Answers to Selected Problems

Chapter 1

1.1

(a) $[-1, 7]$
(b) $[-52, -48]$
(c) $y < 6$ or $y > 8$
(d) $|3-x| = |x-3|$ so same as (a)

1.3

(a) $|x| \le 3$
(b) $-3 \le x \le 3$

1.5 $\frac{2+4+8}{3} = \frac{14}{3}, \quad (2\cdot 4\cdot 8)^{1/3} = (2^6)^{1/3} = 2^2 = 4 < \frac{14}{3}.$

1.7

(a) $(\sqrt{x}-\sqrt{y})(\sqrt{x}+\sqrt{y}) = x-y$ and $0 < \dfrac{1}{\sqrt{x}+\sqrt{y}} \le \dfrac{1}{4}$, so $\sqrt{x}-\sqrt{y} \le \frac{1}{4}(x-y)$.
(b) Since $x > y$, $|\sqrt{x}-\sqrt{y}| = \sqrt{x}-\sqrt{y} \le \frac{1}{4}(0.02) = 0.005$

1.9 $|x| < m$ means $-m < x < m$. So if $|b-a| < \varepsilon$, then:

(a) At least one of $0 \le b-a$ and $0 \le -(b-a)$ is true. The upper bound ε comes from the assumption.
(b) $-\varepsilon < b-a < \varepsilon$ is a restatement of $|b-a| < \varepsilon$
(c) add a to both sides in (b)
(d) $|a-b| = |b-a|$, so $-\varepsilon < a-b < \varepsilon$ is a restatement of $|b-a| < \varepsilon$
(e) add b to both sides in (d)

1.11 $5/3$

1.13

(a) $(1(1)(x))^{1/3} \le \frac{1+1+x}{3}$
(b) $(1\cdots(1)(x))^{1/n} \le \frac{1+\cdots+1+x}{n} = \frac{x+n-1}{n}$
(c) $\frac{2n-1}{n} = 2 - \frac{1}{n} < 2$

P.D. Lax and M.S. Terrell, *Calculus With Applications*, Undergraduate Texts in Mathematics,
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1.15

(a) Since $1 \le 2 \le \cdots \le n$, we get $n! < n^n$. Taking the nth root gives $(n!)^{1/n} \le n$.
(b) By the A-G inequality, $(1 \cdot 2 \cdot 3 \cdots n)^{1/n} \le \frac{1+2+3+\cdots+n}{n}$, so $(n!)^{1/n} \le \frac{\frac{1}{2}n(n+1)}{n} = \frac{n+1}{2}$.

1.16

(a) We have $|ab - a_0b_0| = |ab - ab_0 + ab_0 - a_0b_0|$. By the triangle inequality,

$$|ab - a_0b_0| = |ab - ab_0 + ab_0 - a_0b_0| \le |ab - ab_0| + |ab_0 - a_0b_0|$$

Recall that $|ab| = |a||b|$. Then $|ab - a_0b_0| \le |a||b - b_0| + |b_0||a - a_0|$.
(b) $|a| \le 10$ and $|b_0| \le 10.001$, so $|ab - a_0b_0| \le 10(0.001) + 10.001(0.001)$

1.19 $m = 1$ because $\sqrt{3} = 1.732\cdots = 1.7 + (0.032\cdots)$ and $(0.032\cdots) < 10^{-1}$.

1.21

(a) a_n is part of the area of the 1 by 1 square, so $a_n < 1$
(b) S has an upper bound of 1, therefore has a least upper bound, the area of the quarter-circle of radius 1, which is $\dfrac{\pi}{4}$.

1.27 $s_1 = 1,\ s_2 = (1/2)(s_1 + 3/s_1) = 2,\ s_3 = (1/2)(2 + 3/2) = 7/4 = 1.75$, $s_4 = (1/2)(7/4 + 12/7) = 97/56 = 1.7321\ldots$ If you start with $s_1 = 2$ instead, it just shifts the sequence, since 2 already occurred as s_2.

1.29 If $s > \sqrt{2}$, then $\dfrac{1}{s} < \dfrac{1}{\sqrt{2}}$. Multiply by 2 to get $\dfrac{2}{s} < \dfrac{2}{\sqrt{2}} = \sqrt{2}$.

1.31 Suppose $s \ge \sqrt{2} + q$ for some number q. Then

$$2 + p > s^2 \ge 2 + 2\sqrt{2}q + q^2 \ge 2 + 2\sqrt{2}q.$$

This is possible only if $p > 2\sqrt{2}q$. Therefore, taking $q = \frac{p}{2^{3/2}}$, we get $s < \sqrt{2} + q$.

1.35

(a) a_n is arbitrarily close to a when n is sufficiently large, so in particular, there is N such that a_n is within 1 of a when $n > N$.
(b) $|a_n| = |a + (a_n - a)| \le |a| + |a_n - a| < |a| + 1$
(c) use the definition of α

1.37 $a_1 = s_1 = \frac{1}{3},\ a_2 = s_2 - a_1 = \frac{2}{4} - \frac{1}{3} = \frac{1}{6}$, and the sum is the limit of the s_n, which is 1.

1.39 $\dfrac{1}{1 - \frac{5}{7}}$

1.41 $\left|\dfrac{(n+1)a_{n+1}}{na_n}\right| = \dfrac{n+1}{n}\left|\dfrac{a_{n+1}}{a_n}\right|$ has the same limit as $\left|\dfrac{a_{n+1}}{a_n}\right|$.
The sum $\sum_{n=0}^{\infty}(-1)^n n^5 a_n$ also converges absolutely by the ratio test if $\sum_{n=0}^{\infty} a_n$ does so.

1.47

(a) Converges absolutely by the limit comparison theorem; compare with $\sum \left(\dfrac{2}{3}\right)^n$.
(b) Diverges by the comparison theorem; compare with the harmonic series
(c) Converges by the alternating series theorem
(d) Diverges because the nth term does not tend to 0
(e) Diverges by the limit comparison theorem; compare with the harmonic series
(f) Converges by the ratio test

1.49

(a) The series of absolute values is a convergent geometric series, so the series converges absolutely
(b) Converges by comparing with the geometric series $\sum(10)^{-n}$
(c) For any number b, the series converges absolutely by the ratio test
(d) Converges by comparing with the geometric series $\sum \frac{2}{3^n}$
(e) Sum of two convergent series is convergent
(f) Diverges because the term does not tend to 0

1.51 The sequence converges to x, so it must be Cauchy by the theorem that every convergent sequence is Cauchy. To see this case specifically, a_n is within 10^{-n} of x, so if n and m are both greater than N, then $|a_n - a_m| = |a_n - x + x - a_m| \le |a_n - x| + |x - a_m| < (2)10^{-N}$.

1.53

(a) $\left(1+\dfrac{1}{n-1}\right)^n = e_{n-1}\left(1+\dfrac{1}{n-1}\right) < (3)(2)$.
(b) This is $\left(\dfrac{n}{n-1}\right)^n < 6$, or $n^n < 6(n-1)^n$. Therefore $n^{n-1} < \frac{6}{n}(n-1)^n \le (n-1)^n$ if $n \ge 6$. Take roots to get $n^{1/n} < (n-1)^{1/(n-1)}$.
(c) If $n^{1/n}$ were less than 1, its powers, such as n, would be less than 1. Therefore, we have a decreasing sequence bounded below by 1, which then has a limit $r \ge 1$.
(d) $(2n)^{1/(2n)} = 2^{1/(2n)}\sqrt{n^{1/n}}$ tends to $r = \sqrt{r}$. So $r = 1$.

Chapter 2

2.1

(a) Not bounded, not bounded away from 0
(b) Not bounded, bounded away from 0
(c) Bounded, not bounded away from 0
(d) Not bounded, not bounded away from 0

2.3

(a) Cancel common factors
(b) f is defined except at 0 and -3; g is defined except at -3. h is defined for all numbers.
(c) The graph of h is a line, that of g is the line with one point deleted, and that of f is the line with two points deleted.

2.5 51,116.80

2.7 The change in radius is $\frac{1}{2\pi}$ times the change in circumference, about 3 meters.

2.9

(a) -3
(b) -6
(c) -2

2.13 Numerator is polynomial, continuous on $[-20, 120]$, has a maximum value M and a minimum value m. Denominator is $x^2 + 2 \ge 2$ and has maximum $(120)^2 + 2$. Therefore,

$$\frac{m}{(120)^2+2} \le f(x) \le \frac{M}{2}.$$

So f is bounded.

2.15 It appears to be approximately $(-0.4, 0.4)$.

2.17 No. The truncation of $x = 9.a_1a_2a_3a_4a_5a_6a_7a_8a_9\dots$ is $y = 9.a_1a_2a_3a_4a_5a_6a_7a_8$. The difference is $x - y = 0.00000000a_9\dots < 0.000000010 = 10^{-8}$. Then

$$x^2 - y^2 = (x+y)(x-y) < (20)(10^{-8}) = 2 \times 10^{-7}.$$

In fact, if we take an example with a_9 as large as possible, then

$$(9.000000009)^2 - 9^2 = 0.000000162\dots > 10^{-7}.$$

2.18 f has a minimum value on each closed interval contained in (a,b).

Take an expanding sequence of closed intervals, such as $I_n = \left[a + \frac{b-a}{2n}, b - \frac{1}{n}\right]$.

It might happen that the minimum value of f on I_n decreases with n. If so, there must be locations x_n at which the minimums occur, for which x_n tends to a or to b. This is a contradiction, because the values $f(x_n)$ must tend to infinity.

2.20 (a) 3×10^{-m} (b) $(1/3) \times 10^{-7}$ (c) all x

2.23 We suppose the bottle only has one volume for a given height, $V = f(H)$, and only one height for a given volume, $H = g(V)$. Then $H = g(V) = g(f(H)) = (g \circ f)(H)$ and $V = f(H) = f(g(V)) = (f \circ g)(V)$. So f and g are inverses.

2.25 There are two, x and x^{-1}.

2.27

(a) $k \circ k$
(b) $g \circ k$
(c) $k \circ g$

2.29 Let $f(x) = \sqrt{x^2+1} - \sqrt[3]{x^5+2}$. Then f is continuous because polynomials and roots are continuous and composites of continuous functions are continuous. We have $f(0) = 1 - \sqrt[3]{2} < 0$ and $f(-1) = \sqrt{2} - a > 0$. By the intermediate value theorem, $f(x) = 0$ for some number x in $[-1, 0]$.

2.31

(a) If $a < b$ then $f(a) < f(b)$, so $f(f(a)) < f(f(b))$.
(b) Let $f(x) = -x$, for example. Then f is decreasing, but $f(f(x)) = x$ is increasing.

2.33

(a) By the continuity of f
(b) By the limit of g
(c) Combine parts (a) and (b)
(d) restates part (c)

2.35 Since $x^2 + y^2 = 1$ on the unit circle, $\cos^2 s + \sin^2 s = 1$. This identity does not hold for the first pair, but does for the second.

2.37 The graph has to be very wide if you use the same scale on both axes!

2.39 Maximum height 1.2 reached at $3t = \frac{\pi}{2}$ and $3t = \frac{\pi}{2} + 2\pi$. The t difference is $2\pi/3$.

2.43

(a) $\sin(\tan^{-1}(z))$ is the y-coordinate of the point on the unit circle whose radius points toward $(1,z)$. In order to rescale $(1,z)$ back to the unit circle, multiply by some number c: (c, cz) is on the unit circle if $c^2 + c^2z^2 = 1$, so $c = \frac{1}{\sqrt{1+z^2}}$. Therefore, $\sin(\tan^{-1}(z)) = y = cz = \frac{z}{\sqrt{1+z^2}}$.

(b) $\cos(\sin^{-1}(y))$ is the x-coordinate of the point whose vertical coordinate is y. By the Pythagorean theorem, it is $\sqrt{1-y^2}$.

2.45 $g(x+y) = f(c(x+y)) = f(cx+cy) = f(cx)f(cy) = g(x)g(y)$.

