# Appendix on Set Notation

Consider a set $S$. The notation $x \in S$ means $x$ is an element of $S$; we might also say "$x$ belongs to $S$" or "$x$ is in $S$." The notation $x \notin S$ signifies $x$ is some element but $x$ does not belong to $S$. By $T \subseteq S$ we mean each element of $T$ also belongs to $S$, i.e., $x \in T$ implies $x \in S$. Thus we have $1 \in \mathbb{N}$, $17 \in \mathbb{N}$, $-3 \notin \mathbb{N}$, $\frac{1}{2} \notin \mathbb{N}$, $\sqrt{2} \notin \mathbb{N}$, $\frac{1}{2} \in \mathbb{Q}$, $\frac{1}{2} \in \mathbb{R}$, $\sqrt{2} \in \mathbb{R}$, $\sqrt{2} \notin \mathbb{Q}$, and $\pi \in \mathbb{R}$. Also we have $\mathbb{N} \subseteq \mathbb{R}$, $\mathbb{Q} \subseteq \mathbb{R}$, and $\mathbb{R} \subseteq \mathbb{R}$.

Small finite sets can be listed using braces $\{\quad\}$. For example, $\{2, 3, 5, 7\}$ is the four-element set consisting of the primes less than 10. Sets are often described by properties of their elements via the notation

$$\{\quad : \qquad\}.$$

Before the colon the variable [$n$ or $x$, for instance] is indicated and after the colon the properties are given. For example,

$$\{n : n \in \mathbb{N} \text{ and } n \text{ is odd}\} \tag{1}$$

represents the set of positive odd integers. The colon is always read "such that," so the set in (1) is read "the set of all $n$ *such that* $n$ is

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in $\mathbb{N}$ and $n$ is odd." Likewise

$$\{x : x \in \mathbb{R} \text{ and } 1 \le x < 3\} \tag{2}$$

represents the set of all real numbers greater than or equal to 1 and less than 3. In §4 this set is abbreviated $[1, 3)$. Note that $1 \in [1, 3)$ but $3 \notin [1, 3)$. Just to streamline notation, the expressions (1) and (2) may be written as

$$\{n \in \mathbb{N} : n \text{ is odd}\} \quad \text{and} \quad \{x \in \mathbb{R} : 1 \le x < 3\}.$$

The first set is then read "the set of all $n$ in $\mathbb{N}$ *such that* $n$ is odd."

Another way to list a set is to specify a rule for obtaining its elements using some other set of elements. For example, $\{n^2 : n \in \mathbb{N}\}$ represents the set of all positive integers that are the square of other integers, i.e.,

$$\{n^2 : n \in \mathbb{N}\} = \{m \in \mathbb{N} : m = n^2 \text{ for some } n \in \mathbb{N}\} = \{1, 4, 9, 16, 25, \ldots\}.$$

Similarly $\{\sin \frac{n\pi}{4} : n \in \mathbb{N}\}$ represents the set obtained by evaluating $\sin \frac{n\pi}{4}$ for each positive integer $n$. Actually this set is finite:

$$\left\{\sin \frac{n\pi}{4} : n \in \mathbb{N}\right\} = \left\{\frac{\sqrt{2}}{2}, 1, 0, -\frac{\sqrt{2}}{2}, -1\right\}.$$

The set in (1) can also be written as $\{2n - 1 : n \in \mathbb{N}\}$. One more example: $\{x^3 : x > 3\}$ is the set of all cubes of all real numbers bigger than 3 and of course equals $\{y \in \mathbb{R} : y > 27\}$, i.e., $(27, \infty)$ in the notation of §5.

For sets $S$ and $T$, $S \setminus T$ signifies the set $\{x \in S : x \notin T\}$. For a sequence $(A_n)$ of sets, the union $\cup A_n$ and intersection $\cap A_n$ are defined by

$$\bigcup A_n = \{x : x \in A_n \text{ for at least one } n\},$$
$$\bigcap A_n = \{x : x \in A_n \text{ for all } n\}.$$

The *empty set* $\varnothing$ is the set with no elements at all. Thus, for example, $\{n \in \mathbb{N} : 2 < n < 3\} = \varnothing$, $\{r \in \mathbb{Q} : r^2 = 2\} = \varnothing$, $\{x \in \mathbb{R} : x^2 < 0\} = \varnothing$, and $[0, 2] \cap [5, \infty) = \varnothing$.

For functions $f$ and $g$, the notation $f + g$, $fg$, $f \circ g$, etc. is explained on page 128.

The end of a proof is indicated by a small black box. This replaces the classical QED.

# Selected Hints and Answers

*Notice.* These hints and answers should be consulted only after serious attempts have been made to solve the problems. Students who ignore this advice will only cheat themselves.

Many problems can be solved in several ways. Your solution need not agree with that given here. Often your solution should be more elaborate.

**1.1** *Hint*: The following algebra is needed to verify the induction step:

$$\frac{n(n+1)(2n+1)}{6} + (n+1)^2 = (n+1)\left[\frac{2n^2+n}{6} + n + 1\right] = \cdots$$
$$= \frac{(n+1)(n+2)(2n+3)}{6}.$$

**1.3** *Hint*: Suppose the identity holds for $n$. Then work on the right side of the equation with $n+1$ in place of $n$. Since $(x+y)^2 = x^2+2xy+y^2$,

$$(1+2+\cdots+n+(n+1)) = (1+2+\cdots+n)^2$$
$$+2(n+1)(1+2+\cdots n)+(n+1)^2.$$

Use Example 1 to show the second line has sum $(n+1)^3$; hence

$$(1+2+\cdots+(n+1))^2 = (1+2+\cdots n)^2+(n+1)^3 = 1^3+2^3+\cdots+(n+1)^3.$$

**1.5** *Hint*: $2 - \frac{1}{2^n} + \frac{1}{2^{n+1}} = 2 - \frac{1}{2^{n+1}}$.

**1.7** *Hint*: $7^{n+1} - 6(n+1) - 1 = 7(7^n - 6n - 1) + 36n$.

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**1.9** **(a)** $n \geq 5$ and also $n = 1$.

**(b)** Clearly the inequality holds for $n = 5$. Suppose $2^n > n^2$ for some $n \geq 5$. Then $2^{n+1} = 2 \cdot 2^n > 2n^2$, so $2^{n+1} > (n+1)^2$ provided $2n^2 \geq (n+1)^2$ or $n^2 \geq 2n+1$ for $n \geq 5$. In fact, this holds for $n \geq 3$, which can be verified using calculus or directly: $n^2 \geq 3n = 2n + n > 2n + 1$.

**1.11** **(a)** *Hint*: If $n^2 + 5n + 1$ is even, then so is $(n+1)^2 + 5(n+1) + 1 = n^2 + 5n + 1 + [2n + 6]$.

**(b)** $P_n$ is false for all $n$. *Moral*: The basis for induction ($I_1$) is crucial for mathematical induction.

**2.1** *Hint*: Imitate Example 3. You should, of course, verify your assertions concerning nonsolutions. Note there are 16 rational candidates for solving $x^2 - 24 = 0$.

**2.3** *Hint*: $\sqrt{2 + \sqrt{2}}$ is a solution of $x^4 - 4x^2 + 2 = 0$.

**2.5** *Hint*: $[3 + \sqrt{2}]^{2/3}$ is a solution of $x^6 - 22x^3 + 49 = 0$.

**2.7** **(a)** Show $x = \sqrt{4 + 2\sqrt{3}} - \sqrt{3}$ satisfies the quadratic equation $x^2 + 2x\sqrt{3} - 1 - 2\sqrt{3} = 0$. Alternatively, start with the observation $(\sqrt{4 + 2\sqrt{3}})^2 = 4 + 2\sqrt{3} = 1 + 2\sqrt{3} + 3 = (1 + \sqrt{3})^2$.

**(b)** 2.

**3.1** **(a)** A3 and A4 hold for $a \in \mathbb{N}$, but 0 and $-a$ are not in $\mathbb{N}$. Likewise M4 holds for $a \in \mathbb{N}$, but $a^{-1}$ is not in $\mathbb{N}$ unless $a = 1$. These three properties fail for $\mathbb{N}$ since they implicitly require the numbers 0, $-a$ and $a^{-1}$ to be in the system under scrutiny, namely $\mathbb{N}$ in this case.

**(b)** M4 fails in the sense discussed in (a).

**3.3** **(iv)** Apply (iii), DL, A2, A4, (ii) and A4 again to obtain

$$\begin{aligned}(-a)(-b) + (-ab) &= (-a)(-b) + (-a)b = (-a)[(-b) + b] \\ &= (-a)[b + (-b)] = (-a) \cdot 0 = 0 = ab + (-ab).\end{aligned}$$

Now by (i) we conclude $(-a)(-b) = ab$.

**(v)** Suppose $ac = bc$ and $c \neq 0$. By M4 there exists $c^{-1}$ such that $c \cdot c^{-1} = 1$. Now (supply reasons)

$$a = a \cdot 1 = a(c \cdot c^{-1}) = (ac)c^{-1} = (bc)c^{-1} = b(c \cdot c^{-1}) = b \cdot 1 = b.$$

**3.5** **(a)** If $|b| \leq a$, then $-a \leq -|b|$, so $-a \leq -|b| \leq b \leq |b| \leq a$. Now suppose $-a \leq b \leq a$. If $b \geq 0$, then $|b| = b \leq a$. If $b < 0$, then $|b| = -b \leq a$; the last inequality holds by Theorem 3.2(i) since $-a \leq b$.

**(b)** By (a), it suffices to prove $-|a-b| \leq |a| - |b| \leq |a-b|$. Each of these inequalities follows from the triangle inequality: $|b| = |(b-a)+a| \leq |b-a| + |a| = |a-b| + |a|$ which implies the first inequality; $|a| = |(a-b)+b| \leq |a-b| + |b|$ which implies the second inequality.

**3.7** **(a)** Imitate the answer to Exercise 3.5(a).

**(b)** By (a), $|a-b| < c$ if and only if $-c < a-b < c$, and this obviously holds [see O4] if and only if $b - c < a < b + c$.

**4.1** If the set is bounded above, use any three numbers $\geq$ supremum of the set; see the answers to Exercise 4.3. The sets in (h), (k) and (u) are not bounded above. Note the set in (i) is simply $[0, 1]$.

**4.3** **(a)** 1; **(c)** 7; **(e)** 1; **(g)** 3; **(i)** 1; **(k)** No sup; **(m)** 2; **(o)** 0; **(q)** 16; **(s)** $\frac{1}{2}$; **(u)** No sup; **(w)** $\frac{\sqrt{3}}{2}$. In (s), note 1 is not prime.

**4.5** **Proof** Since $\sup S$ is an upper bound for $S$, we have $\sup S \geq s$ for all $s \in S$. Also $\sup S \in S$ by assumption. Hence $\sup S$ is the maximum of $S$, i.e., $\sup S = \max S$.

**4.7** **(a)** Suppose $S \subseteq T$. Since $\sup T \geq t$ for all $t \in T$ we obviously have $\sup T \geq s$ for all $s \in S$. So $\sup T$ is an upper bound for the set $S$. Hence $\sup T$ is $\geq$ the *least* upper bound for $S$, i.e., $\sup T \geq \sup S$. A similar argument shows $\inf T \leq \inf S$; give it.

**(b)** Since $S \subseteq S \cup T$, $\sup S \leq \sup(S \cup T)$ by (a). Similarly $\sup T \leq \sup(S \cup T)$, so $\max\{\sup S, \sup T\} \leq \sup(S \cup T)$. Since $\sup(S \cup T)$ is the least upper bound for $S \cup T$, we will have equality here provided we show: $\max\{\sup S, \sup T\}$ is an upper bound for the set $S \cup T$. This is easy. If $x \in S$, then $x \leq \sup S \leq \max\{\sup S, \sup T\}$ and if $x \in T$, then $x \leq \sup T \leq \max\{\sup S, \sup T\}$. That is, $x \leq \max\{\sup S, \sup T\}$ for all $x \in S \cup T$.

**4.9** **(1)** If $s \in S$, then $-s \in -S$, so $-s \leq s_0$. Hence we have $s \geq -s_0$ by Theorem 3.2(i).

**(2)** Suppose $t \leq s$ for all $s \in S$. Then $-t \geq -s$ for all $s \in S$, i.e., $-t \geq x$ for all $x \in -S$. So $-t$ is an upper bound for the set $-S$. So $-t \geq \sup(-S)$. That is, $-t \geq s_0$ and hence $t \leq -s_0$.

**4.11** **Proof** By 4.7 there is a rational $r_1$ such that $a < r_1 < b$. By 4.7 again, there is a rational $r_2$ such that $a < r_2 < r_1$. We continue by induction: If rationals $r_1, \ldots, r_n$ have been selected so that $a < r_n < r_{n-1} < \cdots < r_2 < r_1$, then 4.7 applies to $a < r_n$ to yield a rational $r_{n+1}$ such that $a < r_{n+1} < r_n$. This process yields an infinite set $\{r_1, r_2, \ldots\}$ in $\mathbb{Q} \cap (a, b)$.

**Alternative Proof** Assume $\mathbb{Q}\cap(a,b)$ is finite. The set is nonempty by 4.7. Let $c = \min(\mathbb{Q}\cap(a,b))$. Then $a < c$, so by 4.7 there is a rational $r$ such that $a < r < c$. Then $r$ belongs to $\mathbb{Q}\cap(a,b)$, so $c \le r$, a contradiction.

**4.13** By Exercise 3.7(b), we have (i) and (ii) equivalent. The equivalence of (ii) and (iii) is obvious from the definition of an open interval.

**4.15** Assume $a \le b + \frac{1}{n}$ for all $n \in \mathbb{N}$ but $a > b$. Then $a - b > 0$, and by the Archimedean property 4.6 we have $n_0(a-b) > 1$ for some $n_0 \in \mathbb{N}$. Then $a > b + \frac{1}{n_0}$ contrary to our assumption.

