

# Appendix A

## Supplements

### A.1 The Laplace Transform is Injective

In this section, we prove Theorem 1 of Sect. 2.5 that states that the Laplace transform is injective on the set of Laplace transformable continuous functions.<sup>1</sup> Specifically, the statement is

**Theorem 1.** *Suppose  $f_1$  and  $f_2$  are continuous functions on  $[0, \infty)$  and have Laplace transforms. Suppose*

$$\mathcal{L}\{f_1\} = \mathcal{L}\{f_2\}.$$

*Then  $f_1 = f_2$ .*

The proof of this statement is nontrivial. It requires a well-known result from advanced calculus which we will assume: the Weierstrass approximation theorem.

**Theorem 2 (The Weierstrass Approximation Theorem).** *Suppose  $h$  is a continuous function on  $[0, 1]$ . Then for any  $\epsilon > 0$ , there is a polynomial  $p$  such that*

$$|h(t) - p(t)| < \epsilon,$$

*for all  $t$  in  $[0, 1]$ .*

In essence, the Weierstrass approximation theorem states that a continuous function can be approximated by a polynomial to any degree of accuracy.

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<sup>1</sup>The presentation here closely follows that found in *Advanced Calculus* by David Widder, published by Prentice Hall, 1961.

**Lemma 3.** Suppose  $h(t)$  is a continuous function so that

$$\int_0^1 t^n h(t) dt = 0,$$

for each nonnegative integer  $n$ . Then  $h(t) = 0$  for all  $t \in [0, 1]$ .

*Proof.* Let  $\epsilon > 0$ . By the Weierstrass approximation theorem, there is a polynomial  $p$  so that

$$|h(t) - p(t)| < \epsilon,$$

for all  $t$  in  $[0, 1]$ . Since a polynomial is a linear combination of powers of  $t$ , it follows by the linearity of the integral that  $\int_0^1 p(t)h(t) dt = 0$ . Now observe,

$$\begin{aligned} \int_0^1 (h(t))^2 dt &= \int_0^1 h(t)(h(t) - p(t)) dt \\ &\leq \int_0^1 |h(t)| |h(t) - p(t)| dt \\ &\leq \epsilon \int_0^1 |h(t)| dt. \end{aligned}$$

Since  $\epsilon$  is arbitrary, it follows that  $\int_0^1 (h(t))^2 dt$  can be made as small as we like. This forces  $\int_0^1 (h(t))^2 dt = 0$ . Since  $(h(t))^2 \geq 0$ , it follows that  $(h(t))^2 = 0$ , for all  $t \in [0, 1]$ . Therefore,  $h(t) = 0$  for all  $t \in [0, 1]$ .  $\square$

**Theorem 4.** Suppose  $f$  is a continuous function on the interval  $[0, \infty)$ ,  $F(s) = \mathcal{L}\{f(t)\}(s)$  for  $s \geq a$  and  $F(a + nl) = 0$  for all  $n = 0, 1, \dots$ , for some  $l > 0$ . Then  $f \equiv 0$ .

*Proof.* Let  $g(t) = \int_0^t e^{-au} f(u) du$ . Since  $F(a) = 0$ , it follows that  $\lim_{t \rightarrow \infty} g(t) = 0$ . Write

$$F(a + nl) = \int_0^\infty e^{-(a+nl)t} f(t) dt = \int_0^\infty e^{-nlt} e^{-at} f(t) dt$$

and compute using integration by parts with  $u = e^{-nlt}$  and  $dv = e^{-at} f(t)$ . Since  $du = -nle^{-nlt}$  and  $v = \int_0^t e^{-au} f(u) du = g(t)$ , we have

$$\begin{aligned} F(a + nl) &= e^{-nlt} g(t) \Big|_0^\infty + nl \int_0^\infty e^{-nlt} g(t) dt. \\ &= nl \int_0^\infty e^{-nlt} g(t) dt. \end{aligned}$$

Since  $F(a + nl) = 0$  we have

$$\int_0^\infty e^{-nt} g(t) dt = 0,$$

for all  $n = 1, 2, \dots$ . Now let  $x = e^{-lt}$ . Then  $dx = -le^{-lt} dt = -lx dt$  and  $t = -\frac{1}{l} \ln x = \frac{1}{l} \ln \frac{1}{x}$ . Substituting and simplifying, we get

$$\int_0^1 x^{n-1} g\left(\frac{1}{l} \ln \frac{1}{x}\right) dx = 0,$$

for all  $n = 1, 2, \dots$ . By Lemma 3, it follows that  $g\left(\frac{1}{l} \ln \frac{1}{x}\right) = 0$  for all  $x \in [0, 1]$ , and hence,  $g(t) = 0$  on  $[0, \infty)$ . Since  $0 = g'(t) = e^{-at} f(t)$  it follows now that  $f(t) = 0$  for all  $t \in [0, \infty)$ .

*Proof (Proof of Theorem 1).* Suppose  $f(t) = f_1(t) - f_2(t)$ . Then  $\mathcal{L}\{f\}(s) = 0$  for all  $s$ . By Theorem, 4 it follows that  $f$  is zero and so  $f_1 = f_2$ . □

## A.2 Polynomials and Rational Functions

A *polynomial of degree  $n$*  is a function of the form

$$p(s) = a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0,$$

where  $a_n \neq 0$ . The *coefficients*  $a_0, \dots, a_n$  may be real or complex. We refer to  $a_n$  as the *leading coefficient*. If the coefficients are all real, we say  $p(s)$  is a *real polynomial*. The variable  $s$  may also be real or complex. A *root* of  $p(s)$  is a scalar  $r$  such that  $p(r) = 0$ . Again  $r$  may be real or complex. If  $r$  is a root of  $p(s)$ , then there is another polynomial  $p_1(s)$  of degree  $n - 1$  such that

$$p(s) = (s - r)p_1(s).$$

The polynomial  $p_1(s)$  may be obtained by the standard procedure of division of polynomials.

Even though the coefficients may be real, the polynomial  $p(s)$  may only have nonreal complex roots. For example,  $s^2 + 1$  only has  $i$  and  $-i$  as roots. Notice in this example that the roots are complex conjugates. This always happens with real polynomials.

**Proposition 1.** *Suppose  $p(s)$  is a real polynomial and  $r \in \mathbb{C}$  is a root. Then  $\bar{r}$  is also a root.*

*Proof.* Suppose  $p(s) = a_n s^n + \dots + a_1 s + a_0$ , with each coefficient in  $\mathbb{R}$ . We are given that  $p(r) = 0$  from which follows

$$\begin{aligned}
 p(\bar{r}) &= a_n \bar{r}^n + \cdots + a_1 \bar{r} + a_0 \\
 &= \overline{a_n r^n} + \cdots + \overline{a_1 r} + \overline{a_0} \\
 &= \overline{a_n r^n + \cdots + a_1 r + a_0} \\
 &= \overline{p(r)} = \overline{0} = 0.
 \end{aligned}$$

Thus  $\bar{r}$  is also a root. □

The fundamental theorem of algebra addresses the question of whether a polynomial has a root.

**Theorem 2 (Fundamental Theorem of Algebra).** *Let  $p(s)$  be a polynomial of degree greater than 0. Then  $p(s)$  has a root  $r \in \mathbb{C}$ .*

The following corollary follows immediately from the fundamental theorem of algebra.

**Corollary 3.** *Let  $p(s)$  be a polynomial of degree  $n$  and  $n \geq 1$ . Then there are roots  $r_1, \dots, r_n \in \mathbb{C}$  such that*

$$p(s) = a_n(s - r_1) \cdots (s - r_n),$$

where  $a_n$  is the leading coefficient of  $p(s)$ .

Each term of the form  $s - r$  is called a **linear term**: if  $r \in \mathbb{R}$  it is a real linear term, and if  $r \in \mathbb{C}$ , it is a complex linear term. An **irreducible quadratic** is a real polynomial  $p(s)$  of degree 2 that has no real roots. In this case, we may write  $p(s) = as^2 + bs + c$  as a sum of squares by a procedure called **completing the square**:

$$\begin{aligned}
 p(s) &= as^2 + bs + c \\
 &= a \left( s^2 + \frac{b}{a}s + \frac{c}{a} \right) \\
 &= a \left( s^2 + \frac{b}{a}s + \frac{b^2}{4a^2} + \frac{4ca - b^2}{4a^2} \right) \\
 &= a \left( \left( s + \frac{b}{2a} \right) + \left( \frac{\sqrt{4ca - b^2}}{2a} \right)^2 \right) \\
 &= a \left( (s - \alpha)^2 + \beta^2 \right),
 \end{aligned}$$

where we set  $\alpha = -\frac{b}{2a}$  and  $\beta = \frac{\sqrt{4ca - b^2}}{2a}$ . From this form, we may read off the complex roots  $r = \alpha + i\beta$  and  $\bar{r} = \alpha - i\beta$ . We further observe that  $as^2 + bs + c = a(s - (\alpha + i\beta))(s - (\alpha - i\beta))$ .

**Corollary 4.** *If  $p(s)$  is a real polynomial, then  $p(s)$  is a product of real linear terms or irreducible quadratics.*

*Proof.* By fundamental theorem of algebra,  $p(s)$  is a product of linear factors of the form  $s - r$ . By Proposition 1, we have for each nonreal linear factor  $s - r$  a corresponding nonreal factor  $s - \bar{r}$  of  $p(s)$ . As observed above,  $(s - r)(s - \bar{r})$  is an irreducible quadratic. It follows then that  $p(s)$  is a product of real linear terms and irreducible quadratics.

**Corollary 5.** *Suppose  $p(s)$  is a polynomial of degree  $n$  and has  $m > n$  roots. Then  $p(s) = 0$  for all  $s \in \mathbb{R}$ .*

*Proof.* This is an immediate consequence of Corollary 3. □

**Corollary 6.** *Suppose  $p_1(s)$  and  $p_2(s)$  are polynomials and equal for all  $s > A$ , for some real number  $A$ . Then  $p_1(s) = p_2(s)$ , for all  $s \in \mathbb{R}$ .*

*Proof.* The polynomial  $p_1(s) - p_2(s)$  has infinitely many roots so must be zero, identically. Hence,  $p_1(s) = p_2(s)$  for all  $s \in \mathbb{R}$ . □

A **rational function** is a quotient of two polynomials, that is, it takes the form  $\frac{p(s)}{q(s)}$ . A rational function is **proper** if the degree of the numerator is less than the degree of the denominator.

**Corollary 7.** *Suppose  $\frac{p_1(s)}{q_1(s)}$  and  $\frac{p_2(s)}{q_2(s)}$  are rational functions that are equal for all  $s > A$  for some real number  $A$ . Then they are equal for all  $s$  such that  $q_1(s)q_2(s) \neq 0$ .*

*Proof.* Suppose

$$\frac{p_1(s)}{q_1(s)} = \frac{p_2(s)}{q_2(s)}$$

for all  $s > A$ . Then

$$p_1(s)q_2(s) = p_2(s)q_1(s),$$

for all  $s > A$ . Since both sides are polynomials, this implies that  $p_1(s)q_2(s) = p_2(s)q_1(s)$  for all  $s \in \mathbb{R}$ . Dividing by  $q_1(s)q_2(s)$  gives the result. □

### A.3 $\mathcal{B}_q$ Is Linearly Independent and Spans $\mathcal{E}_q$

#### $\mathcal{B}_q$ Spans $\mathcal{E}_q$

This subsection is devoted to a detailed proof of Theorem 2 of Sect. 2.7. To begin, we will need a few helpful lemmas.

**Lemma 1.** *Suppose  $q(s)$  is a polynomial which factors in the following way:  $q(s) = q_1(s)q_2(s)$ . Then*

$$\mathcal{B}_{q_1} \subset \mathcal{B}_q,$$

$$\mathcal{R}_{q_1} \subset \mathcal{R}_q,$$

$$\mathcal{E}_{q_1} \subset \mathcal{E}_q.$$

(Of course, the same inclusions for  $q_2$  are valid.)

*Proof.* Since any irreducible factor (linear or quadratic) of  $q_1$  is a factor of  $q$ , it follows by the way  $\mathcal{B}_q$  is defined using linear and irreducible quadratic factors of  $q(s)$  that  $\mathcal{B}_{q_1} \subset \mathcal{B}_q$ . Suppose  $p_1(s)/q_1(s) \in \mathcal{R}_{q_1}$ . Then  $p_1(s)/q_1(s) = p_1(s)q_2(s)/q_1(s)q_2(s) = p_1(s)q_2(s)/q(s) \in \mathcal{R}_q$ . It follows that  $\mathcal{R}_{q_1} \subset \mathcal{R}_q$ . Finally, if  $f \in \mathcal{E}_{q_1}$ , then  $\mathcal{L}\{f\} \in \mathcal{R}_{q_1} \subset \mathcal{R}_q$ . Hence,  $f \in \mathcal{E}_q$  and therefore  $\mathcal{E}_{q_1} \subset \mathcal{E}_q$ .  $\square$

**Lemma 2.** *Let  $q(s)$  be a polynomial of degree  $n \geq 1$ . Then*

$$\mathcal{B}_q \subset \mathcal{E}_q.$$

*Proof.* We proceed by induction on the degree of  $q(s)$ . If the degree of  $q(s) = 1$ , then we can write  $q(s) = a(s - \lambda)$ , and in this case,  $\mathcal{B}_q = \{e^{\lambda t}\}$ . Since

$$\mathcal{L}\{e^{\lambda t}\} = \frac{1}{s - \lambda} = \frac{a}{a(s - \lambda)} = \frac{a}{q(s)} \in \mathcal{R}_q$$

it follows that  $e^{\lambda t} \in \mathcal{E}_q$ . Hence,  $\mathcal{B}_q \subset \mathcal{E}_q$ . Now suppose  $\deg q(s) > 1$ . According to the fundamental theorem of algebra  $q(s)$  must have a linear or irreducible quadratic factor. Thus,  $q(s)$  factors in one of the following ways:

1.  $q(s) = (s - \lambda)^k q_1(s)$ , where  $k \geq 1$  and  $q_1(s)$  does not contain  $s - \lambda$  as a factor.
2.  $q(s) = ((s - \alpha)^2 + \beta^2)^k q_1(s)$ , where  $k \geq 1$  and  $q_1(s)$  does not contain  $(s - \alpha)^2 + \beta^2$  as a factor.

Since the degree of  $q_1$  is less than the degree of  $q$ , we have in both cases by induction that  $\mathcal{B}_{q_1} \subset \mathcal{E}_{q_1}$ . Lemma 1 implies that  $\mathcal{B}_{q_1} \subset \mathcal{E}_q$ .

*Case 1:*  $q(s) = (s - \lambda)^k q_1(s)$ . Let  $f \in \mathcal{B}_q$  be a simple exponential polynomial. Then either  $f(t) = t^r e^{\lambda t}$ , for some nonnegative integer  $r$  less than  $k$ , or  $f \in \mathcal{B}_{q_1} \subset \mathcal{E}_q$ . If  $f(t) = t^r e^{\lambda t}$  then,  $\mathcal{L}\{f\} = r!/(s - \lambda)^{r+1} \in \mathcal{R}_{(s-\lambda)^{r+1}} \subset \mathcal{R}_q$  by Lemma 1. Thus,  $f \in \mathcal{E}_q$ , and hence  $\mathcal{B}_q \subset \mathcal{E}_q$ .

*Case 2:*  $q(s) = ((s - \alpha)^2 + \beta^2)^k q_1(s)$ . Let  $f \in \mathcal{B}_q$ . Then either  $f(t) = t^r e^{\alpha t} \text{trig } \beta t$ , where  $\text{trig}$  is  $\sin$  or  $\cos$  and  $r$  is a nonnegative integer less than  $k$ , or  $f \in \mathcal{B}_{q_1} \subset \mathcal{E}_q$ . If  $f(t) = t^r e^{\alpha t} \text{trig } \beta t$ , then by Lemma 10 of Sect. 2.6,  $\mathcal{L}\{f(t)\} \in \mathcal{R}_{((s-\alpha)^2 + \beta^2)^k}$  and hence, by Lemma 1,  $\mathcal{L}\{f(t)\} \in \mathcal{R}_q$ . It follows that  $f \in \mathcal{E}_q$ . Hence  $\mathcal{B}_q \subset \mathcal{E}_q$ .  $\square$

*Proof (of Theorem 2 of Sect. 2.7).* Since  $\mathcal{E}_q$  is a linear space by Proposition 2 of Sect. 2.6 and  $\mathcal{B}_q \subset \mathcal{E}_q$  by Lemma 2, we have

$$\text{Span } \mathcal{B}_q \subset \mathcal{E}_q.$$

To show  $\mathcal{E}_q \subset \text{Span } \mathcal{B}_q$ , we proceed by induction on the degree of  $q$ . Suppose  $\deg q(s) = 1$ . Then  $q(s)$  may be written  $q(s) = a(s - \lambda)$ , for some constants  $a \neq 0$  and  $\lambda \in \mathbb{R}$  and  $\mathcal{B}_q = \{e^{\lambda t}\}$ . If  $f \in \mathcal{E}_q$ , then  $\mathcal{L}\{f\} = c/(a(s - \lambda))$ , for some constant  $c$ , and hence,  $f = (c/a)e^{\lambda t}$ . So  $f \in \text{Span } \mathcal{B}_q$  and it follows that  $\mathcal{E}_q = \text{Span } \mathcal{B}_q$ .

Now suppose  $\deg q > 1$ . Then  $q(s)$  factors in one of the following ways:

1.  $q(s) = (s - \lambda)^k q_1(s)$ , where  $k \geq 1$  and  $q_1(s)$  does not contain  $s - \lambda$  as a factor.
2.  $q(s) = ((s - \alpha)^2 + \beta^2)^k q_1(s)$ , where  $k \geq 1$  and  $q_1(s)$  does not contain  $(s - \alpha)^2 + \beta^2$  as a factor.

Since the degree of  $q_1$ , is less than the degree of  $q$  we have in both cases by induction that  $\mathcal{E}_{q_1} = \text{Span } \mathcal{B}_{q_1}$ .

Consider the first case:  $q(s) = (s - \lambda)^k q_1(s)$ . If  $f \in \mathcal{E}_q$ , then  $\mathcal{L}\{f\}$  has a partial fraction decomposition of the form

$$\mathcal{L}\{f\}(s) = \frac{A_1}{s - \lambda} + \cdots + \frac{A_k}{(s - \lambda)^k} + \frac{p_1(s)}{q_1(s)},$$

for some constants  $A_1, \dots, A_k$  and polynomial  $p_1(s)$ . Taking the inverse Laplace transform, it follows that  $f$  is a linear combination of terms in  $\{e^{\lambda t}, \dots, t^{k-1}e^{\lambda t}\}$  and a function in  $\mathcal{E}_{q_1}$ . But since  $\mathcal{E}_{q_1} = \text{Span } \mathcal{B}_{q_1}$  and  $\mathcal{B}_q = \{e^{\lambda t}, \dots, t^{k-1}e^{\lambda t}\} \cup \mathcal{B}_{q_1}$ , it follows that  $f \in \text{Span } \mathcal{B}_q$  and hence  $\mathcal{E}_q = \text{Span } \mathcal{B}_q$ .

