < 0$, then $w(y_1, y_2)(t) = \det\begin{pmatrix} t^3 & -t^3 \\ 3t^2 & -3t^2 \end{pmatrix} = 0$. If $t \geq 0$, then $w(y_1, y_2)(t) = \det\begin{pmatrix} t^3 & t^3 \\ 3t^2 & 3t^2 \end{pmatrix} = 0$. It follows that the Wronskian is zero for all $t \in (-\infty, \infty)$.
3. The condition that the coefficient function $a_2(t)$ be nonzero in Theorem 2 and Proposition 4 is essential. Here the coefficient function, t^2, of y'' is zero at $t = 0$, so Proposition 4 does not apply on $(-\infty, \infty)$. The largest open intervals on which t^2 is nonzero are $(-\infty, 0)$ and $(0, \infty)$. On each of these intervals, y_1 and y_2 are linearly dependent.
4. Consider the cases $t < 0$ and $t \geq 0$. The verification is then straightforward.
5. Again the condition that the coefficient function $a_2(t)$ be nonzero is essential. The uniqueness and existence theorem does not apply.

Section 5.3

1. The general solution is $y(t) = c_1 t + c_2 t^{-2}$.
3. The general solution is $y(t) = c_1 t^{\frac{1}{3}} + c_2 t^{\frac{1}{3}} \ln t$.
5. The general solution is $y(t) = c_1 t^{\frac{1}{2}} + c_2 t^{\frac{1}{2}} \ln t$.
7. $y(t) = c_1 t^{-3} + c_2 t^{-3} \ln t$
9. The general solution is $y(t) = c_1 t^2 + c_2 t^{-2}$.
11. The general solution is $y(t) = c_1 t^2 \cos(3 \ln t) + c_2 t^2 \sin(3 \ln t)$.
13. $y = 2t^{1/2} - t^{1/2} \ln t$
15. No solution is possible.

Section 5.4

1. $\ln\left(\dfrac{s-a}{s-b}\right)$

3. $s\ln\left(\dfrac{s^2+b^2}{s^2+a^2}\right) - 2b\tan^{-1}\left(\dfrac{b}{s}\right) + 2a\tan^{-1}\left(\dfrac{a}{s}\right)$

5. $y = c_1\mathrm{e}^{-t} + c_2(t-1)$

7. $y(t) = c_1\mathrm{e}^{-2t}$.

9. $y(t) = c_1\left((3-t^2)\sin t - 3t\cos t\right)$

11. $y(t) = c_1\dfrac{\mathrm{e}^t - 1}{t}$.

13. $y(t) = c_1\left(\dfrac{\mathrm{e}^{3t} - \mathrm{e}^{2t}}{t}\right)$.

15. $y(t) = c_1\dfrac{\sin t}{t} + c_2\dfrac{1-\cos t}{t}$

Section 5.5

1. $y_2(t) = t^2\ln t$, and the general solution can be written $y(t) = c_1t^2 + c_2t^2\ln t$.

3. $y_2(t) = \sqrt{t}\ln t$, and the general solution can be written $y(t) = c_1\sqrt{t} + c + 2\sqrt{t}\ln t$.

5. $y_2(t) = t\mathrm{e}^t$. The general solution can be written $y(t) = c_1t + c_2t\mathrm{e}^t$.

7. $y_2(t) = \frac{-1}{2}\cos t^2$. The general solution can be written $y(t) = c_1\sin t^2 + c_2\cos t^2$.

9. $y_2(t) = -1 - t\tan t$. The general solution can be written $y(t) = c_1\tan t + c_2(1 + t\tan t)$.

11. $y_2(t) = -\sec t$. The general solution can be written $y(t) = c_1\tan t + c_2\sec t$.

13. $y_2 = -1 - \frac{t\sin 2t}{1+\cos 2t}$. The general solution can be written $y(t) = c_1\frac{\sin 2t}{1+\cos 2t} + c_2\left(1 + \frac{t\sin 2t}{1+\cos 2t}\right)$.

15. $y_2(t) = \frac{1}{2}t + \frac{1}{4}(1-t^2)\ln\left(\frac{1+t}{1-t}\right)$, and the general solution can be written

$$y = c_1(1-t^2) + c_2\left(\frac{1}{2}t + \frac{1}{4}(1-t^2)\ln\left(\frac{1+t}{1-t}\right)\right).$$

Section 5.6

1. The general solution is $y(t) = \frac{-1}{2}t\cos t + c_1\cos t + c_2\sin t$.
3. The general solution is $y(t) = \frac{1}{4}e^t + c_1e^t\cos 2t + c_2e^t\sin 2t$.
5. The general solution is $y(t) = \frac{1}{2}e^{3t} + c_1e^t + c_2e^{2t}$.
7. The general solution is $y(t) = t\ln te^t + c_1e^t + c_2te^t$.
9. The general solution is $y(t) = \frac{t^4}{6} + c_1t + c_2t^2$.
11. The general solution is $y(t) = \frac{t}{2}\ln^2 t + c_1t + c_2t\ln t$.
13. The general solution is $y(t) = \frac{t^2}{2}\tan t + t + c_1\tan t + c_2\sec t$.
15. The general solution is $y(t) = t^2 + c_1\cos t^2 + c_2\sin t^2$.

19. $y_p(t) = \frac{1}{a}f(t) * \sinh at$
21. $y_p(t) = \frac{1}{a-b}f(t) * (e^{at} - e^{bt})$

Section 6.1

1. Graph (c)
3. Graph (e)
5. Graph (f)
7. Graph (h)
9. -22/3
11. 4
13. 11/2

15. 5

17. A, B true. C false.

19. A is true. B and C are false.

21. A and B are true. C and D are false.

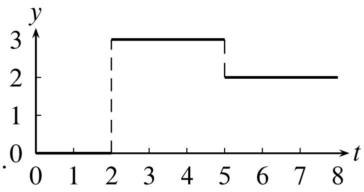
23. A, B, C, D are all true.

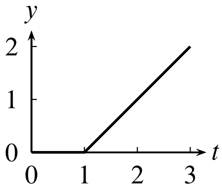
25. $y(t) = \begin{cases} -1 + e^t & \text{if } 0 \le t < 2, \\ 1 - 2e^{t-2} + e^t & \text{if } 2 \le t < 4 \\ e^{t-4} - 2e^{t-2} + e^t & \text{if } 4 \le t < \infty. \end{cases}$

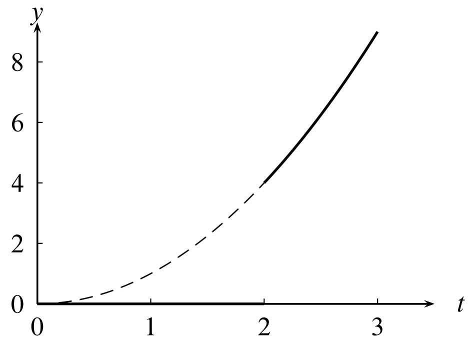
27. $y(t) = \begin{cases} 0 & \text{if } 0 \le t < 1, \\ -t + e^{t-1} & \text{if } 1 \le t < 2, \\ t - 2 - 2e^{t-2} + e^{t-1} & \text{if } 2 \le t < 3 \\ e^{t-3} - 2e^{t-2} + e^{t-1} & \text{if } 3 \le t < \infty. \end{cases}$

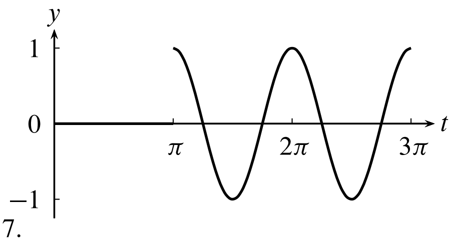
29. $y(t) = \begin{cases} -t + e^t - e^{-t} & \text{if } 0 \le t < 1, \\ e^t - e^{t-1} - e^{-1} & 1 \le t < \infty. \end{cases}$

Section 6.2

1. 

3. 

5. 

7. 

9. (a) $(t-2)\chi_{[2,\infty)}(t)$; (b) $(t-2)h(t-2)$; (c) e^{-2s}/s^2.

11. (a) $(t+2)\chi_{[2,\infty)}(t)$; (b) $(t+2)h(t-2)$; (c) $e^{-2s}\left(\frac{1}{s^2}+\frac{4}{s}\right)$.

