Mathematical Physics

Sadri Hassani

# Mathematical Physics

## A Modern Introduction to Its Foundations

Second Edition

Springer

Sadri Hassani
Department of Physics
Illinois State University
Normal, Illinois, USA

ISBN 978-3-319-01194-3 ISBN 978-3-319-01195-0 (eBook)
DOI 10.1007/978-3-319-01195-0
Springer Cham Heidelberg New York Dordrecht London

Library of Congress Control Number: 2013945405



Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

*To my wife, Sarah,*
*and to my children,*
*Dane Arash and Daisy Bita*

# Preface to Second Edition

Based on my own experience of teaching from the first edition, and more importantly based on the comments of the adopters and readers, I have made some significant changes to the new edition of the book: Part I is substantially rewritten, Part VIII has been changed to incorporate Clifford algebras, Part IX now includes the representation of Clifford algebras, and the new Part X discusses the important topic of fiber bundles.

I felt that a short section on *algebra* did not do justice to such an important topic. Therefore, I expanded it into a comprehensive chapter dealing with the basic properties of algebras and their classification. This required a rewriting of the chapter on operator algebras, including the introduction of a section on the representation of algebras in general. The chapter on *spectral decomposition* underwent a complete overhaul, as a result of which the topic is now more cohesive and the proofs more rigorous and illuminating. This entailed separate treatments of the spectral decomposition theorem for real and complex vector spaces.

The inner product of relativity is non-Euclidean. Therefore, in the discussion of tensors, I have explicitly expanded on the indefinite inner products and introduced a brief discussion of the subspaces of a non-Euclidean (the so-called semi-Riemannian or pseudo-Riemannian) vector space. This inner product, combined with the notion of algebra, leads naturally to *Clifford algebras*, the topic of the second chapter of Part VIII. Motivating the subject by introducing the Dirac equation, the chapter discusses the general properties of Clifford algebras in some detail and completely classifies the Clifford algebras $\mathbf{C}_\mu^\nu(\mathbb{R})$, the generalization of the algebra $\mathbf{C}_3^1(\mathbb{R})$, the Clifford algebra of the Minkowski space. The *representation* of Clifford algebras, including a treatment of *spinors*, is taken up in Part IX, after a discussion of the representation of Lie Groups and Lie algebras.

*Fiber bundles* have become a significant part of the lore of fundamental theoretical physics. The natural setting of gauge theories, essential in describing electroweak and strong interactions, is fiber bundles. Moreover, differential geometry, indispensable in the treatment of gravity, is most elegantly treated in terms of fiber bundles. Chapter 34 introduces fiber bundles and their complementary notion of *connection*, and the *curvature form* arising from the latter. Chapter 35 on *gauge theories* makes contact with physics and shows how connection is related to potentials and curvature to fields. It also constructs the most general gauge-invariant Lagrangian, including its local expression (the expression involving coordinate charts introduced on the underlying manifold), which is the form used by physicists. In Chap. 36, by introducing vector bundles and linear connections, the stage becomes ready for the introduction of *curvature tensor* and *torsion*, two major players in differential geometry. This approach to differential geometry via fiber bundles is, in my opinion, the most elegant and intuitive approach, which avoids the ad hoc introduction of covariant derivative. Continuing with differential geometry, Chap. 37 incorporates the notion of inner product and metric into it, coming up with the *metric connection*, so essential in the general theory of relativity.

All these changes and additions required certain omissions. I was careful not to break the continuity and rigor of the book when omitting topics. Since none of the discussions of numerical analysis was used anywhere else in the book, these were the first casualties. A few mathematical treatments that were too dry, technical, and not inspiring were also removed from the new edition. However, I provided references in which the reader can find these missing details. The only casualty of this kind of omission was the discussion leading to the spectral decomposition theorem for compact operators in Chap. 17.

Aside from the above changes, I have also altered the style of the book considerably. Now all mathematical statements—theorems, propositions, corollaries, definitions, remarks, etc.—and examples are numbered consecutively without regard to their types. This makes finding those statements or examples considerably easier. I have also placed important mathematical statements in boxes which are more visible as they have dark backgrounds. Additionally, I have increased the number of marginal notes, and added many more entries to the index.