**5.1** **(a)** $(-\infty, 0)$; **(b)** $(-\infty, 2]$; **(c)** $[0, \infty)$; **(d)** $(-\sqrt{8}, \sqrt{8})$.

**5.3** *Hint*: The unbounded sets are in (h), (k), (l), (o), (t) and (u).

**5.5** **Proof** Select $s_0 \in S$. Then $\inf S \le s_0 \le \sup S$ whether these symbols represent $\pm\infty$ or not.

**5.7** Use Exercise 5.4 and the fact $-(A+B) = (-A) + (-B)$.

**6.1** **(a)** If $s \le t$, then clearly $s^* \subseteq t^*$. Conversely, assume $s^* \subseteq t^*$ but that $s > t$. Then $t \in s^*$ but $t \notin t^*$, a contradiction.

**(b)** $s = t$ if and only if both $s \le t$ and $t \le s$ if and only if both $s^* \subseteq t^*$ and $t^* \subseteq s^*$ if and only if $s^* = t^*$.

**6.3** **(a)** If $r \in \alpha$ and $s \in 0^*$, then $r + s < r$, so $r + s \in \alpha$. Hence $\alpha + 0^* \subseteq \alpha$. Conversely, suppose $r \in \alpha$. Since $\alpha$ has no largest element, there is a rational $t \in \alpha$ such that $t < r$. Then $r - t$ is in $0^*$, so $r = t + (r - t) \in \alpha + 0^*$. This shows $\alpha \subseteq \alpha + 0^*$.

**(b)** $-\alpha = \{r \in \mathbb{Q} : s \notin \alpha \text{ for some rational } s < -r\}$.

**6.5** **(b)** No; it corresponds to $\sqrt[3]{2}$.

**(c)** This is the Dedekind cut corresponding to $\sqrt{2}$.

**7.1** **(a)** $\frac{1}{4}, \frac{1}{7}, \frac{1}{10}, \frac{1}{13}, \frac{1}{16}$

**(c)** $\frac{1}{3}, \frac{2}{9}, \frac{1}{9}, \frac{4}{81}, \frac{5}{243}$

**7.3** **(a)** converges to 1; **(c)** converges to 0; **(e)** does not converge; **(g)** does not converge; **(i)** converges to 0; **(k)** does not converge; **(m)** converges to 0 [this sequence is $(0, 0, 0, \ldots)$]; **(o)** converges to 0; **(q)** converges to 0 [see Exercise 9.15]; **(s)** converges to $\frac{4}{3}$.

**7.5** **(a)** Has limit 0 since $s_n = 1/(\sqrt{n^2+1} + n)$.

**(c)** $\sqrt{4n^2+n} - 2n = n/(\sqrt{4n^2+n} + 2n)$ and this is close to $\frac{n}{2n+2n}$ for large $n$. So limit appears to be $\frac{1}{4}$; it is.

**8.1** **(a)** **Formal Proof** Let $\epsilon > 0$. Let $N = \frac{1}{\epsilon}$. Then $n > N$ implies $|\frac{(-1)^n}{n} - 0| = \frac{1}{n} < \epsilon$.

**(b)** *Discussion.* We want $n^{-1/3} < \epsilon$ or $\frac{1}{n} < \epsilon^3$ or $1/\epsilon^3 < n$. So for each $\epsilon > 0$, let $N = 1/\epsilon^3$. You should write out the formal proof.

**(c)** *Discussion.* We want $|\frac{2n-1}{3n+2} - \frac{2}{3}| < \epsilon$ or $|\frac{-7}{(3n+2)\cdot 3}| < \epsilon$ or $\frac{7}{3(3n+2)} < \epsilon$ or $\frac{7}{3\epsilon} < 3n + 2$ or $\frac{7}{9\epsilon} - \frac{2}{3} < n$. So set $N$ equal to $\frac{7}{9\epsilon} - \frac{2}{3}$.

**(d)** *Discussion.* We want $(n + 6)/(n^2 - 6) < \epsilon$; we assume $n > 2$ so that absolute values can be dropped. As in Example 3 we observe $n + 6 \le 7n$ and $n^2 - 6 \ge \frac{1}{2}n^2$ provided $n > 3$. So it suffices to get $7n/(\frac{1}{2}n^2) < \epsilon$ [for $n > 3$] or $\frac{14}{\epsilon} < n$. So try $N = \max\{3, \frac{14}{\epsilon}\}$.

**8.3** *Discussion.* We want $\sqrt{s_n} < \epsilon$ or $s_n < \epsilon^2$. But $s_n \to 0$, so we can get $s_n < \epsilon^2$ for large $n$.

**Formal Proof** Let $\epsilon > 0$. Since $\epsilon^2 > 0$ and $\lim s_n = 0$, there exists $N$ so that $|s_n - 0| < \epsilon^2$ for $n > N$. Thus $s_n < \epsilon^2$ for $n > N$, so $\sqrt{s_n} < \epsilon$ for $n > N$. That is, $|\sqrt{s_n} - 0| < \epsilon$ for $n > N$. We conclude $\lim \sqrt{s_n} = 0$.

**8.5** **(a)** Let $\epsilon > 0$. Our goal is to show $s - \epsilon < s_n < s + \epsilon$ for large $n$. Since $\lim a_n = s$, there exists $N_1$ so that $|a_n - s| < \epsilon$ for $n > N_1$. In particular,

$$n > N_1 \quad \text{implies} \quad s - \epsilon < a_n. \tag{1}$$

Likewise there exists $N_2$ so that $|b_n - s| < \epsilon$ for $n > N_2$, so

$$n > N_2 \quad \text{implies} \quad b_n < s + \epsilon. \tag{2}$$

Now

$$n > \max\{N_1, N_2\} \quad \text{implies} \quad s - \epsilon < a_n \le s_n \le b_n < s + \epsilon;$$

hence $|s - s_n| < \epsilon$.

**(b)** It is easy to show $\lim(-t_n) = 0$ if $\lim t_n = 0$. Now apply (a) to the inequalities $-t_n \le s_n \le t_n$.

**8.7** **(a)** Assume $\lim \cos(\frac{n\pi}{3}) = a$. Then there exists $N$ such that $n > N$ implies $|\cos(\frac{n\pi}{3}) - a| < 1$. Consider $n > N$ and $n + 3$ where $n$ is a multiple of 6; substituting these values in the inequality gives $|1 - a| < 1$ and $|-1 - a| < 1$. By the triangle inequality

$$2 = |(1 - a) - (-1 - a)| \le |1 - a| + |-1 - a| < 1 + 1 = 2,$$

a contradiction.

**(b)** Assume $\lim(-1)^n n = a$. Then there exists $N$ such that $n > N$ implies $|(-1)^n n - a| < 1$. For an even $n > N$ and for $n + 2$ this tells us $|n-a| < 1$ and $|n+2-a| < 1$. So $2 = |n+2-a-(n-a)| \le |n + 2 - a| + |n - a| < 2$, a contradiction.

**(c)** Note the sequence takes the values $\pm\frac{\sqrt{3}}{2}$ for large $n$. Assume $\lim \sin(\frac{n\pi}{3}) = a$. Then there is $N$ such that

$$n > N \quad \text{implies} \quad |\sin\left(\frac{n\pi}{3}\right) - a| < \frac{\sqrt{3}}{2}.$$

Substituting suitable $n > N$, we obtain $\left|\frac{\sqrt{3}}{2} - a\right| < \frac{\sqrt{3}}{2}$ and $\left|\frac{-\sqrt{3}}{2} - a\right| < \frac{\sqrt{3}}{2}$. By the triangle inequality

$$\sqrt{3} = \left|\frac{\sqrt{3}}{2} - \left(\frac{-\sqrt{3}}{2}\right)\right| \le \left|\frac{\sqrt{3}}{2} - a\right| + \left|a - \left(\frac{-\sqrt{3}}{2}\right)\right|$$
$$< \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = \sqrt{3},$$

a contradiction.

**8.9** **(a)** *Hint*: There exists $N_0$ in $\mathbb{N}$ such that $s_n \ge a$ for $n > N_0$. Assume $s = \lim s_n$ and $s < a$. Let $\epsilon = a - s$ and select $N \ge N_0$ so that $|s_n - s| < \epsilon$ for $n > N$. Show $s_n < a$ for $n > N$; a picture might help.

**9.1** **(a)** $\lim(\frac{n+1}{n}) = \lim(1 + \frac{1}{n}) = \lim 1 + \lim \frac{1}{n} = 1 + 0 = 1$. The second equality is justified by Theorem 9.3 and the third equality follows from Theorem 9.7(a).

**(b)** $\lim(3n + 7)/(6n - 5) = \lim(3 + 7/n)/(6 - 5/n) = \lim(3 + 7/n)/\lim(6-5/n) = (\lim 3+7\cdot\lim(1/n))/(\lim 6-5\cdot\lim(1/n)) = (3+7\cdot 0)/(6-5\cdot 0) = \frac{1}{2}$. The second equality is justified by Theorem 9.6, the third equality follows from Theorems 9.3 and 9.2, and the fourth equality uses Theorem 9.7(a).

**9.3** First we use Theorem 9.4 twice to obtain $\lim a_n^3 = \lim a_n \cdot \lim a_n^2 = a \cdot \lim a_n^2 = a \cdot \lim a_n \cdot \lim a_n = a \cdot a \cdot a = a^3$. By Theorems 9.3 and 9.2, we have $\lim(a_n^3 + 4a_n) = \lim a_n^3 + 4 \cdot \lim a_n = a^3 + 4a$. Similarly $\lim(b_n^2 + 1) = \lim b_n \cdot \lim b_n + 1 = b^2 + 1$. Since $b^2 + 1 \neq 0$, Theorem 9.6 shows $\lim s_n = (a^3 + 4a)/(b^2 + 1)$.

**9.5** *Hint*: Let $t = \lim t_n$ and show $t = (t^2 + 2)/2t$. Then show $t = \sqrt{2}$.

**9.7** It has been shown that $s_n < \sqrt{2/(n-1)}$ for $n \ge 2$, and we need to prove $\lim s_n = 0$.

*Discussion.* Let $\epsilon > 0$. We want $s_n < \epsilon$, so it suffices to get $\sqrt{2/(n-1)} < \epsilon$ or $2/(n-1) < \epsilon^2$ or $2\epsilon^{-2} + 1 < n$.

**Formal Proof.** Let $\epsilon > 0$ and let $N = 2\epsilon^{-2}+1$. Then $n > N$ implies $s_n < \sqrt{2/(n-1)} < \sqrt{2/(2\epsilon^{-2} + 1 - 1)} = \epsilon$.

**9.9** **(a)** Let $M > 0$. Since $\lim s_n = +\infty$ there exists $N \geq N_0$ such that $s_n > M$ for $n > N$. Then clearly $t_n > M$ for $n > N$, since $s_n \leq t_n$ for all $n$. This shows $\lim t_n = +\infty$.

**(c)** Parts (a) and (b) take care of the infinite limits, so assume $(s_n)$ and $(t_n)$ converge. Since $t_n - s_n \geq 0$ for all $n > N_0$, $\lim(t_n - s_n) \geq 0$ by Exercise 8.9(a). Hence $\lim t_n - \lim s_n \geq 0$ by Theorems 9.3 and 9.2.

**9.11** **(a)** *Discussion.* Let $M > 0$ and let $m = \inf\{t_n : n \in \mathbb{N}\}$. We want $s_n + t_n > M$ for large $n$, but it suffices to get $s_n + m > M$ or $s_n > M - m$ for large $n$. So select $N$ so that $s_n > M - m$ for $n > N$.

**(b)** *Hint*: If $\lim t_n > -\infty$, then $\inf\{t_n : n \in \mathbb{N}\} > -\infty$. Use part (a).

**9.13** If $|a| < 1$, then $\lim a^n = 0$ by Theorem 9.7(b). If $a = 1$, then obviously $\lim a^n = 1$.

Suppose $a > 1$. Then $\frac{1}{a} < 1$, so $\lim(1/a)^n = 0$ as above. Thus $\lim 1/a^n = 0$. Theorem 9.10 [with $s_n = a^n$] now shows $\lim a^n = +\infty$. [This case can also be handled by applying Exercise 9.12.]

Suppose $a \leq -1$ and assume $\lim a^n$ exists. For even $n$, $a^n \geq 1$ and for odd $n$, $a^n \leq -1$. Clearly $\lim a^n = +\infty$ and $\lim a^n = -\infty$ are impossible. Assume $\lim a^n = A$ for a real number $A$. There exists $N$ such that $|a^n - A| < 1$ for $n > N$. For even $n$ this implies $A > 0$ and for odd $n$ this implies $A < 0$, a contradiction.

**9.15** Apply Exercise 9.12 with $s_n = a^n/n!$. Then $L = \lim |s_{n+1}/s_n| = \lim \frac{a}{n+1} = 0$, so $\lim s_n = 0$.

**9.17** *Discussion.* Let $M > 0$. We want $n^2 > M$ or $n > \sqrt{M}$. So let $N = \sqrt{M}$.

**10.1** increasing: (c); decreasing: (a), (f); bounded: (a), (b), (d), (f).

**10.3** The equality in the hint can be verified by induction; compare Exercise 1.5. Now by (1) in Discussion 10.3 we have

$$s_n = K + \frac{d_1}{10} + \cdots + \frac{d_n}{10^n} \leq K + \frac{9}{10} + \cdots + \frac{9}{10^n} < K + 1.$$

**10.7** Let $t = \sup S$. For each $n \in \mathbb{N}$, $t - \frac{1}{n}$ is not an upper bound for $S$, so there exists $s_n$ satisfying $t - \frac{1}{n} < s_n < t$. Now apply the squeeze lemma in Exercise 8.5.

