Appendix A

A1 Vector Analysis

A1.1 Scalars and Vectors

A quantity that is only specified by its magnitude is a **scalar**. Mass, energy, electric charge and temperature are examples of scalars. On the other hand, there is a kind of quantity that needs to be specified not only by magnitude but also direction; such a quantity is a **vector**. Force, velocity and moment in dynamics and the electric field, magnetic flux density and current in electromagnetism are examples of vectors.

A vector is commonly denoted by a bold character such as $\boldsymbol{F}$ or a character with an arrow such as $\vec{F}$. A vector is specified in space by drawing a straight arrow, as shown in Fig. A1.1. The length of the line represents the magnitude of the vector and the direction of the arrow represents the direction of the vector. The magnitude of vector $\boldsymbol{F}$ is written as $|\boldsymbol{F}|$.

In many cases the effect of vector is unchanged even when it is displaced in parallel, i.e. moved without changing the direction in which it points, as shown in Fig. A1.2. Such a vector is called a free vector. On the other hand, there is a kind of vector that gives a different effect when displaced (like a force); such a vector is called a bound vector. For example, when a force is given on the center of gravity of an object it causes a simple translation motion of the object. However, when a force is given on a point other than the center of gravity, it causes both straight motion of the center of gravity and a rotational motion around that center.

A1.2 Addition of Vectors

Suppose the sum of two free vectors $\boldsymbol{A}$ and $\boldsymbol{B}$, as shown in Fig. A1.3. The sum $\boldsymbol{A}+\boldsymbol{B}$ is obtained graphically by translating $\boldsymbol{B}$ so that its starting point reaches the

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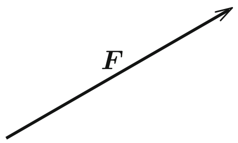

Fig. A1.1 Specifying a vector in space

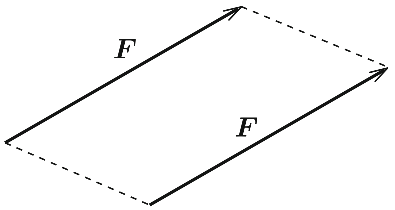

Fig. A1.2 Parallel displacement of vector

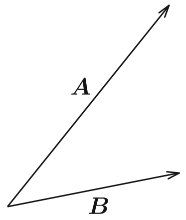

Fig. A1.3 Two vectors

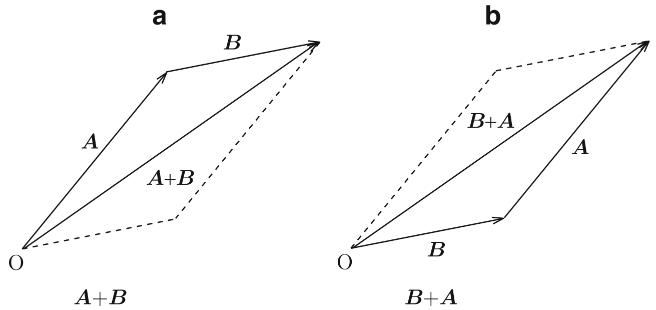

Fig. A1.4 Sum of two vectors: (**a**) $\boldsymbol{A}+\boldsymbol{B}$ and (**b**) $\boldsymbol{B}+\boldsymbol{A}$

end point of $\boldsymbol{A}$, as shown in Fig. A1.4a. In this case the sum is obtained as a vector that connects the starting point of $\boldsymbol{A}$ and the end point of $\boldsymbol{B}$. The sum $\boldsymbol{B}+\boldsymbol{A}$ is similarly obtained by translating $\boldsymbol{A}$, and we can see it is equal to the sum $\boldsymbol{A}+\boldsymbol{B}$ (see Fig. A1.4b). Thus, the relation holds generally:

$$\boldsymbol{A} + \boldsymbol{B} = \boldsymbol{B} + \boldsymbol{A}. \tag{A1.1}$$

This is called the exchange law. For a sum of three vectors the following relation holds:

$$(\boldsymbol{A}+\boldsymbol{B}) + \boldsymbol{C} = \boldsymbol{A} + (\boldsymbol{B}+\boldsymbol{C}), \tag{A1.2}$$

where the operation inside parentheses has priority. This is called the combination law.

A1.3 Products of Vectors and Scalars

The product of a vector $\boldsymbol{A}$ and scalar a gives a vector of magnitude equal to $a|\boldsymbol{A}|$ with the same direction as $\boldsymbol{A}$. For $a < 0$ the direction is reversed. The following relations hold for this kind of product:

$$m(n\boldsymbol{A}) = (mn)\boldsymbol{A}, \tag{A1.3}$$

$$(m+n)\boldsymbol{A} = m\boldsymbol{A} + n\boldsymbol{A}, \tag{A1.4}$$

$$m(\boldsymbol{A} + \boldsymbol{B}) = m\boldsymbol{A} + m\boldsymbol{B}. \tag{A1.5}$$

Equation (A1.5) is called the distribution law.

A1.4 Analytic Expression of a Vector

We use Cartesian coordinates (x, y, z) and denote unit vectors with a unit magnitude along the x- y- and z-axes by $\boldsymbol{i}_x$, $\boldsymbol{i}_y$ and $\boldsymbol{i}_z$, respectively. If vector $\boldsymbol{A}$ is expressed as

$$\boldsymbol{A} = A_x\boldsymbol{i}_x + A_y\boldsymbol{i}_y + A_z\boldsymbol{i}_z \tag{A1.6}$$

(see Fig. A1.5), A_x, A_y and A_z are called the x-, y- and z-components of $\boldsymbol{A}$, respectively. Using these components, $\boldsymbol{A}$ is also expressed as

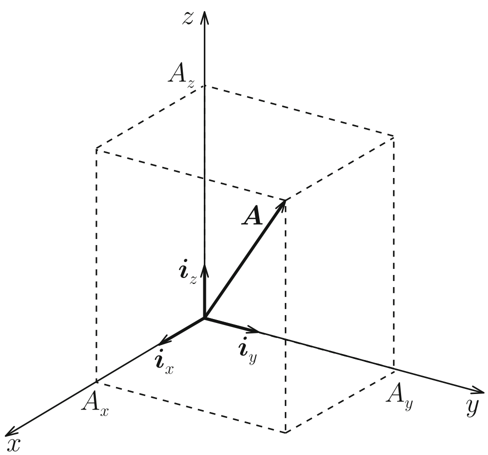

Fig. A1.5 Representation of vector components using Cartesian coordinates

$$(A_x, A_y, A_z). \tag{A1.7}$$

The magnitude of $\boldsymbol{A}$ is given by

$$|\boldsymbol{A}| = A = (A_x^2 + A_y^2 + A_z^2)^{1/2}. \tag{A1.8}$$

When the components of vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ are (A_x, A_y, A_z) and (B_x, B_y, B_z), respectively, the components of $\boldsymbol{A} + \boldsymbol{B}$ are

$$(A_x + B_x,\ A_y + B_y,\ A_z + B_z). \tag{A1.9}$$

The components of $a\boldsymbol{A}$ are

$$(aA_x, aA_y, aA_z). \tag{A1.10}$$

Using these methods, we can prove the laws of exchange, combination and distribution.