Many readers and adopters provided invaluable feedback, both in spotting typos and in clarifying vague and even erroneous statements of the book. I would like to acknowledge the contribution of the following people to the correction of errors and the clarification of concepts: Sylvio Andrade, Salar Baher, Rafael Benguria, Jim Bogan, Jorun Bomert, John Chaffer, Demetris Charalambous, Robert Gooding, Paul Haines, Carl Helrich, Ray Jensen, Jin-Wook Jung, David Kastor, Fred Keil, Mike Lieber, Art Lind, Gary Miller, John Morgan, Thomas Schaefer, Hossein Shojaie, Shreenivas Somayaji, Werner Timmermann, Johan Wild, Bradley Wogsland, and Fang Wu. As much as I tried to keep a record of individuals who gave me feedback on the first edition, fourteen years is a long time, and I may have omitted some names from the list above. To those people, I sincerely apologize. Needless to say, any remaining errors in this new edition is solely my responsibility, and as always, I'll greatly appreciate it if the readers continue pointing them out to me.

I consulted the following three excellent books to a great extent for the addition and/or changes in the second edition:

Greub, W., *Linear Algebra*, 4th ed., Springer-Verlag, Berlin, 1975.
Greub, W., *Multilinear Algebra*, 2nd ed., Springer-Verlag, Berlin, 1978.
Kobayashi, S., and K. Nomizu, *Foundations of Differential Geometry*, vol. 1, Wiley, New York, 1963.

Maury Solomon, my editor at Springer, was immeasurably patient and cooperative on a project that has been long overdue. Aldo Rampioni has

been extremely helpful and cooperative as he took over the editorship of the project. My sincere thanks go to both of them. Finally, I would like to thank my wife Sarah for her unwavering forbearance and encouragement throughout the long-drawn-out writing of the new edition.

Normal, IL, USA Sadri Hassani
November, 2012

# Preface to First Edition

> "Ich kann es nun einmal nicht lassen, in diesem Drama von Mathematik und Physik—die sich im Dunkeln befruchten, aber von Angesicht zu Angesicht so gerne einander verkennen und verleugnen—die Rolle des (wie ich genügsam erfuhr, oft unerwünschten) *Boten* zu spielen."
>
> Hermann Weyl

It is said that mathematics is the language of Nature. If so, then physics is its poetry. Nature started to whisper into our ears when Egyptians and Babylonians were compelled to invent and use mathematics in their day-to-day activities. The faint geometric and arithmetical pidgin of over four thousand years ago, suitable for rudimentary conversations with nature as applied to simple landscaping, has turned into a sophisticated language in which the heart of matter is articulated.

The interplay between mathematics and physics needs no emphasis. What may need to be emphasized is that mathematics is not merely a tool with which the presentation of physics is facilitated, but the only medium in which physics can survive. Just as language is the means by which humans can express their thoughts and without which they lose their unique identity, mathematics is the only language through which physics can express itself and without which it loses its identity. And just as language is perfected due to its constant usage, mathematics develops in the most dramatic way because of its usage in physics. The quotation by Weyl above, an approximation to whose translation is "*In this drama of mathematics and physics—which fertilize each other in the dark, but which prefer to deny and misconstrue each other face to face—I cannot, however, resist playing the role of a messenger, albeit, as I have abundantly learned, often an unwelcome one*," is a perfect description of the natural intimacy between what mathematicians and physicists do, and the unnatural estrangement between the two camps. Some of the most beautiful mathematics has been motivated by physics (differential equations by Newtonian mechanics, differential geometry by general relativity, and operator theory by quantum mechanics), and some of the most fundamental physics has been expressed in the most beautiful poetry of mathematics (mechanics in symplectic geometry, and fundamental forces in Lie group theory).

I do not want to give the impression that mathematics and physics cannot develop independently. On the contrary, it is precisely the independence of each discipline that reinforces not only itself, but the other discipline as well—just as the study of the grammar of a language improves its usage and vice versa. However, the most effective means by which the two camps can accomplish great success is through an intense dialogue. Fortunately, with the advent of gauge and string theories of particle physics, such a dialogue has been reestablished between physics and mathematics after a relatively long lull.

