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Partial Differential Equations

Third Edition

Springer

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Preface

This is the third edition of my textbook intended for students who wish to obtain an introduction to the theory of partial differential equations (PDEs, for short). Why is there a new edition? The answer is simple: I wanted to improve my book. Over the years, I have received much positive feedback from readers from all over the world. Nevertheless, when looking at the book or using it for courses or lectures, I always find some topics that are important, but not yet contained in the book, or I see places where the presentation could be improved. In fact, I also found two errors in Sect. 6.2, and several other corrections have been brought to my attention by attentive and careful readers.

So, what is new? I have completely reorganized and considerably extended Chap. 7 on hyperbolic equations. In particular, it now also contains a treatment of first-order hyperbolic equations. I have written a new Chap. 9 on the relations between different types of PDEs. I have inserted material on the regularity theory for semilinear elliptic equations and systems in various places. In particular, there is a new Sect. 14.3 that shows how to use the Harnack inequality to derive the continuity of bounded weak solutions of semilinear elliptic equations. Such equations play an important role in geometric analysis and elsewhere, and I therefore thought that such an addition should serve a useful purpose. I have also slightly rewritten, reorganized, or extended most other sections of the book, with additional results inserted here and there.

But let me now describe the book in a more systematic manner. As an introduction to the modern theory of PDEs, it does not offer a comprehensive overview of the whole field of PDEs, but tries to lead the reader to the most important methods and central results in the case of elliptic PDEs. The guiding question is how one can find a solution of such a PDE. Such a solution will, of course, depend on given constraints and, in turn, if the constraints are of the appropriate type, be uniquely determined by them. We shall pursue a number of strategies for finding a solution of a PDE; they can be informally characterized as follows:

0. *Write down an* **explicit formula** *for the solution in terms of the given data (constraints).* This may seem like the best and most natural approach, but this

is possible only in rather particular and special cases. Also, such a formula may be rather complicated, so that it is not very helpful for detecting qualitative properties of a solution. Therefore, mathematical analysis has developed other, more powerful, approaches.

1. *Solve a sequence of auxiliary problems that* **approximate** *the given one and show that their solutions converge to a solution of that original problem.* Differential equations are posed in spaces of functions, and those spaces are of infinite dimension. The strength of this strategy lies in carefully choosing finite-dimensional approximating problems that can be solved explicitly or numerically and that still share important crucial features with the original problem. Those features will allow us to control their solutions and to show their convergence.
2. *Start anywhere, with the required constraints satisfied, and let things* **flow** *towards a solution.* This is the diffusion method. It depends on characterizing a solution of the PDE under consideration as an asymptotic equilibrium state for a diffusion process. That diffusion process itself follows a PDE, with an additional independent variable. Thus, we are solving a PDE that is more complicated than the original one. The advantage lies in the fact that we can simply start anywhere and let the PDE control the evolution.
3. *Solve an* **optimization** *problem and identify an optimal state as a solution of the PDE.* This is a powerful method for a large class of elliptic PDEs, namely, for those that characterize the optima of variational problems. In fact, in applications in physics, engineering, or economics, most PDEs arise from such optimization problems. The method depends on two principles. First, one can demonstrate the existence of an optimal state for a variational problem under rather general conditions. Second, the optimality of a state is a powerful property that entails many detailed features: If the state is not very good at every point, it could be improved and therefore could not be optimal.
4. **Connect** *what you want to know to what you know already.* This is the continuity method. The idea is that if you can connect your given problem continuously with another, simpler, problem that you can already solve, then you can also solve the former. Of course, the continuation of solutions requires careful control.

The various existence schemes will lead us to another, more technical, but equally important, question, namely, the one about the regularity of solutions of PDEs. If one writes down a differential equation for some function, then one might be inclined to assume explicitly or implicitly that a solution satisfies appropriate differentiability properties so that the equation is meaningful. The problem, however, with many of the existence schemes described above is that they often only yield a solution in some function space that is so large that it also contains nonsmooth and perhaps even noncontinuous functions. The notion of a solution thus has to be interpreted in some generalized sense. It is the task of regularity theory to show that the equation in question forces a generalized solution to be smooth after all, thus closing the circle. This will be the second guiding problem of this book.

The existence and the regularity questions are often closely intertwined. Regularity is often demonstrated by deriving explicit estimates in terms of the given constraints that any solution has to satisfy, and these estimates in turn can be used for compactness arguments in existence schemes. Such estimates can also often be used to show the uniqueness of solutions, and, of course, the problem of uniqueness is also fundamental in the theory of PDEs.

After this informal discussion, let us now describe the contents of this book in more specific detail.