**10.9** **(a)** $s_2 = \frac{1}{2}$, $s_3 = \frac{1}{6}$, $s_4 = \frac{1}{48}$.

**(b)** First we prove

$$0 < s_{n+1} < s_n \leq 1 \quad \text{for all} \quad n \geq 1. \qquad (1)$$

This is obvious from part (a) for $n = 1, 2, 3$. Assume (1) holds for $n$. Then $s_{n+1} < 1$, so

$$s_{n+2} = \frac{n+1}{n+2} s_{n+1}^2 = \left(\frac{n+1}{n+2} s_{n+1}\right) s_{n+1} < s_{n+1}$$

since $(\frac{n+1}{n+2}) s_{n+1} < 1$. Since $s_{n+1} > 0$ we also have $s_{n+2} > 0$. Hence $0 < s_{n+2} < s_{n+1} \leq 1$ and (1) holds by induction. Assertion (1) shows $(s_n)$ is a bounded monotone sequence, so $(s_n)$ converges by Theorem 10.2.

**(c)** Let $s = \lim s_n$. Using limit theorems we find $s = \lim s_{n+1} = \lim \frac{n}{n+1} \cdot \lim s_n^2 = s^2$. Consequently $s = 1$ or $s = 0$. But $s = 1$ is impossible since $s_n \leq \frac{1}{2}$ for $n \geq 2$. So $s = 0$.

**10.11** **(a)** Show $(t_n)$ is a bounded monotone sequence.

**(b)** The answer is not obvious! It turns out that $\lim t_n$ is a Wallis product and has value $\frac{2}{\pi}$ which is about 0.6366. Observe how much easier part (a) is than part (b).

**11.1** **(a)** 1, 5, 1, 5, 1, 5, 1, 5

**(b)** Let $\sigma(k) = n_k = 2k$. Then $(a_{n_k})$ is the sequence that takes the single value 5. [There are many other possible choices of $\sigma$.]

**11.3** **(b)** For $(s_n)$, the set $S$ of subsequential limits is $\{-1, -\frac{1}{2}, \frac{1}{2}, 1\}$. For $(t_n)$, $S = \{0\}$. For $(u_n)$, $S = \{0\}$. For $(v_n)$, $S = \{-1, 1\}$.

**(c)** $\limsup s_n = 1$, $\liminf s_n = -1$, $\limsup t_n = \liminf t_n = \lim t_n = 0$, $\limsup u_n = \liminf u_n = \lim u_n = 0$, $\limsup v_n = 1$, $\liminf v_n = -1$.

**(d)** $(t_n)$ and $(u_n)$ converge.

**(e)** $(s_n)$, $(t_n)$, $(u_n)$ and $(v_n)$ are all bounded.

**11.5** **(a)** $[0, 1]$; **(b)** $\limsup q_n = 1$, $\liminf q_n = 0$.

**11.7** Apply Theorem 11.2.

**11.9** **(a)** To show $[a, b]$ is closed, we need to consider a limit $s$ of a convergent sequence $(s_n)$ from $[a, b]$ and show $s$ is also in $[a, b]$. But this was done in Exercise 8.9.

**(b)** No! $(0, 1)$ is not closed, i.e., $(0, 1)$ does not have the property described in Theorem 11.9. For example, $t_n = \frac{1}{n}$ defines a sequence in $(0, 1)$ such that $t = \lim t_n$ does *not* belong to $(0, 1)$.

**11.11** Let $t = \sup S$. There are several ways to prove the result. (1) Provide an inductive definition where $s_k \geq \max\{s_{k-1}, t - \frac{1}{k}\}$ for all $k$. (2) Apply Theorem 11.2 directly. (3) Use the sequence $(s_n)$ obtained in Exercise 10.7, and show $t_n = \max\{s_1, \ldots, s_n\}$ defines an increasing sequence $(t_n)$ in $S$ converging to $t$.

**12.1** Let $u_N = \inf\{s_n : n > N\}$ and $w_N = \inf\{t_n : n > N\}$. Then $(u_N)$ and $(w_N)$ are increasing sequences and $u_N \le w_N$ for all $N > N_0$. By Exercise 9.9(c), $\liminf s_n = \lim u_N \le \lim w_N = \liminf t_n$. The inequality $\limsup s_n \le \limsup t_n$ can be shown in a similar way or one can apply Exercise 11.8.

**12.3** **(a)** 0; **(b)** 1; **(c)** 2; **(d)** 3; **(e)** 4; **(f)** 0; **(g)** 2.

**12.5** By Exercise 12.4, $\limsup(-s_n - t_n) \le \limsup(-s_n) + \limsup(-t_n)$, so $-\limsup(-(s_n + t_n)) \ge -\limsup(-s_n) + [-\limsup(-t_n)]$. Now apply Exercise 11.8.

**12.7** Let $(s_{n_j})$ be a subsequence of $(s_n)$ such that $\lim_{j\to\infty} s_{n_j} = +\infty$. [We used $j$ here instead of $k$ to avoid confusion with the given $k > 0$.] Then $\lim_{j\to\infty} ks_{n_j} = +\infty$ by Exercise 9.10(a). Since $(ks_{n_j})$ is a subsequence of $(ks_n)$, we conclude $\limsup(ks_n) = +\infty$.

**12.9** **(a)** Since $\liminf t_n > 0$, there exists $N_1$ such that $m = \inf\{t_n : n > N_1\} > 0$. Now consider $M > 0$. Since $\lim s_n = +\infty$, there exists $N_2$ such that $s_n > \frac{M}{m}$ for $n > N_2$. Then $n > \max\{N_1, N_2\}$ implies $s_n t_n > (\frac{M}{m})t_n \ge (\frac{M}{m})m = M$. Hence $\lim s_n t_n = +\infty$.

**12.11** **Partial Proof** Let $M = \liminf |s_{n+1}/s_n|$ and $\beta = \liminf |s_n|^{1/n}$. To show $M \le \beta$, it suffices to prove $M_1 \le \beta$ for all $M_1 < M$. Since

$$\liminf \left|\frac{s_{n+1}}{s_n}\right| = \lim_{N\to\infty} \inf\left\{\left|\frac{s_{n+1}}{s_n}\right| : n > N\right\} > M_1,$$

there exists $N$ such that

$$\inf\left\{\left|\frac{s_{n+1}}{s_n}\right| : n > N\right\} > M_1.$$

Now imitate the proof in Theorem 12.2, but note that many of the inequalities will be reversed.

**12.13** **Proof** of $\sup A = \liminf s_n$. Consider $N$ in $\mathbb{N}$ and observe $u_N = \inf\{s_n : n > N\}$ is a number in $A$, since $\{n \in \mathbb{N} : s_n < u_N\} \subseteq \{1, 2, \ldots, N\}$. So $u_N \le \sup A$ for all $N$ and consequently $\liminf s_n = \lim u_N \le \sup A$.

Next consider $a \in A$. Let $N_0 = \max\{n \in \mathbb{N} : s_n < a\} < \infty$. Then $s_n \ge a$ for $n > N_0$. Thus for $N \ge N_0$ we have $u_N = \inf\{s_n : n > N\} \ge a$. It follows that $\liminf s_n = \lim u_N \ge a$. We have just shown that $\liminf s_n$ is an upper bound for the set $A$. Therefore $\liminf s_n \ge \sup A$.

**13.1** **(a)** It is clear that $d_1$ and $d_2$ satisfy D1 and D2 of Definition 13.1. If $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^k$, then for each $j = 1, 2, \ldots, k$,

$$|x_j - z_j| \le |x_j - y_j| + |y_j - z_j| \le d_1(\boldsymbol{x}, \boldsymbol{y}) + d_1(\boldsymbol{y}, \boldsymbol{z}),$$

so $d_1(\boldsymbol{x},\boldsymbol{z}) \le d_1(\boldsymbol{x},\boldsymbol{y}) + d_1(\boldsymbol{y},\boldsymbol{z})$. So $d_1$ satisfies the triangle inequality and a similar argument works for $d_2$; give it.

**(b)** For the completeness of $d_1$ we use Theorem 13.4 and the inequalities

$$d_1(\boldsymbol{x},\boldsymbol{y}) \le d(\boldsymbol{x},\boldsymbol{y}) \le \sqrt{k}\, d_1(\boldsymbol{x},\boldsymbol{y}).$$

In fact, if $(\boldsymbol{x}_n)$ is Cauchy for $d_1$, then the second inequality shows $(\boldsymbol{x}_n)$ is Cauchy for $d$. Hence by Theorem 13.4, for some $\boldsymbol{x} \in \mathbb{R}^k$ we have $\lim d(\boldsymbol{x}_n,\boldsymbol{x}) = 0$. By the first inequality, we also have $\lim d_1(\boldsymbol{x}_n,\boldsymbol{x}) = 0$, i.e., $(\boldsymbol{x}_n)$ converges to $\boldsymbol{x}$ in the metric $d_1$. For $d_2$, use the completeness of $d_1$ and the inequalities $d_1(\boldsymbol{x},\boldsymbol{y}) \le d_2(\boldsymbol{x},\boldsymbol{y}) \le k\, d_1(\boldsymbol{x},\boldsymbol{y})$.

**13.3** **(b)** No, because $d^*(\boldsymbol{x},\boldsymbol{y})$ need not be finite. For example, consider the elements $\boldsymbol{x} = (1,1,1,\ldots)$ and $\boldsymbol{y} = (0,0,0,\ldots)$.

**13.7** **Outline of Proof** Consider an open set $U \subseteq \mathbb{R}$. Let $(q_n)$ be an enumeration of the rationals in $U$. For each $n$, let

$$a_n = \inf\{a \in \mathbb{R} : (a,q_n] \subseteq U\}, \quad b_n = \sup\{b \in \mathbb{R} : [q_n,b) \subseteq U\}.$$

Show $(a_n,b_n) \subseteq U$ for each $n$ and $U = \bigcup_{n=1}^{\infty}(a_n,b_n)$. Show

$$(a_n,b_n) \cap (a_m,b_m) \ne \varnothing \quad \text{implies} \quad (a_n,b_n) = (a_m,b_m).$$

Now either there will be only finitely many distinct [and disjoint] intervals or else a subsequence $\{(a_{n_k},b_{n_k})\}_{k=1}^{\infty}$ of $\{(a_n,b_n)\}$ will consist of disjoint intervals for which $\bigcup_{k=1}^{\infty}(a_{n_k},b_{n_k}) = U$.

**13.9** **(a)** $\{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$; **(b)** $\mathbb{R}$; **(c)** $[-\sqrt{2},\sqrt{2}]$.

**13.11** Suppose $E$ is compact, hence closed and bounded by Theorem 13.12. Consider a sequence $(\boldsymbol{x}_n)$ in $E$. By Theorem 13.5, a subsequence of $(x_n)$ converges to some $\boldsymbol{x}$ in $\mathbb{R}^k$. Since $E$ is closed, $\boldsymbol{x}$ must be *in* $E$; see Proposition 13.9(b).

Suppose every sequence in $E$ has a subsequence converging to a point *in* $E$. By Theorem 13.12, it suffices to show $E$ is closed and bounded. If $E$ were unbounded, $E$ would contain a sequence $(\boldsymbol{x}_n)$ where $\lim d(\boldsymbol{x}_n,\mathbf{0}) = +\infty$ and then no subsequence would converge at all. Thus $E$ is bounded. If $E$ were nonclosed, then by Proposition 13.9 there would be a convergent sequence $(\boldsymbol{x}_n)$ in $E$ such that $\boldsymbol{x} = \lim \boldsymbol{x}_n \notin E$. Since every subsequence would also converge to $\boldsymbol{x} \notin E$, we would have a contradiction.

**13.13** Assume, for example, that $\sup E \notin E$. The set $E$ is bounded, so by Exercise 10.7, there exists a sequence $(s_n)$ in $E$ where $\lim s_n = \sup E$. Now Proposition 13.9(b) shows $\sup E \in E$, a contradiction.

**13.15** **(a)** $F$ is bounded because $d(\boldsymbol{x}, \mathbf{0}) \leq 1$ for all $\boldsymbol{x} \in F$ where $\mathbf{0} = (0, 0, 0, \ldots)$. To show $F$ is closed, consider a convergent sequence $(\boldsymbol{x}^{(n)})$ in $F$. We need to show $\boldsymbol{x} = \lim \boldsymbol{x}^{(n)}$ is in $F$. For each $j = 1, 2, \ldots$, it is easy to see $\lim_{n\to\infty} x_j^{(n)} = x_j$. Since each $x_j^{(n)}$ belongs to $[-1, 1]$, $x_j$ belongs to $[-1, 1]$ by Exercise 8.9. It follows that $\boldsymbol{x} \in F$.

**(b)** For the last assertion of the hint, observe $\boldsymbol{x}^{(n)}, \boldsymbol{x}^{(m)}$ in $U(\boldsymbol{x})$ implies $d(\boldsymbol{x}^{(n)}, \boldsymbol{x}^{(m)}) \leq d(\boldsymbol{x}^{(n)}, \boldsymbol{x}) + d(\boldsymbol{x}, \boldsymbol{x}^{(m)}) < 2$ while $d(\boldsymbol{x}^{(n)}, \boldsymbol{x}^{(m)}) = 2$ for $m \neq n$. Now show no finite subfamily of $\mathcal{U}$ can cover $\{\boldsymbol{x}^{(n)} : n \in \mathbb{N}\}$.

**14.1** **(a), (b), (c)** Converge; use Ratio Test.

**(d)** Diverges; use Ratio Test or show $n$th terms don't converge to 0 [see Corollary 14.5].

**(e)** Compare with $\sum 1/n^2$.

**(f)** Compare with $\sum \frac{1}{n}$.

**14.3** All but (e) converge.

**14.5** **(a)** We assume the series begin with $n = 1$. Let $s_n = \sum_{j=1}^{n} a_j$ and $t_n = \sum_{j=1}^{n} b_j$. We are given $\lim s_n = A$ and $\lim t_n = B$. Hence $\lim(s_n + t_n) = A + B$ by Theorem 9.3. Clearly $s_n + t_n = \sum_{j=1}^{n}(a_j + b_j)$ is the $n$th partial sum for $\sum(a_n + b_n)$, so $\sum(a_n + b_n) = \lim(s_n + t_n) = A + B$.