## Level and Philosophy of Presentation

This is a book for physics students interested in the mathematics they use. It is also a book for mathematics students who wish to see some of the abstract ideas with which they are familiar come alive in an applied setting. The level of presentation is that of an advanced undergraduate or beginning graduate course (or sequence of courses) traditionally called "Mathematical Methods of Physics" or some variation thereof. Unlike most existing mathematical physics books intended for the same audience, which are usually lexicographic collections of facts about the diagonalization of matrices, tensor analysis, Legendre polynomials, contour integration, etc., with little emphasis on formal and systematic development of topics, this book attempts to strike a balance between formalism and application, between the abstract and the concrete.

I have tried to include as much of the essential formalism as is necessary to render the book optimally coherent and self-contained. This entails stating and proving a large number of theorems, propositions, lemmas, and corollaries. The benefit of such an approach is that the student will recognize clearly both the power and the limitation of a mathematical idea used in physics. There is a tendency on the part of the novice to universalize the mathematical methods and ideas encountered in physics courses because the limitations of these methods and ideas are not clearly pointed out.

There is a great deal of freedom in the topics and the level of presentation that instructors can choose from this book. My experience has shown that Parts I, II, III, Chap. 12, selected sections of Chap. 13, and selected sections or examples of Chap. 19 (or a large subset of all this) will be a reasonable course content for advanced undergraduates. If one adds Chaps. 14 and 20, as well as selected topics from Chaps. 21 and 22, one can design a course suitable for first-year graduate students. By judicious choice of topics from Parts VII and VIII, the instructor can bring the content of the course to a more modern setting. Depending on the sophistication of the students, this can be done either in the first year or the second year of graduate school.

## Features

To better understand theorems, propositions, and so forth, students need to see them in action. There are over 350 worked-out examples and over 850 problems (many with detailed hints) in this book, providing a vast arena in which students can watch the formalism unfold. The philosophy underlying this abundance can be summarized as "An example is worth a thousand words of explanation." Thus, whenever a statement is intrinsically vague or hard to grasp, worked-out examples and/or problems with hints are provided to clarify it. The inclusion of such a large number of examples is the means by which the balance between formalism and application has been achieved. However, although applications are essential in understanding mathematical physics, they are only one side of the coin. The theorems, propositions, lemmas, and corollaries, being highly condensed versions of knowledge, are equally important.

A conspicuous feature of the book, which is not emphasized in other comparable books, is the attempt to exhibit—as much as it is useful and applicable—interrelationships among various topics covered. Thus, the underlying theme of a vector space (which, in my opinion, is the most primitive concept at this level of presentation) recurs throughout the book and alerts the reader to the connection between various seemingly unrelated topics.

Another useful feature is the presentation of the historical setting in which men and women of mathematics and physics worked. I have gone against the trend of the "ahistoricism" of mathematicians and physicists by summarizing the life stories of the people behind the ideas. Many a time, the anecdotes and the historical circumstances in which a mathematical or physical idea takes form can go a long way toward helping us understand and appreciate the idea, especially if the interaction among—and the contributions of—all those having a share in the creation of the idea is pointed out, and the historical continuity of the development of the idea is emphasized.

To facilitate reference to them, all mathematical statements (definitions, theorems, propositions, lemmas, corollaries, and examples) have been numbered consecutively within each section and are preceded by the section number. For example, *4.2.9 Definition* indicates the ninth mathematical statement (which happens to be a definition) in Sect. 4.2. The end of a proof is marked by an empty square □, and that of an example by a filled square ■, placed at the right margin of each.

Finally, a comprehensive index, a large number of marginal notes, and many explanatory underbraced and overbraced comments in equations facilitate the use and comprehension of the book. In this respect, the book is also useful as a reference.

## Organization and Topical Coverage

Aside from Chap. 0, which is a collection of purely mathematical concepts, the book is divided into eight parts. Part I, consisting of the first four chapters, is devoted to a thorough study of finite-dimensional vector spaces and linear operators defined on them. As the unifying theme of the book, vector spaces demand careful analysis, and Part I provides this in the more accessible setting of finite dimension in a language that is conveniently generalized to the more relevant infinite dimensions, the subject of the next part.

Following a brief discussion of the technical difficulties associated with infinity, Part II is devoted to the two main infinite-dimensional vector spaces of mathematical physics: the classical orthogonal polynomials, and Fourier series and transform.

Complex variables appear in Part III. Chapter 9 deals with basic properties of complex functions, complex series, and their convergence. Chapter 10 discusses the calculus of residues and its application to the evaluation of definite integrals. Chapter 11 deals with more advanced topics such as multivalued functions, analytic continuation, and the method of steepest descent.