A1.5 Products of Vectors

When the angle of vector $\boldsymbol{B}$ measured from vector $\boldsymbol{A}$ is θ $(\pi \leq \theta < \pi)$, $AB\cos\theta$ is called a **scalar product** of $\boldsymbol{A}$ and $\boldsymbol{B}$, and is written as $\boldsymbol{A}\cdot\boldsymbol{B}$. Hence, we have

$$\boldsymbol{A}\cdot\boldsymbol{B} = \boldsymbol{B}\cdot\boldsymbol{A} = AB\cos\theta. \tag{A1.11}$$

If $\boldsymbol{A}$ and $\boldsymbol{B}$ are perpendicular to each other $(\theta = \pm\pi/2)$, $\boldsymbol{A}\cdot\boldsymbol{B} = 0$. When the components of $\boldsymbol{A}$ and $\boldsymbol{B}$ are (A_x, A_y, A_z) and (B_x, B_y, B_z), we have

$$\boldsymbol{A}\cdot\boldsymbol{B} = A_xB_x + A_yB_y + A_zB_z. \tag{A1.12}$$

The **vector product** of $\boldsymbol{A}$ and $\boldsymbol{B}$ is a vector of magnitude $AB\sin\theta$ that points along the direction of a screw when we rotate it from $\boldsymbol{A}$ to $\boldsymbol{B}$, and is written as $\boldsymbol{A}\times\boldsymbol{B}$. Thus,

$$\boldsymbol{A}\times\boldsymbol{B} = -\boldsymbol{B}\times\boldsymbol{A}. \tag{A1.13}$$

Vector $\boldsymbol{A}\times\boldsymbol{B}$ is normal to $\boldsymbol{A}$ and $\boldsymbol{B}$, and when $\boldsymbol{A}$ and $\boldsymbol{B}$ point in the same or opposite direction $(\theta = \pi$ or $0)$, $\boldsymbol{A}\times\boldsymbol{B} = 0$. Using the components, the vector product is expressed as

$$\boldsymbol{A}\times\boldsymbol{B} = (A_yB_z - A_zB_y)\boldsymbol{i}_x + (A_zB_x - A_xB_z)\boldsymbol{i}_y + (A_xB_y - A_yB_x)\boldsymbol{i}_z. \tag{A1.14}$$

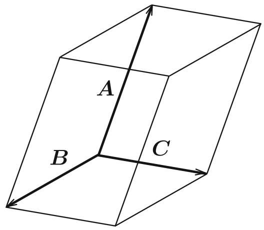

Fig. A1.6 Parallelepiped composed of $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$

We can also represent the vector product with a determinant,

$$\boldsymbol{A} \times \boldsymbol{B} = \begin{vmatrix} \boldsymbol{i}_x & \boldsymbol{i}_y & \boldsymbol{i}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}. \tag{A1.15}$$

Now we treat a product of three vectors. Among the conceivable products of three vectors $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$, it is clear that $\boldsymbol{A}\cdot\boldsymbol{B}\cdot\boldsymbol{C}$ and $(\boldsymbol{A}\cdot\boldsymbol{B})\times\boldsymbol{C}$ do not make sense, and the order of operation is not defined for $\boldsymbol{A}\times\boldsymbol{B}\times\boldsymbol{C}$. One meaningful product is:

$$\boldsymbol{A}(\boldsymbol{B}\cdot\boldsymbol{C}) = a\boldsymbol{A}; \qquad \boldsymbol{B}\cdot\boldsymbol{C} = a. \tag{A1.16}$$

The next one is

$$\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C}) = (A_x\boldsymbol{i}_x + A_y\boldsymbol{i}_y + A_z\boldsymbol{i}_z)\cdot\begin{vmatrix} \boldsymbol{i}_x & \boldsymbol{i}_y & \boldsymbol{i}_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}. \tag{A1.17}$$

This represents the volume of a parallelepiped composed of $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ (see Fig. A1.6). We can easily prove the following equation:

$$\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C}) = \boldsymbol{B}\cdot(\boldsymbol{C}\times\boldsymbol{A}) = \boldsymbol{C}\cdot(\boldsymbol{A}\times\boldsymbol{B}). \tag{A1.18}$$

This product is called a **scalar triple product**.

The final meaningful product of three vectors is a **vector triple product**:

$$\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C}) = (\boldsymbol{A}\cdot\boldsymbol{C})\boldsymbol{B} - (\boldsymbol{A}\cdot\boldsymbol{B})\boldsymbol{C}. \tag{A1.19}$$

A1.6 Differentiation of Vectors

When vector $\boldsymbol{A}$ changes spatially, we express it as a function of coordinates such as $\boldsymbol{A}(x, y, z)$. If position vector $\boldsymbol{r}$ corresponds to point (x, y, z), it is possible to write it as $\boldsymbol{A}(\boldsymbol{r})$. Suppose that only x changes by a small amount, Δx, with no change in y and z. The corresponding change in $\boldsymbol{A}$ is

$$\Delta \boldsymbol{A} = \boldsymbol{A}(x+\Delta x,\ y,\ z) - \boldsymbol{A}(x,\ y,\ z). \tag{A1.20}$$

If the limit of $\Delta \boldsymbol{A}/\Delta x$ exists in the limit $\Delta x \to 0$, this is called a **partial differential coefficient** and is written as

$$\frac{\partial \boldsymbol{A}}{\partial x}. \tag{A1.21}$$

If the components of $\boldsymbol{A}$ are (A_x, A_y, A_z), this coefficient is

$$\frac{\partial \boldsymbol{A}}{\partial x} = \frac{\partial A_x}{\partial x}\boldsymbol{i}_x + \frac{\partial A_y}{\partial x}\boldsymbol{i}_y + \frac{\partial A_z}{\partial x}\boldsymbol{i}_z. \tag{A1.22}$$

The partial differential coefficients with respect to y and z,

$$\frac{\partial \boldsymbol{A}}{\partial y}, \qquad \frac{\partial \boldsymbol{A}}{\partial z} \tag{A1.23}$$

are similarly defined.

When (x, y, z) changes to $(x+\Delta x, y+\Delta y, z+\Delta z)$ or $\boldsymbol{r}$ changes to $\boldsymbol{r}+\Delta \boldsymbol{r}$, the variation in $\boldsymbol{A}$ is

$$\begin{aligned}\Delta \boldsymbol{A} &= \boldsymbol{A}(x+\Delta x,\ y+\Delta y,\ z+\Delta z) - \boldsymbol{A}(x,\ y,\ z),\\ &\simeq \frac{\partial \boldsymbol{A}}{\partial x}\Delta x + \frac{\partial \boldsymbol{A}}{\partial y}\Delta y + \frac{\partial \boldsymbol{A}}{\partial z}\Delta z.\end{aligned} \tag{A1.24}$$

In the limit of small Δx, Δy and Δz, these are written as $\mathrm{d}x$, $\mathrm{d}y$ and $\mathrm{d}z$, and then, $\Delta \boldsymbol{A}$ leads to

$$\mathrm{d}\boldsymbol{A} = \frac{\partial \boldsymbol{A}}{\partial x}\mathrm{d}x + \frac{\partial \boldsymbol{A}}{\partial y}\mathrm{d}y + \frac{\partial \boldsymbol{A}}{\partial z}\mathrm{d}z. \tag{A1.25}$$

This is called **total differentiation**. If $\boldsymbol{A}$ is a function not only of (x, y, z) but also of time t, the total differentiation of $\boldsymbol{A}$ also includes $(\partial \boldsymbol{A}/\partial t)\mathrm{d}t$ on the right side of Eq. (A1.25).

If vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ are functions of scalar φ, the following relations hold:

$$\frac{\mathrm{d}}{\mathrm{d}\varphi}(\boldsymbol{A}+\boldsymbol{B}) = \frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}\varphi} + \frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}\varphi}, \tag{A1.26}$$

$$\frac{\mathrm{d}}{\mathrm{d}\varphi}(m\boldsymbol{A}) = \frac{\mathrm{d}m}{\mathrm{d}\varphi}\boldsymbol{A} + m\frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}\varphi}, \tag{A1.27}$$

$$\frac{\mathrm{d}}{\mathrm{d}\varphi}(\boldsymbol{A}\cdot\boldsymbol{B}) = \frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}\varphi}\cdot\boldsymbol{B} + \boldsymbol{A}\cdot\frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}\varphi}, \tag{A1.28}$$

$$\frac{\mathrm{d}}{\mathrm{d}\varphi}(\boldsymbol{A} \times \boldsymbol{B}) = \frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}\varphi} \times \boldsymbol{B} + \boldsymbol{A} \times \frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}\varphi}. \tag{A1.29}$$

A1.7 Gradient of a Scalar

When $f(x, y, z)$ is a given scalar function, the following vector is called the **gradient** of f:

$$\frac{\partial f}{\partial x}\boldsymbol{i}_x + \frac{\partial f}{\partial y}\boldsymbol{i}_y + \frac{\partial f}{\partial z}\boldsymbol{i}_z. \tag{A1.30}$$

This is written as grad f and the operation, grad, is also called the gradient. The function grad f is a vector that points in the direction of maximum variation with a magnitude equal to the maximum variation. If we use the operator defined by

$$\nabla = \boldsymbol{i}_x\frac{\partial}{\partial x} + \boldsymbol{i}_y\frac{\partial}{\partial y} + \boldsymbol{i}_z\frac{\partial}{\partial z}, \tag{A1.31}$$

Eq. (A1.30) is also written as

$$\mathrm{grad} f = \nabla f = \left(\boldsymbol{i}_x\frac{\partial}{\partial x} + \boldsymbol{i}_y\frac{\partial}{\partial y} + \boldsymbol{i}_z\frac{\partial}{\partial z}\right) f. \tag{A1.32}$$

The operator ∇ is called **nabla**.

If the unit vector along some direction is $\boldsymbol{s}$, the variation rate of a given function f in this direction is $\boldsymbol{s} \cdot \nabla f$. For example, the variation rate along the x-axis is

$$\boldsymbol{i}_x \cdot \nabla f = \frac{\partial f}{\partial x}. \tag{A1.33}$$

Thus, the gradient is an operator that operates on a scalar to result in a vector. An example is the relation between temperature and heat flow. The temperature is a scalar and the heat flows along the opposite direction of its gradient, i.e., from a position with a higher temperature to a position with a lower one. When the temperature and heat conductivity are T and K, the heat that flows across a unit area in unit time because of the temperature gradient is $-K\nabla T$.

A1.8 Divergence of a Vector

When $\boldsymbol{A}(x, y, z)$ is a given vector function, the following scalar is called the **divergence** of $\boldsymbol{A}$:

$$\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}. \tag{A1.34}$$

This is written as div$\boldsymbol{A}$ and the operation, div, is also called the divergence. Using the vector operator ∇, this is also written as $\nabla \cdot \boldsymbol{A}$. Namely,

$$\text{div}\boldsymbol{A} = \nabla \cdot \boldsymbol{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}. \tag{A1.35}$$

The mathematical definition of divergence will be given later in Eq. (A1.67).

Thus, the divergence is an operator that operates on a vector to result in a scalar. For example, if the velocity of a fluid is $\boldsymbol{v}$, $\nabla \cdot \boldsymbol{v}$ is the volume of the fluid that comes out through a unit area in unit time.

A1.9 Rotation of a Vector

When $\boldsymbol{A}(x, y, z)$ is a given vector function, the following vector is called the **rotation** or **curl** of $\boldsymbol{A}$:

$$\boldsymbol{i}_x \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \boldsymbol{i}_y \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \boldsymbol{i}_z \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right). \tag{A1.36}$$

This is written as rot$\boldsymbol{A}$ or curl$\boldsymbol{A}$ and the operation, rot or curl, is also called rotation or curl. Using the vector operator ∇, this is also written as $\nabla \times \boldsymbol{A}$. We can also use a determinant to express Eq. (A1.36) as

$$\begin{aligned} \text{rot}\boldsymbol{A} = \nabla \times \boldsymbol{A} &= \begin{vmatrix} \boldsymbol{i}_x & \boldsymbol{i}_y & \boldsymbol{i}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix} \\ &= \boldsymbol{i}_x \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \boldsymbol{i}_y \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \boldsymbol{i}_z \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right). \end{aligned} \tag{A1.37}$$

The mathematical definition of rotation will be given later in Eq. (A1.76).

Thus, the rotation is an operator that operates on a vector to result in a vector. We consider a rotation of a rigid body. When the rigid body rotates with an angular velocity $\boldsymbol{\omega}$ around an axis through the center of gravity, as shown in Fig. A1.7, the velocity of a point located at $\boldsymbol{r}$ is $\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r}$. Thus, we obtain $\boldsymbol{\omega} = \nabla \times \boldsymbol{v}$.

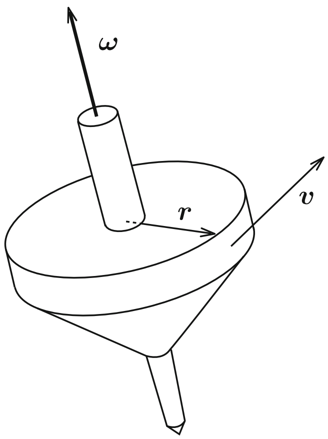

Fig. A1.7 Rotation of rigid body

A1.10 Differentiation of Products of Vectors

The following relations hold for various products:

$$\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi, \tag{A1.38}$$

$$\nabla(\boldsymbol{A}\cdot\boldsymbol{B}) = (\boldsymbol{A}\cdot\nabla)\boldsymbol{B} + (\boldsymbol{B}\cdot\nabla)\boldsymbol{A} + \boldsymbol{A}\times(\nabla\times\boldsymbol{B}) + \boldsymbol{B}\times(\nabla\times\boldsymbol{A}), \tag{A1.39}$$

$$\nabla\cdot(\phi\boldsymbol{A}) = \phi\nabla\cdot\boldsymbol{A} + \nabla\phi\cdot\boldsymbol{A}, \tag{A1.40}$$

$$\nabla\cdot(\boldsymbol{A}\times\boldsymbol{B}) = \boldsymbol{B}\cdot(\nabla\times\boldsymbol{A}) - \boldsymbol{A}\cdot(\nabla\times\boldsymbol{B}), \tag{A1.41}$$

$$\nabla\times(\phi\boldsymbol{A}) = \phi\,\nabla\times\boldsymbol{A} - \boldsymbol{A}\times\nabla\phi, \tag{A1.42}$$

$$\nabla\times(\boldsymbol{A}\times\boldsymbol{B}) = (\boldsymbol{B}\cdot\nabla)\boldsymbol{A} - (\boldsymbol{A}\cdot\nabla)\boldsymbol{B} + \boldsymbol{A}\nabla\cdot\boldsymbol{B} - \boldsymbol{B}\nabla\cdot\boldsymbol{A}. \tag{A1.43}$$