Now consider the second case:  $q(s) = ((s - \alpha)^2 + \beta^2)^k q_1(s)$ . If  $f \in \mathcal{E}_q$ , then  $\mathcal{L}\{f\}$  has a partial fraction decomposition of the form

$$\mathcal{L}\{f\}(s) = \frac{A_1 s + B_1}{(s - \alpha)^2 + \beta^2} + \cdots + \frac{A_k s + B_k}{((s - \alpha)^2 + \beta^2)^k} + \frac{p_1(s)}{q_1(s)},$$

for some constants  $A_1, \dots, A_k, B_1, \dots, B_k$  and polynomial  $p_1(s)$ . Taking the inverse Laplace transform, it follows from Corollary 11 of Sect. 2.5 that  $f$  is a linear combination of terms in

$$\{e^{\alpha t} \cos \beta t, e^{\alpha t} \sin \beta t, \dots, t^{k-1}e^{\alpha t} \cos \beta t, t^{k-1}e^{\alpha t} \sin \beta t\}$$

and a function in  $\mathcal{E}_{q_1}$ . But since  $\mathcal{E}_{q_1} = \text{Span } \mathcal{B}_{q_1}$  and

$$\mathcal{B}_q = \{e^{\alpha t} \cos \beta t, e^{\alpha t} \sin \beta t, \dots, t^{k-1}e^{\alpha t} \cos \beta t, t^{k-1}e^{\alpha t} \sin \beta t\} \cup \mathcal{B}_{q_1}$$

it follows that  $f \in \text{Span } \mathcal{B}_q$  and hence  $\mathcal{E}_q = \text{Span } \mathcal{B}_q$ . □

### The Linear Independence of $\mathcal{B}_q$

Let  $q(s)$  be a nonconstant polynomial. This subsection is devoted to showing that  $\mathcal{B}_q$  is linearly independent. To that end, we derive a sequence of lemmas that will help us reach this conclusion.

**Lemma 3.** *Suppose  $q(s) = (s - \lambda)^m q_1(s)$  where  $(s - \lambda)$  is not a factor of  $q_1(s)$ . Suppose  $f_1 \in \mathcal{E}_{(s-\lambda)^m}$ ,  $f_2 \in \mathcal{E}_{q_1}$ , and*

$$f_1 + f_2 = 0.$$

Then  $f_1 = 0$  and  $f_2 = 0$ .

*Proof.* Let  $F_1(s) = \mathcal{L}\{f_1\}(s)$  and  $F_2(s) = \mathcal{L}\{f_2\}(s)$ . Then  $F_1(s) = \frac{p(s)}{(s-\lambda)^r}$ , for some  $r \leq m$  and some polynomial  $p(s)$  with no factor of  $s - \lambda$ . Similarly,  $F_2(s) = \frac{p_1(s)}{q_1(s)}$ , for some polynomial  $p_1(s)$ . Thus,

$$\frac{p(s)}{(s-\lambda)^r} + \frac{p_1(s)}{q_1(s)} = 0. \quad (1)$$

By Corollary 7 of Appendix A.2, equation (1) holds for all  $s$  different from  $\lambda$  and the roots of  $q_1(s)$ . Since  $q_1(s)$  contains no factor of  $s - \lambda$ , it follows that  $q_1(\lambda) \neq 0$ . Consider the limit as  $s$  approaches  $\lambda$  in (1). The second term approaches the finite value  $p_1(\lambda)/q_1(\lambda)$ . Since the sum is 0, the limit of the left side is 0 and this implies that the limit of the first term is finite as well. But this can only happen if  $p(s) = 0$ . It follows that  $F_1(s) = 0$  and hence  $f_1 = 0$ . This in turn implies that  $f_2 = 0$ .  $\square$

**Lemma 4.** *Suppose  $q(s) = ((s - \alpha)^2 + \beta^2)^m q_1(s)$  where  $(s - \alpha)^2 + \beta^2$  is not a factor of  $q_1(s)$ . Suppose  $f_1 \in \mathcal{E}_{((s-\alpha)+\beta^2)^m}$ ,  $f_2 \in \mathcal{E}_{q_1}$ , and*

$$f_1 + f_2 = 0.$$

Then  $f_1 = 0$  and  $f_2 = 0$ .

*Proof.* Let  $F_1(s) = \mathcal{L}\{f_1\}(s)$  and  $F_2(s) = \mathcal{L}\{f_2\}(s)$ . Then  $F_1(s) = \frac{p(s)}{((s-\alpha)^2 + \beta^2)^r}$ , for some  $r \leq m$  and some polynomial  $p(s)$  with no factor of  $(s - \alpha)^2 + \beta^2$ . Similarly,  $F_2(s) = \frac{p_1(s)}{q_1(s)}$ , for some polynomial  $p_1(s)$ . Thus,

$$\frac{p(s)}{((s-\alpha)^2 + \beta^2)^m} + \frac{p_1(s)}{q_1(s)} = 0. \quad (2)$$

Again, by Corollary 7 of Appendix A.2, equation (2) holds for all  $s$  different from  $\alpha \pm i\beta$  and the roots of  $q_1(s)$ . Since  $q_1(s)$  contains no factor of  $(s - \alpha)^2 + \beta^2$  it follows that  $q_1(\alpha + i\beta) \neq 0$ . Consider the limit as  $s$  approaches the complex number  $\alpha + i\beta$  in (2). The second term approaches a finite value. Since the sum is 0, the limit of the left side is 0 and this implies that the limit of the first term is finite

as well. But this can only happen if  $p(s) = 0$ . It follows that  $F_1(s) = 0$  and hence  $f_1 = 0$ . This in turn implies that  $f_2 = 0$ .  $\square$

**Lemma 5.** *Let  $q(s) = (s - \lambda)^m$ . Then  $\mathcal{B}_q$  is linearly independent.*

*Proof.* By definition,  $\mathcal{B}_q = \{e^{\lambda t}, te^{\lambda t}, \dots, t^{m-1}e^{\lambda t}\}$ . Suppose

$$c_0e^{\lambda t} + \dots + c_{m-1}t^{m-1}e^{\lambda t} = 0.$$

Dividing both sides by  $e^{\lambda t}$  gives

$$c_0 + c_1t + \dots + c_{m-1}t^{m-1} = 0.$$

Evaluating the at  $t = 0$  gives  $c_0 = 0$ . Taking the derivative of both sides and evaluating at  $t = 0$  gives  $c_1 = 0$ . In general, taking the  $r$ th derivative of both sides and evaluating at  $t = 0$  gives  $c_r = 0$ , for  $r = 1, 2, \dots, m - 1$ . It follows that all of the coefficients are necessarily 0, and this implies linear independence.  $\square$

**Lemma 6.** *Let  $q(s) = ((s - \alpha)^2 + \beta^2)^m$ . Then  $\mathcal{B}_q$  is linearly independent.*

*Proof.* By definition,  $\mathcal{B}_q = \{e^{\alpha t} \cos \beta t, e^{\alpha t} \sin \beta t, \dots, t^{m-1}e^{\alpha t} \cos \beta t, t^{m-1}e^{\alpha t} \sin \beta t\}$ . A linear combination of  $\mathcal{B}_q$  set to 0 takes the form

$$p_1(t)e^{\alpha t} \cos \beta t + p_2(t)e^{\alpha t} \sin \beta t = 0, \quad (3)$$

where  $p_1$  and  $p_2$  are polynomials carrying the coefficients of the linear combination. Thus, it is enough to show  $p_1$  and  $p_2$  are 0. Dividing both sides of Equation (3) by  $e^{\alpha t}$  gives

$$p_1(t) \cos \beta t + p_2(t) \sin \beta t = 0. \quad (4)$$

Let  $m$  be an integer. Evaluating at  $t = \frac{2\pi m}{\beta}$  gives  $p_1\left(\frac{2\pi m}{\beta}\right) = 0$ , since  $\sin 2\pi m = 0$  and  $\cos 2\pi m = 1$ . It follows that  $p_1$  has infinitely many roots. By Corollary 5, of Sect. A.2 this implies  $p_1 = 0$ . In a similar way,  $p_2 = 0$ .  $\square$

**Theorem 7.** *Let  $q$  be a nonconstant polynomial. View  $\mathcal{B}_q$  as a set of functions on  $I = [0, \infty)$ . Then  $\mathcal{B}_q$  is linearly independent.*

*Proof.* Let  $k$  be the number of roots of  $q$ . Our proof is by induction on  $k$ . If  $q$  has but one root, then  $q(s) = (s - \lambda)^n$  and this case is taken care of by Lemma 5. Suppose now that  $k > 1$ . We consider two cases. If  $q(s) = (s - \lambda)^m q_1(s)$ , where  $q_1(s)$  has no factor of  $s - \lambda$ , then a linear combination of functions in  $\mathcal{B}_q$  has the form  $f_1(t) + f_2(t)$ , where  $f_1(t) = c_0e^{\lambda t} + \dots + c_{m-1}t^{m-1}e^{\lambda t}$  is a linear combination of functions in  $\mathcal{B}_{(s-\lambda)^m}$  and  $f_2(t)$  is a linear combination of functions in  $\mathcal{B}_{q_1}$ . Suppose

$$f_1(t) + f_2(t) = 0.$$

By Lemma 3,  $f_1$  is identically zero and by Lemma 5, the coefficients  $c_0, \dots, c_{m-1}$  are all zero. It follows that  $f_2 = 0$ . Since  $q_1(s)$  has one fewer root than  $q$ , it follows

by induction the coefficients of the functions in  $\mathcal{B}_{q_1}$  that make up  $f_2$  are all zero. It follows that  $\mathcal{B}_q$  is linearly independent.

Now suppose  $q(s) = ((s - \alpha)^2 + \beta^2)^m q_1(s)$ , where  $q_1(s)$  has no factor of  $(s - \alpha)^2 + \beta^2$ . Then a linear combination of functions in  $\mathcal{B}_q$  has the form  $f_1(t) + f_2(t)$ , where  $f_1(t) = c_0 e^{\alpha t} \cos \beta t + \cdots + c_{m-1} t^{m-1} e^{\alpha t} \cos \beta t + d_0 e^{\alpha t} \sin \beta t + \cdots + d_{m-1} t^{m-1} e^{\alpha t} \sin \beta t$  is a linear combination of functions in  $\mathcal{B}_{((s-\alpha)^2+\beta)^m}$  and  $f_2(t)$  is a linear combination of functions in  $\mathcal{B}_{q_1}$ . Suppose

$$f_1(t) + f_2(t) = 0.$$

By Lemma 4  $f_1$  is identically zero, and by Lemma 6 the coefficients  $c_0, \dots, c_{m-1}$  and  $d_0, \dots, d_{m-1}$  are all zero. It follows that  $f_2 = 0$ . By induction, the coefficients of the functions in  $\mathcal{B}_{q_1}$  that make up  $f_2$  are all zero. It follows that  $\mathcal{B}_q$  is linearly independent.  $\square$

## A.4 The Matrix Exponential

In this section, we verify some properties of the matrix exponential that have been used in Chap. 9.

First, we argue that the matrix exponential

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots + \frac{A^n}{n!} + \cdots$$

converges absolutely for all  $n \times n$  matrices  $A$ . By this we mean that each entry of  $e^A$  converges absolutely. For convenience, let  $E = e^A$ . Let  $a = \max\{|A_{i,j}| : 1 \leq i, j \leq n\}$ . If  $a = 0$ , then  $A = 0$  and the series that defines  $e^A$  reduces to  $I$ , the  $n \times n$  identity. We will thus assume  $a > 0$ . Consider the  $(i, j)$  entry of  $A^2$ . We have

$$\begin{aligned} |(A^2)_{i,j}| &= \left| \sum_{l=1}^n A_{i,l} A_{l,j} \right| \\ &\leq \sum_{l=1}^n |A_{i,l}| |A_{l,j}| \\ &\leq \sum_{l=1}^n a^2 = na^2. \end{aligned}$$

In a similar way, the  $(i, j)$  entry of  $A^3$  satisfies

$$\begin{aligned} |(A^3)_{i,j}| &= \left| \sum_{l=1}^n A_{i,l}(A^2)_{l,j} \right| \\ &\leq \sum_{l=1}^n |A_{i,l}| |(A^2)_{l,j}| \\ &\leq \sum_{l=1}^n ana^2 = n^2a^3. \end{aligned}$$

By induction, we have

$$|(A^k)_{i,j}| \leq n^{k-1}a^k.$$

It follows from this estimate that

$$\begin{aligned} |E_{i,j}| &= \left| (I)_{i,j} + (A)_{i,j} + \frac{(A^2)_{i,j}}{2!} + \frac{(A^3)_{i,j}}{3!} \dots \right| \\ &\leq \delta_{i,j} + |(A)_{i,j}| + \frac{|(A^2)_{i,j}|}{2!} + \frac{|(A^3)_{i,j}|}{3!} \dots \\ &\leq \delta_{i,j} + a + \frac{na^2}{2!} + \frac{n^2a^3}{3!} + \frac{n^3a^4}{4!} + \dots \\ &= \delta_{i,j} + \sum_{k=1}^{\infty} \frac{n^{k-1}a^k}{k!} \\ &\leq 1 + \frac{1}{n} \sum_{k=0}^{\infty} \frac{(na)^k}{k!} = 1 + \frac{1}{n}e^{an}. \end{aligned}$$

It follows now that the series for  $E_{i,j}$  converges absolutely.

If we replace  $A$  by  $At$ , we get that each entry of  $e^{At}$  is an absolutely convergent power series in  $t$  with infinite radius of convergence. Furthermore, the last inequality above shows that the  $(i, j)$  entry of  $e^{At}$  is bounded by  $1 + \frac{1}{n}e^{ant}$  and hence of exponential type.

## A.5 The Cayley–Hamilton Theorem

**Theorem 1 (Cayley–Hamilton).** *Let  $A$  be an  $n \times n$  matrix and  $c_A(s)$  its characteristic polynomial. Then*

$$c_A(A) = 0.$$

The following lemma will be helpful for the proof.

**Lemma 2.** *Let  $j$  be a nonnegative integer and let  $a \in \mathbb{C}$ . Then*

$$\mathbf{D}^l (t^j e^{at})|_{t=0} = \mathbf{D}^j t^l|_{t=a}.$$

*Proof.* The derivative formula  $\mathbf{D}^l (ye^{at}) = (\mathbf{D} + a)^l y e^{at}$  implies

$$\begin{aligned} \mathbf{D}^l (t^j e^{at})|_{t=0} &= (\mathbf{D} + a)^l t^j|_{t=0} \\ &= \sum_{k=0}^l \binom{l}{k} a^{l-k} (\mathbf{D}^k t^j)|_{t=0} \\ &= \begin{cases} 0 & \text{if } l < j \\ \frac{a^{l-j} l!}{(l-j)!} & \text{if } l \geq j \end{cases} \\ &= \mathbf{D}^j t^l|_{t=a}. \quad \square \end{aligned}$$

*Proof (of Cayley-Hamilton Theorem).* Let  $c_A(s)$  be the characteristic polynomial of  $A$ . Suppose  $\lambda_1, \dots, \lambda_m$  are the roots of  $c_A$  with corresponding multiplicities  $r_1, \dots, r_m$ . Then

$$\mathcal{B}_{c_A} = \{t^j e^{\lambda_k t} : j = 0, \dots, r_k - 1, k = 1, \dots, m\}$$

is the standard basis for  $\mathcal{E}_{c_A}$ . As in Fulmer’s method, Sect. 9.4, we may assume that there are  $n \times n$  matrices  $M_{j,k}$ ,  $j = 0, \dots, r_k - 1, k = 1, \dots, m$ , so that

$$e^{At} = \sum_{k=1}^m \sum_{j=0}^{r_k-1} t^j e^{\lambda_k t} M_{j,k}.$$

Differentiating both sides  $l$  times and evaluating at  $t = 0$  gives

$$A^l = \sum_{k=1}^m \sum_{j=0}^{r_k-1} \mathbf{D}^l (t^j e^{\lambda_k t})|_{t=0} M_{j,k} = \sum_{k=1}^m \sum_{j=0}^{r_k-1} \mathbf{D}^j t^l|_{t=\lambda_k} M_{j,k},$$

with the second equality coming from Lemma 2. Now let  $p(s) = c_0 + c_1 s + \dots + c_N s^N = \sum_{l=0}^N c_l s^l$  be any polynomial. Then

$$\begin{aligned} p(A) &= \sum_{l=0}^N c_l A^l \\ &= \sum_{l=0}^N \sum_{k=1}^m \sum_{j=0}^{r_k-1} c_l \mathbf{D}^j t^l|_{t=\lambda_k} M_{j,k} \end{aligned}$$

$$\begin{aligned}
&= \sum_{k=1}^m \sum_{j=0}^{r_k-1} \mathbf{D}^j \left( \sum_{l=0}^N c_l t^l \right) \Big|_{t=\lambda_k} \mathbf{M}_{j,k} \\
&= \sum_{k=1}^m \sum_{j=0}^{r_k-1} p^{(j)}(\lambda_k) \mathbf{M}_{j,k}.
\end{aligned} \tag{1}$$

For the characteristic polynomial, we have  $c_A(s) = \prod_{k=1}^m (s - \lambda_k)^{r_k}$ , and hence,  $c_A^{(j)}(\lambda_k) = 0$  for all  $j = 0, \dots, r_k - 1$  and  $k = 1, \dots, m$ . Now let  $p(s) = c_A(s)$  in Equation (1) to get

$$c_A(A) = \sum_{k=1}^m \sum_{j=0}^{r_k-1} c_A^{(j)}(\lambda_k) \mathbf{M}_{j,k} = 0.$$

□

# Appendix B

## Selected Answers

### Section 1.1

1.  $P' = kP$

3.  $h'(t) = \lambda \sqrt{h(t)}$

5. order: 2; Standard form:  $y'' = t^3/y'$ .

7. order: 2; Standard form:  $y'' = -(3y + ty')/t^2$ .

9. order: 4; standard form:  $y^{(4)} = \sqrt[3]{(1 - (y''')^4)/t}$ .