13. (a) $t^2\chi_{[4,\infty)}(t)$; (b) $t^2h(t-4)$; (c) $e^{-4s}\left(\frac{2}{s^3}+\frac{8}{s^2}+\frac{16}{s}\right)$.

15. (a) $(t-4)^2\chi_{[2,\infty)}(t)$; (b) $(t-4)^2h(t-2)$; (c) $e^{-2s}\left(\frac{2}{s^3}-\frac{4}{s^2}+\frac{4}{s}\right)$.

17. (a) $e^t\chi_{[4,\infty)}(t)$; (b) $e^th(t-4)$; (c) $e^{-4(s-1)}\frac{1}{s-1}$.

19. (a) $te^t\chi_{[4,\infty)}(t)$; (b) $te^th(t-4)$;
(c) $e^{-4(s-1)}\left(\frac{1}{(s-1)^2}+\frac{4}{s-1}\right)$.

21. (a) $t\chi_{[0,1)}(t)+(2-t)\chi_{[1,\infty)}(t)$; (b) $t+(2-2t)h(t-1)$;
(c) $\frac{1}{s^2}-\frac{2e^{-s}}{s^2}$.

23. (a) $t^2\chi_{[0,2)}(t)+4\chi_{[2\,3)}(t)+(7-t)\chi_{[3,\infty)}(t)$;
(b) $t^2+(4-t^2)h(t-2)+(3-t)h(t-3)$;
(c) $\frac{2}{s^3}-e^{-2s}\left(\frac{2}{s^3}+\frac{4}{s^2}\right)-\frac{e^{-3s}}{s^2}$.

25. (a) $\sum_{n=0}^{\infty}(t-n)\chi_{[n,n+1)}(t)$; (b) $t - \sum_{n=1}^{\infty} h(t-n)$;
(c) $\dfrac{1}{s^2} - \dfrac{e^{-s}}{s(1-e^{-s})}$.

27. (a) $\sum_{n=0}^{\infty}(2n+1-t)\chi_{[2n,2n+2)}(t)$; (b) $-(t+1) + 2\sum_{n=0}^{\infty} h(t-2n)$;
(c) $-\dfrac{1}{s^2} - \dfrac{1}{s} + \dfrac{2}{s(1-e^{-2s})}$.

29. $(t-3)h(t-3) = \begin{cases} 0 & \text{if } 0 \le t < 3, \\ t-3 & \text{if } t \ge 3. \end{cases}$

31. $h(t-\pi)\sin(t-\pi)$
$= \begin{cases} 0 & \text{if } 0 \le t < \pi, \\ \sin(t-\pi) & \text{if } t \ge \pi \end{cases} = \begin{cases} 0 & \text{if } 0 \le t < \pi, \\ -\sin t & \text{if } t \ge \pi. \end{cases}$

33. $\frac{1}{2}e^{-(t-\pi)}\sin 2(t-\pi)h(t-\pi) = \begin{cases} 0 & \text{if } 0 \le t < \pi, \\ \frac{1}{2}e^{-(t-\pi)}\sin 2t & \text{if } t \ge \pi. \end{cases}$

35. $\frac{1}{2}h(t-2)\sin 2(t-2) = \begin{cases} 0 & \text{if } 0 \le t < 2, \\ \frac{1}{2}\sin 2(t-2) & \text{if } t \ge 2. \end{cases}$

37. $h(t-4)\left(2e^{-2(t-4)} - e^{-(t-4)}\right)$
$= \begin{cases} 0 & \text{if } 0 \le t < 4, \\ 2e^{-2(t-4)} - e^{-(t-4)} & \text{if } t \ge 4. \end{cases}$

39. $t - (t-5)h(t-5) = \begin{cases} t & \text{if } 0 \le t < 5, \\ 5 & \text{if } t \ge 5. \end{cases}$

41. $h(t-\pi)e^{-3(t-\pi)}\left(2\cos 2(t-\pi) - \frac{5}{2}\sin 2(t-\pi)\right)$
$= \begin{cases} 0 & \text{if } 0 \le t < \pi, \\ e^{-3(t-\pi)}\left(2\cos 2t - \frac{5}{2}\sin 2t\right) & \text{if } t \ge \pi. \end{cases}$

Section 6.3

1. $y = \begin{cases} 0 & \text{if } 0 \le t < 1 \\ -\frac{3}{2}\left(1 - e^{-2(t-1)}\right) & \text{if } 1 \le t < \infty \end{cases}$

3. $y = \begin{cases} 0 & \text{if } 0 \le t < 2 \\ \frac{2}{3}\left(e^{3(t-2)} - 1\right) & \text{if } 2 \le t < 3 \\ \frac{2}{3}\left(e^{3(t-2)} - e^{3(t-3)}\right) & \text{if } 3 \le t < \infty \end{cases}$

5. $y = \begin{cases} 6e^{4t} - 4e^{t} & \text{if } 0 \le t < 1 \\ 6e^{4t} - e^{4t-3} - 3e & \text{if } 1 \le t < \infty \end{cases}$

7. $y = \begin{cases} 0 & \text{if } 0 \le t < 3 \\ \frac{1}{9}(1 - \cos 3(t-3)) & \text{if } 3 \le t < \infty \end{cases}.$

9. $y = \begin{cases} 0 & \text{if } 0 \le t < 1 \\ 1 - 3e^{-2(t-1)} + 2e^{-3(t-1)} & \text{if } 1 \le t < 3 \\ 3e^{-2(t-3)} - 3e^{-2(t-1)} - 2e^{-3(t-3)} + 2e^{-3(t-1)} & \text{if } 3 \le t < \infty \end{cases}$

11. $y = \begin{cases} te^{-t} & \text{if } 0 \le t < 3 \\ 1 + te^{-t} - (t-2)e^{-(t-3)} & \text{if } 3 \le t < \infty \end{cases}$

13. $y(t) = \begin{cases} 4 - 4e^{\frac{-t}{2}} & \text{if } 0 \le t < 3 \\ 20 - 4e^{\frac{-t}{2}} - 16e^{\frac{-(t-3)}{2}} & \text{if } t \ge 3. \end{cases}$

15. $y(t) = \begin{cases} 10 - 8e^{-3t/10} & \text{if } 0 \le t < 2 \\ 10e^{-3(t-2)/10} - 8e^{-3t/10} & \text{if } 2 \le t < 4 \\ 10 - 8e^{-3t/10} + 10e^{-3(t-2)/10} - 10e^{-3(t-4)/10} & \text{if } 4 \le t < \infty \end{cases}$

Section 6.4

1. $y = \begin{cases} 0 & \text{if } 0 \le t < 1 \\ e^{-2(t-1)} & \text{if } 1 \le t < \infty \end{cases}$

3. $y = \begin{cases} 2e^{4t} & \text{if } 0 \le t < 4 \\ 2e^{4t} + e^{4(t-4)} & \text{if } 4 \le t < \infty \end{cases}$

5. $y = \begin{cases} \frac{1}{2}\sin 2t & \text{if } 0 \le t < \pi, \\ \sin 2t & \text{if } t \ge \pi. \end{cases}$

7. $y = \begin{cases} e^{-t} & \text{if } 0 \le t < 2 \\ e^{-t} + e^{-(t-2)} - e^{-3(t-2)} & \text{if } 2 \le t < \infty \end{cases}$.

9. $y = \begin{cases} te^{-2t} - e^{-2t} & \text{if } 0 \le t < 1 \\ te^{-2t} - e^{-2t} + 3(t-1)e^{-2(t-1)} & \text{if } 1 \le t < \infty \end{cases}$

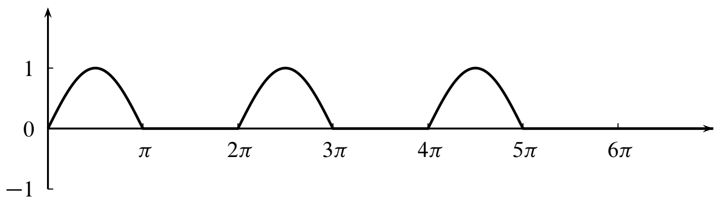
11. $y = \begin{cases} 24 - 24e^{-\frac{1}{4}t} & \text{if } 0 \le t < 3 \\ 24 - 24e^{-\frac{1}{4}t} + 4e^{-\frac{1}{4}(t-3)} & \text{if } 3 \le t < \infty \end{cases}$

13. $y = \begin{cases} e^{-t} & \text{if } 0 \le t < 2 \\ e^{-t} + e^{-(t-2)} & \text{if } 2 \le t < 4 \\ e^{-t} + e^{-(t-2)} + e^{-(t-4)} & \text{if } 4 \le t < 6 \\ e^{-t} + e^{-(t-2)} + e^{-(t-4)} + e^{-(t-6)} & \text{if } 6 \le t < \infty \end{cases}$ and $y(6) = 1.156$ lbs.

