

# Undergraduate Texts in Mathematics

# Undergraduate Texts in Mathematics

---

## Series Editors:

Sheldon Axler

*San Francisco State University, San Francisco, CA, USA*

Kenneth Ribet

*University of California, Berkeley, CA, USA*

## Advisory Board:

Colin C. Adams, *Williams College, Williamstown, MA, USA*

Alejandro Adem, *University of British Columbia, Vancouver, BC, Canada*

Ruth Charney, *Brandeis University, Waltham, MA, USA*

Irene M. Gamba, *The University of Texas at Austin, Austin, TX, USA*

Roger E. Howe, *Yale University, New Haven, CT, USA*

David Jerison, *Massachusetts Institute of Technology, Cambridge, MA, USA*

Jeffrey C. Lagarias, *University of Michigan, Ann Arbor, MI, USA*

Jill Pipher, *Brown University, Providence, RI, USA*

Fadil Santosa, *University of Minnesota, Minneapolis, MN, USA*

Amie Wilkinson, *University of Chicago, Chicago, IL, USA*

**Undergraduate Texts in Mathematics** are generally aimed at third- and fourth-year undergraduate mathematics students at North American universities. These texts strive to provide students and teachers with new perspectives and novel approaches. The books include motivation that guides the reader to an appreciation of interrelations among different aspects of the subject. They feature examples that illustrate key concepts as well as exercises that strengthen understanding.

For further volumes:

<http://www.springer.com/series/666>

Kenneth A. Ross

# Elementary Analysis

The Theory of Calculus

Second Edition

In collaboration with Jorge M. López, University of  
Puerto Rico, Río Piedras



Springer

Kenneth A. Ross  
Department of Mathematics  
University of Oregon  
Eugene, OR, USA

ISSN 0172-6056  
ISBN 978-1-4614-6270-5      ISBN 978-1-4614-6271-2 (eBook)  
DOI 10.1007/978-1-4614-6271-2  
Springer New York Heidelberg Dordrecht London

Library of Congress Control Number: 2013950414

Mathematics Subject Classification: 26-01, 00-01, 26A06, 26A24, 26A27, 26A42

© Springer Science+Business Media New York 2013

This work is subject to copyright. All rights are reserved by the Publisher, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilms or in any other physical way, and transmission or information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed. Exempted from this legal reservation are brief excerpts in connection with reviews or scholarly analysis or material supplied specifically for the purpose of being entered and executed on a computer system, for exclusive use by the purchaser of the work. Duplication of this publication or parts thereof is permitted only under the provisions of the Copyright Law of the Publisher's location, in its current version, and permission for use must always be obtained from Springer. Permissions for use may be obtained through RightsLink at the Copyright Clearance Center. Violations are liable to prosecution under the respective Copyright Law.

The use of general descriptive names, registered names, trademarks, service marks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

While the advice and information in this book are believed to be true and accurate at the date of publication, neither the authors nor the editors nor the publisher can accept any legal responsibility for any errors or omissions that may be made. The publisher makes no warranty, express or implied, with respect to the material contained herein.

Printed on acid-free paper

Springer is part of Springer Science+Business Media ([www.springer.com](http://www.springer.com))

# Preface

**Preface to the First Edition** A study of this book, and especially the exercises, should give the reader a thorough understanding of a few basic concepts in analysis such as continuity, convergence of sequences and series of numbers, and convergence of sequences and series of functions. An ability to read and write proofs will be stressed. A precise knowledge of definitions is essential. The beginner should memorize them; such memorization will help lead to understanding.

Chapter 1 sets the scene and, except for the completeness axiom, should be more or less familiar. Accordingly, readers and instructors are urged to move quickly through this chapter and refer back to it when necessary. The most critical sections in the book are §§7–12 in Chap. 2. If these sections are thoroughly digested and understood, the remainder of the book should be smooth sailing.