**(c)** The conjecture is not even reasonable for series of two terms: $a_1b_1 + a_2b_2 \neq (a_1 + a_2)(b_1 + b_2)$.

**14.7** By Corollary 14.5, there exists $N$ such that $a_n < 1$ for $n > N$. Since $p > 1$, $a_n^p = a_n a_n^{p-1} < a_n$ for $n > N$. Hence $\sum_{n=N+1}^{\infty} a_n^p$ converges by the Comparison Test, so $\sum a_n^p$ also converges.

**14.9** *Hint*: Let $N_0 = \max\{n \in N : a_n \neq b_n\} < \infty$. If $n \geq m > N_0$, then $\sum_{k=m}^{n} a_k = \sum_{k=m}^{n} b_k$.

**14.11** Assume $a_{n+1}/a_n = r$ for $n \geq 1$. Then $a_2 = ra_1$, $a_3 = r^2 a_2$, etc. A simple induction argument shows $a_n = r^{n-1}a_1$ for $n \geq 1$. Thus $\sum a_n = \sum a_1 r^{n-1}$ is a geometric series.

**14.13** **(a)** 2 and $-\frac{2}{5}$.

**(b)** Note

$$s_n = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1}$$

since the intermediate fractions cancel out. Hence $\lim s_n = 1$.

**(d)** 2.

**15.1** **(a)** Converges by Alternating Series Theorem.

**(b)** Diverges; note $\lim(n!/2^n) = +\infty$ by Exercise 9.12(b).

**15.3** *Hint*: Use integral tests. Note

$$\lim_{n\to\infty} \int_3^n \frac{1}{x(\log x)^p}\,dx = \lim_{n\to\infty} \int_{\log 3}^{\log n} \frac{1}{u^p}\,du.$$

**15.5** There is no smallest $p_0 > 1$, so there is no single series $\sum 1/n^{p_0}$ with which all series $\sum 1/n^p$ $[p > 1]$ can be compared.

**15.7** **(a)** **Proof** Let $\epsilon > 0$. By the Cauchy criterion, there exists $N$ such that $n \ge m > N$ implies $|\sum_{k=m}^{n} a_k| < \frac{\epsilon}{2}$. In particular,

$$n > N \quad \text{implies} \quad a_{N+1} + \cdots + a_n < \frac{\epsilon}{2}.$$

So $n > N$ implies

$$(n - N)a_n \le a_{N+1} + \cdots + a_n < \frac{\epsilon}{2}.$$

If $n > 2N$, then $n < 2(n - N)$, so $na_n < 2(n - N)a_n < \epsilon$. This proves $\lim(na_n) = 0$.

**16.1** **(a)** In other words, show

$$2+7\cdot 10^{-1}+4\cdot 10^{-2}+\sum_{j=3}^{\infty} 9\cdot 10^{-j} = 2+7\cdot 10^{-1}+5\cdot 10^{-2} = \frac{11}{4}.$$

The series is a geometric series; see Example 1 of §14.

**(b)** $2.75\overline{0}$

**16.3** Let $A$ and $B$ denote the sums of the series. By Exercise 14.5, we have $B - A = \sum(b_n - a_n)$. Since $b_n - a_n \ge 0$ for all $n$, and $b_n - a_n > 0$ for some $n$, we clearly have $B - A > 0$.

**16.5** **(a)** $.125\overline{0}$ *and* $.124\overline{9}$; **(c)** $.\overline{6}$; **(e)** $.\overline{54}$

**16.7** No.

**16.9** **(a)** $\gamma_n - \gamma_{n+1} = \int_n^{n+1} t^{-1}\,dt - \frac{1}{n+1} > 0$ since $\frac{1}{n+1} < t^{-1}$ for all $t$ in $[n, n + 1)$.

**(b)** For any $n$, $\gamma_n \le \gamma_1 = 1$. Also

$$\gamma_n > \sum_{k=1}^{n} \left( \frac{1}{k} - \int_k^{k+1} t^{-1}\,dt \right) > 0.$$

**(c)** Apply Theorem 10.2.

**17.1** **(a)** $\mathrm{dom}(f+g) = \mathrm{dom}(fg) = (-\infty, 4]$, $\mathrm{dom}(f\circ g) = [-2, 2]$, $\mathrm{dom}(g\circ f) = (-\infty, 4]$.

**(b)** $f \circ g(0) = 2$, $g \circ f(0) = 4$, $f \circ g(1) = \sqrt{3}$, $g \circ f(1) = 3$, $f \circ g(2) = 0$, $g \circ f(2) = 2$.

**(c)** No!

**(d)** $f \circ g(3)$ is not, but $g \circ f(3)$ is.

**17.3** **(a)** We are given $f(x) = \cos x$ and $g(x) = x^4$ $[p = 4]$ are continuous. So $g \circ f$ is continuous by Theorem 17.5, i.e., the function $g \circ f(x) = \cos^4 x$ is continuous. Obviously the function identically 1 is continuous [if you do not find this obvious, check it]. Hence $1 + \cos^4 x$ is continuous by Theorem 17.4(i). Finally $\log_e(1 + \cos^4 x)$ is continuous by Theorem 17.5 since this is $h \circ k(x)$ where $k(x) = 1 + \cos^4 x$ and $h(x) = \log_e x$.

**(b)** Since we are given $\sin x$ and $x^2$ are continuous, Theorem 17.5 shows $\sin^2 x$ is continuous. Similarly, $\cos^6 x$ is continuous. Hence $\sin^2 x + \cos^6 x$ is continuous by Theorem 17.4(i). Since $\sin^2 x + \cos^6 x > 0$ for all $x$ and since $x^\pi$ is given to be continuous for $x > 0$, we use Theorem 17.5 again to conclude $[\sin^2 x + \cos^6 x]^\pi$ is continuous.

**(e)** We are given $\sin x$ and $\cos x$ are continuous at each $x \in \mathbb{R}$. So Theorem 17.4(iii) shows $\frac{\sin x}{\cos x} = \tan x$ is continuous wherever $\cos x \neq 0$, i.e., for $x \neq$ odd multiple of $\frac{\pi}{2}$.

**17.5** **(a)** *Remarks.* An $\epsilon$–$\delta$ proof can be given based on the identity

$$x^m - y^m = (x - y)(x^{m-1} + x^{m-2}y + \cdots + xy^{m-2} + y^{m-1}).$$

Or the result can be proved by induction on $m$, as follows. It is easy to prove $g(x) = x$ is continuous on $\mathbb{R}$. If $f(x) = x^m$ is continuous on $\mathbb{R}$, then so is $(fg)(x) = x^{m+1}$ by Theorem 17.4(ii).

**(b)** Just use (a) and Theorems 17.4(i) and 17.3.

**17.9** **(a)** *Discussion.* Let $\epsilon > 0$. We want $|x^2 - 4| < \epsilon$ for $|x - 2|$ small, i.e., we want $|x - 2| \cdot |x + 2| < \epsilon$ for $|x - 2|$ small. If $|x - 2| < 1$, then $|x + 2| < 5$, so it suffices to get $|x - 2| \cdot 5 < \epsilon$. Set $\delta = \min\{1, \frac{\epsilon}{5}\}$.

**(c)** For $\epsilon > 0$, let $\delta = \epsilon$ and observe

$$|x - 0| < \delta \quad \text{implies} \quad \left| x \sin\left(\frac{1}{x}\right) - 0 \right| < \epsilon.$$

**17.11** If $f$ is continuous at $x_0$ and if $(x_n)$ is a monotonic sequence in $\text{dom}(f)$ converging to $x_0$, then we have $\lim f(x_n) = f(x_0)$ by Definition 17.1.

Now assume

$$\text{if } (x_n) \text{ is monotonic in } \operatorname{dom}(f) \text{ and } \lim x_n = x_0, \quad \text{then } \lim f(x_n) = f(x_0), \tag{1}$$

but $f$ is discontinuous at $x_0$. Then by Definition 17.1, there exists a sequence $(x_n)$ in $\operatorname{dom}(f)$ such that $\lim x_n = x_0$ but $(f(x_n))$ does not converge to $f(x_0)$. Negating Definition 17.1, we see there exists $\epsilon > 0$ such that

$$\text{for each } N \text{ there is } n > N \text{ satisfying } |f(x_n) - f(x_0)| \geq \epsilon. \tag{2}$$

It is easy to use (2) to obtain a subsequence $(x_{n_k})$ of $(x_n)$ such that

$$|f(x_{n_k}) - f(x_0)| \geq \epsilon \quad \text{for all} \quad k. \tag{3}$$

Now Theorem 11.4 shows $(x_{n_k})$ has a monotonic subsequence $(x_{n_{k_j}})$. By (1) we have $\lim_{j\to\infty} f(x_{n_{k_j}}) = f(x_0)$, but by (3) we have $|f(x_{n_{k_j}}) - f(x_0)| \geq \epsilon$ for all $j$, a contradiction.

**17.13** **(a)** *Hint*: Let $x \in \mathbb{R}$. Select a sequence $(x_n)$ such that $\lim x_n = x$, $x_n$ is rational for even $n$, and $x_n$ is irrational for odd $n$. Then $f(x_n)$ is 1 for even $n$ and 0 for odd $n$, so $(f(x_n))$ cannot converge.

**17.15** We abbreviate

- **(i)** $f$ is continuous at $x_0$,
- **(ii)** $\lim f(x_n) = f(x_0)$ for every sequence $(x_n)$ in $\operatorname{dom}(f) \setminus \{x_0\}$ converging to $x_0$.

From Definition 17.1 it is clear that (i) implies (ii). Assume (ii) holds but (i) fails. As in the solution to Exercise 17.11, there is a sequence $(x_n)$ in $\operatorname{dom}(f)$ and an $\epsilon > 0$ such that $\lim x_n = x_0$ and $|f(x_n) - f(x_0)| \geq \epsilon$ for all $n$. Obviously $x_n \neq x_0$ for all $n$, i.e., $(x_n)$ is in $\operatorname{dom}(f) \setminus \{x_0\}$. The existence of this sequence contradicts (ii).

**18.3** This exercise was deliberately poorly stated, as if $f$ must have a maximum and minimum on $[0, 5)$; see the comments following Theorem 18.1. The minimum of $f$ on $[0, 5)$ is $1 = f(0) = f(3)$, but $f$ has *no maximum* on $[0, 5)$ though $\sup\{f(x) : x \in [0, 5)\} = 21$.

**18.5** **(a)** Let $h = f - g$. Then $h$ is continuous [why?] and $h(b) \leq 0 \leq h(a)$. Now apply Theorem 18.2.

**(b)** Use the function $g$ defined by $g(x) = x$ for $x \in [0, 1]$.

**18.7** *Hint*: Let $f(x) = xe^x$; $f$ is continuous, $f(0) = 0$ and $f(1) = e$.

**18.9** Let $f(x) = a_0 + a_1 x + \cdots + a_n x^n$ where $a_n \neq 0$ and $n$ is odd. We may suppose $a_n = 1$; otherwise we would work with $(1/a_n)f$. Since

$f$ is continuous, Theorem 18.2 shows it suffices to show $f(x) < 0$ for some $x$ and $f(x) > 0$ for some other $x$. This is true because $\lim_{x\to\infty} f(x) = +\infty$ and $\lim_{x\to-\infty} f(x) = -\infty$ [remember $a_n = 1$], but we can avoid these limit notions as follows. Observe

$$f(x) = x^n \left[1 + \frac{a_0 + a_1 x + \cdots + a_{n-1}x^{n-1}}{x^n}\right]. \tag{1}$$

Let $c = 1 + |a_0| + |a_1| + \cdots + |a_{n-1}|$. If $|x| > c$, then

$$|a_0 + a_1 x + \cdots + a_{n-1}x^{n-1}| \le (|a_0| + |a_1| + \cdots + |a_{n-1}|)|x|^{n-1} < |x|^n,$$

so the number in brackets in (1) is positive. Now if $x > c$, then $x^n > 0$, so $f(x) > 0$. And if $x < -c$, then $x^n < 0$ [why?], so $f(x) < 0$.

**19.1** *Hints*: To decide (a) and (b), use Theorem 19.2. Parts (c), (e), (f) and (g) can be settled using Theorem 19.5. Theorem 19.4 can also be used to decide (e) and (f); compare Example 6. One needs to resort to the definition to handle (d).

**19.3** **(a)** *Discussion.* Let $\epsilon > 0$. We want

$$\left|\frac{x}{x+1} - \frac{y}{y+1}\right| < \epsilon \quad \text{or} \quad \left|\frac{x-y}{(x+1)(y+1)}\right| < \epsilon$$

for $|x-y|$ small, $x, y \in [0,2]$. Since $x+1 \ge 1$ and $y+1 \ge 1$ for $x, y \in [0,2]$, it suffices to get $|x-y| < \epsilon$. So we let $\delta = \epsilon$.
**Formal Proof** Let $\epsilon > 0$ and let $\delta = \epsilon$. Then $x, y \in [0,2]$ and $|x-y| < \delta = \epsilon$ imply

$$|f(x) - f(y)| = \left|\frac{x-y}{(x+1)(y+1)}\right| \le |x-y| < \epsilon.$$

**(b)** *Discussion.* Let $\epsilon > 0$. We want $|g(x) - g(y)| = |\frac{5y-5x}{(2x-1)(2y-1)}| < \epsilon$ for $|x-y|$ small, $x \ge 1$, $y \ge 1$. For $x, y \ge 1$, we have $2x-1 \ge 1$ and $2y-1 \ge 1$, so it suffices to get $|5y-5x| < \epsilon$. So let $\delta = \frac{\epsilon}{5}$. You should write out the formal proof.