15. $y = \begin{cases} \frac{1}{10}e^{-2t} & \text{if } 0 \le t < 4 \\ \frac{1}{10}e^{-2t} + (t-4)e^{-2(t-4)} & \text{if } 4 \le t < \infty \end{cases}$

17. $y = \begin{cases} \sin t & \text{if } 0 \le t < \pi \\ 0 & \text{if } \pi \le t < 2\pi \\ \sin t & \text{if } 2\pi \le t < 3\pi \\ 0 & \text{if } 3\pi \le t < 4\pi \\ \sin t & \text{if } 4\pi \le t < 5\pi \\ 0 & \text{if } 5\pi \le t < \infty. \end{cases}$

The graph is given below.



At $t = 0$, the hammer imparts a velocity to the system causing harmonic motion. At $t = \pi$, the hammer strikes in precisely the right way to stop the motion. Then at $t = 2\pi$, the process repeats.

19. $y = y_0e^{-at} + ke^{-a(t-c)}$

Section 6.5

1. $f * g(t) = \begin{cases} e^t - 1 & \text{if } 0 \le t < 1 \\ e^t - e^{t-1} & \text{if } 1 \le t < \infty \end{cases}$

5. $f * g = \begin{cases} t & \text{if } 0 \le t < 2 \\ -t + 4 & \text{if } 2 \le t < 4 \\ 0 & \text{if } 4 \le t < \infty \end{cases}$

7. $f * g = \begin{cases} \sin t & \text{if } 0 \le t < \pi \\ 0 & \text{if } \pi \le t < \infty \end{cases}$

9. $\zeta(t) = e^{3t}$, $y = \begin{cases} 2e^{3t} & \text{if } 0 \le t < \infty \\ 2e^{3t} + \frac{1}{3}\left(e^{3(t-2)} - 1\right) & \text{if } 1 \le t < \infty \end{cases}$

11. $\zeta(t) = e^{-8t}$, $y = \begin{cases} -2e^{-8t} & \text{if } 0 \le t < 3 \\ -2e^{-8t} + \frac{1}{8}\left(1 - e^{-8(t-3)}\right) & \text{if } 3 \le t < 5 \\ -2e^{-8t} + \frac{1}{8}\left(e^{-8(t-5)} - e^{-8(t-3)}\right) & \text{if } 5 \le t < \infty \end{cases}$

13. $y = \frac{1}{9}\begin{cases} 1 - \cos 3t & \text{if } 0 \le t < 2\pi \\ 0 & \text{if } 2\pi \le t < \infty \end{cases}$

23. $y = 2e^t - 7te^t$

25. $y = 5 - 4\cos t$

Section 6.6

1. $(\langle t \rangle_1)^2$

3. $\frac{1}{9}([t]_3)^2$

5. $\langle t \rangle_2 + \frac{3}{2}[t]_2$

7. $\mathcal{L}\{f(\langle t \rangle_3)\} = \frac{1-e^{-3(s-1)}}{1-e^{-3s}}\frac{1}{s-1}$

9. $\mathcal{L}\left\{f(\langle t \rangle_{2p})\right\} = \frac{1-e^{-ps}}{1+e^{-ps}}\frac{1}{s}$

11. $\mathcal{L}\left\{[t]_p\right\} = \frac{p}{s(e^{ps}-1)}$

13. $y = \frac{1-e^{-2s}}{1-e^{-2(s+1)}} \frac{1}{s}$

17. $\mathcal{L}^{-1}\left\{\frac{1-e^{-4(s-2)}}{(1-e^{-4s})(s-2)}\right\} = e^{2\langle t\rangle_4}$

Section 6.7

1.
$$y(t) = 10 - 10e^{\frac{-2t}{5}} + 10\,\mathrm{sw}_2(t) - \frac{10e^{\frac{-2t}{5}}}{1+e^{\frac{4}{5}}}\left(1 + e^{\frac{4}{5}}(-1)^{[t/2]_1} e^{\frac{2}{5}[t]_2}\right)$$

The amount of salt fluctuates from 13.10 pounds to 16.90 pounds in the long term.

3. $y(t) = 5e^{-\frac{1}{2}t}\,\frac{e^{\frac{1}{2}[t]_2+1}-1}{e-1}$, and the salt fluctuation in the tank varies between 2.91 and 7.91 pounds for large values of t.

5. The mathematical model is
$$y' = ry - 40\delta_0(\langle t\rangle_1), \quad y(0) = 3000,$$
where $r = \frac{1}{12}\ln\frac{6}{5}$. The solution to the model is
$$y(t) = 3000e^{rt} - 40\frac{e^{rt} - e^{-r([t]_1 - t + 1)}}{1 - e^{-r}}$$
and at the beginning of 60 months, there are $y(60) \approx 3477$ alligators.

Section 6.8

1.
$$y(t) = 2\left(2\,\mathrm{sw}_1(t) - (-1)^{[t]_1}(\cos\langle t\rangle_1 - \alpha\sin\langle t\rangle_1)\right)$$
$$-2\left(\cos t + \alpha\sin t\right),$$
where $\alpha = \frac{-\sin\sqrt{2}}{1+\cos\sqrt{2}}$. Motion is nonperiodic.

3. $y(t) = \frac{1}{\pi^2}\left(2\,\mathrm{sw}_2(t) - \cos\pi t\left((-1)^{[t/2]_1} + 1\right)\right)$, the motion is periodic.

5. $y(t) = \frac{2}{\pi^2}(\mathrm{sw}_1(t) - [t]_1\cos\pi t - \cos\pi t)$, with resonance.

7.

$$\begin{aligned} y(t) &= \sin t + \sin\langle t\rangle_\pi \\ &= (1 + (-1)^{[t/\pi]_1}) \sin t. \end{aligned}$$

Motion is periodic.

9.

$$y(t) = \sin t + \gamma \cos t + \sin\langle t\rangle_1 - \gamma \cos\langle t\rangle_1,$$

where $\gamma = \frac{-\sin 1}{1-\cos 1}$. The motion is nonperiodic.

11.

$$y(t) = 2(\sin t)(1 + [t/2\pi]_1).$$

Resonance occurs.

Section 7.1

1. $R = 1$
3. $R = \infty$
5. $R = 0$
7. $R = \infty$
9. $R = 2$
11. $-\sum_{n=0}^{\infty} \frac{t^n}{a^{n+1}}$
13. $\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n+1)!}$
15. $\sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{2n+1}$
17. $\tan t = 1 + \frac{1}{3}t^3 + \frac{2}{15}t^5 + \frac{17}{315}t^7 + \cdots$
19. $e^t \sin t = t + t^2 + \frac{1}{3}t^3 - \frac{1}{30}t^5 + \cdots$
21. $(1 - t)e^{-t}$

23. $f(t) = \frac{1}{(1-t)^2}$

25. $f(t) = -\frac{t}{2} + \frac{t^2-1}{4} \ln\left(\frac{1+t}{1-t}\right)$

27. The binomial theorem: $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$

Section 7.2

1. $y(t) = c_0 \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!} + c_1 \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!} = c_0 \cosh t + c_1 \sinh t$

3. $y(t) = c_0 \sum_{n=0}^{\infty} (-1)^n \frac{k^{2n} t^{2n}}{(2n)!} + c_1 \sum_{n=0}^{\infty} (-1)^n \frac{k^{2n+1} t^{2n+1}}{(2n+1)!} = c_0 \cos kt + c_1 \sin kt$

5. $y(t) = c_0(1-t^2) - c_1 \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)(2n-1)} = c_0(1-t^2) - c_1 \left(\frac{t}{2} + \frac{t^2-1}{4} \ln\left(\frac{1-t}{1+t}\right)\right)$

7. $y(t) = c_0 \left(1 + \sum_{n=2}^{\infty} \frac{t^n}{n!}\right) + c_1 t = c_0(\mathrm{e}^t - t) + c_1 t$

9. $y(t) = c_0(1-3t^2) + c_1(t - \frac{t^3}{3})$

Section 7.3

1. -1 and 1 are regular singular points.