A1.11 Second Differentiation

There are three formulae for second differentiation. One of them is for an arbitrary vector $\boldsymbol{A}$:

$$\mathrm{div}(\mathrm{rot}\boldsymbol{A}) = \nabla\cdot(\nabla\times\boldsymbol{A}) = 0. \tag{A1.44}$$

The second one is for an arbitrary scalar ϕ:

$$\mathrm{rot}(\mathrm{grad}\phi) = \nabla\times\nabla\phi = 0. \tag{A1.45}$$

The last one is

$$\mathrm{rot}(\mathrm{rot}\boldsymbol{A}) = \nabla\times(\nabla\times\boldsymbol{A}) = \nabla(\nabla\cdot\boldsymbol{A}) - \nabla^2\boldsymbol{A}. \tag{A1.46}$$

In Cartesian coordinates the second term is written as

$$\nabla^2 \boldsymbol{A} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \boldsymbol{A}. \tag{A1.47}$$

Formulae for cylindrical and polar coordinates are given in Sects. A1.17 and A1.18, respectively. It should be noted that the operator ∇^2 is different between scalars and vectors:

$$\nabla^2 \phi = (\nabla \cdot \nabla)\phi = \nabla \cdot (\nabla \phi), \tag{A1.48}$$

$$\nabla^2 \boldsymbol{A} = (\nabla \cdot \nabla)\boldsymbol{A} \neq \nabla(\nabla \cdot \boldsymbol{A}). \tag{A1.49}$$

A1.12 Curvilinear Integral of a Vector

We denote the tangential component of a vector $\boldsymbol{F}(x, y, z)$ on a smooth curve, C, by $F_{\mathrm{t}}(x, y, z)$ and the elementary line vector on C by $\mathrm{d}\boldsymbol{s}$ (see Fig. A1.8). The following integral is called a **curvilinear integral**:

$$\int_{\mathrm{C}} \boldsymbol{F}(x, y, z) \cdot \mathrm{d}\boldsymbol{s} = \int_{\mathrm{C}} F_{\mathrm{t}}(x, y, z)\mathrm{d}s. \tag{A1.50}$$

If we divide this into components, it becomes

$$\int_{\mathrm{C}} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s} = \int_{\mathrm{C}} F_x \mathrm{d}x + \int_{\mathrm{C}} F_y \mathrm{d}y + \int_{\mathrm{C}} F_z \mathrm{d}z. \tag{A1.51}$$

When $\boldsymbol{F}$ is a force on a matter particle, its curvilinear integral given by Eq. (A1.50) is the work to move the particle along C.

When $\boldsymbol{F} = \nabla\phi$, its curvilinear integral is

$$\int_{\mathrm{C}} \nabla\phi \cdot \mathrm{d}\boldsymbol{s} = \int_{\mathrm{C}} \frac{\partial\phi}{\partial x}\mathrm{d}x + \int_{\mathrm{C}} \frac{\partial\phi}{\partial y}\mathrm{d}y + \int_{\mathrm{C}} \frac{\partial\phi}{\partial z}\mathrm{d}z = \int_{\mathrm{C}} \mathrm{d}\phi, \tag{A1.52}$$

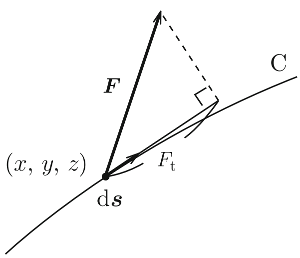

Fig. A1.8 Curvilinear integral of vector $\boldsymbol{F}$ on curve C

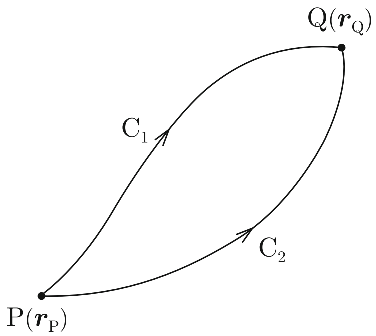

Fig. A1.9 Two integral paths connecting points P and Q

where $\mathrm{d}\phi$ is a total differential. Hence, the curvilinear integral of $\boldsymbol{F}$ from point P of position $\boldsymbol{r}_{\mathrm{P}}$ to point Q of position $\boldsymbol{r}_{\mathrm{Q}}$ along C leads to

$$\int_{\boldsymbol{r}_{\mathrm{P}}}^{\boldsymbol{r}_{\mathrm{Q}}} \nabla\phi \cdot \mathrm{d}\boldsymbol{s} = \int_{\boldsymbol{r}_{\mathrm{P}}}^{\boldsymbol{r}_{\mathrm{Q}}} \mathrm{d}\phi = \phi(\boldsymbol{r}_{\mathrm{Q}}) - \phi(\boldsymbol{r}_{\mathrm{P}}). \quad \text{(A1.53)}$$

Thus, we find that the inverse operation of the gradient is the curvilinear integral. It is obvious that the curvilinear integral of a gradient is determined only by the starting point P and terminating point Q and is independent of the integral path. If there are two integral paths C_1 and C_2 that connect P and Q, as shown in Fig. A1.9, the following relation holds:

$$\int_{\mathrm{C}_1} \nabla\phi \cdot \mathrm{d}\boldsymbol{s} = \int_{\mathrm{C}_2} \nabla\phi \cdot \mathrm{d}\boldsymbol{s} = -\int_{\mathrm{C}_2'} \nabla\phi \cdot \mathrm{d}\boldsymbol{s}, \quad \text{(A1.54)}$$

where C_2' is the integral path from Q to P along the opposite direction on path C_2. Hence, the circular integral on closed line C composed of C_1 and C_2' is

$$\oint_{\mathrm{C}} \nabla\phi \cdot \mathrm{d}\boldsymbol{s} = 0. \quad \text{(A1.55)}$$

This holds for an arbitrary closed line C.