Part IV treats mainly ordinary differential equations. Chapter 12 shows how ordinary differential equations of second order arise in physical problems, and Chap. 13 consists of a formal discussion of these differential equations as well as methods of solving them numerically. Chapter 14 brings in the power of complex analysis to a treatment of the hypergeometric differential equation. The last chapter of this part deals with the solution of differential equations using integral transforms.

Part V starts with a formal chapter on the theory of operator and their spectral decomposition in Chap. 16. Chapter 17 focuses on a specific type of operator, namely the integral operators and their corresponding integral equations. The formalism and applications of Sturm-Liouville theory appear in Chaps. 18 and 19, respectively.

The entire Part VI is devoted to a discussion of Green's functions. Chapter 20 introduces these functions for ordinary differential equations, while Chaps. 21 and 22 discuss the Green's functions in an $m$-dimensional Euclidean space. Some of the derivations in these last two chapters are new and, as far as I know, unavailable anywhere else.

Parts VII and VIII contain a thorough discussion of Lie groups and their applications. The concept of group is introduced in Chap. 23. The theory of group representation, with an eye on its application in quantum mechanics, is discussed in the next chapter. Chapters 25 and 26 concentrate on tensor algebra and tensor analysis on manifolds. In Part VIII, the concepts of group and manifold are brought together in the context of Lie groups. Chapter 27 discusses Lie groups and their algebras as well as their representations, with special emphasis on their application in physics. Chapter 28 is on differential geometry including a brief introduction to general relativity. Lie's original motivation for constructing the groups that bear his name is discussed in Chap. 29 in the context of a systematic treatment of differential equations using their symmetry groups. The book ends in a chapter that blends many of the ideas developed throughout the previous parts in order to treat variational problems and their symmetries. It also provides a most fitting example of the claim made at the beginning of this preface and one of the most beautiful results of mathematical physics: Noether's theorem on the relation between symmetries and conservation laws.

## Acknowledgments

It gives me great pleasure to thank all those who contributed to the making of this book. George Rutherford was kind enough to volunteer for the difficult task of condensing hundreds of pages of biography into tens of extremely informative pages. Without his help this unique and valuable feature of the book would have been next to impossible to achieve. I thank him wholeheartedly. Rainer Grobe and Qichang Su helped me with my rusty

computational skills. (R.G. also helped me with my rusty German!) Many colleagues outside my department gave valuable comments and stimulating words of encouragement on the earlier version of the book. I would like to record my appreciation to Neil Rasband for reading part of the manuscript and commenting on it. Special thanks go to Tom von Foerster, senior editor of physics and mathematics at Springer-Verlag, not only for his patience and support, but also for the extreme care he took in reading the entire manuscript and giving me invaluable advice as a result. Needless to say, the ultimate responsibility for the content of the book rests on me. Last but not least, I thank my wife, Sarah, my son, Dane, and my daughter, Daisy, for the time taken away from them while I was writing the book, and for their support during the long and arduous writing process.

Many excellent textbooks, too numerous to cite individually here, have influenced the writing of this book. The following, however, are noteworthy for both their excellence and the amount of their influence:

Birkhoff, G., and G.-C. Rota, *Ordinary Differential Equations*, 3rd ed., New York, Wiley, 1978.
Bishop, R., and S. Goldberg, *Tensor Analysis on Manifolds*, New York, Dover, 1980.
Dennery, P., and A. Krzywicki, *Mathematics for Physicists*, New York, Harper & Row, 1967.
Halmos, P., *Finite-Dimensional Vector Spaces*, 2nd ed., Princeton, Van Nostrand, 1958.
Hamermesh, M., *Group Theory and its Application to Physical Problems*, Dover, New York, 1989.
Olver, P., *Application of Lie Groups to Differential Equations*, New York, Springer-Verlag, 1986.

Unless otherwise indicated, all biographical sketches have been taken from the following three sources:

Gillispie, C., ed., *Dictionary of Scientific Biography*, Charles Scribner's, New York, 1970.
Simmons, G., *Calculus Gems*, New York, McGraw-Hill, 1992.
History of Mathematics archive at www-groups.dcs.st-and.ac.uk:80.