2.47 $p(0) = 800,\ p(d) = 1600 = 800(1.023)^d$ gives $2 = d\log(1.023)$ so $d = 87.95\ldots$.

2.51 $f(1/2) = 3f(0) = 3 = m$, so $f(1) = 3f(1/2) = 9 = ma$, so $a = 3$.

2.53 $P(N) = P(1)P(N-1) = P(1)P(1)P(N-2) = \cdots = P(1)\cdots P(1) = P(1)^N$. So the sequence is $1+P(1)+(P(1))^2+\cdots+(P(1))^N$.

2.55 $e > 2$, so $e^{10} > 2^{10} = 1024 > 1000 = e^{\log 1000}$. But e^x is increasing; therefore $\log 1000 < 10$. Then $\log(1\,000\,000) = \log(1000)+\log(1000) < 20$

2.57 With $\dfrac{e^x}{x^2} > 1$ for large x, you get $e^x > x^2 = e^{\log(x^2)}$. So $x > \log(x^2)$, or $\sqrt{y} > \log y$.

2.59 $\left|1+x+x^2+x^3+x^4-\dfrac{1}{1-x}\right| = \left|\dfrac{x^5}{1-x}\right|$. If $-\frac{1}{2} \le x \le \frac{1}{2}$, the numerator does not exceed 1/32, and the denominator is at least 1/2, so the error $\le 1/16$.

2.63 Since $\sum_{n=0}^{\infty} a_n(x-2)^n$ converges at $x=4$, it must converge at every x that is closer to 2, i.e., $|x-2| < 2$. So the radius of convergence is at least 2, and f is continuous at least on $(0,4)$.

2.65 We have said that the series in question, which is centered at a, converges for $m < x < M$. We have also said that when a power series centered at a converges at any particular number x, then it converges for every number closer to a. So it converges on intervals *symmetric* about a. Therefore $M-a = a-m$.

2.67

(a) You need to argue that since $p_n^{1/n}$ tends to ℓ, it is eventually smaller than every number such as r that is greater than ℓ. Then compare with a geometric series.
(b) Is similar to part (a)
(c) If $p_n^{1/n} = |a_n|^{1/n}|x|$ tends to $L|x|$. If $L|x| < 1$, the series converges by part (a). If $L|x| > 1$, it diverges by part (b). Therefore $1/L$ is the radius of convergence.

2.69 $\sum_{n=0}^{\infty} nx^n$ converges for $|x| < 1$ by the ratio test. If $\{n^{1/n}\}$ tends to a limit $r > 1$, take x between $1/r$ and 1, so that $rx > 1$. Then the nth root $(nx^n)^{1/n} = n^{1/n}x = \dfrac{n^{1/n}}{r}(rx)$ tends to $rx > 1$. According to the root test, the series diverges, contradiction.

2.71

(a) $\dfrac{1}{1+t^2}$, for $|t| < 1$
(b) $\dfrac{1}{1-x} - 1 - x - x^2$, for $|x| < 1$
(c) We don't have a name for this one.
(d) $\dfrac{1}{1-\frac{1}{2}t} + \dfrac{1}{1-3t^2}$, for $|t| < 1/\sqrt{3}$

2.73 Take the list of $n+1$ numbers $1+\frac{x}{n}$ (n times) and 1. The A-G inequality gives

$$\left(1+\frac{x}{n}\right)^{n/(n+1)} < \frac{n\left(1+\frac{x}{n}\right)+1}{n+1} = 1+\frac{x}{n+1}.$$

The $(n+1)$st power gives the result.

2.75 $e_n(-x)$ will do.

Chapter 3

3.1 A linear function is its own tangent.

3.3 $\ell(x) = 5(x-2)+6 = f(2)+f'(2)(x-2)$.
Use the properties $\ell(2) = f(2) = 6$ and $\ell(2) = 5 = f'(2)$

3.5 (c) $\dfrac{f(a+h)-f(a)}{h} = \dfrac{1}{h}\left(a^3+3a^2h+3ah^2+h^3-a^3\right)$ tends to $3a^2 = 3$.
Tangent line $y = -1+3(x+1)$.

3.7 $y = -x-\frac{1}{4}$ is tangent at both $-\frac{1}{2}$ and $\frac{1}{2}$.

3.9 Average rate of change: $\dfrac{T(a+h)-T(a)}{h}$. If $T'(a)$ is positive, $T(x)$ is locally increasing at a; hence it will be hotter to the right. If it is cooler to the left of a, $T'(a)$ should be positive. If the temperature is constant, $T(a+h)-T(a) = 0$, so $T'(a) = 0$.

3.11 Since $f'(3) = 5,\ g'(3) = 6$, I would say that g is more sensitive to change near 3.

3.15 $(f(h)-f(0))/h = h^{-1/3}$ does not have a limit at h tends to 0. The one-sided derivative does not exist, so f is not differentiable on $[0,1]$.

3.17

(a) $\dfrac{1}{2}(x^3+1)^{-1/2}(3x^2)$
(b) $3\left(x+\dfrac{1}{x}\right)^2\left(1-\dfrac{1}{x^2}\right)$
(c) $\dfrac{1}{2}(1+\sqrt{x})^{-1/2}\left(\dfrac{1}{2}x^{-1/2}\right)$
(d) 1

3.19

(a) Positions $f(0) = 0,\ f(2) = -6$
(b) Velocities $f'(0) = 1,\ f'(2) = -11$
(c) Direction of motion right, left, assuming axis is drawn positive to the right.

3.21 Only in parts (a), (b), and (c) do higher derivatives vanish.

(a) $f(x) = x^3,\ f'(x) = 3x^2,\ f''(x) = 6x,\ f'''(x) = 6,\ 0,\ 0,\ 0$
(b) $t^3+5t^2,\ 3t^2+10t,\ 6t+10,\ 6,\ 0,\ 0,\ 0$
(c) $r^6,\ 6r^5,\ (6)(5)r^4,\ (6)(5)(4)r^3,\ (6)(5)(4)(3)r^2,\ (6)(5)(4)(3)(2)r,\ 6!$
(d) $x^{-1},\ -x^{-2},\ 2x^{-3},\ 3!x^{-4},\ -4!x^{-5},\ 5!x^{-6},\ -6!x^{-7}$
(e) $t^{-3}+t^3,\ -3t^{-4}+3t^2,\ 12t^{-5}+6t,\ -60t^{-6}+6,\ 360t^{-7},\ -2520x^{-8},\ 40\,160x^{-9}$

3.23

(a) $(f^2)' = 2ff' = 2(1+t+t^2)(1+2t)$
(b) 0
(c) $6(5^4)(4)$

3.25

(a) $2GmMr^{-3}$
(b) $2GmMr^{-3}r'(t) = 2GmM(2\,000\,000+1000t)^{-3}(1000)$
(c) $\dfrac{\mathrm{d}F}{\mathrm{d}t} = \dfrac{\mathrm{d}F}{\mathrm{d}r}\dfrac{\mathrm{d}r}{\mathrm{d}t} = 2GmMr^{-3}\dfrac{\mathrm{d}r}{\mathrm{d}t}$

3.27 $V(t) = \frac{4}{3}\pi(r(t))^3$, and for some constant k,
$V'(t) = 4\pi(r(t))^2 r'(t) = k \cdot 4\pi(r(t))^2$. So $r'(t) = k$.

3.29 $\dfrac{dP}{dt} = \dfrac{7}{5}k\rho^{2/5}\dfrac{d\rho}{dt}$.

3.31 $f(1) = 1+2+3+1 = 7$ and $f'(1) = 3+4+3 = 10$,
so $g'(7) = (f^{-1})'(7) = 1/f'(1) = 1/10$.

3.33 $(x^a)^b = x^{ab} = x$ if $ab = 1$. If $1/p + 1/q = 1$ then multiply by pq, giving $q + p = pq$, or $(p-1)(q-1) = 1$.

3.35

(a) $f(x+h) = k + f(x) = k + y$, so $x + h = g(k+y)$
(b) f strictly monotonic means that when $h \neq 0$, $f(x+h) \neq f(x)$
(c) The algebra is correct as long as no denominator is 0, and we have shown that none is, provided $k \neq 0$. The left-hand side tends to $g'(y)$ as k tends to 0. But as k tends to 0, $h = g(k+y) - g(y)$ tends to 0 due to the continuity of g (By Theorem 2.9). Therefore, the right-hand side tends to $1/f'(x)$.

3.37 $(-1/10)^n e^{-t/10}$

3.39 $2x + 0 + 0 + 2^x \log 2 + e^x + ex^{e-1}$

3.41

(a) $1/x$
(b) $2/x$
(c) 0
(d) $-e^x e^{-e^x}$
(e) $\dfrac{1-e^{-x}}{1+e^{-x}}$

3.43 $y = x - 1$

3.45 The rate is 1.5 times the size, $p'(t) = 1.5p(t)$. The solutions are $p(t) = ce^{1.5t}$.
Then $p(1) = 100 = ce^{1.5}$, and $p(3) = ce^{4.5} = 100e^{-1.5}e^{4.5}$.

3.47 $(\log(fg))' = \dfrac{(fg)'}{fg} = (\log f + \log g)' = \dfrac{f'}{f} + \dfrac{g'}{g}$. Multiply by fg to get $(fg)' = f'g + fg'$.

3.49 $\left(\dfrac{1}{2}\log(x^2+1) + \dfrac{1}{3}\log(x^4-1) - \dfrac{1}{5}\log(x^2-1)\right)' = \dfrac{1}{2}\dfrac{2x}{x^2+1} + \dfrac{1}{3}\dfrac{4x^3}{x^4-1} - \dfrac{1}{5}\dfrac{2x}{x^2-1}$

3.53

(a) $\cot x$
(b) $\dfrac{e^{\tan^{-1}(x)}}{x^2+1}$
(c) $\dfrac{10x}{25x^4+1}$
(d) $\dfrac{2e^{2x}}{e^{2x}+1}$
(e) $e^{(\log x)(\cos x)}\left(\dfrac{\cos x}{x} - \sin x \log x\right)$

3.55

(a) $(\sec x)' = -(\cos x)^{-2}(-\sin x)$
(b) $(\csc x)' = -(\sin x)^{-2}(\cos x)$
(c) $(\cot x)' = \dfrac{(\sin x)(-\sin x) - (\cos x)(\cos x)}{\sin^2 x}$

3.57 $y(x) = u\cos x + v\sin x$, $y(0) = -2 = u$, $y'(0) = 3 = v$, so $y(x) = -2\cos x + 3\sin x$.

3.59

(a) For $y > 1$, sketch a triangle with legs 1 and $\sqrt{y^2-1}$ to see that $\cos(\sec^{-1} y) = 1/y$. The derivative gives $-\sin(\sec^{-1} y)(\sec^{-1} y)' = -y^{-2}$. Then

$$(\sec^{-1} y)' = \frac{1}{\sin(\sec^{-1} y)}\frac{1}{y^2} = \frac{1}{\sqrt{1-y^{-2}}}\frac{1}{y^2} = \frac{1}{y\sqrt{y^2-1}}.$$

For $y < -1$, sketch a graph of cosine and secant to see the symmetry $\sec^{-1} y = \pi - \sec^{-1}(-y)$. Then by the chain rule, $(\sec^{-1})'(y) = (\sec^{-1})'(-y)$. So $(\sec^{-1} y)' = \dfrac{1}{|y|\sqrt{y^2-1}}$.

(b) $(\cos^{-1} x)' = -(\sin^{-1} x)' = -1/\sqrt{1-x^2}$.
(c) $(\csc^{-1} x)' = -(\sec^{-1} x)' = -1/(|x|\sqrt{x^2-1}$.
(d) $\left(\tan^{-1}\dfrac{1}{x}\right)' = -(\tan^{-1} x)' = -\dfrac{1}{1+x^2}$. By the chain rule, $\dfrac{1}{1+(x^{-1})^2}\dfrac{-1}{x^2} = -\dfrac{1}{1+x^2}$.

