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Quantum Theory for Mathematicians

Springer

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For as the heavens are higher than the earth, so are my ways higher than your ways, and my thoughts than your thoughts, says the Lord.

Isaiah 55:9

Preface

Ideas from quantum physics play important roles in many parts of modern mathematics. Many parts of representation theory, for example, are motivated by quantum mechanics, including the Wigner–Mackey theory of induced representations, the Kirillov–Kostant orbit method, and, of course, quantum groups. The Jones polynomial in knot theory, the Gromov–Witten invariants in topology, and mirror symmetry in algebraic topology are other notable examples. The awarding of the 1990 Fields Medal to Ed Witten, a physicist, gives an idea of the scope of the influence of quantum theory in mathematics.

Despite the importance of quantum mechanics to mathematics, there is no easy way for mathematicians to learn the subject. Quantum mechanics books in the physics literature are generally not easily understood by most mathematicians. There is, of course, a lower level of mathematical precision in such books than mathematicians are accustomed to. In addition, physics books on quantum mechanics assume knowledge of classical mechanics that mathematicians often do not have. And, finally, there is a subtle difference in "culture"—differences in terminology and notation—that can make reading the physics literature like reading a foreign language for the mathematician. There are few books that attempt to translate quantum theory into terms that mathematicians can understand.

This book is intended as an introduction to quantum mechanics for mathematicians with little prior exposure to physics. The twin goals of the book are (1) to explain the physical ideas of quantum mechanics in language mathematicians will be comfortable with, and (2) to develop the necessary mathematical tools to treat those ideas in a rigorous fashion. I have attempted to give a reasonably comprehensive treatment of nonrelativistic quantum mechanics, including topics found in typical physics texts (e.g., the harmonic oscillator, the hydrogen atom, and the WKB approximation) as well as more mathematical topics (e.g., quantization schemes, the Stone–von Neumann theorem, and geometric quantization). I have also attempted to minimize the mathematical prerequisites. I do not assume, for example, any prior knowledge of spectral theory or unbounded operators, but provide a full treatment of those topics in Chaps. 6 through 10 of the text. Similarly, I do not assume familiarity with the theory of Lie groups and Lie algebras, but provide a detailed account of those topics in Chap. 16. Whenever possible, I provide full proofs of the stated results.

Most of the text will be accessible to graduate students in mathematics who have had a first course in real analysis, covering the basics of L^2 spaces and Hilbert spaces. Appendix A reviews some of the results that are used in the main body of the text. In Chaps. 21 and 23, however, I assume knowledge of the theory of manifolds. I have attempted to provide motivation for many of the definitions and proofs in the text, with the result that there is a fair amount of discussion interspersed with the standard definition-theorem-proof style of mathematical exposition. There are exercises at the end of each chapter, making the book suitable for graduate courses as well as for independent study.

In comparison to the present work, classics such as Reed and Simon [34] and Glimm and Jaffe [14], along with the recent book of Schmüdgen [35], are more focused on the mathematical underpinnings of the theory than on the physical ideas. Hannabuss's text [22] is fairly accessible to mathematicians, but—despite the word "graduate" in the title of the series—uses an undergraduate level of mathematics. The recent book of Takhtajan [39], meanwhile, has an expository bent to it, but provides less physical motivation and is less self-contained than the present book. Whereas, for example, Takhtajan begins with Lagrangian and Hamiltonian mechanics on manifolds, I begin with "low-tech" classical mechanics on the real line. Similarly, Takhtajan assumes knowledge of unbounded operators and Lie groups, while I provide substantial expositions of both of those subjects. Finally, there is the work of Folland [13], which I highly recommend, but which deals with quantum field theory, whereas the present book treats only nonrelativistic quantum mechanics, except for a very brief discussion of quantum field theory in Sect. 20.6.

The book begins with a quick introduction to the main ideas of classical and quantum mechanics. After a brief account in Chap. 1 of the historical origins of quantum theory, I turn in Chap. 2 to a discussion of the necessary background from classical mechanics. This includes Newton's equation in varying degrees of generality, along with a discussion of important physical quantities such as energy, momentum, and angular momentum, and conditions under which these quantities are "conserved" (i.e., constant along each solution of Newton's equation). I give a short treatment here of Poisson brackets and Hamilton's form of Newton's equation, deferring a full discussion of "fancy" classical mechanics to Chap. 21.