**19.5** **(a)** $\tan x$ is uniformly continuous on $[0, \frac{\pi}{4}]$ by Theorem 19.2.
**(b)** $\tan x$ is not uniformly continuous on $[0, \frac{\pi}{2})$ by Exercise 19.4(a), since the function is not bounded on that set.
**(c)** Let $\tilde{h}$ be as in Example 9. Then $(\sin x)\tilde{h}(x)$ is a continuous extension of $(\frac{1}{x})\sin^2 x$ on $(0, \pi]$. Apply Theorem 19.5.
**(e)** $\frac{1}{x-3}$ is not uniformly continuous on $(3,4)$ by Exercise 19.4(a), so it is not uniformly continuous on $(3, \infty)$ either.

**(f)** *Remark.* It is easy to give an $\epsilon$–$\delta$ proof that $\frac{1}{x-3}$ is uniformly continuous on $(4, \infty)$. It is even easier to apply Theorem 19.6.

**19.7** **(a)** We are given $f$ is uniformly continuous on $[k, \infty)$, and $f$ is uniformly continuous on $[0, k+1]$ by Theorem 19.2. Let $\epsilon > 0$. There exist $\delta_1$ and $\delta_2$ so that

$$|x - y| < \delta_1, \ x, y \in [k, \infty) \quad \text{imply} \quad |f(x) - f(y)| < \epsilon,$$

$$|x - y| < \delta_2, \ x, y \in [0, k+1] \quad \text{imply} \quad |f(x) - f(y)| < \epsilon.$$

Let $\delta = \min\{1, \delta_1, \delta_2\}$ and show

$$|x - y| < \delta, \ x, y \in [0, \infty) \quad \text{imply} \quad |f(x) - f(y)| < \epsilon.$$

**19.9** **(c)** This is tricky, but it turns out that $f$ is uniformly continuous on $\mathbb{R}$. A simple modification of Exercise 19.7(a) shows it suffices to show $f$ is uniformly continuous on $[1, \infty)$ and $(-\infty, -1]$. This can be done using Theorem 19.6. Note we cannot apply Theorem 19.6 on $\mathbb{R}$ because $f$ is not differentiable at $x = 0$; also $f'$ is not bounded near $x = 0$.

**19.11** As in the solution to Exercise 19.9(c), it suffices to show $\tilde{h}$ is uniformly continuous on $[1, \infty)$ and $(-\infty, -1]$. Apply Theorem 19.6.

**20.1** $\lim_{x \to \infty} f(x) = \lim_{x \to 0^+} f(x) = 1$; $\lim_{x \to 0^-} f(x) = \lim_{x \to -\infty} f(x) = -1$; $\lim_{x \to 0} f(x)$ does NOT EXIST.

**20.3** $\lim_{x \to \infty} f(x) = \lim_{x \to -\infty} f(x) = 0$; $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x) = \lim_{x \to 0} f(x) = 1$.

**20.5** Let $S = (0, \infty)$. Then $f(x) = 1$ for all $x \in S$. So for any sequence $(x_n)$ in $S$ we have $\lim f(x_n) = 1$. It follows that $\lim_{x \to 0^S} f(x) = \lim_{x \to \infty^S} f(x) = 1$, i.e., $\lim_{x \to 0^+} f(x) = \lim_{x \to \infty} f(x) = 1$. Likewise if $S = (-\infty, 0)$, then $\lim_{x \to 0^S} f(x) = \lim_{x \to -\infty^S} f(x) = -1$, so $\lim_{x \to 0^-} f(x) = \lim_{x \to -\infty} f(x) = -1$. Theorem 20.10 shows $\lim_{x \to 0} f(x)$ does not exist.

**20.7** If $(x_n)$ is a sequence in $(0, \infty)$ and $\lim x_n = +\infty$, then $\lim(1/x_n) = 0$. Since $(\sin x_n)$ is a bounded sequence, we conclude $\lim(\sin x_n)/x_n = 0$ by Exercise 8.4. Hence $\lim_{x \to \infty} f(x) = 0$. Similarly $\lim_{x \to -\infty} f(x) = 0$. The remaining assertion is $\lim_{x \to 0} \frac{\sin x}{x} = 1$ which is discussed in Example 9 of §19.

**20.9** $\lim_{x \to \infty} f(x) = -\infty$; $\lim_{x \to 0^+} f(x) = +\infty$; $\lim_{x \to 0^-} f(x) = -\infty$; $\lim_{x \to -\infty} f(x) = +\infty$; $\lim_{x \to 0} f(x)$ does NOT EXIST.

**20.11** **(a)** $2a$; **(c)** $3a^2$.

**20.13** First note that if $\lim_{x \to a^S} f(x)$ exists and is finite and if $k \in \mathbb{R}$, then $\lim_{x \to a^S}(kf)(x) = k \cdot \lim_{x \to a^S} f(x)$. This is Theorem 20.4(ii) where $f_1$ is the constant $k$ and $f_2 = f$.

**(a)** The remark above and Theorem 20.4 show

$$\lim_{x \to a}[3f(x) + g(x)^2] = 3 \lim_{x \to a} f(x) + [\lim_{x \to a} g(x)]^2 = 3 \cdot 3 + 2^2 = 13.$$

**(c)** As in (a), $\lim_{x \to a}[3f(x) + 8g(x)] = 25$. There exists an open interval $J$ containing $a$ such that $f(x) > 0$ and $g(x) > 0$ for $x \in J \setminus \{a\}$. Theorem 20.5 applies with $S = J \setminus \{a\}$, $3f + 8g$ in place of $f$ and with $g(x) = \sqrt{x}$ to give $\lim_{x \to a} \sqrt{3f(x) + 8g(x)} = \sqrt{25} = 5$.

**20.15** Let $(x_n)$ be a sequence in $(-\infty, 2)$ such that $\lim x_n = -\infty$. We contend

$$\lim(x_n - 2)^{-3} = 0. \qquad (1)$$

We apply Exercises 9.10 and 9.11 and Theorems 9.9 and 9.10 to conclude $\lim(-x_n) = +\infty$, $\lim(2 - x_n) = +\infty$, $\lim(2 - x_n)^3 = +\infty$, $\lim(2 - x_n)^{-3} = 0$, and hence (1) holds.
Now consider a sequence $(x_n)$ in $(2, \infty)$ such that $\lim x_n = 2$. We show

$$\lim(x_n - 2)^{-3} = +\infty. \qquad (2)$$

Since $\lim(x_n - 2) = 0$ and each $x_n - 2 > 0$, Theorem 9.10 shows we have $\lim(x_n - 2)^{-1} = +\infty$ and (2) follows by an application of Theorem 9.9.

**20.17** Suppose first that $L$ is finite. We use (1) in Corollary 20.8. Let $\epsilon > 0$. There exist $\delta_1 > 0$ and $\delta_3 > 0$ such that

$$a < x < a + \delta_1 \quad \text{implies} \quad L - \epsilon < f_1(x) < L + \epsilon$$

and

$$a < x < a + \delta_3 \quad \text{implies} \quad L - \epsilon < f_3(x) < L + \epsilon.$$

If $\delta = \min\{\delta_1, \delta_3\}$, then

$$a < x < a + \delta \quad \text{implies} \quad L - \epsilon < f_2(x) < L + \epsilon.$$

So by Corollary 20.8 we have $\lim_{x \to a^+} f_2(x) = L$.
Suppose $L = +\infty$. Let $M > 0$. In view of Discussion 20.9, there exists $\delta > 0$ such that

$$a < x < a + \delta \quad \text{implies} \quad f_1(x) > M.$$

Then clearly

$$a < x < a + \delta \quad \text{implies} \quad f_2(x) > M,$$

and this shows $\lim_{x \to a^+} f_2(x) = +\infty$. The case $L = -\infty$ is similar.

**20.19** Suppose $L_2 = \lim_{x \to a^S} f(x)$ exists with $S = (a, b_2)$. Consider a sequence $(x_n)$ in $(a, b_1)$ with limit $a$. Then $(x_n)$ is a sequence in $(a, b_2)$ with limit $a$, so $\lim f(x_n) = L_2$. This shows $\lim_{x \to a^S} f(x) = L_2$ with $S = (a, b_1)$.

Suppose $L_1 = \lim_{x \to a^S} f(x)$ exists with $S = (a, b_1)$, and consider a sequence $(x_n)$ in $(a, b_2)$ with limit $a$. There exists $N$ so that $n \geq N$ implies $x_n < b_1$. Then $(x_n)_{n=N}^{\infty}$ is a sequence in $(a, b_1)$ with limit $a$. Hence $\lim f(x_n) = L_1$ whether we begin the sequence at $n = N$ or $n = 1$. This shows $\lim_{x \to a^S} f(x) = L_1$ with $S = (a, b_2)$.

**21.1** Let $\epsilon > 0$. For $j = 1, 2, \ldots, k$, there exist $\delta_j > 0$ such that

$$s, t \in \mathbb{R} \quad \text{and} \quad |s - t| < \delta_j \quad \text{imply} \quad |f_j(s) - f_j(t)| < \frac{\epsilon}{\sqrt{k}}.$$

Let $\delta = \min\{\delta_1, \delta_2, \ldots, \delta_k\}$. Then by (1) in the proof of Proposition 21.2,

$$x, t \in \mathbb{R} \quad \text{and} \quad |s - t| < \delta \quad \text{imply} \quad d^*(\gamma(s), \gamma(t)) < \epsilon.$$

**21.3** *Hint*: Show $|d(s, s_0) - d(t, s_0)| \leq d(s, t)$. Hence if $\epsilon > 0$, then

$$s, t \in S \quad \text{and} \quad d(s, t) < \epsilon \quad \text{imply} \quad |f(s) - f(t)| < \epsilon.$$

**21.5** **(b)** By part (a), there is an unbounded continuous real-valued function $f$ on $E$. Show $h = \frac{|f|}{1+|f|}$ is continuous, bounded and does not assume its supremum 1 on $E$.

**21.7** **(b)** $\gamma$ is continuous at $t_0$ if for each $t_0 \in [a, b]$ and $\epsilon > 0$ there exists $\delta > 0$ such that

$$t \in [a, b] \quad \text{and} \quad |t - t_0| < \delta \quad \text{imply} \quad d^*(\gamma(t), \gamma(t_0)) < \epsilon.$$

*Note*: If $\gamma$ is continuous at each $t_0 \in [a, b]$, then $\gamma$ is uniformly continuous on $[a, b]$ by Theorem 21.4.

**21.9** **(a)** Use $f(x_1, x_2) = x_1$, say.

**(b)** This is definitely *not* obvious, but there do exist continuous mappings of $[0, 1]$ onto the unit square. Such functions must be "wild" and are called Peano curves [after the same Peano with the axioms]; see [16, §5.5] or [55, §6.3].

**21.11** If $\mathbb{Q}$ were equal to $\cap_{n=1}^{\infty} U_n$, where each $U_n$ is open, then $\cap_{n=1}^{\infty} \cap \cap_{r\in\mathbb{Q}}(\mathbb{R} \setminus \{r\})$ would be an intersection of a sequence of open sets that is equal to the empty set, contrary to Theorem 21.7(a).

**21.13** Suppose $\omega_f(x) = 0$. Given $\epsilon > 0$, there is $\delta > 0$ so that

$$\sup\{|f(y) - f(z)| : y, z \in (x - \delta, x + \delta)\} < \epsilon.$$

Then $|f(x) - f(y)| < \epsilon$ for all $y \in (x - \delta, x + \delta)$, i.e., $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$. So $f$ is continuous at $x$. The proof that continuity of $f$ at $x$ implies $\omega_f(x) = 0$ is similar.

**22.1** **(a)** $[0, 1]$ is connected but $[0, 1] \cup [2, 3]$ is not. See Theorem 22.2. Alternatively, apply the Intermediate Value Theorem 18.2.

**22.3** Assume $E$ is connected but $E^-$ is not. Then there exist open sets $U_1$ and $U_2$ that separate $E^-$ as in Definition 22.1(a). Show that $U_1$ and $U_2$ separate $E$, which is a contradiction. Since $E$ satisfies (1) in Definition 22.1(a), it suffices to show $E \cap U_1 \neq \varnothing$ and $E \cap U_2 \neq \varnothing$. In fact, if $E \cap U_1 = \varnothing$, then $E^- \cap (S \setminus U_1)$ would be a closed set containing $E$, and it is smaller than $E^-$ since $E^- \cap U_1 \neq \varnothing$. This contradicts the definition of $E^-$ in Definition 13.8, so $E \cap U_1 \neq \varnothing$. Likewise $E \cap U_2 \neq \varnothing$.

**22.5** **(a)** Assume open sets $U_1$ and $U_2$ separate $E \cup F$ as in Definition 22.1(a). Consider $x \in E \cap F$; $x$ belongs to one of the open sets, say $x \in U_1$. Select $y$ in $(E \cup F) \cap U_2$. Then $y$ is in $E$ or $F$, say $y \in E$. Since $x \in E \cap U_1$ and $y \in E \cap U_2$, these sets are nonempty. Thus $U_1$ and $U_2$ separate $E$, and $E$ is not connected, a contradiction.

**(b)** No such example exists in $\mathbb{R}$ [why?], but many exist in the plane.

**22.9** *Discussion.* Given $\epsilon > 0$, we need $\delta > 0$ so that

$$s, t \in \mathbb{R} \quad \text{and} \quad |s - t| < \delta \quad \text{imply} \quad d(F(s), F(t)) < \epsilon. \qquad (1)$$

Now

$$\begin{aligned} d(F(s), F(t)) &= \sup\{|sf(x) + (1 - s)g(x) - tf(x) - (1 - t)g(x)| : x \in S\} \\ &= \sup\{|sf(x) - tf(x) - sg(x) + tg(x)| : x \in S\} \\ &\leq |s - t| \cdot \sup\{|f(x)| + |g(x)| : x \in S\}. \end{aligned}$$

Since $f$ and $g$ are fixed, the last supremum is a constant $M$. We may assume $M > 0$, in which case $\delta = \frac{\epsilon}{M}$ will make (1) hold.