I would greatly appreciate any comments and suggestions for improvements. Although extreme care was taken to correct all the misprints, the mere volume of the book makes it very likely that I have missed some (perhaps many) of them. I shall be most grateful to those readers kind enough to bring to my attention any remaining mistakes, typographical or otherwise. Please feel free to contact me.

Sadri Hassani
Campus Box 4560
Department of Physics
Illinois State University
Normal, IL 61790-4560, USA
e-mail: hassani@entropy.phy.ilstu.edu

It is my pleasure to thank all those readers who pointed out typographical mistakes and suggested a few clarifying changes. With the exception of a couple that required substantial revision, I have incorporated all the corrections and suggestions in this second printing.

# Note to the Reader

Mathematics and physics are like the game of chess (or, for that matter, like any game)—you will learn only by "playing" them. No amount of reading about the game will make you a master. In this book you will find a large number of examples and problems. Go through as many examples as possible, and try to reproduce them. Pay particular attention to sentences like "The reader may check ..." or "It is straightforward to show ...". These are red flags warning you that for a good understanding of the material at hand, you need to provide the missing steps. The problems often fill in missing steps as well; and in this respect they are essential for a thorough understanding of the book. Do not get discouraged if you cannot get to the solution of a problem at your first attempt. If you start from the beginning and think about each problem hard enough, you *will* get to the solution, and you will see that the subsequent problems will not be as difficult.

The extensive index makes the specific topics about which you may be interested to learn easily accessible. Often the marginal notes will help you easily locate the index entry you are after.

I have included a large collection of biographical sketches of mathematical physicists of the past. These are truly inspiring stories, and I encourage you to read them. They let you see that even under excruciating circumstances, the human mind can work miracles. You will discover how these remarkable individuals overcame the political, social, and economic conditions of their time to let us get a faint glimpse of the truth. They are our true heroes.

# Contents

# List of Symbols

| | |
|---|---|
| *General* | |
| $\forall$ | for all (values of) |
| $\exists$ | there exists (a values of) |
| *iff* | if and only if |
| $A \equiv B$ | $A$ is identical to (equivalent to, defined as) $B$ |
| *Set Theory* | |
| $\in$ | set theoretic membership sign: "belongs to" |
| $\notin$ | set theoretic exclusion sign: "does not belong to" |
| $\subset$ | subset sign |
| $\subseteq$ | subset with the possibility of equality emphasized |
| $\emptyset$ | empty set |
| $\cup$ | union of sets |
| $\cap$ | intersection of sets |
| $\sim A$ | complement of the set $A$ |
| $A \times B$ | set of ordered pairs $(a, b)$ with $a \in A$ and $b \in B$ |
| $\bowtie$ | equivalence relation |
| $[\![a]\!]$ | equivalence class of $a$ |
| $\mathrm{id}_X$ | identity map of the set $X$ |
| $\mathbb{N}$ | the set of natural (non-negative integer) numbers |
| $\mathbb{Z}$ | the set of integers |
| $\mathbb{Q}$ | the set of rational numbers |
| $\mathbb{R}$ | the set of real numbers |
| $\mathbb{R}^+$ | the set of positive real numbers |
| $\mathbb{C}$ | the set of complex numbers |
| $\mathbb{H}$ | the set of quaternions |
| $f : A \to B$ | map $f$ from set $A$ to set $B$ |
| $x \mapsto f(x)$ | $x$ is mapped to $f(x)$ via the map $f$ |
| $g \circ f$ | composition of the two maps $f$ and $g$ |
| *Vector Spaces* | |
| $\mathcal{V}$ | a generic vector space |
| $\lvert a\rangle$ | a generic vector labeled $a$ |
| $\mathbb{C}^n$, $\mathbb{R}^n$ | complex (real) $n$-tuples |
| $\mathcal{M}^{m\times n}$ | $m \times n$ matrices |
| $\mathbb{C}^\infty$ | absolutely convergent complex series |
| $\mathcal{P}^c[t]$ | polynomials in $t$ with complex coefficients |
| $\mathcal{P}^r[t]$ | polynomials in $t$ with real coefficients |
| $\mathcal{P}^c_n[t]$ | complex polynomials of degree $n$ or less |