The first four chapters form a unit for a short course on analysis. I cover these four chapters (except for the enrichment sections and §20) in about 38 class periods; this includes time for quizzes and examinations. For such a short course, my philosophy is that the students are relatively comfortable with derivatives and integrals but do not really understand sequences and series, much less sequences and series of functions, so Chaps. 1–4 focus on these topics. On two

or three occasions, I draw on the Fundamental Theorem of Calculus or the Mean Value Theorem, which appears later in the book, but of course these important theorems are at least discussed in a standard calculus class.

In the early sections, especially in Chap. 2, the proofs are very detailed with careful references for even the most elementary facts. Most sophisticated readers find excessive details and references a hindrance (they break the flow of the proof and tend to obscure the main ideas) and would prefer to check the items mentally as they proceed. Accordingly, in later chapters, the proofs will be somewhat less detailed, and references for the simplest facts will often be omitted. This should help prepare the reader for more advanced books which frequently give very brief arguments.

Mastery of the basic concepts in this book should make the analysis in such areas as complex variables, differential equations, numerical analysis, and statistics more meaningful. The book can also serve as a foundation for an in-depth study of real analysis given in books such as [4, 33, 34, 53, 62, 65] listed in the bibliography.

Readers planning to teach calculus will also benefit from a careful study of analysis. Even after studying this book (or writing it), it will not be easy to handle questions such as “What is a number?” but at least this book should help give a clearer picture of the subtleties to which such questions lead.

The enrichment sections contain discussions of some topics that I think are important or interesting. Sometimes the topic is dealt with lightly, and suggestions for further reading are given. Though these sections are not particularly designed for classroom use, I hope that some readers will use them to broaden their horizons and see how this material fits in the general scheme of things.

I have benefitted from numerous helpful suggestions from my colleagues Robert Freeman, William Kantor, Richard Koch, and John Leahy and from Timothy Hall, Gimli Khazad, and Jorge López. I have also had helpful conversations with my wife Lynn concerning grammar and taste. Of course, remaining errors in grammar and mathematics are the responsibility of the author.

Several users have supplied me with corrections and suggestions that I’ve incorporated in subsequent printings. I thank them all,

---

including Robert Messer of Albion College, who caught a subtle error in the proof of Theorem 12.1.

**Preface to the Second Edition** After 32 years, it seemed time to revise this book. Since the first edition was so successful, I have retained the format and material from the first edition. The numbering of theorems, examples, and exercises in each section will be the same, and new material will be added to some of the sections. Every rule has an exception, and this rule is no exception. In §11, a theorem (Theorem 11.2) has been added, which allows the simplification of four almost-identical proofs in the section: Examples 3 and 4, Theorem 11.7 (formerly Corollary 11.4), and Theorem 11.8 (formerly Theorem 11.7).

Where appropriate, the presentation has been improved. See especially the proof of the Chain Rule 28.4, the shorter proof of Abel's Theorem 26.6, and the shorter treatment of decimal expansions in §16. Also, a few examples have been added, a few exercises have been modified or added, and a couple of exercises have been deleted.

Here are the main additions to this revision. The proof of the irrationality of  $e$  in §16 is now accompanied by an elegant proof that  $\pi$  is also irrational. Even though this is an "enrichment" section, it is especially recommended for those who teach or will teach pre-college mathematics. The Baire Category Theorem and interesting consequences have been added to the enrichment §21. Section 31, on Taylor's Theorem, has been overhauled. It now includes a discussion of Newton's method for approximating zeros of functions, as well as its cousin, the secant method. Proofs are provided for theorems that guarantee when these approximation methods work. Section 35 on Riemann-Stieltjes integrals has been improved and expanded. A new section, §38, contains an example of a continuous nowhere-differentiable function and a theorem that shows "most" continuous functions are nowhere differentiable. Also, each of §§22, 32, and 33 has been modestly enhanced.