11. order: 3; standard form:  $y''' = 2y'' - 3y' + y$ .

13.  $y_1, y_2,$  and  $y_3$

15.  $y_1, y_2,$  and  $y_4$

17.  $y_1, y_2,$  and  $y_4$

19.

$$y'(t) = 3ce^{3t}$$
$$3y + 12 = 3(ce^{3t} - 4) + 12 = 3ce^{3t} - 12 + 12 = 3ce^{3t}.$$

Note that  $y(t)$  is defined for all  $t \in \mathbb{R}$ .

21.

$$y'(t) = \frac{ce^t}{(1 - ce^t)^2}$$

$$y^2(t) - y(t) = \frac{1}{(1 - ce^t)^2} - \frac{1}{1 - ce^t} = \frac{1 - (1 - ce^t)}{(1 - ce^t)^2} = \frac{ce^t}{(1 - ce^t)^2}.$$

If  $c \leq 0$ , then the denominator  $1 - ce^t > 0$  and  $y(t)$  has domain  $\mathbb{R}$ . If  $c > 0$ , then  $1 - ce^t = 0$  if  $t = \ln \frac{1}{c} = -\ln c$ . Thus,  $y(t)$  is defined either on the interval  $(-\infty, -\ln c)$  or  $(-\ln c, \infty)$ .

23.

$$y'(t) = \frac{-ce^t}{ce^t - 1}$$

$$-e^y - 1 = -e^{-\ln(ce^t - 1)} - 1 = \frac{-1}{ce^t - 1} - 1 = \frac{-ce^t}{ce^t - 1}.$$

25.

$$y'(t) = -(c - t)^{-2}(-1) = \frac{1}{(c - t)^2}$$

$$y^2(t) = \frac{1}{(c - t)^2}.$$

The denominator of  $y(t)$  is 0 when  $t = c$ . Thus, the two intervals where  $y(t)$  is defined are  $(-\infty, c)$  and  $(c, \infty)$ .

27.  $y(t) = \frac{e^{2t}}{2} - t + c.$

29.  $y(t) = t + \ln |t| + c$

31.  $y(t) = -\frac{2}{3} \sin 3t + c_1 t + c_2$

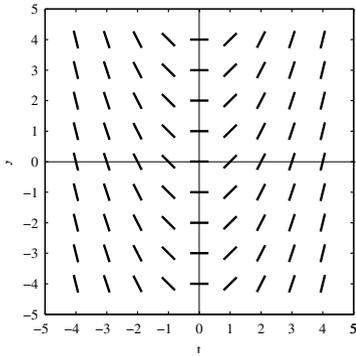
33.  $y(t) = 3e^{-t} + 3t - 3$

35.  $y(t) = -18(t + 1)^{-1}$

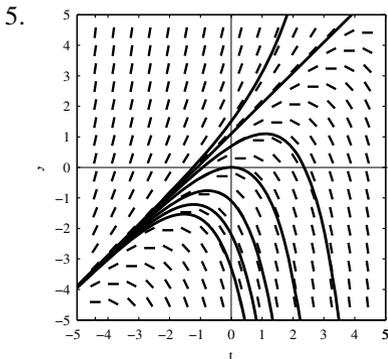
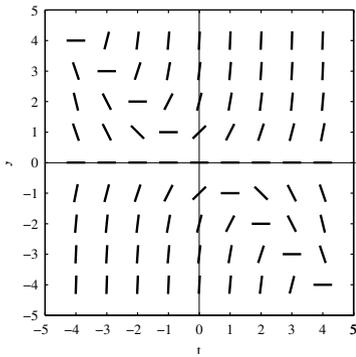
37.  $y(t) = -te^{-t} - e^{-t}.$

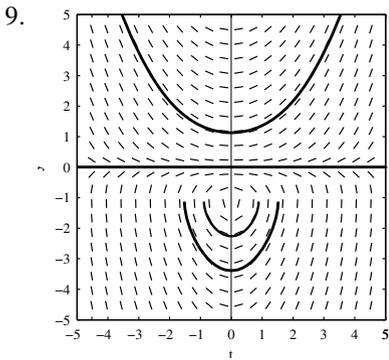
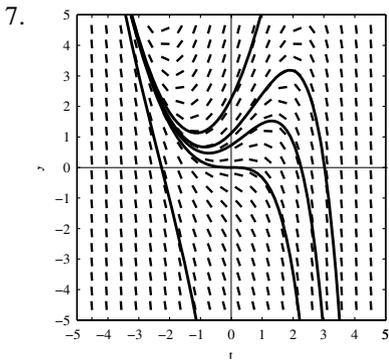
**Section 1.2**

1.  $y' = t$



3.  $y' = y(y + t)$





11.  $y = 0$

13.  $y = \pm 1$

15.  $y = -t + (2n + 1)\pi, n \in \mathbb{Z}$

17.  $2yy' - 2t - 3t^2 = 0$

19.  $y' = \frac{3y}{t} - t.$

### Section 1.3

1. Separable

3. Not separable

5. Separable

7. Not separable

## 9. Separable

11.  $t^2(1 - y^2) = c$ ,  $c$  a real constant

13.  $y^5 = \frac{5}{2}t^2 + 10t + c$ ,  $c$  a real constant

15.  $y = 1 - c \cos t$ ,  $c$  any real constant

17.  $y = \frac{4ce^{4t}}{1+ce^{4t}}$ ,  $c$  a real number, and  $y = 4$

19.  $y = \tan(t + c)$ ,  $c$  a real constant

21.  $\ln(y + 1)^2 - y = \tan^{-1} t + c$ ,  $c$  a real constant, and  $y = -1$

23.  $y = \frac{1}{\ln|1-t|+c}$ ,  $c$  real constant

25.  $y = 0$

27.  $y = 4e^{-t^2}$

29.  $y = 2\sqrt{u^2 + 1}$

31.  $y(t) = \tan\left(-\frac{1}{t} + 1 + \frac{\pi}{3}\right)$ ,  $(a, b) = (6/(6 + 5\pi), 6/(6 - \pi))$ .

33.  $\approx 212$  million years old

35.  $\approx 64$  min

37.  $t \approx 8.2$  min and  $T(20) \approx 67.7^\circ$

39.  $\approx 205.5^\circ$

41. 602

43.  $\approx 3.15$  years

45. 3857

47. 1,400

**Section 1.4**

1.  $y = \frac{1}{4}e^t - \frac{9}{4}e^{-3t}$ .

3.  $y = te^{2t} + 4e^{2t}$ .

5.  $y = \frac{e^t}{t} - \frac{e}{t}$

7.  $y = \frac{\sin(t^2)}{2t} + \frac{c}{t}$ .

9.  $y = 4 \sin 4t - 3 \cos 4t + ce^{3t}$

11.  $z = t^2 + 1 + ce^{t^2}$

13.  $y = 1 - e^{-\sin t}$

15.  $y = -t - \frac{1}{2} - \frac{3}{2}t^{-2}$ .

17.  $y = \frac{1}{a+b}e^{bt} + ce^{-at}$

19.  $y = (t + c) \sec t$

21.  $y = t^n e^t + ct^n$

23.  $y = \frac{1}{5}t^2 - \frac{9}{5}t^{-3}$

25.  $y(t) = \frac{1}{t} + (4a - 2)t^{-2}$

27.  $y = (10 - t) - \frac{8}{10^t}(10 - t)^4$ , for  $0 \leq t \leq 10$ . After 10 min, there is no salt in the tank.

29. (a) 10 min, (b)  $\approx 533.33$  g

31. (a) Differential equation:  $P'(t) + (r/V)P(t) = rc$ . If  $P_0$  denotes the initial amount of pollutant in the lake, then  $P(t) = Vc + (P_0 - Vc)e^{-(r/V)t}$ . The limiting concentration is  $c$ .

(b)  $t_{1/2} = (V/r) \ln 2$ ;  $t_{1/10} = (V/r) \ln 10$

(c) Lake Erie:  $t_{1/2} = 1.82$  years,  $t_{1/10} = 6.05$  years, Lake Ontario:  $t_{1/2} = 5.43$  years,  $t_{1/10} = 18.06$  years

33.  $y_2(t) = 10(5 + t) - \frac{500 \ln(5+t)}{5+t} + \frac{500 \ln 5 - 250}{5+t}$

## Section 1.5

1.  $y = t \tan(\ln |t| + \pi/4)$

3.  $y = 2t$

5.  $y = \pm t \sqrt{1 + kt}$ ,  $k \in \mathbb{R}$

7.  $y = t \sin(\ln |t| + c)$  and  $y = \pm t$

9.  $y = \frac{1}{1-t}$

$$11. y = \pm \frac{1}{\sqrt{1 + ce^{t^2}}} \text{ and } y = 0$$

$$13. y = \frac{1}{-5 + c\sqrt{1 - t^2}} \text{ and } y = 0.$$

$$15. y = -\sqrt{1 + 3e^{-t^2}}$$

$$17. y = \pm \frac{1}{\sqrt{t + \frac{1}{2} + ce^{2t}}} \text{ and } y = 0$$

$$19. -2y + \ln |2t - 2y| = c, c \in \mathbb{R} \text{ and } y = t$$

$$21. y - \tan^{-1}(t + y) = c, c \in \mathbb{R}$$

$$23. y = \pm \sqrt{ke^t - t}, k \in \mathbb{R}$$

$$25. y = e^{t-1+ce^{-t}}, c \in \mathbb{R}$$

## Section 1.6

$$1. t^2 + ty^2 = c$$

3. Not Exact

5. Not Exact

$$7. (y - t^2)^2 - 2t^4 = c$$

$$9. y^4 = 4ty + c$$

## Section 1.7

$$1. y(t) = 1 + \int_1^t uy(u) du$$

$$3. y(t) = 1 + \int_0^t \frac{u - y(u)}{u + y(u)} du$$

5.

$$y_0(t) = 1$$

$$y_1(t) = \frac{1 + t^2}{2}$$

$$y_2(t) = \frac{5}{8} + \frac{t^2}{4} + \frac{t^4}{8}$$

$$y_3(t) = \frac{29}{48} + \frac{5t^2}{16} + \frac{t^4}{16} + \frac{t^6}{48}$$

$$7. y_1(t) = \frac{t^2}{2}; y_2(t) = \frac{t^2}{2} + \frac{t^5}{20}; y_3(t) = \frac{t^2}{2} + \frac{t^5}{20} + \frac{t^8}{160} + \frac{t^{11}}{4400}$$

9.

$$y_0(t) = 0$$

$$y_1(t) = t + \frac{t^3}{3}$$

$$y_2(t) = t + \frac{t^7}{7 \cdot 3^2}$$

$$y_3(t) = t + \frac{t^{15}}{15 \cdot 7^2 \cdot 3^4}$$

$$y_4(t) = t + \frac{t^{31}}{31 \cdot 15^2 \cdot 7^4 \cdot 3^8}$$

$$y_5(t) = t + \frac{t^{63}}{63 \cdot 31^2 \cdot 15^4 \cdot 7^8 \cdot 3^{16}}$$

11. Not guaranteed unique

13. Unique solution

15.

$$y_0(t) = 1$$

$$y_1(t) = 1 + at$$

$$y_2(t) = 1 + at + \frac{a^2 t^2}{2}$$

$$y_3(t) = 1 + at + \frac{a^2 t^2}{2} + \frac{a^3 t^3}{3!}$$

$$\vdots$$

$$y_n(t) = 1 + at + \frac{a^2 t^2}{2} + \cdots + \frac{a^n t^n}{n!}.$$

$y(t) = \lim_{n \rightarrow \infty} y_n(t) = e^{at}$ ; it is a solution; there are no other solutions.

17. Yes

19. 1.  $y(t) = t + ct^2$ .

2. Every solution satisfies  $y(0) = 0$ . There is no contradiction to Theorem 5 since, in standard form, the equation is  $y' = \frac{2}{t}y - 1 = F(t, y)$  and  $F(t, y)$  is not continuous for  $t = 0$ .

21. No

## Section 2.1

1.  $Y(s) = \frac{2}{s-4}$  and  $y(t) = 2e^{4t}$

3.  $Y(s) = \frac{1}{(s-4)^2}$  and  $y(t) = te^{4t}$

5.  $Y(s) = \frac{1}{s+2} + \frac{1}{s-1}$  and  $y(t) = e^{-2t} + e^t$

7.  $Y(s) = \frac{3s+3}{s^2+3s+2} = \frac{3(s+1)}{(s+1)(s+2)} = \frac{3}{s+2}$  and  $y(t) = 3e^{-2t}$

9.  $Y(s) = \frac{s-1}{s^2+25}$  and  $y(t) = \frac{-1}{5} \sin 5t + \cos 5t$

11.  $Y(s) = \frac{1}{s+4}$  and  $y(t) = e^{-4t}$

13.  $Y(s) = \frac{1}{(s+2)^2} + \frac{1}{(s+2)^3} = \frac{1}{(s+2)^2} + \frac{1}{2} \frac{2}{(s+2)^3}$  and  $y(t) = te^{-2t} + \frac{1}{2}t^2e^{-2t}$

## Section 2.2

Compute the Laplace transform of each function given below directly from the integral definition given in (1).

1.  $\frac{3}{s^2} + \frac{1}{s}$

3.  $\frac{1}{s-2} - \frac{3}{s+1}$

5.  $\frac{5}{s-2}$

7.  $\frac{2}{s^3} - \frac{5}{s^2} + \frac{4}{s}$

9.  $\frac{1}{s+3} + \frac{7}{(s+4)^2}$

11.  $\frac{s+2}{s^2+4}$

13.  $\frac{2}{(s+4)^3}$

15.  $\frac{2}{s^3} + \frac{2}{(s-2)^2} + \frac{1}{s-4}$

17.  $\frac{24}{(s+4)^5}$

19.  $\frac{1}{(s-3)^2}$

21.  $\frac{12s^2 - 16}{(s^2 + 4)^3}$

23.  $\frac{2s}{s^2+1} - \frac{2}{s}$

25.  $\frac{\ln(s+6) - \ln 6}{s}$

27.  $\frac{2b^2}{s(s^2+4b^2)}$

29.  $\frac{1}{2} \left( \frac{a-b}{s^2+(a-b)^2} + \frac{a+b}{s^2+(a+b)^2} \right)$

31.  $\frac{b}{s^2-b^2}$

33.  $\frac{b}{s^2-b^2}$

(a) Show that  $\Gamma(1) = 1$ .(b) Show that  $\Gamma$  satisfies the recursion formula  $\Gamma(\beta + 1) = \beta\Gamma(\beta)$ .*(Hint: Integrate by parts.)*(c) Show that  $\Gamma(n + 1) = n!$  when  $n$  is a nonnegative integer.

**Section 2.3**

<i>The s - 1 -chain</i>					
1.	<table border="1" style="width: 100%;"> <tr> <td style="text-align: center;"><math>\frac{5s + 10}{(s - 1)(s + 4)}</math></td> <td style="text-align: center;"><math>\frac{3}{s - 1}</math></td> </tr> <tr> <td style="text-align: center;"><math>\frac{2}{s + 4}</math></td> <td></td> </tr> </table>	$\frac{5s + 10}{(s - 1)(s + 4)}$	$\frac{3}{s - 1}$	$\frac{2}{s + 4}$	
$\frac{5s + 10}{(s - 1)(s + 4)}$	$\frac{3}{s - 1}$				
$\frac{2}{s + 4}$					

<i>The s - 5 -chain</i>					
3.	<table border="1" style="width: 100%;"> <tr> <td style="text-align: center;"><math>\frac{1}{(s + 2)(s - 5)}</math></td> <td style="text-align: center;"><math>\frac{1/7}{(s - 5)}</math></td> </tr> <tr> <td style="text-align: center;"><math>\frac{-1/7}{(s + 2)}</math></td> <td></td> </tr> </table>	$\frac{1}{(s + 2)(s - 5)}$	$\frac{1/7}{(s - 5)}$	$\frac{-1/7}{(s + 2)}$	
$\frac{1}{(s + 2)(s - 5)}$	$\frac{1/7}{(s - 5)}$				
$\frac{-1/7}{(s + 2)}$					

<i>The s - 1 -chain</i>					
5.	<table border="1" style="width: 100%;"> <tr> <td style="text-align: center;"><math>\frac{3s + 1}{(s - 1)(s^2 + 1)}</math></td> <td style="text-align: center;"><math>\frac{2}{s - 1}</math></td> </tr> <tr> <td style="text-align: center;"><math>\frac{-2s + 1}{s^2 + 1}</math></td> <td></td> </tr> </table>	$\frac{3s + 1}{(s - 1)(s^2 + 1)}$	$\frac{2}{s - 1}$	$\frac{-2s + 1}{s^2 + 1}$	
$\frac{3s + 1}{(s - 1)(s^2 + 1)}$	$\frac{2}{s - 1}$				
$\frac{-2s + 1}{s^2 + 1}$					

<i>The s + 3 -chain</i>									
7.	<table border="1" style="width: 100%;"> <tr> <td style="text-align: center;"><math>\frac{s^2 + s - 3}{(s + 3)^3}</math></td> <td style="text-align: center;"><math>\frac{3}{(s + 3)^3}</math></td> </tr> <tr> <td style="text-align: center;"><math>\frac{s - 2}{(s + 3)^2}</math></td> <td style="text-align: center;"><math>\frac{-5}{(s + 3)^2}</math></td> </tr> <tr> <td style="text-align: center;"><math>\frac{1}{s + 3}</math></td> <td style="text-align: center;"><math>\frac{1}{s + 3}</math></td> </tr> <tr> <td style="text-align: center;">0</td> <td></td> </tr> </table>	$\frac{s^2 + s - 3}{(s + 3)^3}$	$\frac{3}{(s + 3)^3}$	$\frac{s - 2}{(s + 3)^2}$	$\frac{-5}{(s + 3)^2}$	$\frac{1}{s + 3}$	$\frac{1}{s + 3}$	0	
$\frac{s^2 + s - 3}{(s + 3)^3}$	$\frac{3}{(s + 3)^3}$								
$\frac{s - 2}{(s + 3)^2}$	$\frac{-5}{(s + 3)^2}$								
$\frac{1}{s + 3}$	$\frac{1}{s + 3}$								
0									

<i>The <math>s + 1</math>-chain</i>		
9.	$\frac{s}{(s+2)^2(s+1)^2}$	$\frac{-1}{(s+1)^2}$
	$\frac{s+4}{(s+2)^2(s+1)}$	$\frac{3}{s+1}$
	$\frac{-3s-8}{(s+2)^2}$	

