

# Undergraduate Texts in Mathematics

# Undergraduate Texts in Mathematics

---

## Series Editors:

Sheldon Axler

*San Francisco State University, San Francisco, CA, USA*

Kenneth Ribet

*University of California, Berkeley, CA, USA*

## Advisory Board:

Colin Adams, *Williams College, Williamstown, MA, USA*

Alejandro Adem, *University of British Columbia, Vancouver, BC, Canada*

Ruth Charney, *Brandeis University, Waltham, MA, USA*

Irene M. Gamba, *The University of Texas at Austin, Austin, TX, USA*

Roger E. Howe, *Yale University, New Haven, CT, USA*

David Jerison, *Massachusetts Institute of Technology, Cambridge, MA, USA*

Jeffrey C. Lagarias, *University of Michigan, Ann Arbor, MI, USA*

Jill Pipher, *Brown University, Providence, RI, USA*

Fadil Santosa, *University of Minnesota, Minneapolis, MN, USA*

Amie Wilkinson, *University of Chicago, Chicago, IL, USA*

**Undergraduate Texts in Mathematics** are generally aimed at third- and fourth-year undergraduate mathematics students at North American universities. These texts strive to provide students and teachers with new perspectives and novel approaches. The books include motivation that guides the reader to an appreciation of interrelations among different aspects of the subject. They feature examples that illustrate key concepts as well as exercises that strengthen understanding.

For further volumes:

<http://www.springer.com/series/666>

Peter D. Lax • Maria Shea Terrell

# Calculus With Applications

Second Edition

 Springer

Peter D. Lax  
Courant Institute of Mathematical Sciences  
New York University  
New York, NY, USA

Maria Shea Terrell  
Department of Mathematics  
Cornell University  
Ithaca, NY, USA

ISSN 0172-6056

ISBN 978-1-4614-7945-1

ISBN 978-1-4614-7946-8 (eBook)

DOI 10.1007/978-1-4614-7946-8

Springer New York Heidelberg Dordrecht London

Library of Congress Control Number: 2013946572

Mathematics Subject Classification: 00-01

© Springer Science+Business Media New York 1976, 2014

This work is subject to copyright. All rights are reserved by the Publisher, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilms or in any other physical way, and transmission or information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed. Exempted from this legal reservation are brief excerpts in connection with reviews or scholarly analysis or material supplied specifically for the purpose of being entered and executed on a computer system, for exclusive use by the purchaser of the work. Duplication of this publication or parts thereof is permitted only under the provisions of the Copyright Law of the Publisher's location, in its current version, and permission for use must always be obtained from Springer. Permissions for use may be obtained through RightsLink at the Copyright Clearance Center. Violations are liable to prosecution under the respective Copyright Law.

The use of general descriptive names, registered names, trademarks, service marks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

While the advice and information in this book are believed to be true and accurate at the date of publication, neither the authors nor the editors nor the publisher can accept any legal responsibility for any errors or omissions that may be made. The publisher makes no warranty, express or implied, with respect to the material contained herein.

Printed on acid-free paper

Springer is part of Springer Science+Business Media ([www.springer.com](http://www.springer.com))

# Preface

Our purpose in writing a calculus text has been to help students learn at first hand that mathematics is the language in which scientific ideas can be precisely formulated, that science is a source of mathematical ideas that profoundly shape the development of mathematics, and that mathematics can furnish brilliant answers to important scientific problems. This book is a thorough revision of the text *Calculus with Applications and Computing* by Lax, Burstein, and Lax. The original text was predicated on a number of innovative ideas, and it included some new and nontraditional material. This revision is written in the same spirit. It is fair to ask what new subject matter or new ideas could possibly be introduced into so old a topic as calculus. The answer is that science and mathematics are growing by leaps and bounds on the research frontier, so what we teach in high school, college, and graduate school must not be allowed to fall too far behind. As mathematicians and educators, our goal must be to simplify the teaching of old topics to make room for new ones.

To achieve that goal, we present the language of mathematics as natural and comprehensible, a language students can learn to use. Throughout the text we offer proofs of all the important theorems to help students understand their meaning; our aim is to foster understanding, not “rigor.” We have greatly increased the number of worked examples and homework problems. We have made some significant changes in the organization of the material; the familiar transcendental functions are introduced before the derivative and the integral. The word “computing” was dropped from the title because today, in contrast to 1976, it is generally agreed that computing is an integral part of calculus and that it poses interesting challenges. These are illustrated in this text in Sects. 4.4, 5.3, and 10.4, and by all of Chap. 8. But the mathematics that enables us to discuss issues that arise in computing when we round off inputs or approximate a function by a sequence of functions, i.e., uniform continuity and uniform convergence, remains. We have worked hard in this revision to show that uniform convergence and continuity are more natural and useful than pointwise convergence and continuity. The initial feedback from students who have used the text is that they “get it.”

This text is intended for a two-semester course in the calculus of a single variable. Only knowledge of high-school precalculus is expected.

Chapter 1 discusses numbers, approximating numbers, and limits of sequences of numbers. Chapter 2 presents the basic facts about continuous functions and describes the classical functions: polynomials, trigonometric functions, exponentials, and logarithms. It introduces limits of sequences of functions, in particular power series.