3.61 $\sinh' x = \frac{1}{2}(e^x - e^{-x})' = \frac{1}{2}(e^x + e^{-x}) = \cosh x$ and similarly for $\cosh'$.

3.63 $\cosh^2 x - \sinh^2 x = \frac{1}{4}\left(2e^x e^{-x}) - 2(-e^x e^{-x})\right) = 1$

3.65

(a) $0 + 0^5 + \sin 0 = 3(1^2) - 3$.
(b) Apply the chain rule to $y(x) + y(x)^5 + \sin(y(x)) = 3x^2 - 3$.
(c) $\dfrac{dy}{dx} = \dfrac{6x}{1+5y^4+\cos y} = 6/2$.
(d) tangent line is $y = 3(x-1)$.
(e) $y(1.01) \approx 3(.01)$

3.67 $y(x) = u\cosh x + v\sinh x$, $y(0) = -1 = u$, $y'(0) = 3 = v$, $y(x) = -\cosh x + 3\sinh x$

3.69 Take $x = 0$ to find $u = 0$. Then evaluate the derivative at 0 to find $0 = v$.

3.71

(a) This is a statement of the binomial theorem applied to $(x+h)^n$.
(b) Apply the triangle inequality to the right-hand side in part (a). In each term, the binomial coefficient is positive, and $|x^{n-k}h^k| = |x|^{n-k}|h|^k$. Then replace $|h|$ by the equal number $\dfrac{|h|}{H}H$, and you get

$$\left|\binom{n}{2}x^{n-2}h^2 + \binom{n}{3}x^{n-3}h^3 + \cdots + h^n\right| \le \left|\binom{n}{2}x^{n-2}h^2\right| + \cdots + |h^n|$$

$$\le \binom{n}{2}|x|^{n-2}|h|^2 + \cdots + |h|^n = \binom{n}{2}|x|^{n-2}\frac{|h|^2}{H^2}H^2 + \cdots + \frac{|h|^n}{H^n}H^n$$

(c) Factor out $\dfrac{|h|^2}{H^2}$

(d) Recognize that the factor $\left(\binom{n}{2}|x|^{n-2}H^2+\cdots+H^n\right)$ consists of all but two (positive) terms of the binomial expansion

$$(|x|+H)^n = |x|^n + h|x|^{n-1}H + \left(\binom{n}{2}|x|^{n-2}H^2 + \binom{n}{3}|x|^{n-3}H^3 + \cdots + H^n\right)$$

and is therefore less than $(|x|+H)^n$.

3.73 Each side is $\left(\dfrac{1}{1-x}\right)^2$ for $|x|<1$.

Chapter 4

4.1 $0.4 \le f'(c) = \dfrac{f(2.1)-6}{0.1} \le 0.5$ gives $6.04 \le f(2.1) \le 6.05$

4.3 $h(x) = \frac{2}{3}\sin(3x) + \frac{3}{2}\cos(2x) + c,\ \ h(x) = \frac{2}{3}\sin(3x) + \frac{3}{2}\cos(2x) - \frac{3}{2}$.

4.5 $f'(x) = (1-x^2)/(1+x^2)$. f increases on $(-1,1)$, decreases on $(-\infty,-1)$ and on $(1,\infty)$. In $[-10,10]$, the minimum has to be either $f(-1) = -1/2$ or $f(10) = 10/101$, so $-1/2$. In $[-10,10]$, the maximum has to be either $f(-10) = -10/101$ or $f(1) = 1/2$, so $1/2$.

4.7

(a) For a rectangle x wide, area $A(x) = x(16-2x)/2 = 8x - x^2$ defined on $[0,8]$. $A'(x) = 8-2x$ is 0 when $x = 4$, $A(0) = A(8) = 0$, so $A(4) = 16$ is the maximum.
(b) Now $A(x) = x(16-2x) = 16x - 2x^2$ defined on $[0,16]$. $A'(x) = 16-4x$ is 0 when $x = 4$, $A(0) = A(16) = 0$, so $A(4) = 32$ is the maximum.

4.9

$$E'(m) = 2(y_1 - mx_1)(-x_1) + \cdots + 2(y_n - mx_n)(-x_n) = -2\sum_{i=1}^{n} x_i y_i + 2\Big(\sum_{i=1}^{n} x_i^2\Big)m$$

is 0 when $m = \dfrac{\sum x_i y_i}{\sum x_i^2}$. This gives a minimum because $E'(m) < 0$ when $m < \dfrac{\sum x_i y_i}{\sum x_i^2}$ and $E'(m) > 0$ when $m > \dfrac{\sum x_i y_i}{\sum x_i^2}$.

4.11 $x'(3/2) = 0$ and $x''(t)$ is negative for all t, and therefore $x(3/2) = 9/4$ is the maximum.

4.13 Let $c(x) = x - x^3$. Then $c'(x) = 1 - 3x^2$ is 0 when $x = \sqrt{1/3}$, and $c''(x) = -6x$ is negative for all $x > 0$. So $c(\sqrt{1/3}) = \sqrt{1/3}(2/3) = 0.384\ldots$ is the largest amount.

4.17 Let $f(x) = h(x) - g(x)$. Then we are given that $f'(x) \ge 0$ for $x > 0$ and $f(0) = 0$. Thus f is nondecreasing for $x > 0$, so if $x > 0$, $f(x) \ge f(0) = 0$. So $h(x) - g(x) \ge 0$, so $h(x) \ge g(x)$.

4.21 Since $e^0 = 1$, $\lim\limits_{x\to 0} \dfrac{e^x - 1}{x}$ is the derivative of e^x at 0, namely 1. The reciprocal $\dfrac{x}{e^x - 1}$ therefore also tends to 1.

4.25 The linear approximation theorem gives $f(t) = 0 + 3t + \frac{1}{2}f''(c)t^2$ for some c between 0 and t. So $3t + 4.9t^2 \le f(t) \le 3t + 4.905t^2$.

4.27 $f'(x) = 6x^2 - 6x + 12 = 6(x-2)(x+1)$ is negative on $(-1,2)$, $f'(-1) = f'(2) = 0$, and positive otherwise; $f(-1) = -17$ is a local minimum and $f(2) = 28$ is a local maximum because of the concavity: $f''(x) = 12x - 6$ is negative when $x < \frac{1}{2}$, so f is concave there, convex when $x > \frac{1}{2}$.

4.29 Tangent is below graph, because the function is convex.

4.31 Set $g(x) = e^{-1/x}$. Then $g(x) > 0$, $g'(x) = x^{-2}g(x)$, and
$g''(x) = (-2x^{-3} + x^{-4})g(x) = (1-2x)x^{-4}g(x)$. So g is convex on $(0, \frac{1}{2})$.

4.33 Yes. $(e^f)' = f'e^f$, $(e^f)'' = f''e^f + (f')^2 e^f$. Therefore if $f'' > 0$, then $(e^f)'' > 0$.

4.35 Write $h = \frac{1}{2}(b-a)$ and $c = \frac{1}{2}(a+b)$. Then $c = a+h = b-h$, and linear approximation gives

$$f(a) = f(c) + f'(c)h + \frac{1}{2}f''(c_1)h^2, \quad f(b) = f(c) - f'(c)h + \frac{1}{2}f''(c_2)h^2,$$

for some c_1 and c_2 in $[a,b]$. Average these to get $\frac{1}{2}(f(a)+f(b)) = f(c) + \frac{1}{4}(f''(c_1)+f''(c_2))h^2$. The last term has $|\frac{1}{4}(f''(c_1)+f''(c_2))h^2| \le \frac{1}{4}2Mh^2 = \frac{M}{8}(b-a)^2$.

4.39 $f' > 0$ on $[-5,-1.8]$ and $[0.5,5]$ $f' < 0$ on $[-1.8,0.5]$
$f'' > 0$ on $[-1.8,2.5]$ $f'' < 0$ on $[-5,-1.8]$ and $[2.5,5]$

4.41

(a) See Fig. 11.5.
(b) $f'(x) = -xe^{-\frac{x^2}{2}}$ is positive when x is negative, so f is increasing on $(-\infty,0)$ and decreasing on $(0,\infty)$. $f''(x) = (-1+x^2)e^{-\frac{x^2}{2}}$ is negative when $-1 < x < 1$, so f is concave on $(-1,1)$ and convex on $(-\infty,-1)$ and on $(1,\infty)$. The only critical point is for the maximum at $x = 0$.
(c) $g'(x) = x^{-2}e^{-1/x}$ is positive and g is increasing on $(0,\infty)$.
$g''(x) = (-2x^{-3} + x^{-4})e^{-1/x} = x^{-4}(1-2x)e^{-1/x}$ is negative on $(0,1/2)$ and positive on $(1/2,\infty)$ so g is convex on $(0,1/2)$ and concave on $(1/2,\infty)$.

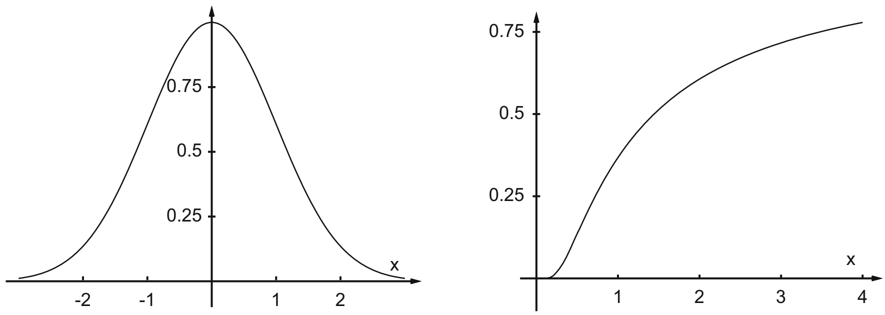

Fig. 11.5 *Left*: the graph of $f(x) = e^{-\frac{x^2}{2}}$. *Right*: the graph of $g(x) = e^{-1/x}$. See Problem 4.41

4.43 $\cos x = 1 - \dfrac{x^2}{2} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \cdots$ converges for all x.

4.45 $t_3 = t_4$ because $\sin''''(0) = 0$. It is better to use t_4 to take advantage of the 5! in the remainder. This gives $|\sin x - t_3(x)| = |\sin x - t_4(x)| \le \dfrac{x^5}{120}$.

4.47 $\cosh' x = \sinh x$, $\sinh' x = \cosh x$, give Taylor polynomials $t_n(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \alpha \frac{x^n}{n!}$, where $\alpha = 1$ when n is even and $\alpha = 0$ when n is odd. The remainder is $\cosh^{(n+1)}(c)\frac{x^{n+1}}{(n+1)!}$ for some c between 0 and x. By definition, $\sinh b$ and $\cosh b$ are each less than e^b. Therefore, for x in $[-b,b]$,

$$|t_n(x) - \cosh x| \le e^b \frac{b^{n+1}}{(n+1)!},$$

and this tends to 0 as b tends to infinity. The convergence is uniform on $[-b,b]$.