**22.11** **(a)** Let $(f_n)$ be a convergent sequence in $\mathcal{E}$. By Proposition 13.9(b), it suffices to show $f = \lim f_n$ is in $\mathcal{E}$. For each $x \in S$,

$$|f(x)| \le |f(x) - f_n(x)| + |f_n(x)| \le d(f, f_n) + 1.$$

Since $\lim d(f, f_n) = 0$, we have $|f(x)| \le 1$.

**(b)** It suffices to show $C(S)$ is path-connected. So use Exercise 22.9.

**23.1** Intervals of convergence: **(a)** $(-1, 1)$; **(c)** $[-\frac{1}{2}, \frac{1}{2}]$; **(e)** $\mathbb{R}$; **(g)** $[-\frac{4}{3}, \frac{4}{3})$.

**23.3** $(-\sqrt[3]{2}, \sqrt[3]{2})$.

**23.5** **(a)** Since $|a_n| \ge 1$ for infinitely many $n$, we have $\sup\{|a_n|^{1/n} : n > N\} \ge 1$ for all $N$. Thus $\beta = \limsup |a_n|^{1/n} \ge 1$; hence $R = \frac{1}{\beta} \le 1$.

**(b)** Select $c$ so that $0 < c < \limsup |a_n|$. Then $\sup\{|a_n| : n > N\} > c$ for all $N$. A subsequence $(a_{n_k})$ of $(a_n)$ has the property that $|a_{n_k}| > c$ for all $k$. Since $|a_{n_k}|^{1/n_k} > (c)^{1/n_k}$ and $\lim_{k\to\infty} c^{1/n_k} = 1$ [by Theorem 9.7(d)], Theorem 12.1 shows $\limsup |a_{n_k}|^{1/n_k} \ge 1$. It follows that $\beta = \limsup |a_n|^{1/n} \ge 1$ [use Theorem 11.8]. Hence $R = \frac{1}{\beta} \le 1$.

**23.9** **(a)** Obviously $\lim f_n(0) = 0$. Consider $0 < x < 1$ and let $s_n = nx^n$. Then $s_{n+1}/s_n = (\frac{n+1}{n})x$, so $\lim |s_{n+1}/s_n| = x < 1$. Exercise 9.12(a) shows $0 = \lim s_n = \lim nx^n = \lim f_n(x)$.

**24.1** *Discussion.* Let $\epsilon > 0$. We want $|f_n(x) - 0| < \epsilon$ for all $x$ and for large $n$. It suffices to arrange for $\frac{3}{\sqrt{n}} < \epsilon$ for large $n$. So consider $n > 9/\epsilon^2 = N$.

**24.3** **(a)** $f(x) = 1$ for $0 \le x < 1$; $f(1) = \frac{1}{2}$; $f(x) = 0$ for $x > 1$. See Exercise 9.13.

**(b)** $(f_n)$ does *not* converge uniformly on $[0, 1]$ by Theorem 24.3.

**24.5** **(a)** $f(x) = 0$ for $x \le 1$ and $f(x) = 1$ for $x > 1$. Note $f_n(x) = 1/(1 + n/x^n)$ and $\lim_{n\to\infty} n/x^n = 0$ for $x > 1$ by Exercise 9.12 or 9.14.

**(b)** $f_n \to 0$ uniformly on $[0, 1]$. *Hint*: Show $|f_n(x)| \le \frac{1}{n}$ for $x \in [0, 1]$.

**(c)** *Hint*: Use Theorem 24.3.

**24.7** **(a)** Yes. $f(x) = x$ for $x < 1$ and $f(1) = 0$.

**(b)** No, by Theorem 24.3 again.

**24.9** **(a)** $f(x) = 0$ for $x \in [0, 1]$. For $x < 1$, $\lim_{n\to\infty} nx^n = 0$ as in Exercise 23.9(a).

**(b)** Use calculus to show $f_n$ takes its maximum at $\frac{n}{n+1}$. Thus $\sup\{|f_n(x)| : x \in [0, 1]\} = f_n(\frac{n}{n+1}) = (\frac{n}{n+1})^{n+1}$. As in Example 8, it turns out $\lim f_n(\frac{n}{n+1}) = 1/e$. So Remark 24.4 shows $(f_n)$ does not converge uniformly to 0.

**(c)** $\int_0^1 f_n(x)\,dx = \frac{n}{(n+1)(n+2)} \to 0 = \int_0^1 f(x)\,dx$.

**24.15** **(a)** $f(0) = 0$ and $f(x) = 1$ for $x > 0$. **(b)** No. **(c)** Yes.

**24.17** *Hint*: Use $|f_n(x_n) - f(x)| \le |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)|$.

**25.3** **(a)** Since $f_n(x) = (1 + (\cos x)/n)/(2 + (\sin^2 x)/n)$, we have $f_n \to \frac{1}{2}$ pointwise. To obtain uniform convergence, show

$$\left|f_n(x) - \frac{1}{2}\right| = \left|\frac{2\cos x - \sin^2 x}{2(2n + \sin^2 x)}\right| \le \frac{3}{2(2n)} < \epsilon$$

for all real numbers $x$ and all $n > \frac{3}{4\epsilon}$.

**(b)** $\int_2^7 \frac{1}{2}\,dx = \frac{5}{2}$, by Theorem 25.2.

**25.5** Since $f_n \to f$ uniformly on $S$, there exists $N \in \mathbb{N}$ such that $n > N$ implies $|f_n(x) - f(x)| < 1$ for all $x \in S$. In particular, we have $|f_{N+1}(x) - f(x)| < 1$ for $x \in S$. If $M$ bounds $|f_{N+1}|$ on $S$ [i.e., if $|f_{N+1}(x)| \le M$ for $x \in S$], then $M + 1$ bounds $|f|$ on $S$ [why?].

**25.7** Let $g_n(x) = n^{-2}\cos nx$. Then we have $|g_n(x)| \le n^{-2}$ for $x \in \mathbb{R}$ and $\sum n^{-2} < \infty$. So $\sum g_n$ converges uniformly on $\mathbb{R}$ by the Weierstrass $M$-Test 25.7. The limit function is continuous by Theorem 25.5.

**25.9** **(a)** The series converges pointwise to $\frac{1}{1-x}$ on $(-1, 1)$ by (2) of Example 1 in §14. The series converges uniformly on $[-a, a]$ by the Weierstrass $M$-Test since $|x^n| \le a^n$ for $x \in [-a, a]$ and since $\sum a^n < \infty$.

**(b)** One can show directly that the sequence of partial sums $s_n(x) = \sum_{k=0}^n x^k = (1 - x^{n+1})/(1 - x)$ does not converge uniformly on $(-1, 1)$. It is easier to observe the partial sums $s_n$ are each bounded on $(-1, 1)$, and hence if $(s_n)$ converges uniformly, then the limit function is bounded by Exercise 25.5. But $\frac{1}{1-x}$ is not bounded on $(-1, 1)$.

**25.11** **(b)** *Hint*: Apply the Weierstrass $M$-Test to $\sum h_n$, where $h_n(x) = (\frac{3}{4})^n g_n(x)$.

**25.13** The series $\sum g_k$ and $\sum h_k$ are uniformly Cauchy on $S$ and it suffices to show $\sum(g_k + h_k)$ is also; see Theorem 25.6. Let $\epsilon > 0$. There exist $N_1$ and $N_2$ such that

$$n \ge m > N_1 \quad \text{implies} \quad \left|\sum_{k=m}^{n} g_k(x)\right| < \frac{\epsilon}{2} \quad \text{for} \quad x \in S, \qquad (1)$$

$$n \ge m > N_2 \quad \text{implies} \quad \left|\sum_{k=m}^{n} h_k(x)\right| < \frac{\epsilon}{2} \quad \text{for} \quad x \in S. \qquad (2)$$

Then

$$n \geq m > \max\{N_1, N_2\} \quad \text{implies} \quad \left|\sum_{k=m}^{n}(g_k + h_k)(x)\right| < \epsilon \quad \text{for} \quad x \in S.$$

**25.15** **(a)** Note $f_n(x) \geq 0$ for all $x$ and $n$. Assume $(f_n)$ does not converge to 0 uniformly on $[a, b]$. Then there exists $\epsilon > 0$ such that

$$\begin{array}{l}\text{for each } N \text{ there exists } n > N \text{ and } x \in [a,b] \\ \text{such that } f_n(x) \geq \epsilon.\end{array} \tag{1}$$

We claim

$$\text{for each } n \in N \text{ there is } x_n \in [a,b] \text{ where } f_n(x_n) \geq \epsilon. \tag{2}$$

If not, there is $n_0 \in \mathbb{N}$ such that $f_{n_0}(x) < \epsilon$ for all $x \in [a, b]$. Since $(f_n(x))$ is decreasing for each $x$, we conclude $f_n(x) < \epsilon$ for all $x \in [a, b]$ and $n \geq n_0$. This clearly contradicts (1). We have now established the hint.
Now by the Bolzano-Weierstrass theorem, the sequence $(x_n)$ given by (2) has a convergent subsequence $(x_{n_k})$: $x_{n_k} \to x_0$. Since $\lim f_n(x_0) = 0$, there exists $m$ such that $f_m(x_0) < \epsilon$. Since $x_{n_k} \to x_0$ and $f_m$ is continuous at $x_0$, we have $\lim_{k\to\infty} f_m(x_{n_k}) = f_m(x_0) < \epsilon$. So there exists $K$ such that

$$k > K \quad \text{implies} \quad f_m(x_{n_k}) < \epsilon.$$

If $k > \max\{K, m\}$, then $n_k \geq k > m$, so

$$f_{n_k}(x_{n_k}) \leq f_m(x_{n_k}) < \epsilon.$$

But $f_n(x_n) \geq \epsilon$ for *all* $n$, so we have a contradiction.

**(b)** *Hint*: Show part (a) applies to the sequence $g_n$ where $g_n = f - f_n$.

**26.3** **(a)** Let $f(x) = \sum_{n=1}^{\infty} nx^n = x/(1-x)^2$ for $|x| < 1$. Then by Theorem 26.5

$$\sum_{n=1}^{\infty} n^2 x^{n-1} = f'(x) = \frac{d}{dx}\left[\frac{x}{(1-x)^2}\right] = (1+x)(1-x)^{-3};$$

therefore $\sum_{n=1}^{\infty} n^2 x^n = (x + x^2)(1-x)^{-3}$.

**(b)** 6 and $\frac{3}{2}$.

**26.5** *Hint*: Apply Theorem 26.5.

**26.7** No! The power series would be differentiable at each $x \in \mathbb{R}$, but $f(x) = |x|$ is not differentiable at $x = 0$.

**27.1** Let $\phi$ be as in the hint. By Theorem 27.4, there is a sequence $(q_n)$ of polynomials such that $q_n \to f \circ \phi$ uniformly on $[0,1]$. Note $\phi$ is one-to-one and $\phi^{-1}(y) = \frac{y-a}{b-a}$. Let $p_n = q_n \circ \phi^{-1}$. Then each $p_n$ is a polynomial and $p_n \to f$ uniformly on $[a,b]$.

**27.3** **(a)** Assume a polynomial $p$ satisfies $|p(x) - \sin x| < 1$ for all $x \in \mathbb{R}$. Clearly $p$ cannot be a constant function. But if $p$ is nonconstant, then $p$ is unbounded on $\mathbb{R}$ and the same is true for $p(x) - \sin x$, a contradiction.

**(b)** Assume $|e^x - \sum_{k=0}^{n-1} a_k x^k| < 1$ for all $x \in \mathbb{R}$. For $x > 0$ we have

$$e^x - \sum_{k=0}^{n-1} a_k x^k \geq \frac{1}{n!} x^n - \sum_{k=0}^{n-1} |a_k| x^k$$

and for large $x$ the right side will exceed 1.

**27.5** **(a)** $B_n f(x) = x$ for all $n$. Use (2) in Lemma 27.2.

**(b)** $B_n f(x) = x^2 + \frac{1}{n} x(1-x)$. Use (4) in Lemma 27.2.

**28.1** **(a)** $\{0\}$; **(b)** $\{0\}$; **(c)** $\{n\pi : n \in \mathbb{Z}\}$; **(d)** $\{0,1\}$; **(e)** $\{-1,1\}$; **(f)** $\{2\}$.

**28.3** **(b)** Since $x - a = (x^{1/3} - a^{1/3})(x^{2/3} + a^{1/3}x^{1/3} + a^{2/3})$,

$$f'(a) = \lim_{x \to a} (x^{2/3} + a^{1/3}x^{1/3} + a^{2/3})^{-1} = (3a^{2/3})^{-1} = \frac{1}{3} a^{-2/3}$$

for $a \neq 0$.

**(c)** $f$ is not differentiable at $x = 0$ since the limit $\lim_{x \to 0} x^{1/3}/x$ does not exist *as a real number*. The limit does exist and equals $+\infty$, which reflects the geometric fact that the graph of $f$ has a vertical tangent at $(0,0)$.

**28.5** **(c)** Let

$$h(x) = \frac{g(f(x)) - g(f(0))}{f(x) - f(0)}.$$

According to Definition 20.3(a), for $\lim_{x \to 0} h(x)$ to be meaningful, $h$ needs to be defined on $J \setminus \{0\}$ for some open interval $J$ containing 0. But the calculation in (b) shows $h$ is undefined at $(\pi n)^{-1}$ for $n = \pm 1, \pm 2, \ldots$.