3. There are no singular points.

5. 0 is a regular singular point.

7. $q(s) = s(s+1)$. The exponents of singularity are 0 and -1. Theorem 2 guarantees one Frobenius solution but there could be two.

9. $q(s) = s^2$. The exponent of singularity is 0 with multiplicity 2. Theorem 2 guarantees that there is one and only one Frobenius solution.

15. $y_1(t) = \sum_{m=0}^{\infty} \frac{(-1)^m (2m+2) t^{2m+3}}{(2m+3)!} = (\sin t - t\cos t)$ and $y_2(t) = \sum_{m=0}^{\infty} \frac{(-1)^{m+1}(2m-1)t^{2m}}{(2m)!} = (t \sin t + \cos t)$

17. $y_1(t) = \sum_{n=0}^{\infty} \frac{1}{n!} t^{n+1} = t\mathrm{e}^t$ and $y_2(t) = \left(t\mathrm{e}^t \ln t - t \sum_{n=1}^{\infty} \frac{s_n t^n}{n!}\right)$, where $s_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots \frac{1}{n}$.

19. $y_1(t) = \sum_{n=0}^{\infty} \frac{(-1)^n (n+1) t^n}{(n+3)!} = \left(\frac{(t+2)\mathrm{e}^{-t}}{t} + \frac{t-2}{t}\right)$ and $y_2(t) = t^{-1}(1 - \frac{t}{2}) = \frac{1}{2}\left(\frac{2-t}{t}\right)$

21. $y_1(t) = t^2$ and $y_2(t) = t^2 \ln t + \left(-1 + 2t - \frac{t^3}{3} + \sum_{n=4}^{\infty} \frac{2(-1)^n t^n}{n!(n-2)}\right)$

23. The complex Frobenius series is $y(t) = (t^i + \left(\frac{1-2i}{1+2i}\right) t^{1+i})$; the real and imaginary parts are $y_1(t) = -3\cos\ln t - 4\sin\ln t + 5t\cos\ln t$ and $y_2(t) = -3\sin\ln t + 4\cos\ln t + 5t\sin\ln t$.

25. The complex Frobenius solution is $y(t) = \sum_{n=0}^{\infty} \frac{t^{n+i}}{n!} = t^i \mathrm{e}^t$; the real and imaginary parts are $y_1(t) = \mathrm{e}^t \cos\ln t$ and $y_2(t) = \mathrm{e}^t \sin\ln t$.

Section 7.4

Section 8.1

1. $B + C = \begin{bmatrix} 1 & 1 \\ -1 & 7 \\ 0 & 3 \end{bmatrix}$, $B - C = \begin{bmatrix} 1 & -3 \\ 5 & -1 \\ -2 & 1 \end{bmatrix}$, and $2B - 3C = \begin{bmatrix} 2 & -8 \\ 13 & -6 \\ -5 & 1 \end{bmatrix}$

3. $A(B + C) = AB + AC = \begin{bmatrix} 3 & 4 \\ 1 & 13 \end{bmatrix}$, $(B + C)A = \begin{bmatrix} 3 & -1 & 7 \\ 3 & 1 & 25 \\ 5 & 0 & 12 \end{bmatrix}$

5. $AB = \begin{bmatrix} 6 & 4 & -1 & -8 \\ 0 & 2 & -8 & 2 \\ 2 & -1 & 9 & -5 \end{bmatrix}$

7. $CA = \begin{bmatrix} 8 & 0 \\ 4 & -5 \\ 8 & 14 \\ 10 & 11 \end{bmatrix}$

9. $ABC = \begin{bmatrix} 8 & 9 & -48 \\ 4 & 0 & -48 \\ -2 & 3 & 40 \end{bmatrix}$.

15. $\begin{bmatrix} 0 & 0 & 1 \\ 3 & -5 & -1 \\ 0 & 0 & 5 \end{bmatrix}$

17. (a) Choose, for example, $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.
(b) $(A + B)^2 = A^2 + 2AB + B^2$ precisely when $AB = BA$.

19. $B^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$

21. (a) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} A = \begin{bmatrix} v_2 \\ v_1 \end{bmatrix}$; the two rows of A are switched. (b) $\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix} A = \begin{bmatrix} v_1 + cv_2 \\ v_2 \end{bmatrix}$; to the first row is added c times the second row while the second row is unchanged. (c) To the second row is added c times the first row while the first row is unchanged. (d) The first row is multiplied by a while the second row is unchanged. (e) The second row is multiplied by a while the first row is unchanged.

Section 8.2

1. $A = \begin{bmatrix} 1 & 4 & 3 \\ 1 & 1 & -1 \\ 2 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 2 \\ 4 \\ 1 \\ 6 \end{bmatrix}$, and $[A|\mathbf{b}] = \left[\begin{array}{ccc|c} 1 & 4 & 3 & 2 \\ 1 & 1 & -1 & 4 \\ 2 & 0 & 1 & 1 \\ 0 & 1 & -1 & 6 \end{array}\right]$.

3.
$$\begin{aligned} x_1 - x_3 + 4x_4 + 3x_5 &= 2 \\ 5x_1 + 3x_2 - 3x_3 - x_4 - 3x_5 &= 1 \\ 3x_1 - 2x_2 + 8x_3 + 4x_4 - 3x_5 &= 3 \\ -8x_1 + 2x_2 + 2x_4 + x_5 &= -4 \end{aligned}$$

5. RREF

7. $m_2(1/2)(A) = \begin{bmatrix} 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

9. $t_{1,3}(-3)(A) = \begin{bmatrix} 1 & 0 & 1 & 0 & 3 \\ 0 & 1 & 3 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

11. $\begin{bmatrix} 1 & 0 & 0 & -11 & -8 \\ 0 & 1 & 0 & -4 & -2 \\ 0 & 0 & 1 & 9 & 6 \end{bmatrix}$

13. $\begin{bmatrix} 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

15. $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

17. $\begin{bmatrix} 1 & 4 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}$

19. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} -3 \\ 1 \\ 5 \end{bmatrix}, \alpha \in \mathbb{R}$

21. $\begin{bmatrix} x \\ y \end{bmatrix} = \alpha \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \alpha \in \mathbb{R}$

23. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 14/3 \\ 1/3 \\ -2/3 \end{bmatrix}$

25. $\begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} + \alpha \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \alpha \in \mathbb{R}$

27. $\emptyset$

29. $\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

31. $\left\{ \begin{bmatrix} -34 \\ -40 \\ 39 \\ 1 \end{bmatrix} \right\}$

33. The equation $\begin{bmatrix} 5 \\ -1 \\ 4 \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + b \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ has solution $a = 2$ and $b = 3$. By Proposition 7 $\begin{bmatrix} 5 \\ -1 \\ 4 \end{bmatrix}$, is a solution.

35. If $\mathbf{x}_i$ is the solution set for $A\mathbf{x} = \mathbf{b}_i$, then $\mathbf{x}_1 = \begin{bmatrix} -7/2 \\ 7/2 \\ -3/2 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} -3/2 \\ 3/2 \\ -1/2 \end{bmatrix}$, and $\mathbf{x}_3 = \begin{bmatrix} 7 \\ -6 \\ 3 \end{bmatrix}$.