In Chap. 3, I attempt to motivate the structures of quantum mechanics in the simplest setting. Although I discuss the "axioms" (in standard physics terminology) of quantum mechanics, I resolutely avoid a strictly axiomatic approach to the subject (using, say, C^*-algebras). Rather, I try to provide some motivation for the position and momentum operators and the Hilbert space approach to quantum theory, as they connect to the probabilistic aspect of the theory. I do not attempt to *explain* the strange probabilistic nature of quantum theory, if, indeed, there is any explanation of it. Rather, I try to elucidate how the wave function, along with the position and momentum operators, *encodes* the relevant probabilities.

In Chaps. 4 and 5, we look into two illustrative cases of the Schrödinger equation in one space dimension: a free particle and a particle in a square well. In these chapters, we encounter such important concepts as the distinction between phase velocity and group velocity and the distinction between a discrete and a continuous spectrum.

In Chaps. 6 through 10, we look into some of the technical mathematical issues that are swept under the carpet in earlier chapters. I have tried to design this section of the book in such a way that a reader can take in as much or as little of the mathematical details as desired. For a reader who simply wants the big picture, I outline the main ideas and results of spectral theory in Chap. 6, including a discussion of the prototypical example of an operator with a continuous spectrum: the momentum operator. For a reader who wants more information, I provide statements of the spectral theorem (in two different forms) for bounded self-adjoint operators in Chap. 7, and an introduction to the notion of unbounded self-adjoint operators in Chap. 9. Finally, for the reader who wants all the details, I give proofs of the spectral theorem for bounded and unbounded self-adjoint operators, in Chaps. 8 and 10, respectively.

In Chaps. 11 through 14, we turn to the vitally important canonical commutation relations. These are used in Chap. 11 to derive algebraically the spectrum of the quantum harmonic oscillator. In Chap. 12, we discuss the uncertainty principle, both in its general form (for arbitrary pairs of noncommuting operators) and in its specific form (for the position and momentum operators). We pay careful attention to subtle domain issues that are usually glossed over in the physics literature. In Chap. 13, we look at different "quantization schemes" (i.e., different ways of ordering products of the noncommuting position and momentum operators). In Chap. 14, we turn to the celebrated Stone–von Neumann theorem, which provides a uniqueness result for representations of the canonical commutation relations. As in the case of the uncertainty principle, there are some subtle domain issues here that require attention.

In Chaps. 15 through 18, we examine some less elementary issues in quantum theory. Chapter 15 addresses the WKB (Wentzel–Kramers–Brillouin) approximation, which gives simple but approximate formulas for the eigenvectors and eigenvalues for the Hamiltonian operator in one dimension. After this, we introduce (Chap. 16) the notion of Lie groups, Lie algebras, and their representations, all of which play an important role in many parts of quantum mechanics. In Chap. 17, we consider the example of angular momentum and spin, which can be understood in terms of the representations of the rotation group $\mathsf{SO}(3)$. Here a more mathematical approach—especially the relationship between Lie group representations and Lie algebra representations—can substantially clarify a topic that is rather mysterious in the physics literature. In particular, the concept of "fractional spin" can be understood as describing a representation of the *Lie algebra* of the rotation group for which there is no associated representation of the rotation group itself. In Chap. 18, we illustrate these ideas by describing the energy levels of the hydrogen atom, including a discussion of the hidden symmetries of hydrogen, which account for the "accidental degeneracy" in the levels. In Chap. 19, we look more closely at the concept of the "state" of a system in quantum mechanics. We look at the notion of subsystems of a quantum system in terms of tensor products of Hilbert spaces, and we see in this setting that the notion of "pure state" (a unit vector in the relevant Hilbert space) is not adequate. We are led, then, to the notion of a mixed state (or density matrix). We also examine the idea that, in quantum mechanics, "identical particles are indistinguishable."

Finally, in Chaps. 21 through 23, we examine some advanced topics in classical and quantum mechanics. We begin, in Chap. 20, by considering the path integral formulation of quantum mechanics, both from the heuristic perspective of the Feynman path integral, and from the rigorous perspective of the Feynman–Kac formula. Then, in Chap. 21, we give a brief treatment of Hamiltonian mechanics on manifolds. Finally, we consider the machinery of geometric quantization, beginning with the Euclidean case in Chap. 22 and continuing with the general case in Chap. 23.

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