Our starting point is the Laplace equation, whose solutions are the harmonic functions. The field of elliptic PDEs is then naturally explored as a generalization of the Laplace equation, and we emphasize various aspects on the way. We shall develop a multitude of different approaches, which in turn will also shed new light on our initial Laplace equation. One of the important approaches is the heat equation method, where solutions of elliptic PDEs are obtained as asymptotic equilibria of parabolic PDEs. In this sense, one chapter treats the heat equation, so that the present textbook definitely is not confined to elliptic equations only. We shall also treat the wave equation as the prototype of a hyperbolic PDE and discuss its relation to the Laplace and heat equations. In general, the behavior of solutions of hyperbolic differential equations can be rather different from that of elliptic and parabolic equations, and we shall use first-order hyperbolic equations to exhibit some typical phenomena. In the context of the heat equation, another chapter develops the theory of semigroups and explains the connection with Brownian motion. There exist many connections between different types of differential equations. For instance, the density function of a system of ordinary differential equations satisfies a first-order hyperbolic equation. Such equations can be studied by semigroup theory, or one can add a small regularizing elliptic term to obtain a so-called viscosity solution.

Other methods for obtaining the existence of solutions of elliptic PDEs, like the difference method, which is important for the numerical construction of solutions, the Perron method; and the alternating method of H.A. Schwarz are based on the maximum principle. We shall present several versions of the maximum principle that are also relevant to applications to nonlinear PDEs.

In any case, it is an important guiding principle of this textbook to develop methods that are also useful for the study of nonlinear equations, as those present the research perspective of the future. Most of the PDEs occurring in applications in the sciences, economics, and engineering are of nonlinear types. One should keep in mind, however, that, because of the multitude of occurring equations and resulting phenomena, there cannot exist a unified theory of nonlinear (elliptic) PDEs, in contrast to the linear case. Thus, there are also no universally applicable methods, and we aim instead at doing justice to this multitude of phenomena by developing very diverse methods.

Thus, after the maximum principle and the heat equation, we shall encounter variational methods, whose idea is represented by the so-called Dirichlet principle. For that purpose, we shall also develop the theory of Sobolev spaces, including fundamental embedding theorems of Sobolev, Morrey, and John–Nirenberg. With the help of such results, one can show the smoothness of the so-called weak solutions obtained by the variational approach. We also treat the regularity theory of the so-called strong solutions, as well as Schauder's regularity theory for solutions in

Hölder spaces. In this context, we also explain the continuity method that connects an equation that one wishes to study in a continuous manner with one that one understands already and deduces solvability of the former from solvability of the latter with the help of a priori estimates.

The final chapter develops the Moser iteration technique, which turned out to be fundamental in the theory of elliptic PDEs. With that technique one can extend many properties that are classically known for harmonic functions (Harnack inequality, local regularity, maximum principle) to solutions of a large class of general elliptic PDEs. The results of Moser will also allow us to prove the fundamental regularity theorem of de Giorgi and Nash for minimizers of variational problems.

At the end of each chapter, we briefly summarize the main results, occasionally suppressing the precise assumptions for the sake of saliency of the statements. I believe that this helps in guiding the reader through an area of mathematics that does not allow a unified structural approach, but rather derives its fascination from the multitude and diversity of approaches and methods and consequently encounters the danger of getting lost in the technical details.

Some words about the logical dependence between the various chapters: Most chapters are composed in such a manner that only the first sections are necessary for studying subsequent chapters. The first—rather elementary—chapter, however, is basic for understanding almost all remaining chapters. Section 3.1 is useful, although not indispensable, for Chap. 4. Sections 5.1 and 5.2 are important for Chaps. 7 and 8. Chapter 9, which partly has some survey character, connects various previous chapters. Sections 10.1–10.4 are fundamental for Chaps. 11 and 14, and Sect. 11.1 will be employed in Chaps. 12 and 14. With those exceptions, the various chapters can be read independently. Thus, it is also possible to vary the order in which the chapters are studied. For example, it would make sense to read Chap. 10 directly after Chap. 2, in order to see the variational aspects of the Laplace equation (in particular, Sect. 10.1) and also the transformation formula for this equation with respect to changes of the independent variables. In this way one is naturally led to a larger class of elliptic equations. In any case, it is usually not very efficient to read a mathematical textbook linearly, and the reader should rather try first to grasp the central statements.

This book can be utilized for a one-year course on PDEs, and if time does not allow all the material to be covered, one could omit certain sections and chapters, for example, Sect. 4.3 and the first part of Sect. 4.4 and Chap. 12. Also, Chap. 9 will not be needed for the rest of the book. Of course, the lecturer may also decide to omit Chap. 14 if he or she wishes to keep the treatment at a more elementary level.

This book is based on various graduate courses that I have given at Bochum and Leipzig. I thank Antje Vandenberg for general logistic support, and of course also all the people who had helped me with the previous editions. They are listed in the previous prefaces, but I should repeat my thanks to Lutz Habermann and Knut Smoczyk here for their help with the first edition.

Concerning corrections for the present edition, I would like to thank Andreas Schäfer for a very detailed and carefully compiled list of corrections. Also, I thank Lei Ni for pointing out that the statement of Lemma 5.3.2 needed a qualification. Finally, I thank my son Leonardo Jost for a discussion that leads to an improvement of the presentation in Sect. 11.3. I am also grateful to Tim Healey and his students Robert Kesler and Aaron Palmer for alerting me to an error in Sect. 13.1.

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