Section 8.3

1. $\begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix}$

3. Not invertible

5. Not invertible

7. $\begin{bmatrix} -6 & 5 & 13 \\ 5 & -4 & -11 \\ -1 & 1 & 3 \end{bmatrix}$

9. $\begin{bmatrix} -29 & 39/2 & -22 & 13 \\ 7 & -9/2 & 5 & -3 \\ -22 & 29/2 & -17 & 10 \\ 9 & -6 & 7 & -4 \end{bmatrix}$

11. $\begin{bmatrix} 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ -1 & -1 & 0 & 1 \end{bmatrix}$

13. $\mathbf{x} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$

15. $\mathbf{x} = \frac{1}{10} \begin{bmatrix} 16 \\ 11 \\ 18 \end{bmatrix}$

17. $\mathbf{x} = \begin{bmatrix} 19 \\ -4 \\ 15 \\ -6 \end{bmatrix}$

19. $(A^t)^{-1} = (A^{-1})^t$

21. $F(\theta)^{-1} = F(-\theta)$

Section 8.4

1. 1
3. 10
5. −21
7. 2
9. 0

11. $\frac{1}{s^2-6s+8} \begin{bmatrix} s-3 & 1 \\ 1 & s-3 \end{bmatrix}$ $s = 2, 4$

13. $\frac{1}{(s-1)^3} \begin{bmatrix} (s-1)^2 & 3 & s-1 \\ 0 & (s-1)^2 & 0 \\ 0 & 3(s-1) & (s-1)^2 \end{bmatrix}$ $s = 1$

15. $\frac{1}{s^3+s^2+4s+4} \begin{bmatrix} s^2+s & 4s+4 & 0 \\ -s-1 & s^2+s & 0 \\ s-4 & 4s+4 & s^2+4 \end{bmatrix}$ $s = -1, \pm 2i$

17. no inverse

19. $\frac{1}{8} \begin{bmatrix} 4 & -4 & 4 \\ -1 & 3 & -1 \\ -5 & -1 & 3 \end{bmatrix}$

21. $\frac{1}{6} \begin{bmatrix} 2 & -98 & 9502 \\ 0 & 3 & -297 \\ 0 & 0 & 6 \end{bmatrix}$

23. $\frac{1}{15}\begin{bmatrix} 55 & -95 & 44 & -171 \\ 50 & -85 & 40 & -150 \\ 70 & -125 & 59 & -216 \\ 65 & -115 & 52 & -198 \end{bmatrix}$

25. $\mathbf{x} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$

27. $\mathbf{x} = \frac{1}{10}\begin{bmatrix} 16 \\ 11 \\ 18 \end{bmatrix}$

Section 8.5

1. The characteristic polynomial is $c_A(s) = (s-1)(s-2)$. The eigenvalues are thus $s = 1, 2$. The eigenspaces are $E_1 = \text{Span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$ and $E_2 = \text{Span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$.

3. The characteristic polynomial is $c_A(s) = s^2 - 2s + 1 = (s-1)^2$. The only eigenvalue is $s = 1$. The eigenspace is $E_1 = \text{Span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$.

5. The characteristic polynomial is $c_A(s) = s^2 + 2s - 3 = (s+3)(s-1)$. The eigenvalues are thus $s = -3, 1$. The eigenspaces are $E_{-3} = \text{Span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$ and $E_1 = \text{Span}\left\{\begin{bmatrix} -3 \\ 2 \end{bmatrix}\right\}$.

7. The characteristic polynomial is $c_A(s) = s^2 + 2s + 10 = (s+1)^2 + 3^2$. The eigenvalues are thus $s = -1 \pm 3i$. The eigenspaces are $E_{-1+3i} = \text{Span}\left\{\begin{bmatrix} 7+i \\ 10 \end{bmatrix}\right\}$ and $E_{-1-3i} = \text{Span}\left\{\begin{bmatrix} 7-i \\ 10 \end{bmatrix}\right\}$.

9. The eigenvalues are $s = -2, 3$. $E_{-2} = \text{Span}\left\{\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right\}$, $E_3 = \text{Span}\left\{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right\}$,

11. The eigenvalues are $s = 0, 2, 3$. $E_0 = \mathrm{NS}(A) = \mathrm{Span}\left\{\begin{bmatrix}0\\2\\1\end{bmatrix}\right\}$, $E_2 = \mathrm{Span}\left\{\begin{bmatrix}2\\2\\1\end{bmatrix}\right\}$, $E_3 = \mathrm{Span}\left\{\begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$.

13. We write $c_A(s) = (s-2)((s-2)^2+1)$ to see that the eigenvalues are $s = 2, 2 \pm i$. $E_2 = \mathrm{Span}\left\{\begin{bmatrix}2\\3\\1\end{bmatrix}\right\}$, $E_{2+i} = \mathrm{Span}\left\{\begin{bmatrix}-4+3i\\4+2i\\5\end{bmatrix}\right\}$, $E_{2-i} = \mathrm{Span}\left\{\begin{bmatrix}-4-3i\\4-2i\\5\end{bmatrix}\right\}$.

Section 9.2

1. Nonlinear

3. $\boldsymbol{y}' = \begin{bmatrix}\sin t & 0\\ 1 & \cos t\end{bmatrix}\boldsymbol{y}$; linear and homogeneous, but not constant coefficient.

5. $\boldsymbol{y}' = \begin{bmatrix}1&0&0&0\\2&0&0&1\\0&0&0&1\\0&1&2&0\end{bmatrix}\boldsymbol{y}$; linear, constant coefficient, homogeneous.

11. $\boldsymbol{y}' = \begin{bmatrix}0&1\\-6&-5\end{bmatrix}\boldsymbol{y} + \begin{bmatrix}0\\e^{2t}\end{bmatrix}, \qquad \boldsymbol{y}(0) = \begin{bmatrix}1\\-2\end{bmatrix}.$

13. $\boldsymbol{y}' = \begin{bmatrix}0&1\\k^2&0\end{bmatrix}\boldsymbol{y} + \begin{bmatrix}0\\\cos\omega t\end{bmatrix}, \quad \boldsymbol{y}(0) = \begin{bmatrix}0\\0\end{bmatrix}.$

15. $\boldsymbol{y}' = \begin{bmatrix}0&1\\-\frac{1}{t^2}&-\frac{2}{t}\end{bmatrix}\boldsymbol{y}, \quad \boldsymbol{y}(1) = \begin{bmatrix}-2\\3\end{bmatrix}.$

17. $A'(t) = \begin{bmatrix}-3e^{-3t} & 1\\ 2t & 2e^{2t}\end{bmatrix}$

19. $\boldsymbol{y}'(t) = \begin{bmatrix}1\\2t\\t^{-1}\end{bmatrix}$

21. $v'(t) = \begin{bmatrix} -2e^{-2t} & \frac{2t}{t^2+1} & -3\sin 3t \end{bmatrix}$

23. $\frac{1}{4}\begin{bmatrix} e^2 - e^{-2} & e^2 + e^{-2} - 2 \\ 2 - e^2 - e^{-2} & e^2 - e^{-2} \end{bmatrix}$

25. $\begin{bmatrix} 4 & 8 \\ 12 & 16 \end{bmatrix}$

27. $\begin{bmatrix} \frac{1}{s} & \frac{1}{s^2} \\ \frac{2}{s^3} & \frac{1}{s-2} \end{bmatrix}$

29. $\begin{bmatrix} \frac{3!}{s^4} & \frac{2s}{(s^2+1)^2} & \frac{1}{(s+1)^2} \\ \frac{2-s}{s^3} & \frac{s-3}{s^2-6s+13} & \frac{3}{s} \end{bmatrix}$

31. $\frac{2}{s^2-1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

33. $\begin{bmatrix} 1 & 2t & 3t^2 \end{bmatrix}$

35. $\begin{bmatrix} e^t + e^{-t} & e^t - e^{-t} \\ e^t - e^{-t} & e^t + e^{-t} \end{bmatrix}$

Section 9.3

1. $e^{At} = \begin{bmatrix} e^t & 0 \\ 0 & e^{-2t} \end{bmatrix}$

3. $e^{At} = \begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix}$

5. $e^{At} = \begin{bmatrix} \frac{1}{2} + \frac{1}{2}e^{2t} & -\frac{1}{2} + \frac{1}{2}e^{2t} \\ -\frac{1}{2} + \frac{1}{2}e^{2t} & \frac{1}{2} + \frac{1}{2}e^{2t} \end{bmatrix}$

7. $e^{At} = \begin{bmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & e^{2t} \end{bmatrix}$

9. $(sI - A)^{-1} = \begin{bmatrix} \frac{s-2}{s(s-3)} & \frac{-1}{s(s-3)} \\ \frac{-2}{s(s-3)} & \frac{s-1}{s(s-3)} \end{bmatrix}$ and $e^{At} = \begin{bmatrix} \frac{2}{3} + \frac{1}{3}e^{3t} & \frac{1}{3} - \frac{1}{3}e^{3t} \\ \frac{2}{3} - \frac{2}{3}e^{3t} & \frac{1}{3} + \frac{2}{3}e^{3t} \end{bmatrix}$