It is a pleasure to thank many people who have helped over the years since the first edition appeared in 1980. This includes David M. Bloom, Robert B. Burckel, Kai Lai Chung, Mark Dalthorp (grandson), M. K. Das (India), Richard Dowds, Ray Hoobler,

Richard M. Koch, Lisa J. Madsen, Pablo V. Negrón Marrero (Puerto Rico), Rajiv Monsurate (India), Theodore W. Palmer, Jürg Rätz (Switzerland), Peter Renz, Karl Stromberg, and Jesús Sueiras (Puerto Rico).

Special thanks go to my collaborator, Jorge M. López, who provided a huge amount of help and support with the revision. Working with him was also a lot of fun. My plan to revise the book was supported from the beginning by my wife, Ruth Madsen Ross. Finally, I thank my editor at Springer, Kaitlin Leach, who was attentive to my needs whenever they arose.

**Epecially for the Student:** Don't be dismayed if you run into material that doesn't make sense, for whatever reason. It happens to all of us. Just tentatively accept the result as true, set it aside as something to return to, and forge ahead. Also, don't forget to use the Index or Symbols Index if some terminology or notation is puzzling.

# Contents

<b>Preface</b>	<b>v</b>
<b>1 Introduction</b>	<b>1</b>
1 The Set $\mathbb{N}$ of Natural Numbers . . . . .	1
2 The Set $\mathbb{Q}$ of Rational Numbers . . . . .	6
3 The Set $\mathbb{R}$ of Real Numbers . . . . .	13
4 The Completeness Axiom . . . . .	20
5 The Symbols $+\infty$ and $-\infty$ . . . . .	28
6 * A Development of $\mathbb{R}$ . . . . .	30
<b>2 Sequences</b>	<b>33</b>
7 Limits of Sequences . . . . .	33
8 A Discussion about Proofs . . . . .	39
9 Limit Theorems for Sequences . . . . .	45
10 Monotone Sequences and Cauchy Sequences . . . . .	56
11 Subsequences . . . . .	66
12 $\limsup$ 's and $\liminf$ 's . . . . .	78
13 * Some Topological Concepts in Metric Spaces . . . . .	83
14 Series . . . . .	95
15 Alternating Series and Integral Tests . . . . .	105
16 * Decimal Expansions of Real Numbers . . . . .	109

<b>3</b>	<b>Continuity</b>	<b>123</b>
17	Continuous Functions . . . . .	123
18	Properties of Continuous Functions . . . . .	133
19	Uniform Continuity . . . . .	139
20	Limits of Functions . . . . .	153
21	* More on Metric Spaces: Continuity . . . . .	164
22	* More on Metric Spaces: Connectedness . . . . .	178
<b>4</b>	<b>Sequences and Series of Functions</b>	<b>187</b>
23	Power Series . . . . .	187
24	Uniform Convergence . . . . .	193
25	More on Uniform Convergence . . . . .	200
26	Differentiation and Integration of Power Series . . . . .	208
27	* Weierstrass's Approximation Theorem . . . . .	216
<b>5</b>	<b>Differentiation</b>	<b>223</b>
28	Basic Properties of the Derivative . . . . .	223
29	The Mean Value Theorem . . . . .	232
30	* L'Hospital's Rule . . . . .	241
31	Taylor's Theorem . . . . .	249
<b>6</b>	<b>Integration</b>	<b>269</b>
32	The Riemann Integral . . . . .	269
33	Properties of the Riemann Integral . . . . .	280
34	Fundamental Theorem of Calculus . . . . .	291
35	* Riemann-Stieltjes Integrals . . . . .	298
36	* Improper Integrals . . . . .	331
<b>7</b>	<b>Capstone</b>	<b>339</b>
37	* A Discussion of Exponents and Logarithms . . . . .	339
38	* Continuous Nowhere-Differentiable Functions . . . . .	347
	<b>Appendix on Set Notation</b>	<b>365</b>
	<b>Selected Hints and Answers</b>	<b>367</b>
	<b>A Guide to the References</b>	<b>394</b>

References	397
Symbols Index	403
Index	405