4.49 Let $f(x) = \cos x$. Then $f(\pi/3) = \frac{1}{2}$, $f'(\pi/3) = -\frac{\sqrt{3}}{2}$, $f''(\pi/3) = -\frac{1}{2}$, $f'''(\pi/3) = \frac{\sqrt{3}}{2}, \dots$, and $\cos x = \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) - \frac{1}{2}\frac{1}{2!}\left(x - \frac{\pi}{3}\right)^2 + \frac{\sqrt{3}}{2}\frac{1}{3!}\left(x - \frac{\pi}{3}\right)^3 - \frac{1}{2}\frac{1}{4!}\left(x - \frac{\pi}{3}\right)^4 + \cdots$

4.51

$$\sqrt{1+y} = 1 + \frac{1}{2}y + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}y^2 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}y^3$$
$$+ \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)}{4!}y^4 + \cdots = 1 + \frac{1}{2}y - \frac{1}{8}y^2 + \frac{1}{16}y^3 - \frac{5}{128}y^4 + \cdots$$

4.53 $\sqrt{x} = 1 + \dfrac{1}{2}(x-1) - \dfrac{1}{8}(x-1)^2 + \dfrac{1}{16}(x-1)^3 - \dfrac{15}{16}c^{-7/2}\dfrac{(x-1)^4}{4!}$. For x in $[1, 1+d]$, $c \ge 1$ and $|x-1| \le d$ give $|\sqrt{x} - t_3(x)| \le \dfrac{15}{16}\dfrac{d^4}{4!} = \dfrac{5d^4}{128}$. This is less than

(a) 0.1 when $d \le ((0.1)128/5)^{1/4} = 1.26\dots$
(b) 0.01 when $d \le ((0.01)128/5)^{1/4} = 0.71\dots$
(c) 0.001 when $d \le ((0.001)128/5)^{1/4} = 0.4$

4.55 $t_6(0.7854) = 1 - 0.308427 + 0.0158545 - 0.000325996 = 0.70710\dots$
This is nearly $\cos\left(\dfrac{\pi}{4}\right) = \dfrac{1}{\sqrt{2}}$.

4.57

(a) $g(0) = f(a)$, $g(-a) = f(0) + af'(0) + \frac{a^2}{2}f''(0) + \cdots + \frac{a^n}{n!}f^{(n)}(0)$
(b) $g'(x) = f'(x+a) - f'(x+a) - xf''(x+a) + xf''(x+a) - \cdots + \frac{(-1)^n x^{n-1}}{(n-1)!}f^{(n)}(x+a)$.
The last term would involve $f^{(n+1)}$, but this is zero because f has degree n. All terms cancel, and $g'(x) = 0$ for all x.
(c) Since $g'(x) = 0$ for all x, $g(x)$ is constant.
(d) Since g is constant, $g(0) = g(-a)$, giving $f(a) = f(0) + af'(0) + \cdots + \frac{a^n}{n!}f^{(n)}(0)$.

4.59 $\dfrac{(x+h)^2 - x^2}{h} = 2x + h$, $\dfrac{(x+h)^2 - (x-h)^2}{2h} = 2x$, $\dfrac{(x+h)^3 - x^3}{h} = 3x^2 + 3xh + h^2$, $\dfrac{(x+h)^3 - (x-h)^3}{2h} = 3x^2 + h^2$.

$\dfrac{(10.1)^2 - 10^2}{0.1} = 20.1 = (\text{derivative}) + 0.1$, $\dfrac{(10.1)^2 - (9.9)^2}{0.2} = 20 = (\text{derivative})$,
$\dfrac{(10.1)^3 - 10^3}{0.1} = 33.01 = (\text{derivative}) + 3.01$, $\dfrac{(10.1)^3 - (9.9)^3}{.2} = 30.01 = (\text{derivative}) + 0.01$.

4.63 (a) $f'(x) = 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$ when $x \neq 0$.

(b) $\dfrac{h^2\sin\left(\frac{1}{h}\right) - 0}{h} = h\sin\left(\frac{1}{h}\right)$ tends to 0 by the squeeze theorem as h tends to 0, because $|\sin| \leq 1$.

(c) $f'\left(\frac{1}{n\pi}\right) = -\cos(n\pi)$ has no limit as n tends to infinity, so $f'(x)$ has no limit as x tends to 0.

Chapter 5

5.3 $y(t) = -4.9t^2 + 10t + 0,\ y(1) = -4.9 + 10t = 5.1,\ y'(1) = -9.8 + 10 = 0.2,$
$y(2) = -4.9(4) + 10(2) = 0.4,\ y'(2) = -9.8(2) + 10 = -9.6,$

5.5 $my'' = g - f_{\text{up}}$.

5.7 Functions with the same derivative on an interval differ by a constant. It is a consequence of the mean value theorem.

5.9

(a) With $x = y^6,\ f(x) = 1 + x^{1/3} - x^{1/2}$ becomes $g(y) = 1 + y^2 - y^3$. Since $g(1) > 0$ and $g(2) < 0$, there is a root y between 1 and 2. Starting from $y_1 = 1$ and iterating

$$y_{\text{new}} = y - \frac{1 + y^2 - y^3}{2y - 3y^2}$$

produces 2, 1.625, 1.4858, 1.4660, 1.4656, 1.4656. So $x = (1.4656)^6 = 9.9093$.

(b) Experiments give $f(-1) = -3,\ f(0) = 1,\ f(2) = -3,\ f(3) = 1$. Since these are $- + - +$, there are three real roots, one in each interval $[-1, 0],\ [0, 2]$, and $[2, 3]$.

(c) $f(x) = \dfrac{x}{x^2+1} + 1 - \sqrt{x}$ has derivative $f'(x) = \dfrac{1 - x^2}{1 + x^2} - \dfrac{1}{2\sqrt{x}}$. The derivative is negative on $[1, \infty)$, so there can be only one zero at most; since $f(1) = 1/2$ and $f(3) = 0.3 + 1 - \sqrt{3} < 0$, the root is in $[1, 3]$.

5.11 There are two statements to be explained, that this finds the maximum, and the statement about a zero of f' in (b).

In (b), there is a zero of f' because f is a continuous function on the closed subinterval $[x_{j-1}, x_{j+1}]$ for which the endpoints do *not* give the maximum.

Reason for large N, and why this finds the maximum, is that only two things could go wrong: (1) If there is more than one zero of f' in a subinterval, then Newton could converge to the wrong one, (2) there might be a maximum at one of the x_j.

Take N so large that f is well approximated by a quadratic function in each subinterval. This solves (1), and replacing N by $N + 1$ solves (2).

5.13

(a) Take $z_1 = 1$, then $z_2 = z_1 - \frac{1}{2}(z_1^2 - 2) = 1.5,\ z_3 = 1.375,\ z_4 = 1.42969,\ z_5 = 1.40768,$ $z_6 = 1.4169$ and in general, $z_{n+1} - \sqrt{2} = (z_n - \sqrt{2})\left(1 - \frac{1}{2}(z_n + \sqrt{2})\right)$; since the second factor is negative, the z_n alternate greater than and less than $\sqrt{2}$.

(b) Take $z_1 = 1$, then $z_2 = z_1 - \frac{1}{3}(z_1^2 - 2) = 1.33333,\ z_3 = 1.40741,\ z_4 = 1.41381,\ z_5 = 1.41381,$ $z_6 = 1.41419$ and in general, $z_{n+1} - \sqrt{2} = (z_n - \sqrt{2})\left(1 - \frac{1}{3}(z_n + \sqrt{2})\right)$; since the second factor is positive, the $z_n - \sqrt{2}$ are all of the same sign.

5.17

(a) Q is fixed, so the cost of producing q in plant 1 plus the cost of producing $Q - q$ in plant 2 is $C_1(q) + C_2(Q - q)$.

(b) If C has a minimum at q, then $C'(q) = 0$ gives $C_1'(q) = C_2'(Q-q)$.

(c) $2aq = 2b(Q-q)$, $q = \dfrac{b}{a+b}Q = \dfrac{1.2a}{a+1.2a}Q = 0.545Q$.

5.19 Marginal cost is $C'(q) = akq^{k-1}$, and average cost is $\dfrac{C(q)}{q} = aq^{k-1} + \dfrac{b}{q}$. These are equal if there is a q for which $a(k-1)q^k = b$. Since a and b are positive, this is possible as long as $k > 1$.

Chapter 6

6.1

(a) Four parts: $1 < 2 < 3 < 4 < 5$.

$$(1)(1)+(2)(1)+(3)(1)+(4)(1) \le R(x,[1,5]) \le (2)(1)+(3)(1)+(4)(1)+(5)(1)$$

gives $10 \le R(x,[1,5]) \le 14$.

(b) Eight parts: $1 < 1.5 < 2 < 2.5 < 3 < 3.5 < 4 < 4.5 < 5$.

$$(1)(0.5)+(1.5)(0.5)+(2)(0.5)+(2.5)(0.5)+(3)(0.5)+(3.5)(0.5)+(4)(0.5)+(4.5)(0.5)$$
$$\le R(x,[1,5]) \le$$
$$(1.5)(0.5)+(2)(0.5)+(2.5)(0.5)+(3)(0.5)+(3.5)(0.5)+(4)(0.5)+(4.5)(0.5)+(5)(0.5)$$

gives $11 \le R(x,[1,5]) \le 13$.

6.3

(a) See Fig. 11.6.

(b) Since f is negative only on $(-1,1)$, we know that $A(x^2-1,[-3,-2])$ is certainly positive, and $A(x^2-1,[-1,0])$ is certainly negative. The other two would require some extra effort.

(c) $I_{\text{upper}} = \big((-3)^2-1+(-2)^2-1+(-1)^2-1+1^2-1+2^2-1\big)1 = 14$.
$I_{\text{lower}} = \big((-2)^2-1+(-1)^2-1+(0)^2-1+0^2-1+1^2-1\big)1 = 1$.

6.5

(a) $I_{\text{approx}}(f,[1,3]) = f(1.2)(0.5)+f(2)(0.5)+f(2.5)(1) = 12.57$

(b) $I_{\text{approx}}(\sin,[0,\pi]) = \big(\sin(0)+\sin(\frac{\pi}{4})+\sin(\frac{\pi}{2})+\sin(\frac{3\pi}{4})\big)\frac{\pi}{4} = 1.896$

6.7 (a) The area of the large rectangle is e, so the area of the shaded region is 1.
(b) If you flip the picture over the diagonal, the shaded region is the region under the graph $\log t$ from $t = 1$ to $t = e$, so the integral is 1.

6.9

(a) left: $(130+75+65+63+61)(3)/15 = 78.8$
right: $(75+65+63+61+60)(3)/15 = 64.8$

(b) left: $(130(1)+120(1)+90(2)+70(2)+65(9))/15$
right: $(120(1)+90(1)+70(2)+65(2)+60(9))/15$

6.11 The graph of f_k is the graph of f stretched horizontally by a factor of k. If we have a set of rectangles approximating the "area under the curve" f with total area A and error ε, stretching them by a factor of k will produce a set of rectangles approximating the "area under the curve" f_k with total area kA and error $k\varepsilon$. Since we can make ε arbitrarily small using narrower rectangles, we can also make $k\varepsilon$ arbitrarily small. The area of the new region is kA.

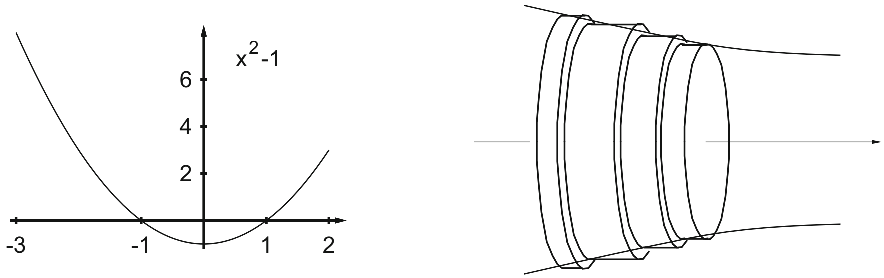

Fig. 11.6 *Left*: The graph for Problem 6.3. *Right*: A stack of cylinders approximates volume in Problem 6.31

6.13 Each approximate integral for f using $a_0 \le t_1 \le a_1 < \cdots < a_n$ becomes an approximate integral for f_- using $a_n < \cdots < a_1 < t_1 < a_0$.