In Chapter 3, the derivative is defined and the basic rules of differentiation are presented. The derivatives of polynomials, the exponential function, the logarithm, and trigonometric functions are calculated. Chapter 4 describes the basic theory of differentiation, higher derivatives, Taylor polynomials and Taylor's theorem, and approximating derivatives by difference quotients. Chapter 5 describes how the derivative enters the laws of science, mainly physics, and how calculus is used to deduce consequences of these laws.

Chapter 6 introduces, through examples of distance, mass, and area, the notion of the integral, and the approximate integrals leading to its definition. The relation between differentiation and integration is proved and illustrated. In Chapter 7, integration by parts and change of variable in integrals are presented, and the integral of the uniform limit of a sequence of functions is shown to be the limit of the integrals of the sequence of functions. Chapter 8 is about the approximation of integrals; Simpson's rule is derived and compared with other numerical approximations of integrals.

Chapter 9 shows how many of the concepts of calculus can be extended to complex-valued functions of a real variable. It also introduces the exponential of complex numbers. Chapter 10 applies calculus to the differential equations governing vibrating strings, changing populations, and chemical reactions. It also includes a very brief introduction to Euler's method. Chapter 11 is about the theory of probability, formulated in the language of calculus.

The material in this book has been used successfully at Cornell in a one-semester calculus II course for students interested in majoring in mathematics or science. The students typically have credit for one semester of calculus from high school. Chapters 1, 2, and 4 have been used to present sequences and series of numbers, power series, Taylor polynomials, and Taylor's theorem. Chapters 6–8 have been used to present the definite integral, application of integration to volumes and accumulation problems, methods of integration, and approximation of integrals. There has been adequate time left in the term then to present Chapter 9, on complex numbers and functions, and to see how complex functions and calculus are used to model vibrations in the first section of Chapter 10.

We are grateful to the many colleagues and students in the mathematical community who have supported our efforts to write this book. The first edition of this book was written in collaboration with Samuel Burstein. We thank him for allowing us to draw on his work. We wish to thank John Guckenheimer for his encouragement and advice on this project. We thank Matt Guay, John Meluso, and Wyatt Deviau, who while they were undergraduates at Cornell, carefully read early drafts of the manuscript, and whose perceptive comments helped us keep our student audience in mind. We also wish to thank Patricia McGrath, a teacher at Maloney High School in Meriden, Connecticut, for her thoughtful review and suggestions, and Thomas

Kern and Chenxi Wu, graduate students at Cornell who assisted in teaching calculus II with earlier drafts of the text, for their help in writing solutions to some of the homework problems. Many thanks go to the students at Cornell who used early drafts of this book in fall 2011 and 2012. Thank you all for inspiring us to work on this project, and to make it better.

This current edition would have been impossible without the support of Bob Terrell, Maria's husband and long-time mathematics teacher at Cornell. From  $\text{\TeX}$ -ing the manuscript to making the figures, to suggesting changes and improvements, at every step along the way we owe Bob more than we can say.

Peter Lax thanks his colleagues at the Courant Institute, with whom he has discussed over 50 years the challenge of teaching calculus.

New York, NY  
Ithaca, NY

Peter Lax  
Maria Terrell



# Contents

<b>1</b>	<b>Numbers and Limits</b> .....	1
1.1	Inequalities .....	1
1.1a	Rules for Inequalities .....	3
1.1b	The Triangle Inequality .....	4
1.1c	The Arithmetic–Geometric Mean Inequality .....	5
1.2	Numbers and the Least Upper Bound Theorem .....	11
1.2a	Numbers as Infinite Decimals .....	11
1.2b	The Least Upper Bound Theorem .....	13
1.2c	Rounding .....	16
1.3	Sequences and Their Limits .....	19
1.3a	Approximation of $\sqrt{2}$ .....	23
1.3b	Sequences and Series .....	24
1.3c	Nested Intervals .....	36
1.3d	Cauchy Sequences .....	37
1.4	The Number $e$ .....	44
<b>2</b>	<b>Functions and Continuity</b> .....	51
2.1	The Notion of a Function .....	51
2.1a	Bounded Functions .....	54
2.1b	Arithmetic of Functions .....	55
2.2	Continuity .....	59
2.2a	Continuity at a Point Using Limits .....	61
2.2b	Continuity on an Interval .....	64
2.2c	Extreme and Intermediate Value Theorems .....	66
2.3	Composition and Inverses of Functions .....	71
2.3a	Composition .....	71
2.3b	Inverse Functions .....	74
2.4	Sine and Cosine .....	81
2.5	Exponential Function .....	86
2.5a	Radioactive Decay .....	86
2.5b	Bacterial Growth .....	87