<i>The <math>s - 5</math>-chain</i>		
11.	$\frac{1}{(s-5)^5(s-6)}$	$\frac{-1}{(s-5)^5}$
	$\frac{1}{(s-5)^4(s-6)}$	$\frac{-1}{(s-5)^4}$
	$\frac{1}{(s-5)^3(s-6)}$	$\frac{-1}{(s-5)^3}$
	$\frac{1}{(s-5)^2(s-6)}$	$\frac{-1}{(s-5)^2}$
	$\frac{1}{(s-5)(s-6)}$	$\frac{-1}{s-5}$
	$\frac{1}{s-6}$	

13.  $\frac{13/8}{s-5} - \frac{5/8}{s+3}$

15.  $\frac{23}{12(s-5)} + \frac{37}{12(s+7)}$

17.  $\frac{25}{8(s-7)} - \frac{9}{8(s+1)}$

19.  $\frac{1}{2(s+5)} - \frac{1}{2(s-1)} + \frac{1}{s-2}$

21.  $\frac{7}{(s+4)^4}$

$$23. \frac{3}{(s+3)^3} - \frac{5}{(s+3)^2} + \frac{1}{s+3}$$

$$25. \frac{1}{54} \left( \frac{5}{s-5} + \frac{21}{(s+1)^2} + \frac{3}{(s-5)^2} - \frac{5}{s+1} \right)$$

$$27. \frac{-2}{(s+2)^2} - \frac{3}{s+2} - \frac{1}{(s+1)^2} + \frac{3}{s+1}$$

$$29. \frac{12}{(s-3)^3} + \frac{-14}{(s-3)^2} + \frac{15}{s-3} + \frac{-16}{s-2} + \frac{1}{s-1}$$

$$31. \frac{2}{(s-2)^2} + \frac{5}{s-2} + \frac{3}{(s-3)^2} - \frac{5}{s-3}$$

$$33. Y(s) = \frac{-3}{(s+1)^2} - \frac{1}{s+1} + \frac{1}{s-2} \text{ and } y(t) = -3te^{-t} - e^{-t} + e^{2t}$$

$$35. Y(s) = \frac{-30}{s^2} + \frac{24}{s} - \frac{26}{s+1} + \frac{1}{s-5} \text{ and } y(t) = -30t + 24 - 26e^{-t} + e^{5t}$$

$$37. Y(s) = \frac{2s-3}{(s-1)(s-2)} + \frac{4}{s(s-1)(s-2)} = \frac{2}{s} + \frac{3}{s-2} - \frac{3}{s-1} \text{ and } y(t) = 2 + 3e^{2t} - 3e^t$$

## Section 2.4

<i>The <math>s^2 + 1</math> -chain</i>	
1.	<div style="text-align: center;"> <math display="block">\frac{1}{(s^2 + 1)^2(s^2 + 2)}</math> <math display="block">\frac{-1}{(s^2 + 1)(s^2 + 2)}</math> <math display="block">\frac{1}{s^2 + 2}</math> </div> <div style="text-align: center;"> <math display="block">\frac{1}{(s^2 + 1)^2}</math> <math display="block">\frac{-1}{(s^2 + 1)}</math> </div>

<i>The <math>s^2 + 3</math> -chain</i>	
$\frac{8s + 8s^2}{(s+3)^3(s^2 + 1)}$	$\frac{12 - 4s}{(s^2 + 3)^3}$
$\frac{4(s - 1)}{(s^2 + 3)^2(s^2 + 1)}$	$\frac{2 - 2s}{(s^2 + 3)^2}$
$\frac{2(s - 1)}{(s^2 + 3)(s^2 + 1)}$	$\frac{1 - s}{s^2 + 3}$
$\frac{s - 1}{s^2 + 1}$	

3.

<i>The <math>s^2 + 2s + 2</math> -chain</i>	
$\frac{1}{(s^2 + 2s + 2)^2(s^2 + 2s + 3)^2}$	$\frac{1}{(s^2 + 2s + 2)^2}$
$\frac{-(s^2 + 2s + 4)}{(s^2 + 2s + 2)(s^2 + 2s + 3)^2}$	$\frac{-2}{s^2 + 2s + 2}$
$\frac{2s^2 + 4s + 7}{(s^2 + 2s + 3)^2}$	

5.

$$7. \frac{1}{10} \left( \frac{3}{s-3} + \frac{1-3s}{s^2+1} \right)$$

$$9. \frac{9s^2}{(s^2+4)^2(s^2+1)} = \frac{12}{(s^2+4)^2} + \frac{1}{s^2+4} - \frac{1}{s+1}$$

$$11. \frac{2}{s-3} + \frac{6-2s}{(s-3)^2+1}$$

$$13. \frac{-5s+15}{(s^2-4s+8)^2} + \frac{-s+3}{s^2-4s+8} + \frac{1}{s-1}$$

$$15. \frac{s+1}{(s^2+4s+6)^2} + \frac{2s+2}{s^2+4s+6} + \frac{s+1}{(s^2+4s+5)^2} - \frac{2s+2}{s^2+4s+5}$$

$$17. Y(s) = \frac{1}{(s+2)^2} + \frac{4s}{(s^2+4)(s+2)^2} = \frac{1}{s^2+4} \text{ and } y(t) = \frac{1}{2} \sin 2t$$

$$19. Y(s) = \frac{1}{s^2+4} + \frac{3}{(s^2+9)(s^2+4)} = \frac{8}{5} \frac{1}{s^2+4} - \frac{3}{5} \frac{1}{s^2+9} \text{ and } y(t) = \frac{4}{5} \sin 2t - \frac{1}{5} \sin 3t$$

**Section 2.5**

1.  $-5$
3.  $3t - 2t^2$
5.  $3 \cos 2t$
7.  $-11te^{-3t} + 2e^{-3t}$
9.  $e^{2t} - e^{-4t}$
11.  $\frac{-1}{6}t^3e^{2t} + \frac{3}{2}t^2e^{2t} + 2te^{2t}$
13.  $te^t + e^t + te^{-t} - e^{-t}$
15.  $4te^{2t} - e^{2t} + \cos 2t - \sin 2t$
17.  $3 - 6t + e^{2t} - 4e^{-t}$
19.  $2e^{-t} \cos 2t - e^{-t} \sin 2t$
21.  $e^{4t} \cos t + 3e^{4t} \sin t$
23.  $e^t \cos 3t$
25.  $2t \sin 2t$
27.  $2te^{-2t} \cos t + (t - 2)e^{-2t} \sin t$
29.  $4te^{-4t} \cos t + (t - 4)e^{-4t} \sin t$
31.  $\frac{1}{256} ((3 - 4t^2)e^t \sin 2t - 6te^t \cos 2t)$
33.  $\frac{1}{48} ((-t^3 + 3t)e^{4t} \sin t - 3t^2e^{4t} \cos t)$
35.  $y(t) = \cos t - \sin t + 2(\sin t - t \cos t) = \cos t + \sin t - 2t \cos t$
37.  $Y(s) = \frac{8s}{(s^2 + 1)^3}$  and  $y(t) = t \sin t - t^2 \cos t$

**Section 2.6**

1.  $\mathcal{B}_q = \{e^{4t}\}$

3.  $\mathcal{B}_q = \{1, e^{-5t}\}$

5.  $\mathcal{B}_q = \{e^{3t}, te^{3t}\}$

7.  $\mathcal{B}_q = \{e^{3t}, e^{-2t}\}$

9.  $\mathcal{B}_q = \{e^{t/2}, e^{4t/3}\}$

11.  $\mathcal{B}_q = \{e^{(2+\sqrt{3})t}, e^{(2-\sqrt{3})t}\}$

13.  $\mathcal{B}_q = \{e^{-3t/2}, te^{-3t/2}\}$

15.  $\mathcal{B}_q = \{\cos(5t/2), \sin(5t/2)\}$

17.  $\mathcal{B}_q = \{e^t \cos 2t, e^t \sin 2t\}$

19.  $\mathcal{B}_q = \{e^{-3t}, te^{-3t}, t^2e^{-3t}, t^3e^{-3t}\}$

21.  $\mathcal{B}_q = \{e^t, te^t, t^2e^t\}$

23.  $\mathcal{B}_q = \{e^{-2t} \cos t, e^{-2t} \sin t, te^{-2t} \cos t, te^{-2t} \sin t\}$

25.  $\mathcal{B}_q = \{\cos t, \sin t, t \cos t, t \sin t, t^2 \cos t, t^2 \sin t, t^3 \cos t, t^3 \sin t\}$

## Section 2.7

1. Yes

3. Yes

5. Yes

7. No

9. No

11. No

13.  $\mathcal{B}_q = \{e^t, e^{-t}, \cos t, \sin t\}$

15.  $\mathcal{B}_q = \{e^t, te^t, t^2e^t, e^{-7t}, te^{-7t}\}$

17.  $\mathcal{B}_q = \{e^{-2t}, te^{-2t}, t^2e^{-2t}, \cos 2t, \sin 2t, t \cos 2t, t \sin 2t\}$

19.  $\mathcal{B}_q = \{e^{2t}, te^{2t}, e^{-3t}, te^{-3t}, t^2e^{-3t}\}$ .

21.  $\mathcal{B}_q = \{e^{-4t}, te^{-4t}, e^{-3t} \cos 2t, e^{-3t} \sin 2t, te^{-3t} \cos 2t, te^{-3t} \sin 2t\}$

$$23. \mathcal{B}_q = \{e^{3t}, te^{3t}, t^2e^{3t}, e^{-t} \cos 3t, e^{-t} \sin 3t, te^{-t} \cos 3t, te^{-t} \sin 3t\}$$

$$25. \mathcal{B}_q = \{e^{t/2}, e^t, te^t\}$$

$$27. \mathcal{B}_q = \{\cos \sqrt{3}t, \sin \sqrt{3}t, \cos \sqrt{2}t, \sin \sqrt{2}t\}$$

## Section 2.8

$$1. \frac{t^3}{6}$$

$$3. 3(1 - \cos t)$$

$$5. \frac{1}{13}(2e^{3t} - 2 \cos 2t - 3 \sin 2t)$$

$$7. \frac{1}{108}(18t^2 - 6t - 6 - e^{-6t})$$

$$9. \frac{1}{6}(e^{2t} - e^{-4t})$$

$$11. \frac{1}{a^2+b^2}(be^{at} - b \cos bt - a \sin bt)$$

$$13. \begin{cases} \frac{b \sin at - a \sin bt}{b^2 - a^2} & \text{if } b \neq a \\ \frac{\sin at - at \cos at}{2a} & \text{if } b = a \end{cases}$$

$$15. \begin{cases} \frac{a \sin at - b \sin bt}{a^2 - b^2} & \text{if } b \neq a \\ \frac{1}{2a}(at \cos at + \sin at) & \text{if } b = a \end{cases}$$

$$17. F(s) = \frac{4}{s^3(s^2 + 4)}$$

$$19. F(s) = \frac{6}{s^4(s + 3)}$$

$$21. F(s) = \frac{4}{(s^2 + 4)^2}$$

$$23. \frac{1}{4}(-e^t + e^{5t})$$

$$25. \frac{1}{2}t \sin t$$

$$27. \frac{1}{13}(2e^{3t} - 2 \cos 2t - 3 \sin 2t)$$

$$29. \frac{e^{at} - e^{bt}}{a - b}$$

$$31. \int_0^t g(x) \cos \sqrt{2}(t - x) dx$$

$$33. t - \sin t$$

$$35. \frac{1}{54}(2 - 6t + 9t^2 - 2e^{-3t})$$

$$37. \frac{1}{162}(2 - 3t \sin 3t - 2 \cos 3t)$$

### Section 3.1

1. No

3. No

5. No

7. Yes;  $(\mathbf{D}^2 - 7\mathbf{D} + 10)(y) = 0$ ,  $q(s) = s^2 - 7s + 10$ , homogeneous

9. Yes;  $\mathbf{D}^2(y) = -2 + \cos t$ ,  $q(s) = s^2$ , nonhomogeneous

11. (a)  $6e^t$

(b) 0

(c)  $\sin t - 3 \cos t$

13. (a) 0

(b) 0

(c) 1

15.  $y(t) = \cos 2t + c_1 e^t + c_2 e^{4t}$  where  $c_1, c_2$  are arbitrary constants.

17.  $y(t) = \cos 2t + e^t - e^{4t}$

19.  $\mathbf{L}(e^{rt}) = (ar^2 + br + c)e^{rt}$

**Section 3.2**

1. Linearly independent
3. Linearly dependent
5. Linearly dependent
7. Linearly independent
9. Linearly independent
11. Linearly independent
13. Linearly independent
15.  $t$
17.  $10t^{29}$
19.  $e^{(r_1+r_2+r_3)t}(r_3 - r_1)(r_3 - r_2)(r_2 - r_1)$
21. 12
23.  $c_1 = -2/5, c_2 = 1$
25.  $a_1 = 3, a_2 = 3$

**Section 3.3**

1.  $y(t) = c_1e^{2t} + c_2e^{-t}$
3.  $y(t) = c_1e^{-4t} + c_2e^{-6t}$
5.  $y(t) = c_1e^{-4t} + c_2te^{-4t}$
7.  $y(t) = c_1e^{-t} \cos 2t + c_2e^{-t} \sin 2t$
9.  $y(t) = c_1e^{-9t} + c_2e^{-4t}$
11.  $y(t) = c_1e^{-5t} + c_2te^{-5t}$
13.  $y = \frac{e^t - e^{-t}}{2}$
15.  $y = te^{5t}$

$$17. q(s) = s^2 + 4s - 21, w(e^{3t}, e^{-7t}) = -10e^{-4t}, K = -10.$$

$$19. q(s) = s^2 - 6s + 9, w(e^{3t}, te^{3t}) = e^{6t}, K = 1.$$

$$21. q(s) = s^2 - 2s + 5, w(e^t \cos 2t, e^t \sin 2t) = 2e^{2t}, K = 2.$$

### Section 3.4

$$1. y_p(t) = a_1 e^{3t}$$

$$3. y_p(t) = a_1 t e^{2t}$$

$$5. y_p(t) = a_1 \cos 5t + a_2 \sin 5t$$

$$7. y_p(t) = a_1 t \cos 2t + a_2 t \sin 2t$$

$$9. y_p(t) = a_1 e^{-2t} \cos t + a_2 e^{-2t} \sin t$$

$$11. y = -te^{-2t} + c_1 e^{-2t} + c_2 e^{5t}$$

$$13. y = \frac{1}{2} t^2 e^{-t} + c_1 e^{-t} + c_2 t e^{-t}$$

$$15. y = \frac{1}{2} e^{-3t} + c_1 e^{-2t} \cos t + c_2 e^{-2t} \sin t$$

$$17. y = -t^2 - 2 + c_1 e^t + c_2 e^{-t}$$

$$19. y = \frac{1}{2} t^2 e^{2t} + c_1 e^{2t} + c_2 t e^{2t}.$$

$$21. y = t e^{2t} - \frac{2}{5} e^{2t} + c_1 e^{-3t} + c_2 t e^{-3t}$$

$$23. y = \frac{1}{4} t e^{-3t} \sin(2t) + c_1 e^{-3t} \cos(2t) + c_2 e^{-3t} \sin(2t)$$

$$25. y = \frac{-1}{12} e^{3t} + \frac{10}{21} e^{6t} + \frac{135}{84} e^{-t}$$

$$27. y = 2e^{2t} - 2 \cos t - 4 \sin t$$

### Section 3.5

$$1. y = \frac{1}{32} e^{-6t} + c_1 e^{2t} + c_2 e^{-2t}$$

$$3. y = t e^{-2t} + c_1 e^{-2t} + c_2 e^{-3t}$$

$$5. y = -t e^{-4t} + c_1 e^{2t} + c_2 e^{-4t}$$

$$7. y = te^{2t} - \frac{2}{5}e^{2t} + c_1e^{-3t} + c_2te^{-3t}$$

$$9. y = -3t^2e^{4t} \cos 3t + te^{4t} \sin 3t + c_1e^{4t} \cos 3t + c_2e^{4t} \sin 3t$$

$$11. y = \frac{1}{2} \sin t + c_1e^{-t} + c_2te^{-t}$$

### Section 3.6

$$1. k = 32 \text{ lbs/ft}$$

$$3. k = 490 \text{ N/m}$$

$$5. \mu = 8 \text{ lbs s/ft}$$

$$7. 400 \text{ lbs}$$

$$9. 6y'' + 20y = 0, \quad y(0) = .1, \quad y'(0) = 0; \quad y = \frac{1}{10} \cos \sqrt{\frac{10}{3}} t, \text{ undamped free or simple harmonic motion; } A = 1/10, \beta = \sqrt{10/3}, \text{ and } \phi = 0$$

$$11. \frac{1}{2}y'' + 2y' + 32y = 0, \quad y(0) = 1, \quad y'(0) = 1; \quad y = e^{-2t} \cos \sqrt{60}t + \frac{3}{\sqrt{60}}e^{-2t} \sin \sqrt{60}t = \sqrt{\frac{23}{20}}e^{-2t} \cos(\sqrt{60}t + \phi), \text{ where } \phi = \arctan \sqrt{60}/20 \approx .3695; \text{ underdamped free motion.}$$

$$13. y'' + 96y = 0, \quad y(0) = 0, \quad y'(0) = 2/3; \quad y = \frac{\sqrt{6}}{36} \sin \sqrt{96}t = \frac{\sqrt{6}}{36} \cos \left( \sqrt{96}t - \frac{\pi}{2} \right); \text{ undamped free or simple harmonic motion and crosses equilibrium.}$$

### Section 3.7

$$1. q(t) = -\frac{3}{100}e^{-20t} \cos 20t - \frac{3}{100}e^{-3t} \sin 20t + \frac{3}{100} \text{ and } I(t) = \frac{6}{5}e^{-20t} \cos(20t - \pi/2)$$

$$3. q(t) = \frac{4}{5}e^{-10t} + 7te^{-10t} - \frac{4}{5} \cos 5t + \frac{3}{5} \sin 5t \text{ and } I(t) = -e^{-10t} - 70te^{-10t} + 4 \sin 5t + 3 \cos 5t$$

$$5. q(t) = \frac{1}{75}(\cos 25t - \cos 100t); \text{ the capacitor will not overcharge.}$$

**Section 4.1**

1. Yes;  $(D^3 - 3D)y = e^t$ , order 3,  $q(s) = s^3 - 3s$ , nonhomogeneous

3. No

5. (a) 0

(b) 0

(c) 0

7. (a)  $10e^{-t}$

(b) 0

(c) 0

9.  $y(t) = te^{2t} + c_1e^{2t} + c_2e^{-2t} + c_3$

11.  $y(t) = te^{2t} + e^{2t} + 2e^{-2t} - 1$

**Section 4.2**

1.  $y(t) = c_1e^{-t} + c_2e^{-\frac{1}{2}t} \cos \frac{\sqrt{3}}{2}t + c_3e^{-\frac{1}{2}t} \sin \frac{\sqrt{3}}{2}t$