6.15

(a) x^3
(b) $x^3 e^{-x}$
(c) $s^6 e^{-s^2}(2s)$
(d) $2\cos\left(\frac{\pi}{2}\right) = 0$

6.17

(a) $\tan^{-1}\left(\frac{\pi}{4}\right) = 1$
(b) $\left[\frac{1}{5}x^5 + \frac{4}{3}x^3 + 4x\right]_0^1$
(c) $\left[4\sqrt{x} - \frac{2}{3}x^{3/2}\right]_1^4$
(d) $\left[2t - 4t^{-1} + 4t^{-2}\right]_{-2}^{-1} = 7$
(e) $\left[s^2 + \log(s+1)\right]_2^6 = 32 + \log 7 - \log 3$

6.19 By the fundamental theorem, $F'(x) = \dfrac{1}{\sqrt{1-x^2}}$. Then the chain rule applied to $t = F(\sin t)$ gives $1 = F'(\sin t)(\sin t)' = \dfrac{1}{\sqrt{1-(\sin t)^2}}(\sin t)'$ or $(\sin t)' = \sqrt{1-(\sin t)^2}$.

6.21 This is the chain rule combined with the fundamental theorem.

6.23

(a) $3t^2(1+t^3)^3 = \left(\frac{1}{4}(1+t^3)^4\right)'$, so this is an example of the fundamental theorem.
(b) $a(t) = \big(v(t)\big)'$, so this is an example of the fundamental theorem

6.25 $k = \dfrac{2000}{0.004} = 500000$. $W = \displaystyle\int_0^{0.004} kx\,dx = k\frac{1}{2}x^2\Big|_0^{0.004} = 4$ joule

6.27 If the pump starts at $t = 0$, the volume drained in T minutes is $\displaystyle\int_0^T (2t+10)\,dt = T^2 + 10T$. This is 200 when $T = 10$ minutes.

6.29 $I_{\text{left}} = 1.237 > \displaystyle\int_1^2 \sqrt{1+x^{-2}}\,dx > I_{\text{right}} = 1.207$ because the integrand is decreasing.

6.31 See Fig. 11.6.

Chapter 7

7.1

(a) $\int_0^1 t^2(e^t)'\,dt = \left[t^2e^t\right]_0^1 - \int_0^1 2t(e^t)'\,dt = \left[t^2e^t - 2te^t\right]_0^1 + 2\int_0^1 e^t\,dt = e - 2e + 2(e-1) = e - 1$

(b) $\dfrac{\pi}{2} - 1$

(c) $\dfrac{\pi^2}{4} - 2$. Integrate twice by parts.

(d) $\int_0^1 x^3(1+x^2)^{1/2}\,dx = \int_0^1 \frac{x^2}{2}\left(\frac{2}{3}(1+x^2)^{3/2}\right)'\,dx = \left[\frac{x^2}{2}\frac{2}{3}(1+x^2)^{3/2}\right]_0^1 - \frac{1}{3}\int_0^1 2x(1+x^2)^{3/2}\,dx$
$= \left[\frac{x^2}{2}\frac{2}{3}(1+x^2)^{3/2}\right]_0^1 - \frac{1}{3}\left[\frac{2}{5}(1+x^2)^{5/2}\right]_0^1 = \frac{1}{3}2^{3/2} - \frac{2}{15}(2^{5/2}-1) = \frac{2}{15}(1+\sqrt{2})$

7.3

(a) Integrate by parts, differentiating $\tan^{-1}x$. Answer $\frac{1}{4}\pi - \frac{1}{2}$.

(b) $u\sin u$

7.5 Set $f = f_1 - f_2$. Then $f'' - vf = 0$, $f(a) = 0$, and $f(b) = 0$. Integrate by parts

$$0 \le \int_a^b v(t)f(t)f(t)\,dt = \int_a^b f''(t)f(t)\,dt = -\int_a^b f'(t)f'(t)\,dt \le 0.$$

Therefore $\int_a^b v(t)(f(t))^2\,dt = 0$. Since $v > 0$, f must be identically 0 on $[a,b]$. Therefore $f_1 = f_2$.

7.7

(a) $\int_0^{\pi/2} \sin^2 t\,dt = \int_0^{\pi/2} \frac{1}{2}(1-\cos(2t))\,dt = \frac{\pi}{4}$

(b) $\int_0^{\pi/2} \sin^3 t\,dt = \int_0^1 (1-u^2)\,du = \frac{2}{3}$ (let $u = \cos t$)

7.9

(a) $e^{-1/x} + C$

(b) $\int x^{-1}e^{-1/x}\,dx = xe^{-1/x} - \int e^{-1/x}\,dx$.

(c) $(x^{-1}+1)e^{-1/x} + C$

(d) $e^{-1/x} - (x^2+2x+1)e^{-x} + C$

7.11

(a) $\sin x - x\cos x + C$

(b) $K_m(x) = -x^m\cos x + \int mx^{m-1}\cos x\,dx = -x^m\cos x + mx^{m-1}\sin x - m(m-1)K_{m-2}(x)$

(c) $K_0(x) = C - \cos x$, $K_2(x) = -x^2\cos x + 2x\sin x - 2K_0(x)$,
$K_4(x) = -x^4\cos x + 4x^3\sin x - 12K_2(x)$. Then $\int_0^{\pi} x^4\sin x\,dx = K_4(x)\Big|_0^{\pi} = \pi^4 - 12\pi^2 + 48$.

(d) $K_3(x) = -x^3\cos x + 3x^2\sin x - 6K_1(x)$.

7.15

(a) Let $u = t^2+1$, then $\int_0^1 \frac{t}{t^2+1}\,dt = \int_1^2 \frac{1}{2u}\,du = \frac{\log 2}{2}$.

(b) $\frac{1}{4}$

(c) Let $t = \tan u$, then $\int_0^1 \frac{1}{(t^2+1)^2}\,dt = \int_0^{\pi/4} \frac{\sec^2 u}{\tan^2 u + 1}\,du = \int_0^{\pi/4} \cos^2 u\,du$
$= \int_0^{\pi/4} \frac{1}{2}(1+\cos 2u)\,du = \left[\frac{1}{2}u + \frac{1}{4}\sin 2u\right]_0^{\pi/4} = \frac{\pi}{8} + \frac{1}{4}$.

(d) $\displaystyle\int_{-1}^{1} x^2 e^{x^3}\,dx = \frac{1}{3}\int_{-1}^{1}\left(e^{x^3}\right)'\,dx = \frac{1}{3}(e - e^{-1})$.

(e) $\displaystyle\int_{-1}^{1}\frac{2t+3}{t^2+9}\,dt = 2\tan^{-1}\left(\frac{1}{3}\right)$ (rewrite the integrand as $\dfrac{2t}{t^2+9}+\dfrac{3}{t^2+9}$. Let $u = t^2+9$ on the left and $v = \frac{t}{3}$ on the right)

(f) With $t = \sqrt{2}\sinh u$ we have $2+t^2 = 2+2\sinh^2 u = 2+2(\cosh^2 u - 1) = 2\cosh^2 u$. So

$$\int_0^1 \sqrt{2+t^2}\,dt = \int_0^b \sqrt{2}\cosh u\sqrt{2}\cosh u\,du,$$

where $1 = \sqrt{2}\sinh b$. But $\cosh^2 u = \frac{1}{4}(e^{2u}+2+e^{-2u})$, so the integral is equal to

$$\frac{2}{4}\left[\frac{e^{2u}}{2}+2u+\frac{e^{-2u}}{-2}\right]_0^b = \frac{e^{2b}}{4}+b-\frac{e^{-2b}}{4}.$$

Then since $b = \sinh^{-1}\dfrac{1}{\sqrt{2}} = \log\left(\dfrac{1}{\sqrt{2}}+\sqrt{1+\dfrac{1}{2}}\right)$, the integral is

$$\frac{1}{4}\left(\frac{1}{\sqrt{2}}+\sqrt{\frac{3}{2}}\right)^2 + \log\left(\frac{1}{\sqrt{2}}+\sqrt{1+\frac{1}{2}}\right) - \frac{1}{4}\left(\frac{1}{\sqrt{2}}+\sqrt{\frac{3}{2}}\right)^{-2} = 1.5245043\ldots.$$

7.17 With $x = t^2$, $0 \le t \le 1$, and change of variables gives $\displaystyle\int_0^1\sqrt{1+\sqrt{x}}\,dx = \int_0^1\sqrt{1+t}\,2t\,dt$. Then integrate by parts to get

$$\left[\frac{2}{3}(1+t)^{3/2}\,2t\right]_0^1 - \int_0^1\frac{2}{3}(1+t)^{3/2}\,2\,dt = \frac{4}{3}2^{3/2} - \frac{4}{15}\left[(1+t)^{5/2}\right]_0^1 = \frac{4}{3}2^{3/2} - \frac{4}{15}2^{7/2}$$

7.19 f has an antiderivative F. Then

$$\int_a^b f(g(t))|g'(t)|\,dt = -\int_a^b F'(g(t))g'(t)\,dt = -F\circ g\Big|_a^b = -\int_{g(a)}^{g(b)}F'(u)\,du = \int_{g(b)}^{g(a)}F'(u)\,du$$

7.21

(a) Set $u = x - r$. Then $u = a$ when $x = a+r$, and $u = b$ when $x = b+r$, so

$$\int_a^b f(u)\,du = \int_{a+r}^{b+r} f(x-r)\,dx.$$

(b) Set $u = -x$. Then $u = -a$ when $x = a$, and $u = -b$ when $x = b$, so

$$\int_a^b f(u)\,du = \int_{-a}^{-b} f(-x)(-1)\,dx = \int_{-b}^{-a} f(-x)\,dx.$$

7.23

(a) Yes: $\dfrac{na_n}{nb_n} = \dfrac{a_n}{b_n}$ tends to 1. Yes: $\dfrac{na_n}{\sqrt{1+n^2b_n}} = \dfrac{n}{\sqrt{1+n^2}}\dfrac{a_n}{b_n}$ tends to 1.

(b) $\displaystyle\lim_{n\to\infty}(\log a_n - \log b_n) = \log 1 = 0$.

7.25

(a) $\sum_{n=1}^{\infty} \frac{1}{n^2} \le 1 + \int_1^{\infty} x^{-2} dx = -x^{-1}\Big|_1^{\infty} = 1$. The series converges.

(b) $\sum_{n=1}^{\infty} \frac{1}{n^{1.2}} \le 1 + \int_1^{\infty} x^{-1.2} dx = \frac{x^{-.2}}{-.2}\Big|_1^{\infty}$. The series converges.

(c) $\sum_{n=2}^{\infty} \frac{1}{n \log n} \ge \int_2^{\infty} \frac{1}{x \log x} dx = \log(\log x)\Big|_2^{\infty}$. This limit does not exist; the series diverges.

(d) $\sum_{n=1}^{\infty} \frac{1}{n^{.9}} \ge \int_1^{\infty} x^{-.9} dx = 10x^{.1}\Big|_1^{\infty} = \infty$. This limit does not exist; the series diverges.

7.27

(a)

$$\int_1^b \frac{p_0 + \cdots + p_{n-2}x^{n-2}}{q_0 + \cdots + q_n x^n}\,\mathrm{d}x = \int_{1/b}^1 \frac{p_0 + \cdots + p_{n-2}z^{-(n-2)}}{q_0 + \cdots + q_n z^{-n}} \left| -z^{-2} \right| \frac{z^n}{z^n}\,\mathrm{d}z$$
$$= \int_{1/b}^1 \frac{p_0 z^{n-2} + \cdots + p_{n-2}}{q_0 z^{n+2} + \cdots + q_n}\,\mathrm{d}z$$

tends to a proper integral on $[0,1]$, that is, it is proper and remains proper as b tends to infinity because the denominator is never 0 in $[0,1]$. The result: $\int_1^{\infty} f(x)\,\mathrm{d}x = \int_0^1 f(z^{-1})z^{-2}\,\mathrm{d}z$.

(b) $f(x) = \frac{1}{1+x^3}$ has denominator of degree $3 = n \ge 2$, numerator of degree $0 \le 3-2$, and $1+x^3 \ne 0$ for $x \ge 1$, so part (a) applies. Then $x = z^{-1}$ gives $f(x) = \frac{1}{1+z^{-3}}$, $\mathrm{d}x = -z^{-2}\,\mathrm{d}z$, so $\int_1^{\infty} \frac{1}{1+x^3}\,\mathrm{d}x = -\int_1^0 \frac{1}{1+z^{-3}} z^{-2}\,\mathrm{d}z = \int_0^1 \frac{z}{z^3+1}\,\mathrm{d}z$.