**28.7** **(d)** $f'$ is continuous on $\mathbb{R}$, but $f'$ is not differentiable at $x = 0$.

**28.9** **(b)** $f(x) = x^4 + 13x$ and $g(y) = y^7$. Then

$$h'(x) = g'(f(x)) \cdot f'(x) = 7(x^4 + 13x)^6 \cdot (4x^3 + 13).$$

**28.11** With the stated hypotheses, $h \circ g \circ f$ is differentiable at $a$ and $(h \circ g \circ f)'(a) = h'(g \circ f(a)) \cdot g'(f(a)) \cdot f'(a)$. **Proof** By 28.4, $g \circ f$ is differentiable at $a$ and $(g \circ f)'(a) = g'(f(a)) \cdot f'(a)$. Again by 28.4,

$$(h \circ (g \circ f))'(a) = h'((g \circ f)(a)) \cdot (g \circ f)'(a).$$

**28.13** There exist positive numbers $\delta_1$ and $\epsilon$ so that $f$ is defined on the interval $(a - \delta_1, a + \delta_1)$ and $g$ is defined on $(f(a) - \epsilon, f(a) + \epsilon)$. By Theorem 17.2, there exists $\delta_2 > 0$ so that

$$x \in \text{dom}(f) \quad \text{and} \quad |x - a| < \delta_2 \quad \text{imply} \quad |f(x) - f(a)| < \epsilon.$$

If $|x - a| < \min\{\delta_1, \delta_2\}$, then $x \in \text{dom}(f)$ and $|f(x) - f(a)| < \epsilon$, so $f(x) \in \text{dom}(g)$, i.e., $x \in \text{dom}(g \circ f)$.

**29.1** **(a)** $x = \frac{1}{2}$

**(c)** If $f(x) = |x|$, then $f'(x) = \pm 1$ except at 0. So no $x$ satisfies the equation $f'(x) = \frac{f(2)-f(-1)}{2-(-1)} = \frac{1}{3}$. Missing hypothesis: $f$ is not differentiable on $(-1, 2)$, since $f$ is not differentiable at $x = 0$.

**(e)** $x = \sqrt{3}$

**29.3** **(a)** Apply Mean Value Theorem to $[0, 2]$.

**(b)** By the Mean Value Theorem, $f'(y) = 0$ for some $y \in (1, 2)$. In view of this and part (a), Theorem 29.8 shows $f'$ takes all values between 0 and $\frac{1}{2}$.

**29.5** For any $a \in \mathbb{R}$ we have $|\frac{f(x)-f(a)}{x-a}| \leq |x-a|$. It follows easily that $f'(a)$ exists and equals 0 for all $a \in \mathbb{R}$. So $f$ is constant by Corollary 29.4.

**29.7** **(a)** Applying 29.4 to $f'$, we find $f'(x) = a$ for some constant $a$. If $g(x) = f(x) - ax$, then $g'(x) = 0$ for $x \in I$, so by 29.4 there is a constant $b$ such that $g(x) = b$ for $x \in I$.

**29.9** *Hint*: Let $f(x) = e^x - ex$ for $x \in \mathbb{R}$. Use $f'$ to show $f$ is increasing on $[1, \infty)$ and decreasing on $(-\infty, 1]$. Hence $f$ takes its minimum at $x = 1$.

**29.13** Let $h(x) = g(x) - f(x)$ and show $h(x) \geq 0$ for $x \geq 0$.

**29.15** As in Example 2, let $g(x) = x^{1/n}$. Since $\text{dom}(g) = [0, \infty)$ if $n$ is even and $\text{dom}(g) = \mathbb{R}$ if $n$ is odd, we have $\text{dom}(g) = \text{dom}(h) \cup \{0\}$. Also $h = g^m$. Use the Chain Rule to calculate $h'(x)$.

**29.17** Suppose $f(a) = g(a)$. Then

$$\lim_{x \to a^+} \frac{h(x) - h(a)}{x - a} = g'(a) \quad \text{and} \quad \lim_{x \to a^-} \frac{h(x) - h(a)}{x - a} = f'(a). \quad (1)$$

If also $f'(a) = g'(a)$, then Theorem 20.10 shows $h'(a)$ exists and, in fact, $h'(a) = f'(a) = g'(a)$.

Now suppose $h$ is differentiable at $a$. Then $h$ is continuous at $a$ and so $f(a) = \lim_{x \to a^-} f(x) = \lim_{x \to a^-} h(x) = h(a) = g(a)$. Hence (1) holds. But the limits in (1) both equal $h'(a)$, so $f'(a) = g'(a)$.

**30.1** **(a)** 2; **(b)** $\frac{1}{2}$; **(c)** 0; **(d)** 1. Sometimes L'Hospital's rule can be avoided. For example, for (d) note that

$$\frac{\sqrt{1+x}-\sqrt{1-x}}{x} = \frac{2}{\sqrt{1+x}+\sqrt{1-x}}.$$

**30.3** **(a)** 0; **(b)** 1; **(c)** $+\infty$ **(d)** $-\frac{2}{3}$.

**30.5** **(a)** $e^2$; **(b)** $e^2$; **(c)** $e$.

**31.1** Differentiate the power series for $\sin x$ term-by-term and cite Theorem 26.5.

**31.3** The derivatives do not have a *common* bound on any interval containing 1.

**31.5** **(a)** $g(x) = f(x^2)$ for $x \in \mathbb{R}$ where $f$ is as in Example 3. Use induction to prove there exist polynomials $p_{kn}$, $1 \le k \le n$, so that

$$g^{(n)}(x) = \sum_{k=1}^{n} f^{(k)}(x^2)p_{kn}(x) \quad \text{for} \quad x \in \mathbb{R}, n \ge 1.$$

**31.9** Use $x_n = x_{n-1} - \frac{x_{n-1}-\cos(x_{n-1})}{1+\sin(x_{n-1})}$. To six places, the answer is 0.739085. As always, use radians.

**31.11** For some $C > 0$, we have $|f'(x)| \le C$ for $x$ in $(a, b)$. Then $|f(x_{n-1})| \le |x_n - x_{n-1}| \cdot C$ for all $n$. Let $n \to \infty$.

**32.1** Use the partition $P$ in Example 1 to calculate $U(f, P) = b^4n^2(n+1)^2/(4n^4)$ and $L(f, P) = b^4(n-1)^2n^2/(4n^4)$. Conclude $U(f) = b^4/4$ and $L(f) = b^4/4$.

**32.3** **(a)** The upper sums are the same as in Example 1, so $U(g) = b^3/3$. Show $L(g) = 0$.

**(b)** No.

**32.5** $S$ is all the numbers $L(f, P)$, and $T$ is all $U(f, P)$.

**32.7** Assume $f(x) = 0$ for all $x \in [a, b]$. A simple induction shows we may assume $g(x) = 0$ except at one point $u \in [a, b]$. Clearly all lower sums $L(g, P) = 0$, so $L(g) = 0$. Since $u$ belongs to at most two intervals of any partition $P$, we have

$$U(g, P) \le 2|g(u)| \max_k [t_k - t_{k-1}].$$

Infer $U(g) = 0$; hence $\int_a^b g = 0 = \int_a^b f$.

**33.1** If $f$ is decreasing on $[a, b]$, then $-f$ is increasing on $[a, b]$, so $-f$ is integrable as proved in Theorem 33.1. Now apply Theorem 33.3 with $c = -1$.

**33.3** **(b)** $4A + 6B$

**33.7** **(a)** For any set $S \subseteq [a,b]$ and $x_0,\ y_0 \in S$, we have

$$f(x_0)^2 - f(y_0)^2 \le |f(x_0) + f(y_o)| \cdot |f(x_0) - f(y_0)| \\ \le 2B|f(x_0) - f(y_0)| \le 2B[M(f,S) - m(f,S)].$$

It follows that $M(f^2,S) - m(f^2,S) \le 2B[M(f,S) - m(f,S)]$. Use this to show $U(f^2,P) - L(f^2,P) \le 2B[U(f,P) - L(f,P)]$.

**(b)** Use Theorem 32.5 and part (a).

**33.9** Select $m \in \mathbb{N}$ so that $|f(x) - f_m(x)| < \frac{\epsilon}{2(b-a)}$ for all $x \in [a,b]$. Then for any partition $P$

$$-\frac{\epsilon}{2} \le L(f - f_m, P) \le U(f - f_m, P) \le \frac{\epsilon}{2}.$$

Select a partition $P_0$ so that $U(f_m,P_0) - L(f_m,P_0) < \frac{\epsilon}{2}$. Since $f = (f - f_m) + f_m$, we can use inequalities from the proof of Theorem 33.3 to conclude $U(f,P_0) - L(f,P_0) < \epsilon$. Now Theorem 32.5 shows $f$ is integrable. To complete the exercise, proceed as in the proof of Theorem 25.2.

**33.11** **(a)** and **(b)**: Show $f$ is neither continuous nor monotonic on any interval containing 0.

**(c)** Let $\epsilon > 0$. Since $f$ is piecewise continuous on $[\frac{\epsilon}{8}, 1]$, there is a partition $P_1$ of $[\frac{\epsilon}{8}, 1]$ such that $U(f,P_1) - L(f,P_1) < \frac{\epsilon}{4}$. Likewise there is a partition $P_2$ of $[-1, -\frac{\epsilon}{8}]$ such that $U(f,P_2) - L(f,P_2) < \frac{\epsilon}{4}$. Let $P = P_1 \cup P_2$, a partition of $[-1,1]$. Since

$$\left\{M\left(f,\left[-\frac{\epsilon}{8},\frac{\epsilon}{8}\right]\right) - m\left(f,\left[-\frac{\epsilon}{8},\frac{\epsilon}{8}\right]\right)\right\} \cdot \left\{\frac{\epsilon}{8} - \left(-\frac{\epsilon}{8}\right)\right\} \le \frac{\epsilon}{2},$$

we conclude $U(f,P) - L(f,P) < \epsilon$. Now Theorem 32.5 shows $f$ is integrable.

**33.13** Apply Theorem 33.9 to $f - g$.

**34.3** **(a)** $F(x) = 0$ for $x < 0$; $F(x) = x^2/2$ for $0 \le x \le 1$; $F(x) = 4x - \frac{7}{2}$ for $x > 1$.

**(c)** $F$ is differentiable except possibly at $x = 1$ by Theorem 34.3. To show $F$ is not differentiable at $x = 1$, use Exercise 29.17.

**34.5** $F'(x) = f(x+1) - f(x-1)$.

**34.9** Use $a = 0$, $b = \frac{\pi}{6}$ and $g(x) = \sin x$.

**35.3** **(a)** 21; **(b)** 14; **(c)** 0.

**35.5** **(a)** Every upper sum is $F(b) - F(a)$ and every lower sum is 0. Hence $U_F(f) = F(b) - F(a) \ne 0 = L_F(f)$.

**35.7** **(a)** Imitate solution to Exercise 33.7.

**(b)** and **(c)**: Use hints in Exercise 33.8.

**35.9** **(a)** Let $m$ and $M$ be the [assumed] minimum and maximum of $f$ on $[a,b]$. Then $\int_a^b m\,dF \leq \int_a^b f\,dF \leq \int_a^b M\,dF$ or $m \leq [F(b) - F(a)]^{-1} \int_a^b f\,dF \leq M$. Apply Theorem 18.2.

**(b)** Consider $f$ and $g$ as in Exercise 33.14, and let $F$ be as in Exercise 35.8. By part (a), for some $x \in [a,b]$ we have

$$\int_a^b f(t)g(t)\,dt = \int_a^b f\,dF = f(x)[F(b) - F(a)] = f(x)\int_a^b g(t)\,dt.$$

**35.11** Let $\epsilon > 0$ and select a partition

$$P = \{a = t_0 < t_1 < \cdots < t_n = b\}$$

satisfying $U_F(f,P) - L_F(f,P) < \epsilon$. Let $u_k = \phi^{-1}(t_k)$ and

$$Q = \{c = u_0 < u_1 < \cdots < u_n = d\}.$$

Show $U_G(g,Q) = U_F(f,P)$ and $L_G(g,Q) = L_F(f,P)$. Then $U_G(g,Q) - L_G(g,Q) < \epsilon$, so $g$ is $G$-integrable. The equality of the integrals follows easily.

**36.1** *Hint*: If $B$ bounds $|f|$, then

$$\left| \int_a^d f(x)\,dx - \int_a^b f(x)\,dx \right| \leq B(b-d).$$

**36.3** **(b)** Use part (a) and Examples 1 and 2.

**36.7** **(a)** It suffices to show $\int_1^\infty e^{-x^2}\,dx < \infty$. But $e^{-x^2} \leq e^{-x}$ for $x \geq 1$ and $\int_1^\infty e^{-x}\,dx = \frac{1}{e}$.

**(b)** The double integral equals $[\int_{-\infty}^\infty e^{-x^2}\,dx]^2$, and it also equals

$$\int_0^\infty \int_0^{2\pi} e^{-r^2} r\,d\theta\,dr = 2\pi \int_0^\infty e^{-r^2} r\,dr = \pi.$$

**36.9** **(a)** *Hint*: Use Theorem 35.13.

**(b)** 1; **(c)** $+\infty$; **(d)** $\sqrt{2/\pi}$; **(e)** 0.

**36.13** *Claim*: If $f$ is continuous on $\mathbb{R}$ and $\int_{-\infty}^\infty |f|\,dF < \infty$, then $f$ is $F$-integrable. **Proof** Since $0 \leq f + |f|$, the integral $\int_{-\infty}^\infty [f + |f|]\,dF$ exists, and since $f + |f| \leq 2|f|$, this integral is finite, i.e., $f + |f|$ is $F$-integrable. Since $-|f|$ is $F$-integrable, Exercise 36.10 shows the sum of $f + |f|$ and $-|f|$ is $F$-integrable.