11. $(sI-A)^{-1} = \begin{bmatrix} \frac{s+1}{(s-1)^2+1} & \frac{5}{(s-1)^2+1} \\ \frac{-1}{(s-1)^2+1} & \frac{s-3}{(s-1)^2+1} \end{bmatrix}$

and $e^{At} = \begin{bmatrix} e^t \cos t + 2e^t \sin t & 5e^t \sin t \\ -e^t \sin t & e^t \cos t - 2e^t \sin t \end{bmatrix}$

13. $(sI-A)^{-1} = \begin{bmatrix} \frac{1}{s} & \frac{1}{s^2} & \frac{s+1}{s^3} \\ 0 & \frac{1}{s} & \frac{1}{s^2} \\ 0 & 0 & \frac{1}{s} \end{bmatrix}$ and $e^{At} = \begin{bmatrix} 1 & t & t+\frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$

15. $e^{At} = \begin{bmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & e^{2t} \end{bmatrix}$

Section 9.4

1. $e^{At} = \begin{bmatrix} e^t + te^t & -te^t \\ te^t & e^t - te^t \end{bmatrix}$

3. $e^{At} = \begin{bmatrix} 1+2t & t \\ -4t & 1-2t \end{bmatrix}$

5. $e^{At} = \begin{bmatrix} e^t \cos t + 3e^t \sin t & -10e^t \sin t \\ e^t \sin t & e^t \cos t - 3e^t \sin t \end{bmatrix}$

7. $e^{At} = \frac{1}{4} \begin{bmatrix} -7e^{2t} + 11e^{-2t} & 11e^{2t} - 11e^{-2t} \\ -7e^{2t} + 7e^{-2t} & 11e^{2t} - 7e^{-2t} \end{bmatrix}$

9. $e^{At} = \begin{bmatrix} e^{2t} \cos 3t + 8e^{2t} \sin 3t & 13e^{2t} \sin 3t \\ -5e^{2t} \sin 3t & e^{2t} \cos 3t - 8e^{2t} \sin 3t \end{bmatrix}$

11. $e^{At} = \begin{bmatrix} e^{-2t} - te^{-2t} & te^{-2t} \\ -te^{-2t} & e^{-2t} + te^{-2t} \end{bmatrix}$

13. $e^{At} = \begin{bmatrix} 2-e^{-t} & 0 & -1+e^{-t} \\ 0 & e^t & 0 \\ 2-2e^{-t} & 0 & -1+2e^{-t} \end{bmatrix}$

15. $e^{At} = \frac{1}{2} \begin{bmatrix} e^t + e^t \cos t & -e^t \sin t & e^t - e^t \cos t \\ 2e^t \sin t & 2e^t \cos t & -2e^t \sin t \\ e^t - e^t \cos t & e^t \sin t & e^t + e^t \cos t \end{bmatrix}$

17. $e^{At} = \begin{bmatrix} -e^t + 2\cos 2t & -\sin 2t & e^t - \cos 2t \\ 2\sin 2t & \cos 2t & -\sin 2t \\ -2e^t + 2\cos 2t & -\sin 2t & 2e^t - \cos 2t \end{bmatrix}$

19. $e^{At} = \begin{bmatrix} \cos t - t\cos t & \sin t - t\sin t & t\cos t & t\sin t \\ -\sin t + t\sin t & \cos t - t\cos t & -t\sin t & t\cos t \\ -t\cos t & -t\sin t & \cos t + t\cos t & \sin t + t\sin t \\ t\sin t & -t\cos t & -\sin t - t\sin t & \cos t + t\cos t \end{bmatrix}$

Section 9.5

1. $y(t) = \begin{bmatrix} e^{-t} \\ -2e^{3t} \end{bmatrix}$

3. $y(t) = \begin{bmatrix} -e^{2t} + 2te^{2t} \\ 2e^{2t} \end{bmatrix}$

5. $y(t) = \begin{bmatrix} e^{-t} \\ 3e^{-t} \end{bmatrix}$

7. $y(t) = \begin{bmatrix} e^t - 2te^t \\ e^t - te^t \end{bmatrix}$

9. $y(t) = \begin{bmatrix} 2\cos 2t + 2\sin 2t \\ \cos 2t - \sin 2t \\ 2\cos 2t + 2\sin 2t \end{bmatrix}$

11. $y(t) = \begin{bmatrix} 1 + 2e^{-t}\sin 2t \\ -2 + 2e^{-t}\cos 2t \end{bmatrix}$

13. $y(t) = \begin{bmatrix} 2e^{-t}\cos 2t + 4e^{-t}\sin 2t \\ 1 + e^{-t}\sin 2t - 2e^{-t}\cos 2t \end{bmatrix}$

15. $y(t) = \begin{bmatrix} 2te^t + e^t - t - 1 \\ -4te^t - e^t + 2t + 2 \end{bmatrix}$

17. $y(t) = \begin{bmatrix} te^t \\ 2te^{2t} - e^{2t} + e^t \\ -2te^{2t} + te^t \end{bmatrix}$

19. $y_1(t) = 1 + 3e^{-2t}$, $y_2(t) = 2 + 4e^{-t} - 6e^{-2t}$, $t = 9.02$ seconds.

21. $y_1(t) = y_2(t) = 1 - e^{-2t}$

Section 9.6

1. $P = \begin{bmatrix} 1 & 3 \\ -1 & -1 \end{bmatrix}$, $J = P^{-1}AP = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, and the critical point is saddle.

3. $P = \begin{bmatrix} -3 & -1 \\ 5 & 0 \end{bmatrix}$, $J = \begin{bmatrix} -2 & -1 \\ 1 & -2 \end{bmatrix}$, and the origin is a stable spiral node.

5. A is of type J_3. The origin is an unstable star node.

7. $P = \begin{bmatrix} 1 & 3 \\ -1 & -1 \end{bmatrix}$, $J = P^{-1}AP = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$, and the origin is an unstable node.

9. $P = \begin{bmatrix} -1 & 1 \\ 4 & 0 \end{bmatrix}$, $J = P^{-1}AP = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$, and the origin is an unstable star node.

Section 9.7

1. $\boldsymbol{\Phi}(t)$ is a fundamental matrix, $\boldsymbol{y}(t) = \begin{bmatrix} 2e^{-t} - e^{2t} \\ 2e^{-t} - 4e^{2t} \end{bmatrix}$, and the standard fundamental matrix at $t = 0$ is $\boldsymbol{\Psi}(t) = \frac{1}{3}\begin{bmatrix} 4e^{-t} - e^{2t} & -e^{-t} + e^{2t} \\ 4e^{-t} - 4e^{2t} & -e^{-t} + 4e^{2t} \end{bmatrix}$.

3. $\boldsymbol{\Phi}(t)$ is a fundamental matrix, $\boldsymbol{y}(t) = \begin{bmatrix} \cos(t^2/2) \\ -\sin(t^2/2) \end{bmatrix}$, and the standard fundamental matrix at $t = 0$ is $\boldsymbol{\Psi}(t) = \begin{bmatrix} \cos(t^2/2) & \sin(t^2/2) \\ -\sin(t^2/2) & \cos(t^2/2) \end{bmatrix}$.

5. $\boldsymbol{\Phi}(t)$ is a fundamental matrix, $\boldsymbol{y}(t) = \frac{t}{\pi}\begin{bmatrix} -\cos t + \sin t \\ \cos t + \sin t \end{bmatrix}$, and the standard fundamental matrix at $t = \pi$ is $\boldsymbol{\Psi}(t) = \frac{1}{\pi}\begin{bmatrix} -t\cos t & -t\sin t \\ t\sin t & -t\cos t \end{bmatrix}$.

7. $\boldsymbol{\Phi}(t)$ is a fundamental matrix, $\boldsymbol{y}(t) = \begin{bmatrix} (t-1)e^{t-1} - 3 \\ e^{t-1} + 3 \end{bmatrix}$, and the standard fundamental matrix at $t = 1$ is $\boldsymbol{\Psi}(t) = \begin{bmatrix} 1 + (t-1)e^{t-1} & (t-1)e^{t-1} \\ -1 + e^{t-1} & e^{t-1} \end{bmatrix}$.