A1.13 Surface Integral of a Vector

We denote the normal component of vector $\boldsymbol{F}(x, y, z)$ on a curved surface, S, by $F_{\mathrm{n}}(x, y, z)$ and the elementary surface vector on S by $\mathrm{d}\boldsymbol{S}$ (see Fig. A1.10). In this case the following integral is called a **surface integral**:

$$\int_{\mathrm{S}} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S} = \int_{\mathrm{S}} F_{\mathrm{n}} \mathrm{d}S. \quad \text{(A1.56)}$$

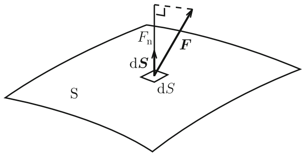

Fig. A1.10 Surface integral of vector $\boldsymbol{F}$ on curved surface S

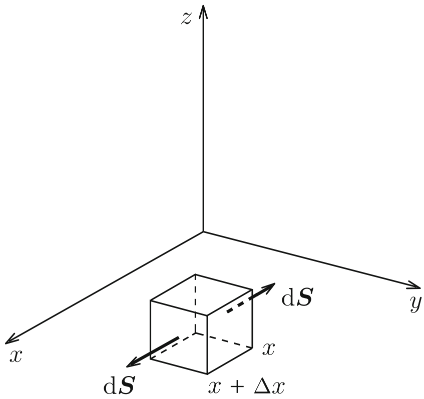

Fig. A1.11 Small parallelepiped with sides parallel to x-, y- and z-axes

When $\boldsymbol{F}$ is the velocity of a fluid and S is a cross-section, the surface integral given by Eq. (A1.56) is the amount of fluid that flows through the cross-section in unit time.

A1.14 Gauss' Theorem

We suppose a small parallelepiped in the region x to $x+\Delta x$, y to $y+\Delta y$ and z to $z+\Delta z$ of volume $\Delta V = \Delta x \Delta y \Delta z$ and integrate vector $\boldsymbol{A}$ on the surface of this region:

$$\int_{\Delta \mathrm{S}} \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{S}, \tag{A1.57}$$

where $\mathrm{d}\boldsymbol{S}$ is directed outward from this region. This integral is divided into six surface integrals. First, we treat integrals on two surfaces parallel to the y-z plane at x and $x+\Delta x$ (see Fig. A1.11). On the surface at x, $\mathrm{d}\boldsymbol{S} = -\boldsymbol{i}_x \mathrm{d}S$, and the integral on this surface is

$$-\int A_x(x, y, z)\mathrm{d}S \simeq -A_x\left(x,\ y+\frac{\Delta y}{2},\ z+\frac{\Delta z}{2}\right)\Delta y \Delta z, \tag{A1.58}$$

where we have used the mean value of A_x on this surface. On the surface at $x+\Delta x$, $\mathrm{d}\boldsymbol{S} = \boldsymbol{i}_x \mathrm{d}S$ and the integral on this surface is

$$\int A_x(x+\Delta x,\ y,\ z)\mathrm{d}S \simeq A_x\left(x+\Delta x,\ y+\frac{\Delta y}{2},\ z+\frac{\Delta z}{2}\right)\Delta y\Delta z. \tag{A1.59}$$

We expand this as

$$A_x\left(x+\Delta x,\ y+\frac{\Delta y}{2},\ z+\frac{\Delta z}{2}\right) \simeq A_x\left(x,\ y+\frac{\Delta y}{2},\ z+\frac{\Delta z}{2}\right) + \frac{\partial A_x}{\partial x}\Delta x. \tag{A1.60}$$

Then, the sum of both surface integrals yields

$$\left[A_x\left(x+\Delta x,\ y+\frac{\Delta y}{2},\ z+\frac{\Delta z}{2}\right) - A_x\left(x,\ y+\frac{\Delta y}{2},\ z+\frac{\Delta z}{2}\right)\right]\Delta y\Delta z$$
$$= \frac{\partial A_x}{\partial x}\Delta x\Delta y\Delta z = \frac{\partial A_x}{\partial x}\Delta V. \tag{A1.61}$$

We similarly obtain the contributions from the sets of two surfaces parallel to the z-x and x-y planes as

$$\frac{\partial A_y}{\partial y}\Delta V, \qquad \frac{\partial A_z}{\partial z}\Delta V. \tag{A1.62}$$

Thus, we have

$$\int_{\Delta \mathrm{S}} \boldsymbol{A}\cdot\mathrm{d}\boldsymbol{S} = \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)\Delta V = \nabla\cdot\boldsymbol{A}\,\Delta V = \int_{\Delta \mathrm{V}} \nabla\cdot\boldsymbol{A}\ \mathrm{d}V, \tag{A1.63}$$

where ΔV is the region surrounded by ΔS.

Here we divide region V surrounded by closed surface S into a set of small regions $\{\Delta \mathrm{V}_i\}$ and denote the surface of the i-th region $\Delta \mathrm{V}_i$ as $\Delta \mathrm{S}_i$. The surface integral of $\boldsymbol{A}$ on S is

$$\int_{\mathrm{S}} \boldsymbol{A}\cdot\mathrm{d}\boldsymbol{S} = \sum_i \int_{\Delta \mathrm{S}_i} \boldsymbol{A}\cdot\mathrm{d}\boldsymbol{S}, \tag{A1.64}$$

since the surface integrals on a common surface between two adjacent regions cancel each other because $\mathrm{d}\boldsymbol{S}$ has an opposite direction in each integral (see Fig. A1.12). On the other hand, using Eq. (A1.63), this is equal to

$$\sum_i \int_{\Delta \mathrm{V}_i} \nabla\cdot\boldsymbol{A}\ \mathrm{d}V = \int_{\mathrm{V}} \nabla\cdot\boldsymbol{A}\ \mathrm{d}V. \tag{A1.65}$$

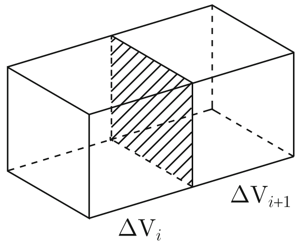

Fig. A1.12 Two adjacent regions with common surface

Thus, we have

$$\int_S \boldsymbol{A}\cdot \mathrm{d}\boldsymbol{S} = \int_V \nabla\cdot\boldsymbol{A}\,\mathrm{d}V. \tag{A1.66}$$

This is called **Gauss' theorem**.

In the limit $\Delta V \to 0$ in Eq. (A1.63), the right side leads to $\Delta V \nabla\cdot\boldsymbol{A}$. Hence, we obtain the relationship,

$$\lim_{\Delta V\to 0} \frac{1}{\Delta V}\int_{\Delta S} \boldsymbol{A}\cdot \mathrm{d}\boldsymbol{S} = \nabla\cdot\boldsymbol{A}. \tag{A1.67}$$

This gives the definition of divergence.

A1.15 Stokes' Theorem

We consider a small rectangle in the region y to $y + \Delta y$ and z to $z + \Delta z$ on the y-z plane. We integrate vector $\boldsymbol{A}$ along this small rectangle. The integral path is directed counterclockwise, as shown in Fig. A1.13, so that a screw points in the direction of the x-axis when we rotate a screwdriver along the direction of the integral. On the path from P to Q, $\mathrm{d}\boldsymbol{s} = \boldsymbol{i}_y \mathrm{d}y$, and the contribution from this region to the integral is $A_y(x, y + (1/2)\Delta y, z)\Delta y$ using the mean value. On the path from R to S, $\mathrm{d}\boldsymbol{s} = -\boldsymbol{i}_y \mathrm{d}y$, and the contribution from this region is $-A_y(x, y + (1/2)\Delta y, z + \Delta z)\Delta y$. Their sum is

$$-\left[A_y\left(x,\ y+\frac{\Delta y}{2},\ z+\Delta z\right) - A_y\left(x,\ y+\frac{\Delta y}{2},\ z\right)\right]\Delta y$$

$$\simeq -\frac{\partial A_y}{\partial z}\Delta y\Delta z = -\frac{\partial A_y}{\partial z}\Delta S, \tag{A1.68}$$

where $\Delta S = \Delta y \Delta z$ is the area of the small rectangle.