3.  $y(t) = c_1e^t + c_2e^{-t} + c_3 \sin t + c_4 \cos t$

5.  $y(t) = c_1e^t + c_2e^{-t} + c_3e^{2t} + c_4e^{-2t}$

7.  $y(t) = c_1e^{-2t} + c_2 \cos 5t + c_3 \sin 5t$

9.  $y(t) = c_1e^t + c_2e^{-3t} + c_3te^{-3t} + c_4t^2e^{-3t}$

11.  $y = e^t - 3e^{-t} + \cos t + 2 \sin t$

**Section 4.3**

1.  $y = cte^{-t}$

3.  $y = ce^{2t}$

5.  $y = \frac{1}{2}te^t + c_1e^{-t} + c_2e^t + c_3$

7.  $y = \frac{1}{12}te^{2t} + c_1e^t + c_2e^{-t} + c_3e^{2t} + c_4e^{-2t}$

9.  $y = \frac{1}{2}te^t + c_1e^{-t} + c_2e^t + c_3$

11.  $y = \frac{t^2}{8} + c_1 + c_2 \cos 2t + c_3 \sin 2t$

13.  $y = -t(\sin t + \cos t) + c_1 e^t + c_2 \cos t + c_3 \sin t$

**Section 4.4**

1.  $y_1(t) = -4e^{2t} + 6e^{4t}$  and  $y_2(t) = -4e^{2t} + 3e^{4t}$

3.  $y_1(t) = \cos 2t - \sin 2t$  and  $y_2(t) = -\cos 2t + \sin 2t$

5.  $y_1(t) = -20e^{-t} + 40e^t - 20e^{2t}$  and  $y_2(t) = -6e^{-t} + 20e^t - 12e^{2t}$

7.  $y_1(t) = 3e^{-t} - 3 \cos 3t + \sin 3t$  and  $y_2(t) = 3e^{-t} + 3 \cos 3t + 3 \sin 3t$

9.  $y_1(t) = \cos t + \sin t + 2 \cos 2t + \sin 2t$  and  $y_2(t) = 2 \cos t + 2 \sin t - 2 \cos 2t - \sin 2t$

13.  $y_1(t) = 2e^t \cos 2t + 2e^t \sin 2t$  and  $y_2(t) = -2e^t \cos 2t + 2e^t \sin 2t$

15.  $y_1(t) = 20 - 19 \cos t - 2 \sin t$  and  $y_2(t) = 8 - 8 \cos t + 3 \sin t$

**Section 4.5**

1.  $y(t) = 10e^{-5t}$ , asymptotically stable

3.  $y(t) = e^t + e^{3t}$ , unstable

5.  $y(t) = e^{-2t} \sin(t)$ , stable

7.  $y(t) = e^{-3t} + 4te^{-3t}$ , stable

9.  $y(t) = e^t(\sin(t) + \cos(t))$ , unstable

11.  $y(t) = e^{-t}$ , marginally stable

13.  $y(t) = e^{-t}$

15.  $y(t) = \frac{1}{4}(e^{2t} - e^{-2t})$

17.  $y(t) = 1 - \cos(t)$

## Section 5.1

1. Not linear
3. Yes, nonhomogeneous, yes
5. Yes, nonhomogeneous, no
7. Yes, nonhomogeneous, no
9. Not linear
11. Yes, homogeneous, no
13.
  1.  $L\left(\frac{1}{t}\right) = 0$
  2.  $L(1) = -1$
  3.  $L(t) = 0$
  4.  $L(t^r) = (r^2 - 1)t^r$
15.  $C = \frac{-1}{2}$
17.
  - (3)a.  $y(t) = e^{-t} - e^t + 2t$
  - (3)b.  $y(t) = e^{-t} + (0)e^t + (1)t = e^{-t} + t$
  - (3)c.  $y(t) = e^{-t} + -e^t + 3t$
  - (3)d.  $y(t) = e^{-t} + (a - 1)e^t + (b - a + 2)t$
19.  $(-\infty, 0)$
21.  $(0, \pi)$
23.  $(3, \infty)$
25. The initial condition occurs at  $t = 0$  which is precisely where  $a_2(t) = t^2$  has a zero. Theorem 6 does not apply.
27. The assumptions say that  $y_1(t_0) = y_2(t_0)$  and  $y_1'(t_0) = y_2'(t_0)$ . Both  $y_1$  and  $y_2$  therefore satisfies the same initial conditions. By the uniqueness part of Theorem 6  $y_1 = y_2$ .

## Section 5.2

1. Dependent;  $2t$  and  $5t$  are multiples of each other.
3. Independent
5. Independent

11. 1. Suppose  $at^3 + b|t^3| = 0$  on  $(-\infty, \infty)$ . Then for  $t = 1$  and  $t = -1$  we get

$$a + b = 0$$

$$-a + b = 0.$$

These equations imply  $a = b = 0$ . So  $y_1$  and  $y_2$  are linearly independent.

2. Observe that  $y_1'(t) = 3t^2$  and  $y_2'(t) = \begin{cases} -3t^2 & \text{if } t < 0 \\ 3t^2 & \text{if } t \geq 0. \end{cases}$  If  $t < 0$ ,

then  $w(y_1, y_2)(t) = \det \begin{pmatrix} t^3 & -t^3 \\ 3t^2 & -3t^2 \end{pmatrix} = 0$ . If  $t \geq 0$ , then  $w(y_1, y_2)(t) =$

$\det \begin{pmatrix} t^3 & t^3 \\ 3t^2 & 3t^2 \end{pmatrix} = 0$ . It follows that the Wronskian is zero for all  $t \in (-\infty, \infty)$ .

3. The condition that the coefficient function  $a_2(t)$  be nonzero in Theorem 2 and Proposition 4 is essential. Here the coefficient function,  $t^2$ , of  $y''$  is zero at  $t = 0$ , so Proposition 4 does not apply on  $(-\infty, \infty)$ . The largest open intervals on which  $t^2$  is nonzero are  $(-\infty, 0)$  and  $(0, \infty)$ . On each of these intervals,  $y_1$  and  $y_2$  are linearly dependent.

4. Consider the cases  $t < 0$  and  $t \geq 0$ . The verification is then straightforward.

5. Again the condition that the coefficient function  $a_2(t)$  be nonzero is essential. The uniqueness and existence theorem does not apply.

### Section 5.3

1. The general solution is  $y(t) = c_1t + c_2t^{-2}$ .

3. The general solution is  $y(t) = c_1t^{\frac{1}{3}} + c_2t^{\frac{1}{3}} \ln t$ .

5. The general solution is  $y(t) = c_1t^{\frac{1}{2}} + c_2t^{\frac{1}{2}} \ln t$ .

7.  $y(t) = c_1t^{-3} + c_2t^{-3} \ln t$

9. The general solution is  $y(t) = c_1t^2 + c_2t^{-2}$ .

11. The general solution is  $y(t) = c_1t^2 \cos(3 \ln t) + c_2t^2 \sin(3 \ln t)$ .

13.  $y = 2t^{1/2} - t^{1/2} \ln t$

15. No solution is possible.

**Section 5.4**

1.  $\ln\left(\frac{s-a}{s-b}\right)$

3.  $s \ln\left(\frac{s^2+b^2}{s^2+a^2}\right) - 2b \tan^{-1}\left(\frac{b}{s}\right) + 2a \tan^{-1}\left(\frac{a}{s}\right)$

5.  $y = c_1 e^{-t} + c_2(t-1)$

7.  $y(t) = c_1 e^{-2t}$ .

9.  $y(t) = c_1((3-t^2)\sin t - 3t \cos t)$

11.  $y(t) = c_1 \frac{e^t - 1}{t}$ .

13.  $y(t) = c_1 \left(\frac{e^{3t} - e^{2t}}{t}\right)$ .

15.  $y(t) = c_1 \frac{\sin t}{t} + c_2 \frac{1 - \cos t}{t}$

**Section 5.5**

1.  $y_2(t) = t^2 \ln t$ , and the general solution can be written  $y(t) = c_1 t^2 + c_2 t^2 \ln t$ .

3.  $y_2(t) = \sqrt{t} \ln t$ , and the general solution can be written  $y(t) = c_1 \sqrt{t} + c_2 + 2\sqrt{t} \ln t$ .

5.  $y_2(t) = te^t$ . The general solution can be written  $y(t) = c_1 t + c_2 te^t$ .

7.  $y_2(t) = \frac{-1}{2} \cos t^2$ . The general solution can be written  $y(t) = c_1 \sin t^2 + c_2 \cos t^2$ .

9.  $y_2(t) = -1 - t \tan t$ . The general solution can be written  $y(t) = c_1 \tan t + c_2(1 + t \tan t)$ .

11.  $y_2(t) = -\sec t$ . The general solution can be written  $y(t) = c_1 \tan t + c_2 \sec t$ .

13.  $y_2 = -1 - \frac{t \sin 2t}{1 + \cos 2t}$ . The general solution can be written  $y(t) = c_1 \frac{\sin 2t}{1 + \cos 2t} + c_2 \left(1 + \frac{t \sin 2t}{1 + \cos 2t}\right)$ .

15.  $y_2(t) = \frac{1}{2}t + \frac{1}{4}(1 - t^2) \ln\left(\frac{1+t}{1-t}\right)$ , and the general solution can be written

$$y = c_1(1 - t^2) + c_2 \left( \frac{1}{2}t + \frac{1}{4}(1 - t^2) \ln\left(\frac{1+t}{1-t}\right) \right).$$

## Section 5.6

1. The general solution is  $y(t) = \frac{-1}{2}t \cos t + c_1 \cos t + c_2 \sin t$ .
3. The general solution is  $y(t) = \frac{1}{4}e^t + c_1 e^t \cos 2t + c_2 e^t \sin 2t$ .
5. The general solution is  $y(t) = \frac{1}{2}e^{3t} + c_1 e^t + c_2 e^{2t}$ .
7. The general solution is  $y(t) = t \ln t e^t + c_1 e^t + c_2 t e^t$ .
9. The general solution is  $y(t) = \frac{t^4}{6} + c_1 t + c_2 t^2$ .
11. The general solution is  $y(t) = \frac{t}{2} \ln^2 t + c_1 t + c_2 t \ln t$ .
13. The general solution is  $y(t) = \frac{t^2}{2} \tan t + t + c_1 \tan t + c_2 \sec t$ .
15. The general solution is  $y(t) = t^2 + c_1 \cos t^2 + c_2 \sin t^2$ .
19.  $y_p(t) = \frac{1}{a} f(t) * \sinh at$
21.  $y_p(t) = \frac{1}{a-b} f(t) * (e^{at} - e^{bt})$

## Section 6.1

1. Graph (c)
3. Graph (e)
5. Graph (f)
7. Graph (h)
9.  $-22/3$
11. 4
13.  $11/2$

15. 5

17. A, B true. C false.

19. A is true. B and C are false.

21. A and B are true. C and D are false.

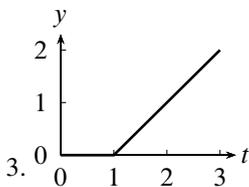
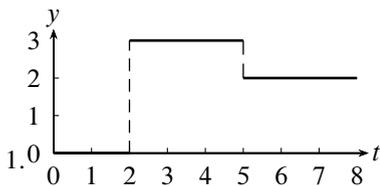
23. A, B, C, D are all true.

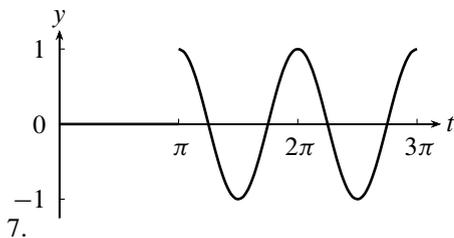
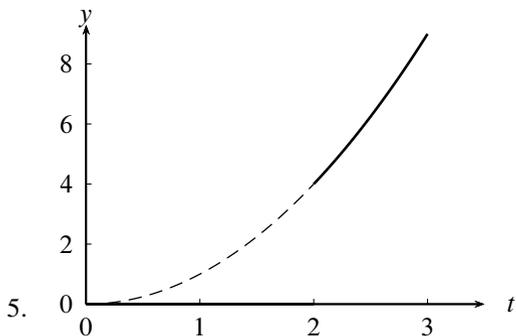
$$25. y(t) = \begin{cases} -1 + e^t & \text{if } 0 \leq t < 2, \\ 1 - 2e^{t-2} + e^t & \text{if } 2 \leq t < 4 \\ e^{t-4} - 2e^{t-2} + e^t & \text{if } 4 \leq t < \infty. \end{cases}$$

$$27. y(t) = \begin{cases} 0 & \text{if } 0 \leq t < 1, \\ -t + e^{t-1} & \text{if } 1 \leq t < 2, \\ t - 2 - 2e^{t-2} + e^{t-1} & \text{if } 2 \leq t < 3 \\ e^{t-3} - 2e^{t-2} + e^{t-1} & \text{if } 3 \leq t < \infty. \end{cases}$$

$$29. y(t) = \begin{cases} -t + e^t - e^{-t} & \text{if } 0 \leq t < 1, \\ e^t - e^{t-1} - e^{-1} & 1 \leq t < \infty. \end{cases}$$

## Section 6.2





9. (a)  $(t - 2)\chi_{[2, \infty)}(t)$ ; (b)  $(t - 2)h(t - 2)$ ; (c)  $e^{-2s}/s^2$ .
11. (a)  $(t + 2)\chi_{[2, \infty)}(t)$ ; (b)  $(t + 2)h(t - 2)$ ; (c)  $e^{-2s} \left( \frac{1}{s^2} + \frac{4}{s} \right)$ .
13. (a)  $t^2\chi_{[4, \infty)}(t)$ ; (b)  $t^2h(t - 4)$ ; (c)  $e^{-4s} \left( \frac{2}{s^3} + \frac{8}{s^2} + \frac{16}{s} \right)$ .
15. (a)  $(t - 4)^2\chi_{[2, \infty)}(t)$ ; (b)  $(t - 4)^2h(t - 2)$ ; (c)  $e^{-2s} \left( \frac{2}{s^3} - \frac{4}{s^2} + \frac{4}{s} \right)$ .
17. (a)  $e^t\chi_{[4, \infty)}(t)$ ; (b)  $e^t h(t - 4)$ ; (c)  $e^{-4(s-1)} \frac{1}{s - 1}$ .
19. (a)  $te^t\chi_{[4, \infty)}(t)$ ; (b)  $te^t h(t - 4)$ ;  
 (c)  $e^{-4(s-1)} \left( \frac{1}{(s - 1)^2} + \frac{4}{s - 1} \right)$ .
21. (a)  $t\chi_{[0,1)}(t) + (2 - t)\chi_{[1, \infty)}(t)$ ; (b)  $t + (2 - 2t)h(t - 1)$ ;  
 (c)  $\frac{1}{s^2} - \frac{2e^{-s}}{s^2}$ .
23. (a)  $t^2\chi_{[0,2)}(t) + 4\chi_{[2,3)}(t) + (7 - t)\chi_{[3, \infty)}(t)$ ;  
 (b)  $t^2 + (4 - t^2)h(t - 2) + (3 - t)h(t - 3)$ ;  
 (c)  $\frac{2}{s^3} - e^{-2s} \left( \frac{2}{s^3} + \frac{4}{s^2} \right) - \frac{e^{-3s}}{s^2}$ .

25. (a)  $\sum_{n=0}^{\infty} (t-n)\chi_{[n,n+1)}(t)$ ; (b)  $t - \sum_{n=1}^{\infty} h(t-n)$ ;

(c)  $\frac{1}{s^2} - \frac{e^{-s}}{s(1-e^{-s})}$ .

27. (a)  $\sum_{n=0}^{\infty} (2n+1-t)\chi_{[2n,2n+2)}(t)$ ; (b)  $-(t+1) + 2\sum_{n=0}^{\infty} h(t-2n)$ ;

(c)  $-\frac{1}{s^2} - \frac{1}{s} + \frac{2}{s(1-e^{-2s})}$ .