9. $\boldsymbol{\Psi}(t) = \begin{bmatrix} t - t\ln t & -t\ln t \\ t\ln t & t + t\ln t \end{bmatrix}$ and $\boldsymbol{y}(t) = \begin{bmatrix} 2t - t\ln t \\ t\ln t \end{bmatrix}$

11. $\boldsymbol{\Psi}(t) = \begin{bmatrix} \sec^2 t + 3\sec t\tan t + \tan^2 t & 5\sec t\tan t \\ -\sec t\tan t & \sec^2 t - 3\sec t\tan t + \tan^2 t \end{bmatrix}$ and $\boldsymbol{y}(t) = \begin{bmatrix} 2\sec^2 t + 11\sec t\tan t + 2\tan^2 t \\ \sec^2 t - 5\sec t\tan t + \tan^2 t \end{bmatrix}$.

13. $y_1(t) = 4(2-t) + (2-t)^2 - \frac{3}{4}(2-t)^4$ and $y_2(t) = 2(2-t) + (2-t)^2 + \frac{3}{4}(2-t)^3$. The concentration (grams/L) of salt in Tank 1 after 1 min is $\frac{17}{4}$ and in Tank 2 is $\frac{15}{4}$.

Appendix C
Tables

C.1 Laplace Transforms

Table C.1 Laplace transform rules

$f(t)$	$F(s)$	Page
Definition of the Laplace transform		
1. $f(t)$	$F(s) = \int_0^\infty e^{-st} f(t)\, dt$	111
Linearity		
2. $a_1 f_1(t) + a_2 f_2(t)$	$a_1 F_1(s) + a_2 F_2(s)$	114
Dilation principle		
3. $f(at)$	$\frac{1}{a} F\left(\frac{s}{a}\right)$	122
Translation principle		
4. $e^{at} f(t)$	$F(s-a)$	120
Input derivative principle: first order		
5. $f'(t)$	$sF(s) - f(0)$	115
Input derivative principle: second order		
6. $f''(t)$	$s^2 F(s) - sf(0) - f'(0)$	115
Input derivative principle: nth order		
7. $f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$	116
Transform derivative principle: first order		
8. $tf(t)$	$-F'(s)$	121
Transform derivative principle: second order		
9. $t^2 f(t)$	$F''(s)$	
Transform derivative principle: nth order		
10. $t^n f(t)$	$(-1)^n F^{(n)}(s)$	121
Convolution principle		
11. $(f * g)(t) = \int_0^t f(\tau) g(t-\tau)\, d\tau$	$F(s)G(s)$	188
Input integral principle		
12. $\int_0^t f(v) dv$	$\frac{F(s)}{s}$	190

(continued)

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Table C.1 (continued)

$f(t)$		$F(s)$	Page
Transform integral formula			
13.	$\dfrac{f(t)}{t}$	$\int_s^\infty F(\sigma)\,d\sigma$	357
	$\dfrac{f(t)}{t}$ has a continuous extension to 0		
Second translation principle			
14.	$f(t-c)h(t-c)$	$e^{-sc}F(s)$	405
Corollary to the second translation principle			
15.	$g(t)h(t-c)$	$e^{-sc}\mathcal{L}\{g(t+c)\}$	405
Periodic functions			
16.	$f(t)$, periodic with period p	$\dfrac{\int_0^p e^{-st}f(t)\,dt}{1-e^{-sp}}$	455
17.	$f(\langle t\rangle_p)$	$\dfrac{\mathcal{L}\{f(t)-f(t)h(t-p)\}}{1-e^{-sp}}$	458
Staircase functions			
18.	$f([t]_p)$	$\dfrac{1-e^{-ps}}{s}\sum_{n=0}^{\infty} f(np)e^{-nps}$	463
Transforms involving $\dfrac{1}{1\pm e^{-sp}}$			
19.	$\sum_{N=0}^{\infty}\sum_{n=0}^{N} f(t-np)\chi_{[Np,(N+1)p)}$	$\dfrac{1}{1-e^{-sp}}F(s)$	461
20.	$\sum_{N=0}^{\infty}\sum_{n=0}^{N} (-1)^n f(t-np)\chi_{[Np,(N+1)p)}$	$\dfrac{1}{1+e^{-sp}}F(s)$	461

Table C.2 Laplace transforms

	$f(t)$	$F(s)$	Page
1.	1	$\dfrac{1}{s}$	116
2.	t	$\dfrac{1}{s^2}$	
3.	$t^n \quad (n = 0, 2, 3, \ldots)$	$\dfrac{n!}{s^{n+1}}$	116
4.	$t^\alpha \quad (\alpha > 0)$	$\dfrac{\Gamma(\alpha+1)}{s^{\alpha+1}}$	118
5.	e^{at}	$\dfrac{1}{s-a}$	118
6.	te^{at}	$\dfrac{1}{(s-a)^2}$	
7.	$t^n e^{at} \quad (n = 1, 2, 3, \ldots)$	$\dfrac{n!}{(s-a)^{n+1}}$	119
8.	$\sin bt$	$\dfrac{b}{s^2+b^2}$	118
9.	$\cos bt$	$\dfrac{s}{s^2+b^2}$	118

Table C.2 (continued)

	$f(t)$	$F(s)$	Page
10.	$e^{at}\sin bt$	$\dfrac{b}{(s-a)^2+b^2}$	120
11.	$e^{at}\cos bt$	$\dfrac{s-a}{(s-a)^2+b^2}$	120
12.	$\dfrac{\sin t}{t}$	$\tan^{-1}\dfrac{1}{s}$	358
13.	$\dfrac{\sin at}{t}$	$\tan^{-1}\left(\dfrac{a}{s}\right)$	365
14.	$\dfrac{e^{bt}-e^{at}}{t}$	$\ln\left(\dfrac{s-a}{s-b}\right)$	365
15.	$2\dfrac{\cos bt-\cos at}{t}$	$\ln\left(\dfrac{s^2+a^2}{s^2+b^2}\right)$	365
16.	$2\dfrac{\cos bt-\cos at}{t^2}$	$s\ln\left(\dfrac{s^2+b^2}{s^2+a^2}\right)-2b\tan^{-1}\left(\dfrac{b}{s}\right)$	
		$+2a\tan^{-1}\left(\dfrac{a}{s}\right)$	365
Laguerre polynomials			
17.	$\ell_n(t)=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{t^k}{k!}$	$\dfrac{(s-1)^n}{s^{n+1}}$	361
18.	$\ell_n(at)$	$\dfrac{(s-a)^n}{s^{n+1}}$	366
The Heaviside function			
19.	$h(t-c)$	$\dfrac{e^{-sc}}{s}$	404
The on-off switch			
20.	$\chi_{[a,b)}$	$\dfrac{e^{-as}}{s}-\dfrac{e^{-bs}}{s}$	405
The Dirac delta function			
21.	δ_c	e^{-cs}	428
The square-wave function			
22.	sw_c	$\dfrac{1}{1+e^{-cs}}\dfrac{1}{s}$	456
The sawtooth function			
23.	$\langle t\rangle_p$	$\dfrac{1}{s^2}\left(1-\dfrac{spe^{-sp}}{1-e^{-sp}}\right)$	457
Periodic dirac delta functions			
24.	$\delta_0(\langle t\rangle_p)$	$\dfrac{1}{1-e^{-ps}}$	459
Alternating periodic dirac delta functions			
25.	$(\delta_0-\delta_p)(\langle t\rangle_{2p})$	$\dfrac{1}{1+e^{-ps}}$	459
The matrix exponential			
26.	e^{At}	$(sI-A)^{-1}$	459