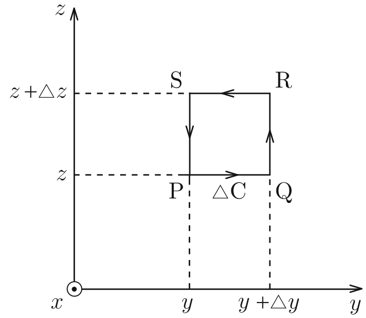

Fig. A1.13 Small rectangle with sides parallel to y- and z-axes

On the path from Q to R, $\mathrm{d}\boldsymbol{s} = \boldsymbol{i}_z \mathrm{d}z$ with a contribution of $A_z(x, y+\Delta y, z+(1/2)\Delta z)\Delta z$, and on the path from S to P, $\mathrm{d}\boldsymbol{s} = -\boldsymbol{i}_z \mathrm{d}z$ with a contribution of $-A_z(x, y, z+(1/2)\Delta z)\Delta z$. Their sum is

$$\left[A_z\left(x,\ y+\Delta y,\ z+\frac{\Delta z}{2}\right) - A_z\left(x,\ y,\ z+\frac{\Delta z}{2}\right)\right]\Delta z \simeq \frac{\partial A_z}{\partial y}\Delta S. \quad \text{(A1.69)}$$

Thus, the curvilinear integral on ΔC is

$$\oint_{\Delta \mathrm{C}} \boldsymbol{A}\cdot\mathrm{d}\boldsymbol{s} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\Delta S = (\nabla\times\boldsymbol{A})_x \Delta S = \int_{\Delta \mathrm{S}} (\nabla\times\boldsymbol{A})_x \mathrm{d}S_x. \quad \text{(A1.70)}$$

We consider a curvilinear integral of $\boldsymbol{A}$ on an arbitrary closed line C. The surface surrounded by C is divided into a set of infinitesimal rectangular regions normal to each of the x- y- and z-axes (see Fig. A1.14). We can show that the curvilinear integral on C is equal to the sum of the curvilinear integrals of all rectangular regions. That is,

$$\oint_{\mathrm{C}} \boldsymbol{A}\cdot\mathrm{d}\boldsymbol{s} = \sum_i \oint_{\Delta \mathrm{C}_i} \boldsymbol{A}\cdot\mathrm{d}\boldsymbol{s}. \quad \text{(A1.71)}$$

This is because every curvilinear integral on the common side of two adjacent regions cancel out. Using Eq. (A1.70), the right side of Eq. (A1.71) leads to

$$\sum_i \int_{\Delta \mathrm{S}_i} (\nabla\times\boldsymbol{A})\cdot\mathrm{d}\boldsymbol{S} = \int_{\mathrm{S}} (\nabla\times\boldsymbol{A})\cdot\mathrm{d}\boldsymbol{S}, \quad \text{(A1.72)}$$

where $\Delta \mathrm{S}_i$ is the surface surrounded by $\Delta \mathrm{C}_i$, and S is the curved surface surrounded by C. Thus, we have

$$\oint_{\mathrm{C}} \boldsymbol{A}\cdot\mathrm{d}\boldsymbol{s} = \int_{\mathrm{S}} (\nabla\times\boldsymbol{A})\cdot\mathrm{d}\boldsymbol{S}. \quad \text{(A1.73)}$$

This is called **Stokes' theorem**.

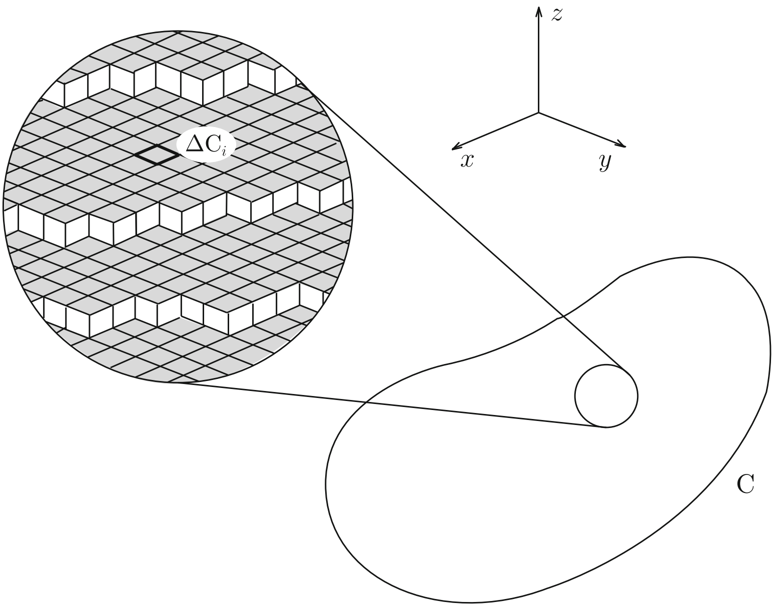

Fig. A1.14 Closed line C and small rectangular segments $\{\Delta C_i\}$

In the limit $\Delta S \to 0$ in Eq. (A1.70), its right side leads to $(\nabla \times \boldsymbol{A})_x \Delta S$. Hence, if the unit vector normal to the small surface is $\boldsymbol{n}$, we have

$$\lim_{\Delta S \to 0} \frac{1}{\Delta S} \oint_{\Delta C} \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{s} = (\nabla \times \boldsymbol{A}) \cdot \boldsymbol{n}. \tag{A1.74}$$

This gives the definition of rotation.

A1.16 Green's Theorem

Substituting $\boldsymbol{A} = \nabla \psi$ into Eq. (A1.40) leads to

$$\nabla \cdot (\phi \nabla \psi) = \phi \nabla^2 \psi + (\nabla \phi) \cdot (\nabla \psi). \tag{A1.75}$$

Integrating this over region V and transforming the left side into a surface integral using Gauss' theorem, we have

$$\int_{\mathrm{S}} \phi \nabla \psi \cdot \mathrm{d}\boldsymbol{S} = \int_{\mathrm{V}} \phi \nabla^2 \psi \, \mathrm{d}V + \int_{\mathrm{V}} (\nabla \phi) \cdot (\nabla \psi) \mathrm{d}V. \tag{A1.76}$$

Then, the subtraction between this and the quantity in which ϕ and ψ are exchanged yields:

$$\int_S (\phi \nabla \psi - \psi \nabla \phi) \cdot \mathrm{d}\boldsymbol{S} = \int_V (\phi \nabla^2 \psi - \psi \nabla^2 \phi) \mathrm{d}V. \tag{A1.77}$$

Equations (A1.76) and (A1.77) are called **Green's theorem**.