7.29 $\lim_{b\to\infty} \int_1^b \frac{\sin x}{x}\,\mathrm{d}x = \lim_{b\to\infty} \left(-\frac{1}{x}\cos x\Big|_1^b - \int_1^b \frac{\cos x}{x^2}\,\mathrm{d}x \right)$ The limit of the first term is $\cos 1$, and the integral of $\frac{\cos x}{x^2}$ converges by comparison with $\int_1^{\infty} \frac{1}{x^2}\,\mathrm{d}x$, which converges. For the integral $\lim_{b\to\infty} \int_1^b \frac{|\sin x|}{x}\,\mathrm{d}x$, we have

(a) ignore integral from 1 to π,
(b) in each subinterval $\frac{1}{x} \ge \frac{1}{k\pi}$,
(c) each integral is equal to $\int_0^{\pi} \sin x\,\mathrm{d}x = 2$, and
(d) the integral has been shown to be larger than each partial sum of the divergent harmonic series.

7.31 $\int_s^b \frac{1}{x \log x}\,\mathrm{d}x = [\log(\log x)]_s^b$ tends to infinity with b because $\log b$ tends to infinity.

7.33 An antiderivative for $\frac{1}{y-y^2} = \frac{1}{y} + \frac{1}{1-y}$ is $\log|y| - \log|1-y|$. Therefore,

$$\int_2^b \frac{1}{y-y^2}\,\mathrm{d}y = \log\left|\frac{b}{1-b}\right| - \log\left|\frac{2}{1-2}\right|.$$

Since $\frac{b}{1-b}$ tends to -1 as b tends to infinity, the integral tends to $-\log 2$.

7.35 Denote the integrals by I_1, I_2 respectively. Then

$$I_1 = \int_0^\infty \sin(at)e^{-pt}\,\mathrm{d}t = -\frac{1}{p}\sin(at)e^{-pt}\Big|_0^\infty + \frac{1}{p}\int_0^\infty a\cos(at)e^{-pt}\,\mathrm{d}t = \frac{a}{p}I_2$$

$$I_1 = \int_0^\infty \sin(at)e^{-pt}\,\mathrm{d}t = -\frac{1}{a}\cos(at)e^{-pt}\Big|_0^\infty - \frac{p}{a}\int_0^\infty \cos(at)e^{-pt}\,\mathrm{d}t = \frac{1}{a} - \frac{p}{a}I_2.$$

Then solve for I_1 and I_2 from these relations.

7.37 Since $(x^n)' = nx^{n-1}$ is true for real positive n, integration by parts gives

$$\int_0^b x^n \mathrm{e}^{-x}\,\mathrm{d}x = \left[-\mathrm{e}^{-x}x^n\right]_0^b - \int_0^b \left(-\mathrm{e}^{-x}\right)nx^{n-1}\,\mathrm{d}x.$$

As b tends to infinity, this becomes $n! = 0 + n\big((n-1)!\big)$.

7.39 Change $x = y^2$ in $\left(\frac{1}{2}\right)! = \int_0^\infty x^{1/2}\mathrm{e}^{-x}\,\mathrm{d}x = \int_0^\infty y\mathrm{e}^{-y^2}2y\,\mathrm{d}y = 2\int_0^\infty y^2\mathrm{e}^{-y^2}\,\mathrm{d}y = \frac{1}{2}\sqrt{\pi}.$

7.41 (a) $10 + \frac{1000}{3} + 10t$ (b) 10 (c) 10 (d) They are equal.

Chapter 8

8.1

(a) $I_{\text{left}}(x^3,[1,2]) = 1^3(2-1) = 1,\ (1^3+(1.5)^3)(2-1)/2 = 2.1875,$
$(1^3+(1.25)^3+(1.5)^3+(1.75)^3)(2-1)/4 = 2.92188$
$I_{\text{right}}(x^3),[1,2] = 2^3(2-1) = 8,\ ((1.5)^3+2^3)(2-1)/2 = 5.6875,$
$((1.25)^3+(1.5)^3+(1.75)^3+2^3)(2-1)/4 = 4.67188$
$I_{\text{mid}}(x^3),[1,2] = (1.5)^3(2-1) = 3.375,\ ((1.25)^3+(1.75)^3)(2-1)/2 = 3.35625,$
$((1.125)^3+(1.375)^3+(1.625)^3+(1.875)^3)(2-1)/4 = 3.72656$

(b) pseudocode to compute $I_{\text{mid}}(f,[a,b])$ with n subdivisions.

```
function iapprox = Imid(a,b,n)
 h = (b-a)/n;                [ width of subinterval ]
 x = a+h/2;                  [ midpoint of 1st subinterval ]
 iapprox = f(x);
for k = 2 up to n
 x = x+h;                    [ move to next midpoint ]
 iapprox = iapprox+f(x);     [ add value of f there ]
endfor
iapprox = iapprox*h;         [ multiply by width h last ]

function y = f(x)
 y = sqrt(1-x*x);
```

$I_{\text{left}}(\sqrt{1-x^2},\left[0,\frac{1}{\sqrt{2}}\right]) = 0.70711, 0.68427, 0.66600$
$I_{\text{right}}(\sqrt{1-x^2},\left[0,\frac{1}{\sqrt{2}}\right]) = 0.5, 0.58072, 0.61422;$
$I_{\text{mid}}(\sqrt{1-x^2},\left[0,\frac{1}{\sqrt{2}}\right]) = 0.66144, 0.66722, 0.66399$

(c) $I_{\text{left}}(\frac{1}{1+x^2},[0,1]) = 1.00000, 0.90000, 0.84529$
$I_{\text{right}}(\frac{1}{1+x^2},[0,1]) = 0.5, 0.65000, 0.72079;$ $I_{\text{mid}}(\frac{1}{1+x^2},[0,1]) = 0.80000, 0.79059, 0.78670$

8.3

(a) $n = 1 : 0.447214,\ \ n = 5 : 0.415298,\ \ n = 10 : 0.414483,\ \ n = 100 : 0.414216$
actual value $\sqrt{2} - 1 \approx 0.414214$
(b) $n = 1 : 0.8,\ \ n = 5 : 0.78623,\ \ n = 10 : 0.78561,\ \ n = 100 : 0.78540$
actual value $\tan^{-1} 1 = \frac{\pi}{4} \approx 0.78540$
(c) $n = 1 : 0.5,\ \ n = 5 : 0.65449,\ \ n = 10 : 0.66350,\ \ n = 100 : 0.66663$
actual value $\frac{2}{3} \approx 0.66666$

8.5

(a) The graph of a convex function lies above each tangent line, in particular the tangent line at the midpoint of each subinterval $[c,d]$. Since then $f \geq \ell$ for that linear function ℓ on the subinterval, $\int_c^d f(x)\,\mathrm{d}x \geq \int_c^d \ell(x)\,\mathrm{d}x$. But the integral of ℓ is the midpoint rule for f.

(b) The graph of a convex function lies above each tangent line, in particular the secant line on each subinterval $[c,d]$. Since then $f \leq \ell$ for that linear secant function ℓ on the subinterval, $\int_c^d f(x)\,\mathrm{d}x \leq \int_c^d \ell(x)\,\mathrm{d}x$. But the integral of ℓ is the trapezoidal rule for f.

8.7

(a) The midpoint rule with one interval $[-h,h]$ gives $2hf(0)$.
(b) The derivatives for Taylor are
$K(h) - K(-h)$; at $h = 0$ it is 0.
$K'(h) + K'(-h) = f(0) - f(h) + \big(f(0) - f(-h)\big)$; at $h = 0$ it is 0.
$K''(h) - K''(-h) = -f'(h) + f'(-h)$; at $h = 0$ it is 0.
$K'''(h) + K'''(-h) = -f''(h) - f''(-h)$.

(c) In the last step recognize that the subinterval width is $2h = (b-a)/n$, and use the triangle inequality, $|-f''(c_2) - f''(-c_2)| \leq 2M_2$.

8.9 $(1/2)((1/4)^2 + (3/4)^2) + (1/24)(1/2)^2(4-0) = 0.2002\ldots$

8.11 $n = 100 : 0.7853575,\quad n = 1000 : 0.7853968\ ,\quad \frac{\pi}{4} = 0.7853981\ .$ The error term depends on the maximum value of $|f^{(4)}(x)|$ on the interval, which is unbounded as $x \to 1$.

Chapter 9

9.1

(a) $\sqrt{2^2 + 3^2} = \sqrt{13},\ \ \sqrt{4^2 + (-1)^2} = \sqrt{17}$
(b) $2 - 3\mathrm{i},\ \ 4 + \mathrm{i}$
(c) $\frac{1}{2+3\mathrm{i}} = \frac{1}{2+3\mathrm{i}}\frac{2-3\mathrm{i}}{2-3\mathrm{i}} = \frac{2-3\mathrm{i}}{13} = \frac{2}{13} - \frac{3}{13}\mathrm{i}$, and the reciprocal of the conjugate is the conjugate of the reciprocal: $\frac{2}{13} + \frac{3}{13}\mathrm{i}$.
$\frac{1}{4-\mathrm{i}} = \frac{1}{4-\mathrm{i}}\frac{4+\mathrm{i}}{4+\mathrm{i}} = \frac{4+\mathrm{i}}{17} = \frac{4}{17} + \frac{1}{17}\mathrm{i}$, and $\frac{4}{17} - \frac{1}{17}\mathrm{i}$
(d) $(2+3\mathrm{i}) + (2-3\mathrm{i}) = 4 = 2(2),\ \ (4-\mathrm{i}) + (4+\mathrm{i}) = 8 = 2(4)$
(e) $(2+3\mathrm{i})(2-3\mathrm{i}) = (4+9) = \sqrt{13}^2,\ \ (4-\mathrm{i})(4+\mathrm{i}) = (16+1) = \sqrt{17}^2$

9.3 $z = (4-1) + 2\mathrm{i} = 3 + 2\mathrm{i},\ \ \bar{z} = (3 - 2\mathrm{i})$

9.5 (a) 5 (b) $\sqrt{61}$ (c) $5/\sqrt{61}$ (d) 1

9.7 If $z = x + \mathrm{i}y$ then these say $y = \dfrac{x + \mathrm{i}y - (x - \mathrm{i}y)}{2\mathrm{i}}$ and $x = \dfrac{x + \mathrm{i}y + (x - \mathrm{i}y)}{2}$, which are true.

9.9 $(a^2+b^2)(c^2+d^2) = a^2c^2+a^2d^2+b^2c^2+b^2d^2$ and

$$\begin{aligned}(ac-bd)^2+(ad+bc)^2 &= (ac)^2-2acbd+(bd)^2+(ad)^2+2adbc+(bc)^2\\ &= a^2c^2+a^2d^2+b^2c^2+b^2d^2\end{aligned}$$

Then if $z = a+b\mathrm{i}$ and $w = c+d\mathrm{i}$, the left-hand side is $|z|^2|w|^2$ and $zw = (ac-bd)+(ad+bc)\mathrm{i}$, so the right-hand side is $|zw|^2$. Taking the square root of both sides gives $|z||w| = \pm|zw|$. However, since absolute values are only positive, $|z||w| = |zw|$.

9.11

(a) $|z_1-z_2|^2 = (z_1-z_2)\overline{(z_1-z_2)} = (z_1-z_2)(\overline{z_1}-\overline{z_2}) = z_1\overline{z_1}-z_1\overline{z_2}-z_2\overline{z_1}+z_2\overline{z_2}$
$= |z_1|^2+|z_2|^2-(z_1\overline{z_2}+\overline{z_1\overline{z_2}}) = |z_1|^2+|z_2|^2-2\mathrm{Re}\,(z_1\overline{z_2})$

(b) By the triangle inequality, $|z_1| = |z_2+(z_1-z_2)| \le |z_2|+|z_1-z_2|$.
Thus, $|z_1|-|z_2| \le |z_1-z_2|$. Similarly, $|z_2|-|z_1| \le |z_2-z_1| = |z_1-z_2|$. These combine to give us the desired inequality.