Table C.3 Heaviside formulas

	$f(t)$	$F(s)$
Heaviside formulas of the first kind		
1.	$\dfrac{e^{at}}{a-b}+\dfrac{e^{bt}}{b-a}$	$\dfrac{1}{(s-a)(s-b)}$
2.	$\dfrac{ae^{at}}{a-b}+\dfrac{be^{bt}}{b-a}$	$\dfrac{s}{(s-a)(s-b)}$
3.	$\dfrac{e^{at}}{(a-b)(a-c)}+\dfrac{e^{bt}}{(b-a)(b-c)}+\dfrac{e^{ct}}{(c-a)(c-b)}$	$\dfrac{1}{(s-a)(s-b)(s-c)}$
4.	$\dfrac{ae^{at}}{(a-b)(a-c)}+\dfrac{be^{bt}}{(b-a)(b-c)}+\dfrac{ce^{ct}}{(c-a)(c-b)}$	$\dfrac{s}{(s-a)(s-b)(s-c)}$
5.	$\dfrac{a^2e^{at}}{(a-b)(a-c)}+\dfrac{b^2e^{bt}}{(b-a)(b-c)}+\dfrac{c^2e^{ct}}{(c-a)(c-b)}$	$\dfrac{s^2}{(s-a)(s-b)(s-c)}$
6.	$\dfrac{r_1^k e^{r_1 t}}{q'(r_1)}+\cdots+\dfrac{r_n^k e^{r_n t}}{q'(r_n)}$, $q(s)=(s-r_1)\cdots(s-r_n)$	$\dfrac{s^k}{(s-r_1)\cdots(s-r_n)}$, $r_1,\ldots,r_n$, distinct
Heaviside formulas of the second kind		
7.	te^{at}	$\dfrac{1}{(s-a)^2}$
8.	$(1+at)e^{at}$	$\dfrac{s}{(s-a)^2}$
9.	$\dfrac{t^2}{2}e^{at}$	$\dfrac{1}{(s-a)^3}$
10.	$\left(t+\dfrac{at^2}{2}\right)e^{at}$	$\dfrac{s}{(s-a)^3}$
11.	$\left(1+2at+\dfrac{a^2t^2}{2}\right)e^{at}$	$\dfrac{s^2}{(s-a)^3}$
12.	$\boldsymbol{\left(\sum_{l=0}^{k}\binom{k}{l}a^{k-l}\frac{t^{n-l-1}}{(n-l-1)!}\right)e^{at}}$	$\boldsymbol{\dfrac{s^k}{(s-a)^n}}$

In each case, a, b, and c are distinct. See Page 165.

Table C.4 Laplace transforms involving irreducible quadratics

	$f(t)$	$F(s)$
1.	$\sin bt$	$\dfrac{b}{(s^2+b^2)}$
2.	$\dfrac{1}{2b^2}(\sin bt - bt\cos bt)$	$\dfrac{b}{(s^2+b^2)^2}$
3.	$\dfrac{1}{8b^4}\left((3-(bt)^2)\sin bt - 3bt\cos bt\right)$	$\dfrac{b}{(s^2+b^2)^3}$
4.	$\dfrac{1}{48b^6}\left((15-6(bt)^2)\sin bt-(15bt-(bt)^3)\cos bt\right)$	$\dfrac{b}{(s^2+b^2)^4}$
5.	$\cos bt$	$\dfrac{s}{(s^2+b^2)}$
6.	$\dfrac{1}{2b^2}bt\sin bt$	$\dfrac{s}{(s^2+b^2)^2}$
7.	$\dfrac{1}{8b^4}\left(bt\sin bt-(bt)^2\cos bt\right)$	$\dfrac{s}{(s^2+b^2)^3}$
8.	$\dfrac{1}{48b^6}\left((3bt-(bt)^3)\sin bt-3(bt)^2\cos bt\right)$	$\dfrac{s}{(s^2+b^2)^4}$

Table C.5 Reduction of order formulas

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+b^2)^{k+1}}\right\} = \frac{-t}{2kb^2}\mathcal{L}^{-1}\left\{\frac{s}{(s^2+b^2)^k}\right\} + \frac{2k-1}{2kb^2}\mathcal{L}^{-1}\left\{\frac{1}{(s^2+b^2)^k}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+b^2)^{k+1}}\right\} = \frac{t}{2k}\mathcal{L}^{-1}\left\{\frac{1}{(s^2+b^2)^k}\right\}$$

See Page 155.

Table C.6 Laplace transforms involving quadratics

	$f(t)$	$F(s)$	Page
Laplace transforms involving the quadratic s^2+b^2			
1.	$\frac{\sin bt}{(2b)^{2k}}\sum_{m=0}^{\lfloor \frac{k}{2}\rfloor}(-1)^m\binom{2k-2m}{k}\frac{(2bt)^{2m}}{(2m)!}$ $-\frac{\cos bt}{(2b)^{2k}}\sum_{m=0}^{\lfloor \frac{k-1}{2}\rfloor}(-1)^m\binom{2k-2m-1}{k}\frac{(2bt)^{2m+1}}{(2m+1)!}$	$\frac{b}{(s^2+b^2)^{k+1}}$	549
2.	$\frac{2bt\sin bt}{k\cdot(2b)^{2k}}\sum_{m=0}^{\lfloor \frac{k-1}{2}\rfloor}(-1)^m\binom{2k-2m-2}{k-1}\frac{(2bt)^{2m}}{(2m)!}$ $-\frac{2bt\cos bt}{k\cdot(2b)^{2k}}\sum_{m=0}^{\lfloor \frac{k-2}{2}\rfloor}(-1)^m\binom{2k-2m-3}{k-1}\frac{(2bt)^{2m+1}}{(2m+1)!}$	$\frac{s}{(s^2+b^2)^{k+1}}$	549
Laplace transforms involving the quadratic s^2-b^2			
3.	$\frac{(-1)^k}{2^{2k+1}k!}\sum_{n=0}^{k}\frac{(2k-n)!}{n!(k-n)!}((-2t)^n e^t-(2t)^n e^{-t})$	$\frac{1}{(s^2-1)^{k+1}}$	553
4.	$\frac{(-1)^k}{2^{2k+1}k!}\sum_{n=1}^{k}\frac{(2k-n-1)!}{(n-1)!(k-n)!}((-2t)^n e^t+(2t)^n e^{-t})$	$\frac{s}{(s^2-1)^{k+1}}$	553

C.2 Convolutions

Table C.7 Convolutions

	$f(t)$	$g(t)$	$(f*g)(t)$	Page
1.	$f(t)$	$g(t)$	$f*g(t)=\int_0^t f(u)g(t-u)\,du$	187
2.	1	$g(t)$	$\int_0^t g(\tau)\,d\tau$	190
3.	t^m	t^n	$\frac{m!n!}{(m+n+1)!}t^{m+n+1}$	193
4.	t	$\sin at$	$\frac{at-\sin at}{a^2}$	
5.	t^2	$\sin at$	$\frac{2}{a^3}(\cos at-(1-\frac{a^2t^2}{2}))$	
6.	t	$\cos at$	$\frac{1-\cos at}{a^2}$	
7.	t^2	$\cos at$	$\frac{2}{a^3}(at-\sin at)$	

(continued)

Table C.7 (continued)

	$f(t)$	$g(t)$	$(f * g)(t)$	Page
8.	t	e^{at}	$\dfrac{e^{at} - (1 + at)}{a^2}$	
9.	t^2	e^{at}	$\dfrac{2}{a^3}(e^{at} - (a + at + \frac{a^2t^2}{2}))$	
10.	e^{at}	e^{bt}	$\dfrac{1}{b-a}(e^{bt} - e^{at}) \quad a \neq b$	192
11.	e^{at}	e^{at}	te^{at}	192
12.	e^{at}	$\sin bt$	$\dfrac{1}{a^2 + b^2}(be^{at} - b\cos bt - a\sin bt)$	195
13.	e^{at}	$\cos bt$	$\dfrac{1}{a^2 + b^2}(ae^{at} - a\cos bt + b\sin bt)$	195
14.	$\sin at$	$\sin bt$	$\dfrac{1}{b^2 - a^2}(b\sin at - a\sin bt) \quad a \neq b$	195
15.	$\sin at$	$\sin at$	$\dfrac{1}{2a}(\sin at - at\cos at)$	195
16.	$\sin at$	$\cos bt$	$\dfrac{1}{b^2 - a^2}(a\cos at - a\cos bt) \quad a \neq b$	195
17.	$\sin at$	$\cos at$	$\dfrac{1}{2}t\sin at$	195
18.	$\cos at$	$\cos bt$	$\dfrac{1}{a^2 - b^2}(a\sin at - b\sin bt) \quad a \neq b$	195
19.	$\cos at$	$\cos at$	$\dfrac{1}{2a}(at\cos at + \sin at)$	195
20.	f	$\delta_c(t)$	$f(t-c)h(t-c)$	444
21.	f	$\delta_0(t)$	$f(t)$	445

Symbol Index

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Index

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