29.  $(t-3)h(t-3) = \begin{cases} 0 & \text{if } 0 \leq t < 3, \\ t-3 & \text{if } t \geq 3. \end{cases}$

31.  $h(t-\pi)\sin(t-\pi)$   
 $= \begin{cases} 0 & \text{if } 0 \leq t < \pi, \\ \sin(t-\pi) & \text{if } t \geq \pi \end{cases} = \begin{cases} 0 & \text{if } 0 \leq t < \pi, \\ -\sin t & \text{if } t \geq \pi. \end{cases}$

33.  $\frac{1}{2}e^{-(t-\pi)}\sin 2(t-\pi)h(t-\pi) = \begin{cases} 0 & \text{if } 0 \leq t < \pi, \\ \frac{1}{2}e^{-(t-\pi)}\sin 2t & \text{if } t \geq \pi. \end{cases}$

35.  $\frac{1}{2}h(t-2)\sin 2(t-2) = \begin{cases} 0 & \text{if } 0 \leq t < 2, \\ \frac{1}{2}\sin 2(t-2) & \text{if } t \geq 2. \end{cases}$

37.  $h(t-4)(2e^{-2(t-4)} - e^{-(t-4)})$   
 $= \begin{cases} 0 & \text{if } 0 \leq t < 4, \\ 2e^{-2(t-4)} - e^{-(t-4)} & \text{if } t \geq 4. \end{cases}$

39.  $t - (t-5)h(t-5) = \begin{cases} t & \text{if } 0 \leq t < 5, \\ 5 & \text{if } t \geq 5. \end{cases}$

41.  $h(t-\pi)e^{-3(t-\pi)}(2\cos 2(t-\pi) - \frac{5}{2}\sin 2(t-\pi))$   
 $= \begin{cases} 0 & \text{if } 0 \leq t < \pi, \\ e^{-3(t-\pi)}(2\cos 2t - \frac{5}{2}\sin 2t) & \text{if } t \geq \pi. \end{cases}$

### Section 6.3

1.  $y = \begin{cases} 0 & \text{if } 0 \leq t < 1 \\ -\frac{3}{2}(1 - e^{-2(t-1)}) & \text{if } 1 \leq t < \infty \end{cases}$

$$3. y = \begin{cases} 0 & \text{if } 0 \leq t < 2 \\ \frac{2}{3}(e^{3(t-2)} - 1) & \text{if } 2 \leq t < 3 \\ \frac{2}{3}(e^{3(t-2)} - e^{3(t-3)}) & \text{if } 3 \leq t < \infty \end{cases}$$

$$5. y = \begin{cases} 6e^{4t} - 4e^t & \text{if } 0 \leq t < 1 \\ 6e^{4t} - e^{4t-3} - 3e & \text{if } 1 \leq t < \infty \end{cases}$$

$$7. y = \begin{cases} 0 & \text{if } 0 \leq t < 3 \\ \frac{1}{9}(1 - \cos 3(t-3)) & \text{if } 3 \leq t < \infty \end{cases}$$

$$9. y = \begin{cases} 0 & \text{if } 0 \leq t < 1 \\ 1 - 3e^{-2(t-1)} + 2e^{-3(t-1)} & \text{if } 1 \leq t < 3 \\ 3e^{-2(t-3)} - 3e^{-2(t-1)} - 2e^{-3(t-3)} + 2e^{-3(t-1)} & \text{if } 3 \leq t < \infty \end{cases}$$

$$11. y = \begin{cases} te^{-t} & \text{if } 0 \leq t < 3 \\ 1 + te^{-t} - (t-2)e^{-(t-3)} & \text{if } 3 \leq t < \infty \end{cases}$$

$$13. y(t) = \begin{cases} 4 - 4e^{-\frac{t}{2}} & \text{if } 0 \leq t < 3 \\ 20 - 4e^{-\frac{t}{2}} - 16e^{-\frac{(t-3)}{2}} & \text{if } t \geq 3. \end{cases}$$

$$15. y(t) = \begin{cases} 10 - 8e^{-3t/10} & \text{if } 0 \leq t < 2 \\ 10e^{-3(t-2)/10} - 8e^{-3t/10} & \text{if } 2 \leq t < 4 \\ 10 - 8e^{-3t/10} + 10e^{-3(t-2)/10} - 10e^{-3(t-4)/10} & \text{if } 4 \leq t < \infty \end{cases}$$

## Section 6.4

$$1. y = \begin{cases} 0 & \text{if } 0 \leq t < 1 \\ e^{-2(t-1)} & \text{if } 1 \leq t < \infty \end{cases}$$

$$3. y = \begin{cases} 2e^{4t} & \text{if } 0 \leq t < 4 \\ 2e^{4t} + e^{4(t-4)} & \text{if } 4 \leq t < \infty \end{cases}$$

$$5. y = \begin{cases} \frac{1}{2} \sin 2t & \text{if } 0 \leq t < \pi, \\ \sin 2t & \text{if } t \geq \pi. \end{cases}$$

$$7. y = \begin{cases} e^{-t} & \text{if } 0 \leq t < 2 \\ e^{-t} + e^{-(t-2)} - e^{-3(t-2)} & \text{if } 2 \leq t < \infty \end{cases}.$$

$$9. y = \begin{cases} te^{-2t} - e^{-2t} & \text{if } 0 \leq t < 1 \\ te^{-2t} - e^{-2t} + 3(t-1)e^{-2(t-1)} & \text{if } 1 \leq t < \infty \end{cases}$$

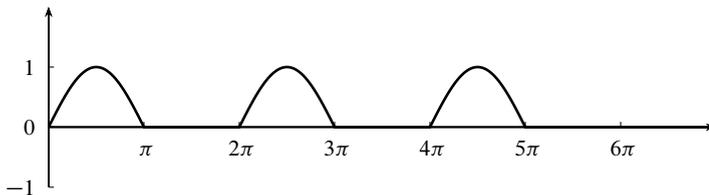
$$11. y = \begin{cases} 24 - 24e^{-\frac{1}{4}t} & \text{if } 0 \leq t < 3 \\ 24 - 24e^{-\frac{1}{4}t} + 4e^{-\frac{1}{4}(t-3)} & \text{if } 3 \leq t < \infty \end{cases}$$

$$13. y = \begin{cases} e^{-t} & \text{if } 0 \leq t < 2 \\ e^{-t} + e^{-(t-2)} & \text{if } 2 \leq t < 4 \\ e^{-t} + e^{-(t-2)} + e^{-(t-4)} & \text{if } 4 \leq t < 6 \\ e^{-t} + e^{-(t-2)} + e^{-(t-4)} + e^{-(t-6)} & \text{if } 6 \leq t < \infty \end{cases} \text{ and } y(6) = 1.156 \text{ lbs.}$$

$$15. y = \begin{cases} \frac{1}{10}e^{-2t} & \text{if } 0 \leq t < 4 \\ \frac{1}{10}e^{-2t} + (t-4)e^{-2(t-4)} & \text{if } 4 \leq t < \infty \end{cases}$$

$$17. y = \begin{cases} \sin t & \text{if } 0 \leq t < \pi \\ 0 & \text{if } \pi \leq t < 2\pi \\ \sin t & \text{if } 2\pi \leq t < 3\pi \\ 0 & \text{if } 3\pi \leq t < 4\pi \\ \sin t & \text{if } 4\pi \leq t < 5\pi \\ 0 & \text{if } 5\pi \leq t < \infty. \end{cases}$$

The graph is given below.



At  $t = 0$ , the hammer imparts a velocity to the system causing harmonic motion. At  $t = \pi$ , the hammer strikes in precisely the right way to stop the motion. Then at  $t = 2\pi$ , the process repeats.

$$19. y = y_0e^{-at} + ke^{-a(t-c)}$$

**Section 6.5**

$$1. f * g(t) = \begin{cases} e^t - 1 & \text{if } 0 \leq t < 1 \\ e^t - e^{t-1} & \text{if } 1 \leq t < \infty \end{cases}$$

$$5. f * g = \begin{cases} t & \text{if } 0 \leq t < 2 \\ -t + 4 & \text{if } 2 \leq t < 4 \\ 0 & \text{if } 4 \leq t < \infty \end{cases}$$

$$7. f * g = \begin{cases} \sin t & \text{if } 0 \leq t < \pi \\ 0 & \text{if } \pi \leq t < \infty \end{cases}$$

$$9. \zeta(t) = e^{3t}, y = \begin{cases} 2e^{3t} & \text{if } 0 \leq t < \infty \\ 2e^{3t} + \frac{1}{3}(e^{3(t-2)} - 1) & \text{if } 1 \leq t < \infty \end{cases}$$

$$11. \zeta(t) = e^{-8t}, y = \begin{cases} -2e^{-8t} & \text{if } 0 \leq t < 3 \\ -2e^{-8t} + \frac{1}{8}(1 - e^{-8(t-3)}) & \text{if } 3 \leq t < 5 \\ -2e^{-8t} + \frac{1}{8}(e^{-8(t-5)} - e^{-8(t-3)}) & \text{if } 5 \leq t < \infty \end{cases}$$

$$13. y = \frac{1}{9} \begin{cases} 1 - \cos 3t & \text{if } 0 \leq t < 2\pi \\ 0 & \text{if } 2\pi \leq t < \infty \end{cases}$$

$$23. y = 2e^t - 7te^t$$

$$25. y = 5 - 4 \cos t$$

**Section 6.6**

$$1. (\langle t \rangle_1)^2$$

$$3. \frac{1}{9}(\langle t \rangle_3)^2$$

$$5. \langle t \rangle_2 + \frac{3}{2}\langle t \rangle_2$$

$$7. \mathcal{L}\{f(\langle t \rangle_3)\} = \frac{1 - e^{-3(s-1)}}{1 - e^{-3s}} \frac{1}{s-1}$$

$$9. \mathcal{L}\{f(\langle t \rangle_{2p})\} = \frac{1 - e^{-ps}}{1 + e^{-ps}} \frac{1}{s}$$

$$11. \mathcal{L}\{\langle t \rangle_p\} = \frac{p}{s(e^{ps} - 1)}$$

$$13. y = \frac{1-e^{-2s}}{1-e^{-2(s+1)}} \frac{1}{s}$$

$$17. \mathcal{L}^{-1} \left\{ \frac{1-e^{-4(s-2)}}{(1-e^{-4s})(s-2)} \right\} = e^{2(t)} 4$$

## Section 6.7

1.

$$y(t) = 10 - 10e^{-\frac{2t}{5}} + 10 \operatorname{sw}_2(t) - \frac{10e^{-\frac{2t}{5}}}{1 + e^{\frac{4}{5}}} \left( 1 + e^{\frac{4}{5}} (-1)^{\lfloor t/2 \rfloor} e^{\frac{2}{5} \lfloor t \rfloor} \right)$$

The amount of salt fluctuates from 13.10 pounds to 16.90 pounds in the long term.

3.  $y(t) = 5e^{-\frac{1}{2}t} \frac{e^{\frac{1}{2}\lfloor t \rfloor} - 1}{e^{-1} - 1}$ , and the salt fluctuation in the tank varies between 2.91 and 7.91 pounds for large values of  $t$ .

5. The mathematical model is

$$y' = ry - 40\delta_0(\langle t \rangle_1), \quad y(0) = 3000,$$

where  $r = \frac{1}{12} \ln \frac{6}{5}$ . The solution to the model is

$$y(t) = 3000e^{rt} - 40 \frac{e^{rt} - e^{-r(\lfloor t \rfloor_1 - t + 1)}}{1 - e^{-r}}$$

and at the beginning of 60 months, there are  $y(60) \approx 3477$  alligators.

## Section 6.8

1.

$$y(t) = 2 \left( 2 \operatorname{sw}_1(t) - (-1)^{\lfloor t \rfloor_1} (\cos \langle t \rangle_1 - \alpha \sin \langle t \rangle_1) \right) - 2 (\cos t + \alpha \sin t),$$

where  $\alpha = \frac{-\sin \sqrt{2}}{1 + \cos \sqrt{2}}$ . Motion is nonperiodic.

3.  $y(t) = \frac{1}{\pi^2} (2 \operatorname{sw}_2(t) - \cos \pi t ((-1)^{\lfloor t/2 \rfloor_1} + 1))$ , the motion is periodic.

5.  $y(t) = \frac{2}{\pi^2} (\operatorname{sw}_1(t) - \lfloor t \rfloor_1 \cos \pi t - \cos \pi t)$ , with resonance.

7.

$$\begin{aligned} y(t) &= \sin t + \sin\langle t \rangle_\pi \\ &= (1 + (-1)^{\lfloor t/\pi \rfloor}) \sin t. \end{aligned}$$

Motion is periodic.

9.

$$y(t) = \sin t + \gamma \cos t + \sin\langle t \rangle_1 - \gamma \cos\langle t \rangle_1,$$

where  $\gamma = \frac{-\sin 1}{1 - \cos 1}$ . The motion is nonperiodic.

11.

$$y(t) = 2(\sin t)(1 + \lfloor t/2\pi \rfloor_1).$$

Resonance occurs.

## Section 7.1

1.  $R = 1$

3.  $R = \infty$

5.  $R = 0$

7.  $R = \infty$

9.  $R = 2$

11.  $-\sum_{n=0}^{\infty} \frac{t^n}{a^{n+1}}$

13.  $\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n+1)!}$

15.  $\sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{2n+1}$

17.  $\tan t = 1 + \frac{1}{3}t^3 + \frac{2}{15}t^5 + \frac{17}{315}t^7 + \dots$

19.  $e^t \sin t = t + t^2 + \frac{1}{3}t^3 - \frac{1}{30}t^5 + \dots$

21.  $(1 - t)e^{-t}$

23.  $f(t) = \frac{1}{(1-t)^2}$

25.  $f(t) = -\frac{t}{2} + \frac{t^2-1}{4} \ln\left(\frac{1+t}{1-t}\right)$

27. The binomial theorem:  $(a+b)^n = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k}$

**Section 7.2**

1.  $y(t) = c_0 \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!} + c_1 \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!} = c_0 \cosh t + c_1 \sinh t$

3.  $y(t) = c_0 \sum_{n=0}^{\infty} (-1)^n \frac{k^{2n} t^{2n}}{(2n)!} + c_1 \sum_{n=0}^{\infty} (-1)^n \frac{k^{2n+1} t^{2n+1}}{(2n+1)!} = c_0 \cos kt + c_1 \sin kt$

5.  $y(t) = c_0(1-t^2) - c_1 \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)(2n-1)} = c_0(1-t^2) - c_1 \left( \frac{t}{2} + \frac{t^2-1}{4} \ln\left(\frac{1-t}{1+t}\right) \right)$

7.  $y(t) = c_0 \left( 1 + \sum_{n=2}^{\infty} \frac{t^n}{n!} \right) + c_1 t = c_0(e^t - t) + c_1 t$

9.  $y(t) = c_0(1-3t^2) + c_1 \left( t - \frac{t^3}{3} \right)$

**Section 7.3**1.  $-1$  and  $1$  are regular singular points.

3. There are no singular points.

5.  $0$  is a regular singular point.7.  $q(s) = s(s+1)$ . The exponents of singularity are  $0$  and  $-1$ . Theorem 2 guarantees one Frobenius solution but there could be two.9.  $q(s) = s^2$ . The exponent of singularity is  $0$  with multiplicity  $2$ . Theorem 2 guarantees that there is one and only one Frobenius solution.

15.  $y_1(t) = \sum_{m=0}^{\infty} \frac{(-1)^m (2m+2)t^{2m+3}}{(2m+3)!} = (\sin t - t \cos t)$  and  $y_2(t) = \sum_{m=0}^{\infty} \frac{(-1)^{m+1} (2m-1)t^{2m}}{(2m)!} = (t \sin t + \cos t)$

$$17. y_1(t) = \sum_{n=0}^{\infty} \frac{1}{n!} t^{n+1} = te^t \text{ and } y_2(t) = \left( te^t \ln t - t \sum_{n=1}^{\infty} \frac{s_n t^n}{n!} \right), \text{ where } s_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

$$19. y_1(t) = \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)t^n}{(n+3)!} = \left( \frac{(t+2)e^{-t}}{t} + \frac{t-2}{t} \right) \text{ and } y_2(t) = t^{-1} \left( 1 - \frac{t}{2} \right) = \frac{1}{2} \left( \frac{2-t}{t} \right)$$

$$21. y_1(t) = t^2 \text{ and } y_2(t) = t^2 \ln t + \left( -1 + 2t - \frac{t^3}{3} + \sum_{n=4}^{\infty} \frac{2(-1)^n t^n}{n!(n-2)} \right)$$

23. The complex Frobenius series is  $y(t) = (t^i + (\frac{1-2i}{1+2i})t^{1+i})$ ; the real and imaginary parts are  $y_1(t) = -3 \cos \ln t - 4 \sin \ln t + 5t \cos \ln t$  and  $y_2(t) = -3 \sin \ln t + 4 \cos \ln t + 5t \sin \ln t$ .

25. The complex Frobenius solution is  $y(t) = \sum_{n=0}^{\infty} \frac{t^{n+i}}{n!} = t^i e^t$ ; the real and imaginary parts are  $y_1(t) = e^t \cos \ln t$  and  $y_2(t) = e^t \sin \ln t$ .

## Section 7.4

## Section 8.1

$$1. B + C = \begin{bmatrix} 1 & 1 \\ -1 & 7 \\ 0 & 3 \end{bmatrix}, B - C = \begin{bmatrix} 1 & -3 \\ 5 & -1 \\ -2 & 1 \end{bmatrix}, \text{ and } 2B - 3C = \begin{bmatrix} 2 & -8 \\ 13 & -6 \\ -5 & 1 \end{bmatrix}$$

$$3. A(B + C) = AB + AC = \begin{bmatrix} 3 & 4 \\ 1 & 13 \end{bmatrix}, (B + C)A = \begin{bmatrix} 3 & -1 & 7 \\ 3 & 1 & 25 \\ 5 & 0 & 12 \end{bmatrix}$$

$$5. AB = \begin{bmatrix} 6 & 4 & -1 & -8 \\ 0 & 2 & -8 & 2 \\ 2 & -1 & 9 & -5 \end{bmatrix}$$

$$7. CA = \begin{bmatrix} 8 & 0 \\ 4 & -5 \\ 8 & 14 \\ 10 & 11 \end{bmatrix}$$

$$9. ABC = \begin{bmatrix} 8 & 9 & -48 \\ 4 & 0 & -48 \\ -2 & 3 & 40 \end{bmatrix}.$$

$$15. \begin{bmatrix} 0 & 0 & 1 \\ 3 & -5 & -1 \\ 0 & 0 & 5 \end{bmatrix}$$

$$17. \text{(a) Choose, for example, } A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

(b)  $(A + B)^2 = A^2 + 2AB + B^2$  precisely when  $AB = BA$ .

$$19. B^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$$

21. (a)  $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} A = \begin{bmatrix} v_2 \\ v_1 \end{bmatrix}$ ; the two rows of  $A$  are switched. (b)  $\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix} A = \begin{bmatrix} v_1 + cv_2 \\ v_2 \end{bmatrix}$ ; to the first row is added  $c$  times the second row while the second row is unchanged. (c) To the second row is added  $c$  times the first row while the first row is unchanged. (d) The first row is multiplied by  $a$  while the second row is unchanged. (e) The second row is multiplied by  $a$  while the first row is unchanged.

## Section 8.2

$$1. A = \begin{bmatrix} 1 & 4 & 3 \\ 1 & 1 & -1 \\ 2 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 2 \\ 4 \\ 1 \\ 6 \end{bmatrix}, \text{ and } [A|\mathbf{b}] = \begin{bmatrix} 1 & 4 & 3 & 2 \\ 1 & 1 & -1 & 4 \\ 2 & 0 & 1 & 1 \\ 0 & 1 & -1 & 6 \end{bmatrix}.$$

$$3. \begin{array}{rcl} x_1 - & x_3 + 4x_4 + 3x_5 = & 2 \\ 5x_1 + 3x_2 - 3x_3 - & x_4 - 3x_5 = & 1 \\ 3x_1 - 2x_2 + 8x_3 + 4x_4 - & 3x_5 = & 3 \\ -8x_1 + 2x_2 & + 2x_4 + x_5 = & -4 \end{array}$$

5. RREF

$$7. m_2(1/2)(A) = \begin{bmatrix} 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$9. t_{1,3}(-3)(A) = \begin{bmatrix} 1 & 0 & 1 & 0 & 3 \\ 0 & 1 & 3 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$11. \begin{bmatrix} 1 & 0 & 0 & -11 & -8 \\ 0 & 1 & 0 & -4 & -2 \\ 0 & 0 & 1 & 9 & 6 \end{bmatrix}$$

$$13. \begin{bmatrix} 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$15. \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$17. \begin{bmatrix} 1 & 4 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}$$

$$19. \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} -3 \\ 1 \\ 5 \end{bmatrix}, \alpha \in \mathbb{R}$$

$$21. \begin{bmatrix} x \\ y \end{bmatrix} = \alpha \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \alpha \in \mathbb{R}$$

$$23. \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 14/3 \\ 1/3 \\ -2/3 \end{bmatrix}$$

$$25. \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} + \alpha \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \alpha \in \mathbb{R}$$

$$27. \emptyset$$

$$29. \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$31. \left\{ \begin{bmatrix} -34 \\ -40 \\ 39 \\ 1 \end{bmatrix} \right\}$$

33. The equation  $\begin{bmatrix} 5 \\ -1 \\ 4 \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + b \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$  has solution  $a = 2$  and  $b = 3$ . By

Proposition 7  $\begin{bmatrix} 5 \\ -1 \\ 4 \end{bmatrix}$ , is a solution.