2.5c	Algebraic Definition	87
2.5d	Exponential Growth	89
2.5e	Logarithm	91
2.6	Sequences of Functions and Their Limits	96
2.6a	Sequences of Functions	96
2.6b	Series of Functions	103
2.6c	Approximating the Functions $\sqrt{x}$ and $e^x$	107
<b>3</b>	<b>The Derivative and Differentiation</b>	<b>117</b>
3.1	The Concept of Derivative	117
3.1a	Graphical Interpretation	120
3.1b	Differentiability and Continuity	123
3.1c	Some Uses for the Derivative	125
3.2	Differentiation Rules	133
3.2a	Sums, Products, and Quotients	133
3.2b	Derivative of Compositions of Functions	138
3.2c	Higher Derivatives and Notation	141
3.3	Derivative of $e^x$ and $\log x$	146
3.3a	Derivative of $e^x$	146
3.3b	Derivative of $\log x$	148
3.3c	Power Rule	149
3.3d	The Differential Equation $y' = ky$	150
3.4	Derivatives of the Trigonometric Functions	154
3.4a	Sine and Cosine	154
3.4b	The Differential Equation $y'' + y = 0$	156
3.4c	Derivatives of Inverse Trigonometric Functions	159
3.4d	The Differential Equation $y'' - y = 0$	161
3.5	Derivatives of Power Series	166
<b>4</b>	<b>The Theory of Differentiable Functions</b>	<b>171</b>
4.1	The Mean Value Theorem	171
4.1a	Using the First Derivative for Optimization	174
4.1b	Using Calculus to Prove Inequalities	179
4.1c	A Generalized Mean Value Theorem	181
4.2	Higher Derivatives	186
4.2a	Second Derivative Test	191
4.2b	Convex Functions	192
4.3	Taylor's Theorem	197
4.3a	Examples of Taylor Series	202
4.4	Approximating Derivatives	209
<b>5</b>	<b>Applications of the Derivative</b>	<b>217</b>
5.1	Atmospheric Pressure	217
5.2	Laws of Motion	220
5.3	Newton's Method for Finding the Zeros of a Function	225
5.3a	Approximation of Square Roots	226

5.3b	Approximation of Roots of Polynomials	227
5.3c	The Convergence of Newton’s Method	229
5.4	Reflection and Refraction of Light	234
5.5	Mathematics and Economics	240
<b>6</b>	<b>Integration</b>	<b>245</b>
6.1	Examples of Integrals	245
6.1a	Determining Mileage from a Speedometer	245
6.1b	Mass of a Rod	247
6.1c	Area Below a Positive Graph	249
6.1d	Negative Functions and Net Amount	252
6.2	The Integral	254
6.2a	The Approximation of Integrals	257
6.2b	Existence of the Integral	261
6.2c	Further Properties of the Integral	265
6.3	The Fundamental Theorem of Calculus	271
6.4	Applications of the Integral	281
6.4a	Volume	281
6.4b	Accumulation	284
6.4c	Arc Length	284
6.4d	Work	287
<b>7</b>	<b>Methods for Integration</b>	<b>291</b>
7.1	Integration by Parts	291
7.1a	Taylor’s Formula, Integral Form of Remainder	295
7.1b	Improving Numerical Approximations	297
7.1c	Application to a Differential Equation	299
7.1d	Wallis Product Formula for $\pi$	299
7.2	Change of Variables in an Integral	302
7.3	Improper Integrals	310
7.4	Further Properties of Integrals	326
7.4a	Integrating a Sequence of Functions	326
7.4b	Integrals Depending on a Parameter	329
<b>8</b>	<b>Approximation of Integrals</b>	<b>333</b>
8.1	Approximating Integrals	333
8.1a	The Midpoint Rule	335
8.1b	The Trapezoidal Rule	336
8.2	Simpson’s Rule	339
8.2a	An Alternative to Simpson’s Rule	343
<b>9</b>	<b>Complex Numbers</b>	<b>347</b>
9.1	Complex Numbers	347
9.1a	Arithmetic of Complex Numbers	348
9.1b	Geometry of Complex Numbers	352

9.2	Complex-Valued Functions	361
9.2a	Continuity	362
9.2b	Derivative	362
9.2c	Integral of Complex-Valued Functions	364
9.2d	Functions of a Complex Variable	365
9.2e	The Exponential Function of a Complex Variable	368
<b>10</b>	<b>Differential Equations</b>	<b>375</b>
10.1	Using Calculus to Model Vibrations	375
10.1a	Vibrations of a Mechanical System	375
10.1b	Dissipation and Conservation of Energy	379
10.1c	Vibration Without Friction	381
10.1d	Linear Vibrations Without Friction	385
10.1e	Linear Vibrations with Friction	387
10.1f	Linear Systems Driven by an External Force	391
10.2	Population Dynamics	398
10.2a	The Differential Equation $\frac{dN}{dt} = R(N)$	399
10.2b	Growth and Fluctuation of Population	405
10.2c	Two Species	409
10.3	Chemical Reactions	420
10.4	Numerical Solution of Differential Equations	428
<b>11</b>	<b>Probability</b>	<b>435</b>
11.1	Discrete Probability	436
11.2	Information Theory: How Interesting Is Interesting?	446
11.3	Continuous Probability	452
11.4	The Law of Errors	463
	<b>Answers to Selected Problems</b>	<b>475</b>
	<b>Index</b>	<b>501</b>