9.13 $-1 = \cos(\pi)$ has cube roots $\cos\left(\frac{\pi+2k\pi}{3}\right)+\mathrm{i}\sin\left(\frac{\pi+2k\pi}{3}\right)$ for $k = 0,1,2$.
These are $\frac{1}{2}+\frac{\sqrt{3}}{2}\mathrm{i}$, -1, and $\frac{1}{2}-\frac{\sqrt{3}}{2}\mathrm{i}$. These three roots are equally spaced around the unit circle.

9.15 The area formula says that triangle $(0,a,b)$ has area $A(0,a,b) = \frac{1}{2}|\mathrm{Im}\,(\overline{a}b)|$. If $a = a_1+\mathrm{i}a_2$, $b = b_1+\mathrm{i}b_2$, you obtain

$$\overline{a}b = (a_1-\mathrm{i}a_2)(b_1+\mathrm{i}b_2) = a_1b_1+a_2b_2+\mathrm{i}(a_1b_2-a_2b_1).$$

Therefore, the area is $\frac{1}{2}|a_1b_2-a_2b_1|$.

9.17 On the unit circle, $\overline{p} = \frac{1}{p}$. Therefore,

(a) $(p-1)^2\overline{p} = (p^2-2p+1)\overline{p} = p-2+\overline{p}$. This is real because $z+\overline{z}$ is twice the real part of z.

(b) When q is on the unit circle, so is $\overline{q}$. According to part (a), $(\overline{q}-1)^2q$ is real.
Then $\big((p-1)(\overline{q}-1)\big)^2\overline{p}q$ is the product of two real numbers, which is real.

(c) From the figure, and using Problem 9.16 part (b), β is the argument of $q\overline{p}$, and α is minus the argument of $(\overline{q-1})(p-1)$. But since $\big((p-1)(\overline{q}-1)\big)^2\overline{p}q$ is real, its argument, which is $-2\alpha+\beta$, must be 0.

9.19

(a) Because of the identity $(x-1)w(x) = x^5-1$, and because $w(1) = 5 \neq 0$, it follows that four of the five fifth roots of 1 are the roots of w.

(b) The n roots of 1 are equally spaced around the unit circle and one of them is 1.

(c) If $r^n = 1$, then every integer power of r is also a root: $(r^p)^n = r^{np} = (r^n)^p = 1$. The only issue is whether these powers are *all* of the roots. [For example, powers of i^2 do not give all the fourth roots of 1.] Taking the one with smallest argument makes

$$r = \cos\left(\frac{2\pi}{n}\right)+\mathrm{i}\sin\left(\frac{2\pi}{n}\right),$$

and the powers have arguments $\frac{2\pi}{n}$, $2\frac{2\pi}{n}$, etc., so this gives all the roots. The identity $(x-1)(x^{n-1}+\cdots+x^2+x+1) = x^n-1$ and the same argument as in part (a) show that $r,\ldots,r^{n-1}$ are the roots of w and $r^n = 1$ is the other root of $x^n = 1$.

9.21

(a) $e^t + i\cos t$
(b) $-\dfrac{1}{(t-i)^2} - \dfrac{1}{(t+i)^2}$
(c) $ie^{t^2}2t$
(d) $i\cos t - (t+3+i)^{-2}$

9.23 Since $\overline{e^{it}} = e^{-it} = \dfrac{1}{e^{it}}$, $\cos t = \operatorname{Re} e^{it} = \dfrac{e^{it}+e^{-it}}{2}$ and $\sin t = \operatorname{Im} e^{it} = \dfrac{e^{it}-e^{-it}}{2i}$.

9.25 Define $\cosh z = \frac{1}{2}(e^z + e^{-z})$. Then $\cosh(it) = \frac{1}{2}(e^{it}+e^{-it}) = \cos t$.

9.27 $\int_0^b e^{ikx-x}\,dx = \dfrac{e^{ikb}-1}{ik-1}$ tends to $\dfrac{1}{1-ik}$. (e^{-b} tends to 0 as b tends to infinity, $|e^{ikb}| = 1$.)

Chapter 10

10.1 Only (b), and (c) if we allow f_{re} to be 0. The others do not match the descriptions of the frictional and restoring forces.

10.3 We need $r^2 + r = 0$, so $r = 0$ or -1. Then trying $x(t) = c_1e^0 + c_2e^{-t}$, we obtain

(a) $x(t) = 12 - 7e^{-t}$ tends to 12
(b) $x(t) = -2 + 7e^{-t}$ tends to -2

10.5 $2r^2 + 7r + 3 = 0$ gives $r = (-7 \pm \sqrt{49-24})/4 = -\frac{1}{2}, -3$. So $e^{-\frac{1}{2}t}$ and e^{-3t} are solutions.

10.7 Look at solutions e^{rt} where $mr^2 + hr + k = 0$, $r = \dfrac{-h \pm \sqrt{h^2-4km}}{2m}$. When h is small, this is roughly $r \approx -\dfrac{h}{2m} \pm i\sqrt{\dfrac{k}{m}}$, and you have a solution $x \approx e^{-\frac{h}{2m}} \cos\left(\sqrt{\dfrac{k}{m}}\right)$ with a gradually decreasing amplitude. When h is large, one value of r is a negative number close to 0 by the binomial theorem,

$$-h + \sqrt{h^2-4km} = -h + h\sqrt{1-\frac{4km}{h^2}} \approx -h + h(1 - \frac{2km}{h^2}),$$

and you have a solution $x \approx e^{-(\text{small})t}$, which is a gradually decreasing exponential.

10.9 We use $(cf)' = cf'$ and $(f+g)' = f' + g'$ repeatedly.

(a) Let $x = cx_1$. Then $A_n x^{(n)} + \cdots = A_n c x_1^{(n)} + \cdots = c\left(A_n x_1^{(n)} + \cdots\right) = c(0) = 0$.
(b) $A_n y^{(n)} + \cdots = \left(A_n x_1^{(n)} + \cdots\right) + \left(A_n x_2^{(n)} + \cdots\right) = 0 + 0 = 0$.

10.11 $m(c_1x_1 + c_2x_2)'' + h(c_1x_1 + c_2x_2)' + k(c_1x_1 + c_2x_2)$
$= m(c_1x_1'' + c_2x_2'') + h(c_1x_1' + c_2x_2') + k(c_1x_1 + c_2x_2)$
$= c_1(mx_1'' + hx_1' + kx_1) + c_2(mx_2'' + hx_2' + kx_2) = 0 + 0 = 0$.

10.13

(a) $mr^2 + 2\sqrt{mk}r + k = (\sqrt{m}r + \sqrt{k})^2$, so $r = -\sqrt{k/m}$
(b) $x' = (1+rt)e^{rt}$, $x'' = (2r + r^2t)e^{rt}$. Then

$$mx'' + 2\sqrt{mk}x' + kx = \left(2mr + mr^2t + 2\sqrt{km}(1+rt) + kt\right)e^{rt} = \left(-2\sqrt{mk} + 0t + 2\sqrt{km}\right)e^{rt} = 0$$

10.15 Suppose $z(t) = ae^{6it}$. Then $z'' + z' + 6z - 52e^{6it} = \big(a(-36+6i+6) - 52\big)e^{6it}$ is zero if $a = \dfrac{52}{-30+6i} = \dfrac{52}{6}\dfrac{-5-i}{26} = \dfrac{-5-i}{3}$. Then the real part of $z(t) = \dfrac{-5-i}{3}e^{6it}$ is $x(t) = -\frac{5}{3}\cos(6t) + \frac{1}{3}\sin(6t)$.

10.17 Try $x = \operatorname{Re} z$, where $z = ae^{it}$ solves $z'' + z' + z = e^{it}$.
We need $(-1+i+1)ae^{it} = e^{it}$, so $a = -i$. Then $z = i\cos t + \sin t$ and $x_1(t) = \sin t$.
With $x_2 = y + x_1$, you have $x_2'' + x_2' + x_2 = y'' + y' + y + x_1'' + x_1' + x_1 = 0 + \cos t$.

10.19 The inequality holds if its square holds: $\dfrac{1}{h^2}\dfrac{1}{\frac{k}{m} - \frac{h^2}{4m^2}} > \dfrac{1}{k^2}$,
if the reciprocals $k^2 > h^2\left(\dfrac{k}{m} - \dfrac{h^2}{4m^2}\right)$, if $4m^2$ times it $4m^2k^2 > h^2(4mk - h^2)$.
But that is true because since $h < \sqrt{2mk}$, we have $h^2 < 2mk$, $0 > 4m^2k^2 - 2mkh^2$, $4m^2k^2 > 8m^2k^2 - 2mkh^2 = 2mk(4mk - h^2) > h^2(4mk - h^2)$.

10.21

(a) Use $w'' = y'' - x''$ and subtract the differential equations.
(b) This is due to the mean value theorem.
(c) Using (b), $mw''w' - f_{\text{re}}'(v)ww' = f_{\text{fr}}'(u)(w')^2 \le 0$ because f_{fr} is decreasing.
(d) Due to the law of decrease of energy, $x(t), y(t)$ are both bounded with values in some interval $[-M, M]$. Let k be the bound of $|f_{\text{re}}'|$ in $[-M, M]$. Because v is between x and y, $v \in [-M, M]$, so $f_{\text{re}}'(v)$ is bounded above by $-k$. Therefore,

$$mw''w' + kww' \le mw''w' - f_{\text{re}}'(v)ww' \le 0,$$

so its antiderivative $\frac{1}{2}m(w')^2 + \frac{1}{2}kw^2$ is nonincreasing.
(e) The function $\frac{1}{2}m(w')^2 + \frac{1}{2}kw^2$ is nonincreasing, nonnegative, and is zero when $t = s$. Therefore it is zero for all $t > s$. Therefore w is zero for all $t > s$.

10.23 For $N(t) = 0$, $N' = 0 = \sqrt{0}$, so that is a solution. For $N(t) = \frac{1}{4}t^2$, if $t \ge 0$ then $\sqrt{N} = \frac{1}{2}t$. Then $N' = \frac{2}{4}t = \sqrt{N}$, so that is a solution. (Note: $\frac{1}{4}t^2$ is not a solution when $t < 0$ because the derivative has the wrong sign.)

There is no contradiction: the existence theorem does not apply to $N' = \sqrt{N}$ with $N(0) = 0$ because the function $\sqrt{N}$ is not defined in an interval containing $N(0)$.

10.25 $N' = N^2 - N$, $\dfrac{1}{N^2 - N}N' = 1$, $\dfrac{-N^{-2}}{N^{-1} - 1}N' = 1$, $\log(N^{-1} - 1) = t + c$, $N^{-1} - 1 = e^{t+c}$, $N_0^{-1} - 1 = e^c$. So $N(t) = \dfrac{1}{1 + e^{t+c}} = \dfrac{1}{1 + (N_0^{-1} - 1)e^t} = \dfrac{N_0}{N_0 + (1 - N_0)e^t}$. With N_0 between 0 and 1, the denominator is more than 1 and tends to infinity, so $N(t)$ is less than N_0 and tends to 0.

10.27

(a) Where $P > P_m$, K is increasing, so has an inverse. Then $P = K^{-1}(c - H(N))$ defines one function P_+. Similarly for P_- with values less than P_m.
(b) is by the chain rule
(c) For P_+, K is increasing, so in the formula for $\dfrac{d^2P_+}{dN^2}$, the denominator is positive. All numbers in the numerator are positive because H and K are convex. Therefore $\dfrac{d^2P_+}{dN^2}$ is negative. The denominator reverses sign for P_-.