| | |
|---|---|
| $\mathcal{C}^0(a,b)$ | continuous functions on real interval $(a,b)$ |
| $\mathcal{C}^n(a,b)$ | $n$-times differentiable functions on real interval $(a,b)$ |
| $\mathcal{C}^\infty(a,b)$ | infinitely differentiable functions on real interval $(a,b)$ |
| $\text{Span}\{S\}$ | the span of a subset $S$ of a vector space |
| $\langle a \mid b\rangle$ | inner product of vectors $\lvert a\rangle$ and $\lvert b\rangle$ |
| $\lVert a\rVert$ | norm of the vector $\lvert a\rangle$ |
| $\oplus$ | direct sum of vectors or vector spaces |
| $\bigoplus_{k=1}^n$ | direct sum of $n$ vectors or vector spaces |
| $\mathcal{V}^*$ | dual of the vector space $\mathcal{V}$ |
| $\langle a, \boldsymbol{\alpha}\rangle$ | pairing of the vector $\lvert a\rangle$ with its dual $\boldsymbol{\alpha}$ |

*Algebras*

| | |
|---|---|
| $\mathcal{A}$ | a generic algebra |
| $\mathcal{L}(\mathcal{V})$ | the algebra of linear maps of the vector space $\mathcal{V}$; same as $\text{End}(\mathcal{V})$ |
| $\text{End}(\mathcal{V})$ | the algebra of linear maps of the vector space $\mathcal{V}$; same as $\mathcal{L}(\mathcal{V})$ |
| $\oplus$ | direct sum of algebras |
| $\oplus_V$ | direct sum of algebras only as vector spaces |
| $\otimes$ | tensor product of algebras |
| $\mathcal{M}(\mathbb{F})$ | total matrix algebra over the field $\mathbb{F}$ |
| $\mathcal{C}(\mathcal{V})$ | Clifford algebra of the inner-product space $\mathcal{V}$ |
| $\vee$ | Clifford product symbol |

*Groups*

| | |
|---|---|
| $S_n$ | symmetric group; group of permutations of $1, 2, \ldots, n$ |
| $GL(\mathcal{V})$ | general linear group of the vector space $\mathcal{V}$ |
| $GL(n,\mathbb{C})$ | general linear group of the vector space $\mathbb{C}^n$ |
| $GL(n,\mathbb{R})$ | general linear group of the vector space $\mathbb{R}^n$ |
| $SL(\mathcal{V})$ | special linear group; subgroup of $GL(\mathcal{V})$ with unit determinant |
| $O(n)$ | orthogonal group; group of orthogonal $n \times n$ matrices |
| $SO(n)$ | special orthogonal group; subgroup of $O(n)$ with unit determinant |
| $U(n)$ | unitary group; group of unitary $n \times n$ matrices |
| $SU(n)$ | special unitary group; subgroup of $U(n)$ with unit determinant |
| $\mathfrak{g}$ | Lie algebra of the Lie group $G$ |
| $\cong$ | isomorphism of groups, vector spaces, and algebras |

*Tensors*

| | |
|---|---|
| $\mathcal{T}_s^r(\mathcal{V})$ | set of tensors of type $(r,s)$ in vector space $\mathcal{V}$ |
| $\mathcal{S}^r(\mathcal{V})$ | set of symmetric tensors of type $(r,0)$ in vector space $\mathcal{V}$ |
| $\Lambda^p(\mathcal{V})$ | set of $p$-forms in vector space $\mathcal{V}$ |
| $\Lambda^p(\mathcal{V}^*)$ | set of $p$-vectors in vector space $\mathcal{V}$ |
| $\wedge$ | wedge (exterior) product symbol |
| $\Lambda^p(\mathcal{V},\mathcal{U})$ | set of $p$-forms in $\mathcal{V}$ with values in vector space $\mathcal{U}$ |
| $F^\infty(P)$ | set of infinitely differentiable functions at point $P$ of a manifold |
| $\mathcal{T}_P(M)$ | set of vectors tangent to manifold $M$ at point $P \in M$ |

| | |
|---|---|
| $T(M)$ | tangent bundle of the manifold $M$ |
| $T^*(M)$ | cotangent bundle of the manifold $M$ |
| $\mathfrak{X}(M)$ | set of vector fields on the manifold $M$ |
| $\exp(t\mathbf{X})$ | flow of the vector field $\mathbf{X}$ parametrized by $t$ |
| $T_s^r(M)$ | set of tensor fields of type $(r, s)$ in the manifold $M$ |