35. If  $\mathbf{x}_i$  is the solution set for  $A\mathbf{x} = \mathbf{b}_i$ , then  $\mathbf{x}_1 = \begin{bmatrix} -7/2 \\ 7/2 \\ -3/2 \end{bmatrix}$ ,  $\mathbf{x}_2 = \begin{bmatrix} -3/2 \\ 3/2 \\ -1/2 \end{bmatrix}$ , and

$$\mathbf{x}_3 = \begin{bmatrix} 7 \\ -6 \\ 3 \end{bmatrix}.$$

### Section 8.3

$$1. \begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix}$$

3. Not invertible

5. Not invertible

$$7. \begin{bmatrix} -6 & 5 & 13 \\ 5 & -4 & -11 \\ -1 & 1 & 3 \end{bmatrix}$$

$$9. \begin{bmatrix} -29 & 39/2 & -22 & 13 \\ 7 & -9/2 & 5 & -3 \\ -22 & 29/2 & -17 & 10 \\ 9 & -6 & 7 & -4 \end{bmatrix}$$

$$11. \begin{bmatrix} 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ -1 & -1 & 0 & 1 \end{bmatrix}$$

13.  $\mathbf{x} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$

15.  $\mathbf{x} = \frac{1}{10} \begin{bmatrix} 16 \\ 11 \\ 18 \end{bmatrix}$

17.  $\mathbf{x} = \begin{bmatrix} 19 \\ -4 \\ 15 \\ -6 \end{bmatrix}$

19.  $(A^t)^{-1} = (A^{-1})^t$

21.  $F(\theta)^{-1} = F(-\theta)$

**Section 8.4**

1. 1

3. 10

5. -21

7. 2

9. 0

11.  $\frac{1}{s^2-6s+8} \begin{bmatrix} s-3 & 1 \\ 1 & s-3 \end{bmatrix} s = 2, 4$

13.  $\frac{1}{(s-1)^3} \begin{bmatrix} (s-1)^2 & 3 & s-1 \\ 0 & (s-1)^2 & 0 \\ 0 & 3(s-1) & (s-1)^2 \end{bmatrix} s = 1$

15.  $\frac{1}{s^3+s^2+4s+4} \begin{bmatrix} s^2+s & 4s+4 & 0 \\ -s-1 & s^2+s & 0 \\ s-4 & 4s+4 & s^2+4 \end{bmatrix} s = -1, \pm 2i$

17. no inverse

19.  $\frac{1}{8} \begin{bmatrix} 4 & -4 & 4 \\ -1 & 3 & -1 \\ -5 & -1 & 3 \end{bmatrix}$

21.  $\frac{1}{6} \begin{bmatrix} 2 & -98 & 9502 \\ 0 & 3 & -297 \\ 0 & 0 & 6 \end{bmatrix}$

$$23. \frac{1}{15} \begin{bmatrix} 55 & -95 & 44 & -171 \\ 50 & -85 & 40 & -150 \\ 70 & -125 & 59 & -216 \\ 65 & -115 & 52 & -198 \end{bmatrix}$$

$$25. \mathbf{x} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$$

$$27. \mathbf{x} = \frac{1}{10} \begin{bmatrix} 16 \\ 11 \\ 18 \end{bmatrix}$$

## Section 8.5

1. The characteristic polynomial is  $c_A(s) = (s-1)(s-2)$ . The eigenvalues are thus  $s = 1, 2$ . The eigenspaces are  $E_1 = \text{Span} \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$  and  $E_2 = \text{Span} \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ .

3. The characteristic polynomial is  $c_A(s) = s^2 - 2s + 1 = (s-1)^2$ . The only eigenvalue is  $s = 1$ . The eigenspace is  $E_1 = \text{Span} \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ .

5. The characteristic polynomial is  $c_A(s) = s^2 + 2s - 3 = (s+3)(s-1)$ . The eigenvalues are thus  $s = -3, 1$ . The eigenspaces are  $E_{-3} = \text{Span} \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$  and  $E_1 = \text{Span} \left\{ \begin{bmatrix} -3 \\ 2 \end{bmatrix} \right\}$ .

7. The characteristic polynomial is  $c_A(s) = s^2 + 2s + 10 = (s+1)^2 + 3^2$ . The eigenvalues are thus  $s = -1 \pm 3i$ . The eigenspaces are  $E_{-1+3i} = \text{Span} \left\{ \begin{bmatrix} 7+i \\ 10 \end{bmatrix} \right\}$  and  $E_{-1-3i} = \text{Span} \left\{ \begin{bmatrix} 7-i \\ 10 \end{bmatrix} \right\}$ .

9. The eigenvalues are  $s = -2, 3$ .  $E_{-2} = \text{Span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ ,  $E_3 = \text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$ ,

11. The eigenvalues are  $s = 0, 2, 3$ .  $E_0 = \text{NS}(A) = \text{Span} \left\{ \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right\}$ ,  $E_2 = \text{Span} \left\{ \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \right\}$ ,  $E_3 = \text{Span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ .

13. We write  $c_A(s) = (s - 2)((s - 2)^2 + 1)$  to see that the eigenvalues are  $s = 2, 2 \pm i$ .  $E_2 = \text{Span} \left\{ \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \right\}$ ,  $E_{2+i} = \text{Span} \left\{ \begin{bmatrix} -4 + 3i \\ 4 + 2i \\ 5 \end{bmatrix} \right\}$ ,  $E_{2-i} = \text{Span} \left\{ \begin{bmatrix} -4 - 3i \\ 4 - 2i \\ 5 \end{bmatrix} \right\}$ .

## Section 9.2

### 1. Nonlinear

3.  $y' = \begin{bmatrix} \sin t & 0 \\ 1 & \cos t \end{bmatrix} y$ ; linear and homogeneous, but not constant coefficient.

5.  $y' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 \end{bmatrix} y$ ; linear, constant coefficient, homogeneous.

11.  $y' = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} y + \begin{bmatrix} 0 \\ e^{2t} \end{bmatrix}$ ,  $y(0) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ .

13.  $y' = \begin{bmatrix} 0 & 1 \\ k^2 & 0 \end{bmatrix} y + \begin{bmatrix} 0 \\ \cos \omega t \end{bmatrix}$ ,  $y(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ .

15.  $y' = \begin{bmatrix} 0 & 1 \\ -\frac{1}{t^2} & -\frac{2}{t} \end{bmatrix} y$ ,  $y(1) = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ .

17.  $A'(t) = \begin{bmatrix} -3e^{-3t} & 1 \\ 2t & 2e^{2t} \end{bmatrix}$

19.  $y'(t) = \begin{bmatrix} 1 \\ 2t \\ t^{-1} \end{bmatrix}$

$$21. \mathbf{v}'(t) = \begin{bmatrix} -2e^{-2t} & \frac{2t}{t^2+1} & -3 \sin 3t \end{bmatrix}$$

$$23. \frac{1}{4} \begin{bmatrix} e^2 - e^{-2} & e^2 + e^{-2} - 2 \\ 2 - e^2 - e^{-2} & e^2 - e^{-2} \end{bmatrix}$$

$$25. \begin{bmatrix} 4 & 8 \\ 12 & 16 \end{bmatrix}$$

$$27. \begin{bmatrix} \frac{1}{s} & \frac{1}{s^2} \\ \frac{2}{s^3} & \frac{1}{s-2} \end{bmatrix}$$

$$29. \begin{bmatrix} \frac{3!}{s^4} & \frac{2s}{(s^2+1)^2} & \frac{1}{(s+1)^2} \\ \frac{2-s}{s^3} & \frac{s-3}{s^2-6s+13} & \frac{3}{s} \end{bmatrix}$$

$$31. \frac{2}{s^2-1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$33. [1 \ 2t \ 3t^2]$$

$$35. \begin{bmatrix} e^t + e^{-t} & e^t - e^{-t} \\ e^t - e^{-t} & e^t + e^{-t} \end{bmatrix}$$

### Section 9.3

$$1. e^{At} = \begin{bmatrix} e^t & 0 \\ 0 & e^{-2t} \end{bmatrix}$$

$$3. e^{At} = \begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix}$$

$$5. e^{At} = \begin{bmatrix} \frac{1}{2} + \frac{1}{2}e^{2t} & -\frac{1}{2} + \frac{1}{2}e^{2t} \\ -\frac{1}{2} + \frac{1}{2}e^{2t} & \frac{1}{2} + \frac{1}{2}e^{2t} \end{bmatrix}$$

$$7. e^{At} = \begin{bmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & e^{2t} \end{bmatrix}$$

$$9. (sI - A)^{-1} = \begin{bmatrix} \frac{s-2}{s(s-3)} & \frac{-1}{s(s-3)} \\ \frac{-2}{s(s-3)} & \frac{s-1}{s(s-3)} \end{bmatrix} \text{ and } e^{At} = \begin{bmatrix} \frac{2}{3} + \frac{1}{3}e^{3t} & \frac{1}{3} - \frac{1}{3}e^{3t} \\ \frac{2}{3} - \frac{2}{3}e^{3t} & \frac{1}{3} + \frac{2}{3}e^{3t} \end{bmatrix}$$

$$11. (sI - A)^{-1} = \begin{bmatrix} \frac{s+1}{(s-1)^2+1} & \frac{5}{(s-1)^2+1} \\ \frac{-1}{(s-1)^2+1} & \frac{s-3}{(s-1)^2+1} \end{bmatrix}$$

$$\text{and } e^{At} = \begin{bmatrix} e^t \cos t + 2e^t \sin t & 5e^t \sin t \\ -e^t \sin t & e^t \cos t - 2e^t \sin t \end{bmatrix}$$

$$13. (sI - A)^{-1} = \begin{bmatrix} \frac{1}{s} & \frac{1}{s^2} & \frac{s+1}{s^3} \\ 0 & \frac{1}{s} & \frac{1}{s^2} \\ 0 & 0 & \frac{1}{s} \end{bmatrix} \text{ and } e^{At} = \begin{bmatrix} 1 & t & t + \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$

$$15. e^{At} = \begin{bmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & e^{2t} \end{bmatrix}$$

## Section 9.4

$$1. e^{At} = \begin{bmatrix} e^t + te^t & -te^t \\ te^t & e^t - te^t \end{bmatrix}$$

$$3. e^{At} = \begin{bmatrix} 1 + 2t & t \\ -4t & 1 - 2t \end{bmatrix}$$

$$5. e^{At} = \begin{bmatrix} e^t \cos t + 3e^t \sin t & -10e^t \sin t \\ e^t \sin t & e^t \cos t - 3e^t \sin t \end{bmatrix}$$

$$7. e^{At} = \frac{1}{4} \begin{bmatrix} -7e^{2t} + 11e^{-2t} & 11e^{2t} - 11e^{-2t} \\ -7e^{2t} + 7e^{-2t} & 11e^{2t} - 7e^{-2t} \end{bmatrix}$$

$$9. e^{At} = \begin{bmatrix} e^{2t} \cos 3t + 8e^{2t} \sin 3t & 13e^{2t} \sin 3t \\ -5e^{2t} \sin 3t & e^{2t} \cos 3t - 8e^{2t} \sin 3t \end{bmatrix}$$

$$11. e^{At} = \begin{bmatrix} e^{-2t} - te^{-2t} & te^{-2t} \\ -te^{-2t} & e^{-2t} + te^{-2t} \end{bmatrix}$$

$$13. e^{At} = \begin{bmatrix} 2 - e^{-t} & 0 & -1 + e^{-t} \\ 0 & e^t & 0 \\ 2 - 2e^{-t} & 0 & -1 + 2e^{-t} \end{bmatrix}$$

$$15. e^{At} = \frac{1}{2} \begin{bmatrix} e^t + e^t \cos t & -e^t \sin t & e^t - e^t \cos t \\ 2e^t \sin t & 2e^t \cos t & -2e^t \sin t \\ e^t - e^t \cos t & e^t \sin t & e^t + e^t \cos t \end{bmatrix}$$

$$17. e^{At} = \begin{bmatrix} -e^t + 2 \cos 2t - \sin 2t & e^t - \cos 2t \\ 2 \sin 2t & \cos 2t - \sin 2t \\ -2e^t + 2 \cos 2t - \sin 2t & 2e^t - \cos 2t \end{bmatrix}$$

$$19. e^{At} = \begin{bmatrix} \cos t - t \cos t & \sin t - t \sin t & t \cos t & t \sin t \\ -\sin t + t \sin t & \cos t - t \cos t & -t \sin t & t \cos t \\ -t \cos t & -t \sin t & \cos t + t \cos t & \sin t + t \sin t \\ t \sin t & -t \cos t & -\sin t - t \sin t & \cos t + t \cos t \end{bmatrix}$$

## Section 9.5

$$1. \mathbf{y}(t) = \begin{bmatrix} e^{-t} \\ -2e^{3t} \end{bmatrix}$$

$$3. \mathbf{y}(t) = \begin{bmatrix} -e^{2t} + 2te^{2t} \\ 2e^{2t} \end{bmatrix}$$

$$5. \mathbf{y}(t) = \begin{bmatrix} e^{-t} \\ 3e^{-t} \end{bmatrix}$$

$$7. \mathbf{y}(t) = \begin{bmatrix} e^t - 2te^t \\ e^t - te^t \end{bmatrix}$$

$$9. \mathbf{y}(t) = \begin{bmatrix} 2 \cos 2t + 2 \sin 2t \\ \cos 2t - \sin 2t \\ 2 \cos 2t + 2 \sin 2t \end{bmatrix}$$

$$11. \mathbf{y}(t) = \begin{bmatrix} 1 + 2e^{-t} \sin 2t \\ -2 + 2e^{-t} \cos 2t \end{bmatrix}$$

$$13. \mathbf{y}(t) = \begin{bmatrix} 2e^{-t} \cos 2t + 4e^{-t} \sin 2t \\ 1 + e^{-t} \sin 2t - 2e^{-t} \cos 2t \end{bmatrix}$$

$$15. \mathbf{y}(t) = \begin{bmatrix} 2te^t + e^t - t - 1 \\ -4te^t - e^t + 2t + 2 \end{bmatrix}$$

$$17. \mathbf{y}(t) = \begin{bmatrix} te^t \\ 2te^{2t} - e^{2t} + e^t \\ -2te^{2t} + te^t \end{bmatrix}$$

19.  $y_1(t) = 1 + 3e^{-2t}$ ,  $y_2(t) = 2 + 4e^{-t} - 6e^{-2t}$ ,  $t = 9.02$  seconds.

21.  $y_1(t) = y_2(t) = 1 - e^{-2t}$

## Section 9.6

1.  $P = \begin{bmatrix} 1 & 3 \\ -1 & -1 \end{bmatrix}$ ,  $J = P^{-1}AP = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ , and the critical point is saddle.

3.  $P = \begin{bmatrix} -3 & -1 \\ 5 & 0 \end{bmatrix}$ ,  $J = \begin{bmatrix} -2 & -1 \\ 1 & -2 \end{bmatrix}$ , and the origin is a stable spiral node.

5.  $A$  is of type  $J_3$ . The origin is an unstable star node.

7.  $P = \begin{bmatrix} 1 & 3 \\ -1 & -1 \end{bmatrix}$ ,  $J = P^{-1}AP = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$ , and the origin is an unstable node.

9.  $P = \begin{bmatrix} -1 & 1 \\ 4 & 0 \end{bmatrix}$ ,  $J = P^{-1}AP = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$ , and the origin is an unstable star node.

## Section 9.7

1.  $\Phi(t)$  is a fundamental matrix,  $\mathbf{y}(t) = \begin{bmatrix} 2e^{-t} - e^{2t} \\ 2e^{-t} - 4e^{2t} \end{bmatrix}$ , and the standard fundamental matrix at  $t = 0$  is  $\Psi(t) = \frac{1}{3} \begin{bmatrix} 4e^{-t} - e^{2t} & -e^{-t} + e^{2t} \\ 4e^{-t} - 4e^{2t} & -e^{-t} + 4e^{2t} \end{bmatrix}$ .

3.  $\Phi(t)$  is a fundamental matrix,  $\mathbf{y}(t) = \begin{bmatrix} \cos(t^2/2) \\ -\sin(t^2/2) \end{bmatrix}$ , and the standard fundamental matrix at  $t = 0$  is  $\Psi(t) = \begin{bmatrix} \cos(t^2/2) & \sin(t^2/2) \\ -\sin(t^2/2) & \cos(t^2/2) \end{bmatrix}$ .

5.  $\Phi(t)$  is a fundamental matrix,  $\mathbf{y}(t) = \frac{t}{\pi} \begin{bmatrix} -\cos t + \sin t \\ \cos t + \sin t \end{bmatrix}$ , and the standard fundamental matrix at  $t = \pi$  is  $\Psi(t) = \frac{1}{\pi} \begin{bmatrix} -t \cos t & -t \sin t \\ t \sin t & -t \cos t \end{bmatrix}$ .

7.  $\Phi(t)$  is a fundamental matrix,  $\mathbf{y}(t) = \begin{bmatrix} (t-1)e^{t-1} - 3 \\ e^{t-1} + 3 \end{bmatrix}$ , and the standard fundamental matrix at  $t = 1$  is  $\Psi(t) = \begin{bmatrix} 1 + (t-1)e^{t-1} & (t-1)e^{t-1} \\ -1 + e^{t-1} & e^{t-1} \end{bmatrix}$ .

9.  $\Psi(t) = \begin{bmatrix} t - t \ln t & -t \ln t \\ t \ln t & t + t \ln t \end{bmatrix}$  and  $\mathbf{y}(t) = \begin{bmatrix} 2t - t \ln t \\ t \ln t \end{bmatrix}$

11.  $\Psi(t) = \begin{bmatrix} \sec^2 t + 3 \sec t \tan t + \tan^2 t & 5 \sec t \tan t \\ -\sec t \tan t & \sec^2 t - 3 \sec t \tan t + \tan^2 t \end{bmatrix}$  and  
 $\mathbf{y}(t) = \begin{bmatrix} 2 \sec^2 t + 11 \sec t \tan t + 2 \tan^2 t \\ \sec^2 t - 5 \sec t \tan t + \tan^2 t \end{bmatrix}$ .