**36.15** **(a)** For example, let $f_n(x) = \frac{1}{n}$ for $x \in [0,n]$ and $f_n(x) = 0$ elsewhere.

**(b)** **Outline of Proof** First, $f$ is $F$-integrable on each $[a,b]$ by Exercise 35.6. An elaboration of Exercise 25.5 shows there is a *common* bound $B$ for $|f|$ *and* all $|f_n|$. Consider any $b > 0$

such that $1 - F(b) < \frac{\epsilon}{2B}$. There exists a number $N$ so that $|\int_0^b f\,dF - \int_0^b f_n\,dF| < \frac{\epsilon}{2}$ for $n > N$. Then

$$n > N \quad \text{implies} \quad \left| \int_0^b f\,dF - \int_0^\infty f_n\,dF \right| < \epsilon. \tag{1}$$

In particular, $m, n > N$ implies $|\int_0^\infty f_n\,dF - \int_0^\infty f_m\,dF| < 2\epsilon$, so $(\int_0^\infty f_n\,dF)_{n\in\mathbb{N}}$ is a Cauchy sequence with a finite limit $L$. From (1) it follows that

$$1 - F(b) < \frac{\epsilon}{2B} \quad \text{implies} \quad \left| \int_0^b f\,dF - L \right| \le \epsilon,$$

so $\lim_{b\to\infty} \int_0^b f\,dF = L$. Hence $\int_0^\infty f\,dF$ exists, is finite, and equals $\lim_{n\to\infty} \int_0^\infty f_n\,dF$. A similar argument handles $\int_{-\infty}^0 f\,dF$.

**37.1** *Hint*:

$$\int_1^{yz} \frac{1}{t}\,dt - \int_1^y \frac{1}{t}\,dt = \int_y^{yz} \frac{1}{t}\,dt.$$

**37.7** **(a)** $B(x) = E(xL(b))$, so by the Chain Rule, we have $B'(x) = E(xL(b)) \cdot L(b) = L(b)b^x = (\log_e b)b^x$.

**37.9** **(a)** $\log_e y = L(y) = \int_1^y \frac{1}{t}\,dt \le y - 1 < y$.

**38.3** Apply the Fundamental Theorem of Calculus 34.3 to a continuous nowhere-differentiable function $f$.

# A Guide to the References

There are many books with goals similar to ours, including: Brannan [12], Clark [16], and Pedrick [51]. Zorn [73] is a gentle, well written introduction with a strong first chapter on "numbers, sets and proofs." Lay [41] also starts with sections on logic and proofs, and provides good answers. Abbott [1] appears to be a fine book, though lots of the text is left to the exercises. Bauldry [3] is a nice book designed for a last course in real analysis for those intending to go into teaching. Morgan [47] gives a concise introduction to the key concepts.

Beardon [6] gives a coherent treatment of limits by defining the notion just once, in terms of directed sets. Bressoud [13] is a wonderful book that introduces analysis through its history. Gardiner [23] is a more challenging book, though the subject is at the same level; there is a lot about numbers and there is a chapter on geometry. Hijab [32] is another interesting, idiosyncratic and challenging book. Note many of these books do not provide answers to some of the exercises, though [32] provides answers to **all** of the exercises.

Rotman [61] is a very nice book that serves as an intermediate course between the standard calculus sequence and the first course in **both** abstract algebra and real analysis. I always shared Rotman's doubts about inflicting logic on undergraduates, until I saw Wolf's wonderful book [70] that introduces logic as a mathematical tool in an **interesting** way.

More encyclopedic real analysis books that are presented at the same level with great detail and numerous examples are: Lewin and Lewin [43], Mattuck [46] and Reed [56]. I find Mattuck's book [46] very thoughtfully written; he shares many of his insights (acquired over many years) with the reader.

There are several superb texts at a more sophisticated level: Beals [4], Bear [5], Hoffman [33], Johnsonbaugh and Pfaffenberger [34], Protter and Morrey [53], Rudin [62] and Stromberg [65]. Any of these books can be used to obtain a really thorough understanding of analysis and to prepare for various advanced graduate-level topics in analysis. The possible directions for study after this are too numerous to enumerate here. However, a reader who has no specific needs or goals but who would like an introduction to several important ideas in several branches of mathematics would enjoy and profit from Garding [24].

# References

[1] Abbott, S.D.: Understanding Analysis. Springer, New York (2010)

[2] Bagby, R.J.: Introductory Analysis – A Deeper View of Calculus. Academic, San Diego (2001)

[3] Bauldry, W.C.: Introduction to Real Analysis – An Educational Approach. Wiley (2010)

[4] Beals, R.: Advanced Mathematical Analysis. Graduate Texts in Mathematics, vol. 12. Springer, New York/Heidelberg/Berlin (1973). Also, Analysis–an Introduction, Cambridge University Press 2004

[5] Bear, H.S.: An Introduction to Mathematical Analysis. Academic, San Diego (1997)

[6] Beardon, A.F.: Limits – A New Approach to Real Analysis. Undergraduate Texts in Mathematics. Springer, New York/Heidelberg/Berlin (1997)

[7] Berberian, S.K.: A First Course in Real Analysis. Springer, New York (1994)

[8] Birkhoff, G., Mac Lane, S.: A Survey of Modern Algebra. Macmillan, New York (1953). A. K. Peters/CRC 1998

[9] Boas, R.P. Jr.: A Primer of Real Functions, 4th edn. Revised and updated by Harold P. Boas. Carus Monograph, vol. 13. Mathematical Association of America, Washington, DC (1996)

K.A. Ross, *Elementary Analysis: The Theory of Calculus*, Undergraduate Texts in Mathematics, DOI 10.1007/978-1-4614-6271-2,

[10] Borman, J.L.: A remark on integration by parts. Amer. Math. Monthly **51**, 32–33 (1944)

[11] Botsko, M.W.: Quicky problem. Math. Mag. **85**, 229 (2012)

[12] Brannan, D.: A First Course in Mathematical Analysis. Cambridge University Press, Cambridge/New York (2006)

[13] Bressoud, D.: A Radical Approach to Real Analysis, 2nd edn. The Mathematical Association of America, Washington, DC (2007)

[14] Burckel, R.B.: An Introduction to Classical Complex Analysis, vol. 1. Birkhäuser, Basel (1979)

[15] Burgess, C.E.: Continuous functions and connected graphs. Amer. Math. Monthly **97**, 337–339 (1990)

[16] Clark, C.: The Theoretical Side of Calculus. Wadsworth, Belmont (1972). Reprinted by Krieger, New York 1978

[17] Corominas, E., Sunyer Balaguer, F.: Conditions for an infinitely differentiable function to be a polynomial, Rev. Mat. Hisp.-Amer. (4) **14**, 26–43 (1954). (Spanish)

[18] Cunningham, F. Jr.: The two fundamental theorems of calculus. Amer. Math. Monthly **72**, 406–407 (1975)

[19] Dangello, F., Seyfried, M.: Introductory Real Analysis. Houghton Mifflin, Boston, (2000)

[20] Donoghue, W.F. Jr.: Distributions and Fourier Transforms. Academic, New York (1969)

[21] Dunham, W.: The Calculus Gallery: Masterpieces from Newton to Lebesgue. Princeton University Press, Princeton/Woodstock (2008)

[22] Fitzpatrick, P.M.: Real Analysis. PWS, Boston (1995)

[23] Gardiner, A.: Infinite Processes, Background to Analysis. Springer, New York/Heidelberg/Berlin (1982). Republished as Understanding Infinity – The Mathematics of Infinite Processes. Dover 2002

[24] Garding, L.: Encounter with Mathematics. Springer, New York/Heidelberg/Berlin (1977)

[25] Gaskill, H.S., Narayanaswami, P.P.: Elements of Real Analysis. Prentice-Hall, Upper Saddle River (1998)

[26] Gaughan, E.D.: Introduction to Analysis, 5th edn. American Mathematical Society, Providence (2009)

[27] Gordon, R.A.: Real Analyis – A First Course, 2nd edn. Addison-Wesley, Boston (2002)

[28] Greenstein, D.S.: A property of the logarithm. Amer. Math. Monthly **72**, 767 (1965)

[29] Hewitt, E.: Integration by parts for Stieltjes integrals. Amer. Math. Monthly **67**, 419–423 (1960)

[30] Hewitt, E.: The role of compactness in analysis. Amer. Math. Monthly **67**, 499–516 (1960)

[31] Hewitt, E., Stromberg, K.: Real and Abstract Analysis. Graduate Texts in Mathematics, vol. 25. Springer, New York/Heidelberg/Berlin (1975)

[32] Hijab, O.: Introduction to Calculus and Classical Analysis. Undergraduate Texts in Mathematics, 2nd edn. Springer, New York/Heidelberg/Berlin (2007)

[33] Hoffman, K.: Analysis in Euclidean Space. Prentice-Hall, Englewood Cliffs (1975). Republished by Dover 2007

[34] Johnsonbaugh, R., Pfaffenberger, W.E.: Foundations of Mathematical Analysis. Marcel Dekker, New York (1980). Republished by Dover 2010

[35] Kantrowitz, R.: Series that converge absolutely but don't converge. Coll. Math. J. **43**, 331–333 (2012)

[36] Kenton, S.: A natural proof of the chain rule. Coll. Math. J. **30**, 216–218 (1999)

[37] Kosmala, W.: Advanced Calculus – A Friendly Approach. Prentice-Hall, Upper Saddle River (1999)

[38] Krüppel, M.: On the zeros of an infinitely often differentiable function and their derivatives. Rostock. Math. Kolloq. **59**, 63–70 (2005)

[39] Landau, E.: Foundations of Analysis. Chelsea, New York (1951). Republished by American Mathematical Society 2001

[40] Lang, S.: Undergraduate Analysis. Undergraduate Texts in Mathematics, 2nd edn. Springer, New York/Heidelberg/Berlin (2010)

[41] Lay, S.R.: Analysis – An Introduction to Proof, 4th edn. Prentice-Hall (2004)

[42] Lewin, J.: A truly elementary approach to the bounded convergence theorem. Amer. Math. Monthly **93**, 395–397 (1986)

[43] Lewin, J., Lewin, M.: An Introduction to Mathematical Analysis, 2nd edn. McGraw-Hill, New York (1993)

[44] Lynch, M.: A continuous nowhere differentiable function. Amer. Math. Monthly **99**, 8–9 (1992)

[45] Lynch, M.: A continuous function which is differentiable only at the rationals. Math. Mag. **86**, April issue (2013)

[46] Mattuck, A.: Introduction to Analysis. Prentice-Hall, Upper Saddle River (1999)

[47] Morgan, F.: Real Analysis. American Mathematical Society, Providence (2005)

[48] Newman, D.J.: A Problem Seminar. Springer, New York/Berlin/Heidelberg (1982)

[49] Niven, I.: Irrational Numbers. Carus Monograph, vol. 11. Mathematical Association of America, Washington, DC (1956)

[50] Niven, I., Zuckerman, H.S., Montgomery, H.I.: An Introduction to the Theory of Numbers, 5th edn. Wiley, New York (1991)

[51] Pedrick, G.: A First Course in Analysis. Undergraduate Texts in Mathematics. Springer, New York/Heidelberg/Berlin (1994)

[52] Phillips, E.: An Introduction to Analysis and Integration Theory. Intext Educational Publishers, Scranton/Toronto/London (1971)

[53] Protter, M.H., Morrey, C.B.: A First Course in Real Analysis. Undergraduate Texts in Mathematics, 2nd edn. Springer, New York/Heidelberg/Berlin (1997)

[54] Pugh, C.: Real Mathematical Analysis. Springer, New York/Heidelberg/Berlin (2002)

[55] Randolph, J.F.: Basic Real and Abstract Analysis. Academic, New York (1968)

[56] Reed, M.: Fundamental Ideas of Analysis. Wiley, New York (1998)

[57] Robdera, M.A.: A Concise Approach to Mathematical Analysis. Springer, London/New York (2003)

[58] Rosenlicht, M.: Introduction to Analysis. Dover, New York (1985)

[59] Ross, K.A.: First digits of squares and cubes. Math. Mag. **85**, 36–42 (2012)

[60] Ross, K.A., Wright, C.R.B.: Discrete Mathematics, 5th edn. Prentice-Hall, Upper Saddle River (2003)

[61] Rotman, J.: Journey into Mathematics – An Introduction to Proofs. Prentice-Hall, Upper Saddle River (1998)

[62] Rudin, W.: Principles of Mathematical Analysis, 3rd edn. McGraw-Hill, New York (1976)

[63] Schramm, M.J.: Introduction to Real Analysis. Prentice-Hall, Upper Saddle River (1996). Dover 2008

[64] Stolz, O.: Über die Grenzwerthe der Quotienten. Math. Ann. **15**, 556–559 (1879)

[65] Stromberg, K.: An Introduction to Classical Real Analysis. Prindle, Weber & Schmidt, Boston (1980)

[66] Thim, J.: Continuous Nowhere Differentiable Functions, Master's thesis (2003), Luleå University of Technology (Sweden). http://epubl.luth.se/1402-1617/2003/320/LTU-EX-03320-SE.pdf

[67] Thomson, B.S.: Monotone convergence theorem for the Riemann integral. Amer. Math. Monthly **117**, 547–550 (2010)

[68] van der Waerden, B.L.: Ein einfaches Beispiel einer nicht-differenzierbare Stetige Funktion. Math. Z. **32**, 474–475 (1930)

[69] Weierstrass, K.: Über continuirliche Funktionen eines reellen Arguments, die für keinen Werth des letzteren einen bestimmten Differentialquotienten besitzen, Gelesen Akad. Wiss. 18 July 1872, and J. für Mathematik **79**, 21–37 (1875)

[70] Wolf, R.S.: Proof, Logic, and Conjecture: The Mathematician's Toolbox. W. H. Freeman, New York (1998)

[71] Wolfe, J.: A proof of Taylor's formula. Amer. Math. Monthly **60**, 415 (1953)

[72] Zhou, L., Markov, L.: Recurrent proofs of the irrationality of certain trigonometric values. Amer. Math. Monthly **117**, 360–362 (2010)

[73] Zorn, P.: Understanding Real Analysis. A. K. Peters, Natick (2010)

# Symbols Index




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# Index



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