A1.17 Cylindrical Coordinates

In Sect. A1.4, we used Cartesian coordinates. However, it is convenient to use **cylindrical coordinates** when we calculate electromagnetic properties for long cylindrical objects. When we use cylindrical coordinates, we first define the central axis (the z-axis). Then, we express the target position with the distance from this axis (R), the azimuthal angle (φ) on the plane normal to the z-axis and the position on this axis (z): (R, φ, z). When we use the common z-axis with Cartesian coordinates, as shown in Fig. A1.15, the relationships between the two sets of coordinates are:

$$R = (x^2 + y^2)^{1/2}, \qquad \varphi = \tan^{-1} \frac{y}{x}, \qquad z = z. \tag{A1.78}$$

If we use $\boldsymbol{i}_R$, $\boldsymbol{i}_\varphi$ and $\boldsymbol{i}_z$ to denote the unit vectors along the radial, azimuthal and z-axial directions, respectively, these are perpendicular to each other and follow the right-hand rule in the order $\boldsymbol{i}_R \to \boldsymbol{i}_\varphi \to \boldsymbol{i}_z \to \boldsymbol{i}_R$.

The gradient, divergence and rotation in cylindrical coordinates are

$$\nabla f = \boldsymbol{i}_R \frac{\partial f}{\partial R} + \boldsymbol{i}_\varphi \frac{1}{R} \cdot \frac{\partial f}{\partial \varphi} + \boldsymbol{i}_z \frac{\partial f}{\partial z}, \tag{A1.79}$$

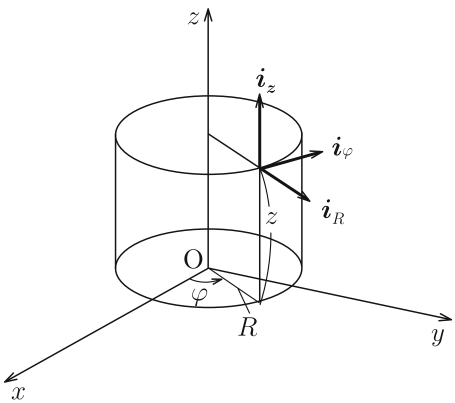

Fig. A1.15 Cylindrical coordinates

$$\nabla \cdot \boldsymbol{A} = \frac{1}{R} \cdot \frac{\partial (R A_R)}{\partial R} + \frac{1}{R} \cdot \frac{\partial A_\varphi}{\partial \varphi} + \frac{\partial A_z}{\partial z}, \quad \text{(A1.80)}$$

$$\nabla \times \boldsymbol{A} = \boldsymbol{i}_R \left(\frac{1}{R} \cdot \frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z} \right) + \boldsymbol{i}_\varphi \left(\frac{\partial A_R}{\partial z} - \frac{\partial A_z}{\partial R} \right) + \boldsymbol{i}_z \frac{1}{R} \left[\frac{\partial (R A_\varphi)}{\partial R} - \frac{\partial A_R}{\partial \varphi} \right]. \quad \text{(A1.81)}$$

The second differentiation of a scalar function is given by

$$\nabla^2 f = \frac{1}{R} \cdot \frac{\partial}{\partial R} \left(R \frac{\partial f}{\partial R} \right) + \frac{1}{R^2} \cdot \frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2}$$
$$= \frac{\partial^2 f}{\partial R^2} + \frac{1}{R} \cdot \frac{\partial f}{\partial R} + \frac{1}{R^2} \cdot \frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2}. \quad \text{(A1.82)}$$

A1.18 Polar Coordinates

It is convenient to use **polar coordinates** when we calculate electromagnetic properties for spherical objects. When we use polar coordinates, we first define the center with an axis that determines the two poles. Then, we express the target position with the distance from the center (r), the zenithal angle (θ) measured from the north pole and the azimuthal angle (φ) on the plane normal to the axis: (r, θ, φ). When we use the common center and z-axis with Cartesian coordinates, as shown in Fig. A1.16, the relationships between the two sets of coordinates are:

$$r = (x^2 + y^2 + z^2)^{1/2}, \qquad \theta = \tan^{-1} \frac{(x^2 + y^2)^{1/2}}{z}, \qquad \varphi = \tan^{-1} \frac{y}{x}. \quad \text{(A1.83)}$$

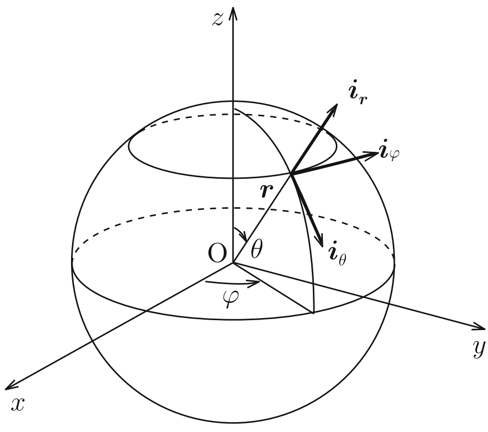

Fig. A1.16 Polar coordinates

If we use $\boldsymbol{i}_r$, $\boldsymbol{i}_\theta$ and $\boldsymbol{i}_\varphi$ to denote the unit vectors along the radial, zenithal and azimuthal directions, respectively, these are perpendicular to each other and follow the right-hand rule in the order $\boldsymbol{i}_r \to \boldsymbol{i}_\theta \to \boldsymbol{i}_\varphi \to \boldsymbol{i}_r$.

The gradient, divergence and rotation in polar coordinates are

$$\nabla f = \boldsymbol{i}_r \frac{\partial f}{\partial r} + \boldsymbol{i}_\theta \frac{1}{r} \cdot \frac{\partial f}{\partial \theta} + \boldsymbol{i}_\varphi \frac{1}{r \sin\theta} \cdot \frac{\partial f}{\partial \varphi}, \tag{A1.84}$$

$$\nabla \cdot \boldsymbol{A} = \frac{1}{r^2} \cdot \frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r \sin\theta} \cdot \frac{\partial}{\partial \theta}(\sin\theta A_\theta) + \frac{1}{r \sin\theta} \cdot \frac{\partial A_\varphi}{\partial \varphi}, \tag{A1.85}$$

$$\begin{aligned}\nabla \times \boldsymbol{A} &= \boldsymbol{i}_r \frac{1}{r \sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta A_\varphi) - \frac{\partial A_\theta}{\partial \varphi}\right] + \boldsymbol{i}_\theta \frac{1}{r}\left[\frac{1}{\sin\theta} \cdot \frac{\partial A_r}{\partial \varphi} - \frac{\partial}{\partial r}(r A_\varphi)\right] \\ &\quad + \boldsymbol{i}_\varphi \frac{1}{r}\left[\frac{\partial}{\partial r}(r A_\theta) - \frac{\partial A_r}{\partial \theta}\right].\end{aligned} \tag{A1.86}$$