13.  $y_1(t) = 4(2-t) + (2-t)^2 - \frac{3}{4}(2-t)^4$  and  $y_2(t) = 2(2-t) + (2-t)^2 + \frac{3}{4}(2-t)^3$ .

The concentration (grams/L) of salt in Tank 1 after 1 min is  $\frac{17}{4}$  and in Tank 2 is  $\frac{15}{4}$ .

# Appendix C

## Tables

### C.1 Laplace Transforms

**Table C.1** Laplace transform rules

$f(t)$	$F(s)$	Page
<i>Definition of the Laplace transform</i>		
1. $f(t)$	$F(s) = \int_0^{\infty} e^{-st} f(t) dt$	111
<i>Linearity</i>		
2. $a_1 f_1(t) + a_2 f_2(t)$	$a_1 F_1(s) + a_2 F_2(s)$	114
<i>Dilation principle</i>		
3. $f(at)$	$\frac{1}{a} F\left(\frac{s}{a}\right)$	122
<i>Translation principle</i>		
4. $e^{at} f(t)$	$F(s - a)$	120
<i>Input derivative principle: first order</i>		
5. $f'(t)$	$sF(s) - f(0)$	115
<i>Input derivative principle: second order</i>		
6. $f''(t)$	$s^2 F(s) - sf(0) - f'(0)$	115
<i>Input derivative principle: nth order</i>		
7. $f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - sf^{(n-2)}(0) - f^{(n-1)}(0)$	116
<i>Transform derivative principle: first order</i>		
8. $tf(t)$	$-F'(s)$	121
<i>Transform derivative principle: second order</i>		
9. $t^2 f(t)$	$F''(s)$	
<i>Transform derivative principle: nth order</i>		
10. $t^n f(t)$	$(-1)^n F^{(n)}(s)$	121
<i>Convolution principle</i>		
11. $(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$	$F(s)G(s)$	188
<i>Input integral principle</i>		
12. $\int_0^t f(v)dv$	$\frac{F(s)}{s}$	190

(continued)

**Table C.1** (continued)

$f(t)$	$F(s)$	Page
<i>Transform integral formula</i>		
13. $\frac{f(t)}{t}$ $\frac{f(t)}{t}$ has a continuous extension to 0	$\int_s^\infty F(\sigma) d\sigma$	357
<i>Second translation principle</i>		
14. $f(t - c)h(t - c)$	$e^{-sc} F(s)$	405
<i>Corollary to the second translation principle</i>		
15. $g(t)h(t - c)$	$e^{-sc} \mathcal{L}\{g(t + c)\}$	405
<i>Periodic functions</i>		
16. $f(t)$ , periodic with period $p$	$\frac{\int_0^p e^{-st} f(t) dt}{1 - e^{-sp}}$	455
17. $f((t)_p)$	$\frac{\mathcal{L}\{f(t) - f(t)h(t - p)\}}{1 - e^{-sp}}$	458
<i>Staircase functions</i>		
18. $f([t]_p)$	$\frac{1 - e^{-ps}}{s} \sum_{n=0}^\infty f(np)e^{-nps}$	463
<i>Transforms involving <math>\frac{1}{1 \pm e^{-sp}}</math></i>		
19. $\sum_{N=0}^\infty \sum_{n=0}^N f(t - np)\chi_{[Np, (N+1)p)}$	$\frac{1}{1 - e^{-sp}} F(s)$	461
20. $\sum_{N=0}^\infty \sum_{n=0}^N (-1)^n f(t - np)\chi_{[Np, (N+1)p)}$	$\frac{1}{1 + e^{-sp}} F(s)$	461

**Table C.2** Laplace transforms

	$f(t)$	$F(s)$	Page
1.	1	$\frac{1}{s}$	116
2.	$t$	$\frac{1}{s^2}$	
3.	$t^n \quad (n = 0, 2, 3, \dots)$	$\frac{n!}{s^{n+1}}$	116
4.	$t^\alpha \quad (\alpha > 0)$	$\frac{\Gamma(\alpha + 1)}{s^{\alpha+1}}$	118
5.	$e^{at}$	$\frac{1}{s - a}$	118
6.	$te^{at}$	$\frac{1}{(s - a)^2}$	
7.	$t^n e^{at} \quad (n = 1, 2, 3, \dots)$	$\frac{n!}{(s - a)^{n+1}}$	119
8.	$\sin bt$	$\frac{b}{s^2 + b^2}$	118
9.	$\cos bt$	$\frac{s}{s^2 + b^2}$	118

**Table C.2** (continued)

	$f(t)$	$F(s)$	Page
10.	$e^{at} \sin bt$	$\frac{b}{(s-a)^2 + b^2}$	120
11.	$e^{at} \cos bt$	$\frac{s-a}{(s-a)^2 + b^2}$	120
12.	$\frac{\sin t}{t}$	$\tan^{-1} \frac{1}{s}$	358
13.	$\frac{\sin at}{t}$	$\tan^{-1} \left(\frac{a}{s}\right)$	365
14.	$\frac{e^{bt} - e^{at}}{t}$	$\ln \left(\frac{s-a}{s-b}\right)$	365
15.	$2 \frac{t \cos bt - \cos at}{\cos^2 bt - \cos^2 at}$	$\ln \left(\frac{s^2 + a^2}{s^2 + b^2}\right)$	365
16.	$2 \frac{\cos bt - \cos at}{t^2}$	$s \ln \left(\frac{s^2 + a^2}{s^2 + b^2}\right) - 2b \tan^{-1} \left(\frac{b}{s}\right) + 2a \tan^{-1} \left(\frac{a}{s}\right)$	365
<i>Laguerre polynomials</i>			
17.	$\ell_n(t) = \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{t^k}{k!}$	$\frac{(s-1)^n}{s^{n+1}}$	361
18.	$\ell_n(at)$	$\frac{(s-a)^n}{s^{n+1}}$	366
<i>The Heaviside function</i>			
19.	$h(t-c)$	$\frac{e^{-sc}}{s}$	404
<i>The on-off switch</i>			
20.	$\chi_{[a,b]}$	$\frac{e^{-as}}{s} - \frac{e^{-bs}}{s}$	405
<i>The Dirac delta function</i>			
21.	$\delta_c$	$e^{-cs}$	428
<i>The square-wave function</i>			
22.	$\text{sw}_c$	$\frac{1}{1+e^{-cs}} \frac{1}{s}$	456
<i>The sawtooth function</i>			
23.	$\langle t \rangle_p$	$\frac{1}{s^2} \left(1 - \frac{sp e^{-sp}}{1 - e^{-sp}}\right)$	457
<i>Periodic dirac delta functions</i>			
24.	$\delta_0(\langle t \rangle_p)$	$\frac{1}{1 - e^{-ps}}$	459
<i>Alternating periodic dirac delta functions</i>			
25.	$(\delta_0 - \delta_p)(\langle t \rangle_{2p})$	$\frac{1}{1 + e^{-ps}}$	459
<i>The matrix exponential</i>			
26.	$e^{At}$	$(sI - A)^{-1}$	459

**Table C.3** Heaviside formulas

$f(t)$	$F(s)$
<i>Heaviside formulas of the first kind</i>	
1. $\frac{e^{at}}{a-b} + \frac{e^{bt}}{b-a}$	$\frac{1}{(s-a)(s-b)}$
2. $\frac{ae^{at}}{a-b} + \frac{be^{bt}}{b-a}$	$\frac{s}{(s-a)(s-b)}$
3. $\frac{e^{at}}{(a-b)(a-c)} + \frac{e^{bt}}{(b-a)(b-c)} + \frac{e^{ct}}{(c-a)(c-b)}$	$\frac{1}{(s-a)(s-b)(s-c)}$
4. $\frac{ae^{at}}{(a-b)(a-c)} + \frac{be^{bt}}{(b-a)(b-c)} + \frac{ce^{ct}}{(c-a)(c-b)}$	$\frac{s}{(s-a)(s-b)(s-c)}$
5. $\frac{a^2e^{at}}{(a-b)(a-c)} + \frac{b^2e^{bt}}{(b-a)(b-c)} + \frac{c^2e^{ct}}{(c-a)(c-b)}$	$\frac{s^2}{(s-a)(s-b)(s-c)}$
6. $\frac{r_1^k e^{r_1 t}}{q'(r_1)} + \dots + \frac{r_n^k e^{r_n t}}{q'(r_n)}$ , $q(s) = (s-r_1)\dots(s-r_n)$	$\frac{s^k}{(s-r_1)\dots(s-r_n)}$ , $r_1, \dots, r_n$ , distinct
<i>Heaviside formulas of the second kind</i>	
7. $te^{at}$	$\frac{1}{(s-a)^2}$
8. $(1+at)e^{at}$	$\frac{s}{(s-a)^2}$
9. $\frac{t^2}{2}e^{at}$	$\frac{1}{(s-a)^3}$
10. $\left(t + \frac{at^2}{2}\right)e^{at}$	$\frac{s}{(s-a)^3}$
11. $\left(1 + 2at + \frac{a^2t^2}{2}\right)e^{at}$	$\frac{s^2}{(s-a)^3}$
12. $\left(\sum_{l=0}^k \binom{k}{l} a^{k-l} \frac{t^{n-l-1}}{(n-l-1)!}\right)e^{at}$	$\frac{s^k}{(s-a)^n}$

In each case,  $a$ ,  $b$ , and  $c$  are distinct. See Page 165.

**Table C.4** Laplace transforms involving irreducible quadratics

$f(t)$	$F(s)$
1. $\sin bt$	$\frac{b}{(s^2 + b^2)}$
2. $\frac{1}{2b^2} (\sin bt - bt \cos bt)$	$\frac{b}{(s^2 + b^2)^2}$
3. $\frac{1}{8b^4} ((3 - (bt)^2) \sin bt - 3bt \cos bt)$	$\frac{b}{(s^2 + b^2)^3}$
4. $\frac{1}{48b^6} ((15 - 6(bt)^2) \sin bt - (15bt - (bt)^3) \cos bt)$	$\frac{b}{(s^2 + b^2)^4}$
5. $\cos bt$	$\frac{s}{(s^2 + b^2)}$
6. $\frac{1}{2b^2} bt \sin bt$	$\frac{s}{(s^2 + b^2)^2}$
7. $\frac{1}{8b^4} (bt \sin bt - (bt)^2 \cos bt)$	$\frac{s}{(s^2 + b^2)^3}$
8. $\frac{1}{48b^6} ((3bt - (bt)^3) \sin bt - 3(bt)^2 \cos bt)$	$\frac{s}{(s^2 + b^2)^4}$

**Table C.5** Reduction of order formulas

$$\mathcal{L}^{-1} \left\{ \frac{1}{(s^2 + b^2)^{k+1}} \right\} = \frac{-t}{2kb^2} \mathcal{L}^{-1} \left\{ \frac{s}{(s^2 + b^2)^k} \right\} + \frac{2k-1}{2kb^2} \mathcal{L}^{-1} \left\{ \frac{1}{(s^2 + b^2)^k} \right\}$$

$$\mathcal{L}^{-1} \left\{ \frac{s}{(s^2 + b^2)^{k+1}} \right\} = \frac{t}{2k} \mathcal{L}^{-1} \left\{ \frac{1}{(s^2 + b^2)^k} \right\}$$

See Page 155.

**Table C.6** Laplace transforms involving quadratics

$f(t)$	$F(s)$	Page
<i>Laplace transforms involving the quadratic <math>s^2 + b^2</math></i>		
1. $\frac{\sin bt}{(2b)^{2k}} \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} (-1)^m \binom{2k-2m}{k} \frac{(2bt)^{2m}}{(2m)!}$ $-\frac{\cos bt}{(2b)^{2k}} \sum_{m=0}^{\lfloor \frac{k-1}{2} \rfloor} (-1)^m \binom{2k-2m-1}{k} \frac{(2bt)^{2m+1}}{(2m+1)!}$	$\frac{b}{(s^2 + b^2)^{k+1}}$	549
2. $\frac{2bt \sin bt}{k \cdot (2b)^{2k}} \sum_{m=0}^{\lfloor \frac{k-1}{2} \rfloor} (-1)^m \binom{2k-2m-2}{k-1} \frac{(2bt)^{2m}}{(2m)!}$ $-\frac{2bt \cos bt}{k \cdot (2b)^{2k}} \sum_{m=0}^{\lfloor \frac{k-2}{2} \rfloor} (-1)^m \binom{2k-2m-3}{k-1} \frac{(2bt)^{2m+1}}{(2m+1)!}$	$\frac{s}{(s^2 + b^2)^{k+1}}$	549
<i>Laplace transforms involving the quadratic <math>s^2 - b^2</math></i>		
3. $\frac{(-1)^k}{2^{2k+1}k!} \sum_{n=0}^k \frac{(2k-n)!}{n!(k-n)!} ((-2t)^n e^t - (2t)^n e^{-t})$	$\frac{1}{(s^2 - 1)^{k+1}}$	553
4. $\frac{(-1)^k}{2^{2k+1}k!} \sum_{n=1}^k \frac{(2k-n-1)!}{(n-1)!(k-n)!} ((-2t)^n e^t + (2t)^n e^{-t})$	$\frac{s}{(s^2 - 1)^{k+1}}$	553

## C.2 Convolutions

**Table C.7** Convolutions

	$f(t)$	$g(t)$	$(f * g)(t)$	Page
1.	$f(t)$	$g(t)$	$f * g(t) = \int_0^t f(u)g(t-u) du$	187
2.	1	$g(t)$	$\int_0^t g(\tau) d\tau$	190
3.	$t^m$	$t^n$	$\frac{m!n!}{(m+n+1)!} t^{m+n+1}$	193
4.	$t$	$\sin at$	$\frac{at - \sin at}{a^2}$	
5.	$t^2$	$\sin at$	$\frac{2}{a^3} (\cos at - (1 - \frac{a^2 t^2}{2}))$	
6.	$t$	$\cos at$	$\frac{1 - \cos at}{a^2}$	
7.	$t^2$	$\cos at$	$\frac{2}{a^3} (at - \sin at)$	

(continued)

**Table C.7** (continued)

	$f(t)$	$g(t)$	$(f * g)(t)$	Page
8.	$t$	$e^{at}$	$\frac{e^{at} - (1 + at)}{a^2}$	
9.	$t^2$	$e^{at}$	$\frac{2}{a^3}(e^{at} - (a + at + \frac{a^2 t^2}{2}))$	
10.	$e^{at}$	$e^{bt}$	$\frac{1}{b-a}(e^{bt} - e^{at}) \quad a \neq b$	192
11.	$e^{at}$	$e^{at}$	$t e^{at}$	192
12.	$e^{at}$	$\sin bt$	$\frac{1}{a^2 + b^2}(b e^{at} - b \cos bt - a \sin bt)$	195
13.	$e^{at}$	$\cos bt$	$\frac{1}{a^2 + b^2}(a e^{at} - a \cos bt + b \sin bt)$	195
14.	$\sin at$	$\sin bt$	$\frac{1}{b^2 - a^2}(b \sin at - a \sin bt) \quad a \neq b$	195
15.	$\sin at$	$\sin at$	$\frac{1}{2a}(\sin at - at \cos at)$	195
16.	$\sin at$	$\cos bt$	$\frac{1}{b^2 - a^2}(a \cos at - a \cos bt) \quad a \neq b$	195
17.	$\sin at$	$\cos at$	$\frac{1}{2}t \sin at$	195
18.	$\cos at$	$\cos bt$	$\frac{1}{a^2 - b^2}(a \sin at - b \sin bt) \quad a \neq b$	195
19.	$\cos at$	$\cos at$	$\frac{1}{2a}(at \cos at + \sin at)$	195
20.	$f$	$\delta_c(t)$	$f(t - c)h(t - c)$	444
21.	$f$	$\delta_0(t)$	$f(t)$	445

# Symbol Index

$\mathcal{B}_q$	standard basis of $\mathcal{E}_q$	171
$c_A(s)$	characteristic polynomial of $A$	652
$\chi_{[a, b]}$	characteristic function or on-off switch	402
$\mathbb{C}$	complex numbers	559
$d_{c, \epsilon}$	approximation to the Dirac delta function	427
$\mathbf{D}$	derivative operator	205
$\delta_c(t)$	Dirac delta function	428
$\mathcal{E}$	linear space of exponential polynomials	179
$e^{At}$	matrix exponential	649
$\mathcal{E}_q$	exponential polynomials whose Laplace transform is in $\mathcal{R}_q$	168
$\mathcal{F}$	a generic linear space of functions	112
$F_{\text{net}}$	net force acting on a body	2
$f(t)$	forcing function	2
$F(s)$	Laplace transform of $f(t)$	111
$f * g$	$f$ convolved with $g$	439
$f^{*k}$	convolution of $f$ , $k$ times	194
$\mathcal{H}$	Heaviside class	399
$h(t - c)$	translate of the Heaviside function	401
$h(t)$	Heaviside function	401
$\mathbf{L}$	differential operator	205
$\mathcal{L}$	The Laplace transform	111
$\mathcal{L}\{f(t)\}$	Laplace transform of $f(t)$	111
$\ell_n$	Laguerre polynomial of order $n$	361
$\mathcal{L}^{-1}$	inverse Laplace transform	151
$\mathcal{L}^{-1}\{F(s)\}$	inverse Laplace transform of $F(s)$	151
$\text{NS}(A)$	null space of $A$	570
$\Phi(t)$	fundamental matrix	706
$\Psi(t)$	standard fundamental matrix	706
$\mathbb{Q}$	rational numbers	464
$q(\mathbf{D})$	polynomial differential operator	205

$\mathbb{R}$	Real numbers	6
$\mathcal{R}_q$	rational functions with $q$ in the denominator	168
$\mathcal{S}$	generic spanning set	171
$sw_c$	square wave function	456
$T_n$	Chebyshev polynomial	511
$[t]_p$	Stair case function on intervals of length $p$	454
$\langle t \rangle_p$	sawtooth function with period $p$	453
$U_n$	Chebyshev polynomial	511
$W(f_1, \dots, f_n)$	Wronskian matrix of $f_1, \dots, f_n$	222
$w(f_1, \dots, f_n)$	Wronskian	222
$\mathbf{y}$	generic unknown function to a differential system	633
$y$	generic unknown function in a differential equation	3
$y_g$	general solution to a differential equation	11
$\mathbf{y}_h$	homogeneous solution for a linear differential system	667
$y_h$	homogeneous solution to a linear differential equation	53
$\mathbf{y}_p$	particular solution for a linear differential system	667
$y_p$	particular solution to a linear differential equation	53

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