10.29

(a) If $y(0)$ is positive, then $y(t)$, being continuous, will remain positive for some interval of time. Divide by $y(t)$ to get $-y^{-2}y' = 1$. Then we integrate from 0 to t to obtain $y^{-1}(t) - y^{-1}(0) = t$. Rearrange to

$$y(t) = \frac{1}{t + y^{-1}(0)} = \frac{y(0)}{y(0)t + 1}.$$

Note that in fact, $y(t)$ remains positive for all t.

(b) We know this equation; the answer is $y(t) = y(0)\mathrm{e}^{-t}$.

For the second equation, y tends exponentially to zero, faster than the $1/t$ rate for the first one.

10.31 $\dfrac{\mathrm{d}a}{\mathrm{d}t} = ap$, which is negative when $a > 0$ and $p < 0$.

$\dfrac{\mathrm{d}b}{\mathrm{d}t} = -ap - db$, which is negative when $a > 0,\ b > 0$, and $p > 0$.

10.33 Using subintervals of length h, and $y_{n+1} = y_n + hf(nh)$, we obtain
$y_1 = y_0 + f(0)h = I_{\text{left}}(f, [0, h]),\ y_2 = y_1 + f(h)h = \big(f(0) + f(h)\big)h = I_{\text{left}}(f, [0, 2h])$
$y_3 = y_2 + f(2h)h = \big(f(0) + f(h) + f(2h)\big)h = I_{\text{left}}(f, [0, 3h])$ and so forth.

Chapter 11

11.1 The expected value of the die roll is $\frac{1}{6}(1+2+3+4+5+6) = 3.5$. The expected value of the squared difference between 3.5 and the value rolled is: $\frac{1}{6}(2.5^2 + 1.5^2 + 0.5^2 + 0.5^2 + 1.5^2 + 2.5^2) = \frac{35}{12}$ This is the variance.

11.3 Let $S(E)$ be the number of instances among the first N experiments when E occurred. Then $S(E') = N - S(E)$. So $p(E) + p(E') = \lim_{N\to\infty} \left(\frac{S(E)}{N} + \frac{S(E')}{N}\right) = \lim 1 = 1$.

11.5 Each time an outcome of E occurs, it is also an outcome of F, so $S(E) \le S(F)$.
Then $P(E) = \lim_{n\to\infty} \dfrac{S(E)}{N} \le \lim_{n\to\infty} \frac{S(F)}{N} = P(F)$.

11.7 Since each outcome that occurs in an experiment can belong to only one of the events, the count $S(E_1 \cup \cdots \cup E_m)$ increases by exactly one each time an event occurs in some E_j, and only that one $S(E_j)$ increases by one. So $S(E_1 \cup \cdots \cup E_m) = S(E_1) + \cdots + S(E_m)$.

11.9 $\displaystyle\sum_{k=0}^{\infty} k\frac{u^k}{k!}\mathrm{e}^{-u} = \sum_{k=1}^{\infty} u\frac{u^{k-1}}{(k-1)!}\mathrm{e}^{-u} = u\sum_{k=0}^{\infty} \frac{u^k}{k!}\mathrm{e}^{-u} = u\mathrm{e}^{u}\mathrm{e}^{-u} = u$

11.11 We need to show that

$$q\log q + r\log r < (1-p)\log(1-p) = (q+r)\log(q+r) = q\log(q+r) + r\log(q+r).$$

But this is true because log is increasing, that is, $\log q < \log(q+r)$ and $\log r < \log(q+r)$.

11.13 We need to show that $p_2 \log p_2 + \cdots + p_n \log p_n < (1 - p_1)\log(1 - p_1)$

$$= (p_2 + \cdots + p_n)\log(p_2 + \cdots + p_n) = p_2 \log(p_2 + \cdots + p_n) + \cdots + p_n \log(p_2 + \cdots + p_n),$$

which is true since log is increasing.

11.15

(a) $\int_{-\infty}^{\infty} p(x)\,\mathrm{d}x = \int_0^A \frac{2}{A}\big(1-\frac{x}{A}\big)\,\mathrm{d}x = \frac{2}{A}\big(A - \frac{A^2}{2A}\big) = 1$

(b) $\overline{x} = \int_{-\infty}^{\infty} xp(x)\,\mathrm{d}x = \int_0^A \frac{2}{A}\big(x - \frac{x^2}{A}\big)\,\mathrm{d}x = \frac{2}{A}\big(\frac{A^2}{2} - \frac{A^3}{3A}\big) = \frac{A}{3}$

(c) $\overline{x^2} = \int_{-\infty}^{\infty} x^2 p(x)\,\mathrm{d}x = \int_0^A \frac{2}{A}\big(x^2 - \frac{x^3}{A}\big)\,\mathrm{d}x = \frac{2}{A}\big(\frac{A^3}{3} - \frac{A^4}{4A}\big) = \frac{A^2}{6}$

(d) Using $\sqrt{\overline{x^2} - (\overline{x})^2}$ we get $\sqrt{\frac{A^2}{6} - \left(\frac{A}{3}\right)^2} = \frac{1}{18}A.$

11.17

(a) $\int_0^A \frac{1}{A}\,\mathrm{d}x = 1$, similarly for p.

(b) $u(x) = \int_{-\infty}^{\infty} p(t)q(x-t)\,\mathrm{d}t = \int_0^A p(t)q(x-t)\,\mathrm{d}t$. In this integral, $-A \le -t \le 0$. If $x < 0$, then $x - t < 0$, so $q(x-t) = 0$. So $u(x) = 0$ when $x < 0$. If $x > A+B$, then $x - t > A + B - A = B$, so again $q(x-t) = 0$, showing that $u(x) = 0$ when $x > A+B$.

(c) If you graph $q(x-t)$ as a function of t, you see that the convolution integral gives the area of a rectangle whose size is constant when $B < x < A$. The width is B, height $\frac{1}{AB}$, so $u = 1/A$.

(d) $u(x) = \frac{x}{AB}$ on $[0,B]$, $\frac{1}{A}$ on $[B,A]$, and $\frac{1}{A} - \frac{x}{AB}(x-A)$ in $[A, A+B]$.

11.19

(a) $|w|_1 = \int_a^b 5\,\mathrm{d}x = 5(b-a).$

(b) $|cu|_1 = \int_{-\infty}^{\infty} |cu(x)|\,\mathrm{d}x = \int_{-\infty}^{\infty} |c||u(x)|\,\mathrm{d}x = |c||u|_1,$

$|u+v|_1 = \int_{-\infty}^{\infty} |u(x)+v(x)|\,\mathrm{d}x \le \int_{-\infty}^{\infty} \big(|u(x)|+|v(x)|\big)\,\mathrm{d}x = |u|_1 + |v|_1.$

(c)

$$|u*v|_1 = \int_{-\infty}^{\infty} |u*v(x)|\,\mathrm{d}x = \int_{-\infty}^{\infty} \left|\int_{-\infty}^{\infty} u(y)v(x-y)\,\mathrm{d}y\right| \mathrm{d}x$$
$$\le \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |u(y)v(x-y)|\,\mathrm{d}y\,\mathrm{d}x = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |u(y)||v(x-y)|\,\mathrm{d}y\,\mathrm{d}x$$

This inner integral is the convolution of $|u|$ and $|v|$. The integral of a convolution is the product of the integrals. Therefore, we get

$$|u*v|_1 = \left(\int_{-\infty}^{\infty} |u(x)|\,\mathrm{d}x\right)\left(\int_{-\infty}^{\infty} |v(x)|\,\mathrm{d}x\right) = |u|_1|v|_1.$$

11.21

(a) $n^2 > kn$, then use a geometric series times $\mathrm{e}^{-K^2h^2}$. Then $\sum_4^{\infty} \le \mathrm{e}^{-16}/(1-\mathrm{e}^{-4}) = (1.1463)10^{-7}$.

(b) $1 + 2\mathrm{e}^{-1} + 2\mathrm{e}^{-4} + 2\mathrm{e}^{-9} = 1.7726369797\ldots$

11.23

(a) Use the fact that if $p_n \sim q_n$ and $r_n \sim s_n$, then $p_n r_n \sim q_n s_n$. Then

$$2^{-n}\binom{n}{k} = 2^{-n}\frac{n!}{k!(n-k)!} \sim 2^{-n}\frac{\sqrt{2\pi n}\left(\frac{n}{\mathrm{e}}\right)^n}{\sqrt{2\pi k}\left(\frac{k}{\mathrm{e}}\right)^k\sqrt{2\pi(n-k)}\left(\frac{n-k}{\mathrm{e}}\right)^{n-k}}$$

All the powers of e factor out as $\mathrm{e}^{-n+k+n-k} = 1$.

(b) Part of this step is just substitution, but the hard part is

$$\frac{n^n}{2^n k^k} = \left(\frac{n}{2}\right)^{n/2} \left(\frac{n}{2}\right)^{n/2} \frac{1}{k^k} = \left(\frac{n}{2}\right)^{n/2} \frac{1}{\left(\frac{2}{n}\right)^{n/2} \left(\frac{n}{2}+\sqrt{n}y\right)^{\frac{n}{2}} \left(\frac{n}{2}+\sqrt{n}y\right)^{\sqrt{n}y}}$$

$$= \left(\frac{n}{2}\right)^{n/2} \frac{1}{\left(1+\frac{2y}{\sqrt{n}}\right)^{\frac{n}{2}} \left(\frac{n}{2}+\sqrt{n}y\right)^{\sqrt{n}y}}$$

Then handle the $(n-k)^{n-k}$ factor similarly:

$$\frac{n^n}{2^n k^k (n-k)^{n-k}} = \frac{1}{\left(1+\frac{2y}{\sqrt{n}}\right)^{\frac{n}{2}} \left(\frac{n}{2}+\sqrt{n}y\right)^{\sqrt{n}y}} \left(\frac{n}{2}\right)^{n/2} \frac{1}{\left(\frac{n}{2}-\sqrt{n}y\right)^{\frac{n}{2}} \left(\frac{n}{2}-\sqrt{n}y\right)^{-\sqrt{n}y}}$$

$$= \frac{1}{\left(1+\frac{2y}{\sqrt{n}}\right)^{\frac{n}{2}} \left(\frac{n}{2}+\sqrt{n}y\right)^{\sqrt{n}y}} \frac{1}{\left(1-\frac{2y}{\sqrt{n}}\right)^{\frac{n}{2}} \left(\frac{n}{2}-\sqrt{n}y\right)^{-\sqrt{n}y}}$$

and combine the $n/2$ powers to get $\dfrac{1}{\left(1-\frac{4y^2}{n}\right)^{\frac{n}{2}} \left(\frac{n}{2}+\sqrt{n}y\right)^{\sqrt{n}y} \left(\frac{n}{2}-\sqrt{n}y\right)^{-\sqrt{n}y}}$.

(c) Two of the factors in the denominator,

$$\left(\frac{n}{2}+\sqrt{n}y\right)^{\sqrt{n}y} \left(\frac{n}{2}-\sqrt{n}y\right)^{-\sqrt{n}y} = \left(\frac{\frac{n}{2}+\sqrt{n}y}{\frac{n}{2}-\sqrt{n}y}\right)^{\sqrt{n}y} = \left(\frac{1+\frac{2y}{\sqrt{n}}}{1+\frac{2y}{\sqrt{n}}}\right)^{\sqrt{n}y}$$

which tends to $\dfrac{(e^{2y})^y}{(e^{-2y})^y}$. The factor $\left(1-\dfrac{4y^2}{n}\right)^{\frac{n}{2}}$ tends to $\left(e^{-4y^2}\right)^{1/2}$.

(d) The coefficient, $\dfrac{n}{\frac{n^2}{4}-ny^2} = \dfrac{4}{n-4y^2} \sim \dfrac{4}{n}$, giving $\dfrac{1}{\sqrt{2\pi}}\sqrt{\dfrac{n}{\frac{n^2}{4}-ny^2}} \sim \dfrac{1}{\sqrt{n}}\sqrt{\dfrac{2}{\pi}}$.

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