The second differentiation of a scalar function is given by

$$\begin{aligned}\nabla^2 f &= \frac{1}{r^2} \cdot \frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta} \cdot \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta} \cdot \frac{\partial^2 f}{\partial \varphi^2} \\ &= \frac{\partial^2 f}{\partial r^2} + \frac{2}{r} \cdot \frac{\partial f}{\partial r} + \frac{1}{r^2} \cdot \frac{\partial^2 f}{\partial \theta^2} + \frac{\cos\theta}{r^2 \sin\theta} \cdot \frac{\partial f}{\partial \theta} + \frac{1}{r^2 \sin^2\theta} \cdot \frac{\partial^2 f}{\partial \varphi^2}.\end{aligned} \tag{A1.87}$$

A2 Proofs

A2.1 Proof of Eq. (1.37)

For the electric potential given by Eq. (1.27) we have

$$\Delta \phi(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0} \int_{\mathrm{V}} \left(\Delta \frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|}\right) \rho(\boldsymbol{r}') \mathrm{d}V'.$$

Here we put $\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r}'$ with $R = |\boldsymbol{R}|$ and define Δ_R as the Laplacian with respect to $\boldsymbol{R}$. Then, applying the formula might suggest

$$\Delta \frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} = \Delta_R \frac{1}{R} = \frac{1}{R^2} \cdot \frac{\partial}{\partial R}\left(R^2 \frac{\partial}{\partial R} \cdot \frac{1}{R}\right) = 0.$$

However, the above is not valid in the vicinity of $\boldsymbol{R} = 0$ ($\boldsymbol{r} = \boldsymbol{r}'$), since this position is an abnormal point at which the function $1/|\boldsymbol{r} - \boldsymbol{r}'|$ diverges. Hence, the integral has a finite contribution in the vicinity of this abnormal point. We denote the surface and volume of a small region around the abnormal point as ΔS and ΔV, respectively. Rewriting $\Delta\phi$ with the definition of divergence, Eq. (A1.67), we have

$$\Delta\phi(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\nabla\cdot\left(\nabla\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\right)\rho(\boldsymbol{r})\,\Delta V = \lim_{\Delta V\to 0}\frac{\rho(\boldsymbol{r})}{4\pi\epsilon_0}\int_{\Delta S}\nabla\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\cdot\mathrm{d}\boldsymbol{S}.$$

Here we assume a spherical surface of radius a with the center at $\boldsymbol{r}'$ for small ΔS; then,

$$\int_{\Delta S}\nabla\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\cdot\mathrm{d}\boldsymbol{S} = -\int_{\Delta S}\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\cdot\mathrm{d}\boldsymbol{S} = -\frac{4\pi a^2}{a^2} = -4\pi.$$

Thus, we obtain Eq. (1.37),

$$\Delta\phi(\boldsymbol{r}) = -\frac{\rho(\boldsymbol{r})}{\epsilon_0}.$$

We find that the Laplacian of $1/|\boldsymbol{r}-\boldsymbol{r}'|$ can be expressed in terms of the three-dimensional delta function as

$$\Delta\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} = -4\pi\delta(\boldsymbol{r}-\boldsymbol{r}').$$

A2.2 Proof of Eq. (6.21)

Since the divergence is a differentiation with respect to $\boldsymbol{r}$ for the magnetic flux density in Eq. (6.7), we have

$$\nabla\cdot\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int_{V}\nabla\cdot\left[\frac{\boldsymbol{i}(\boldsymbol{r}')\times(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\right]\mathrm{d}V'.$$

Using Eq. (A1.41) with

$$\boldsymbol{A}\to\boldsymbol{i}(\boldsymbol{r}'),\quad \boldsymbol{B}\to\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3},$$

the integrand in the above equation leads to

$$\nabla\cdot\left[\frac{\boldsymbol{i}(\boldsymbol{r}')\times(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\right] = -\boldsymbol{i}(\boldsymbol{r}')\cdot\left(\nabla\times\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\right),$$

where we have used $\nabla\times\boldsymbol{i}(\boldsymbol{r}') = 0$. If we use

$$\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3} = -\nabla\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}$$

and Eq. (A1.45), we obtain Eq. (6.21),

$$\nabla \cdot \boldsymbol{B} = 0.$$

A2.3 Proof of Eq. (6.27)

The rotation of Eq. (6.7) is

$$\nabla \times \boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \int_{\mathrm{V}} \nabla \times \left[\frac{\boldsymbol{i}(\boldsymbol{r}') \times (\boldsymbol{r} - \boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|^3} \right] \mathrm{d}V'.$$

Substituting

$$\boldsymbol{A} \to \boldsymbol{i}(\boldsymbol{r}'), \qquad \boldsymbol{B} \to \frac{\boldsymbol{r} - \boldsymbol{r}'}{|\boldsymbol{r} - \boldsymbol{r}'|^3} = -\nabla \frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|}$$

into Eq. (A1.43) yields

$$\nabla \times \left[\frac{\boldsymbol{i}(\boldsymbol{r}') \times (\boldsymbol{r} - \boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|^3} \right] = -\boldsymbol{i}(\boldsymbol{r}') \nabla \cdot \left(\nabla \frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} \right) + [\boldsymbol{i}(\boldsymbol{r}') \cdot \nabla] \nabla \frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|}.$$

Here we denote the differential operator with respect to $\boldsymbol{r}'$ by ∇'. Then,

$$\nabla \frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} = -\nabla' \frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|}.$$

That is, ∇ is equivalent to $-\nabla'$. Using this relationship in part of the above equation, we have

$$\nabla \times \left[\frac{\boldsymbol{i}(\boldsymbol{r}') \times (\boldsymbol{r} - \boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|^3} \right] = -\boldsymbol{i}(\boldsymbol{r}') \nabla'^2 \frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} - [\boldsymbol{i}(\boldsymbol{r}') \cdot \nabla'] \nabla \frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|}.$$

Here we use the relationship shown in Sect. A2.1:

$$-\nabla'^2 \frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} = 4\pi \delta(\boldsymbol{r}' - \boldsymbol{r}).$$

Then, the integral of the first term on the right side leads to

$$-\int_{\mathrm{V}} \boldsymbol{i}(\boldsymbol{r}') \nabla'^2 \frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} \mathrm{d}V' = 4\pi \boldsymbol{i}(\boldsymbol{r}).$$

Changing the order of differentials in the second term, we have

$$-[\boldsymbol{i}(\boldsymbol{r}')\cdot\nabla']\nabla\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} = -\nabla[\boldsymbol{i}(\boldsymbol{r}')\cdot\nabla']\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} = -\nabla\left(\boldsymbol{i}(\boldsymbol{r}')\cdot\nabla'\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\right).$$

The scalar function operated on by ∇ is written as

$$\boldsymbol{i}(\boldsymbol{r}')\cdot\nabla'\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} = \nabla'\cdot\frac{\boldsymbol{i}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|} - \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\nabla'\cdot\boldsymbol{i}(\boldsymbol{r}').$$

The condition of a steady current gives $\nabla'\cdot\boldsymbol{i}(\boldsymbol{r}') = 0$. Hence, the integral of the second term becomes

$$-\nabla\int_{\mathrm{V}}\nabla'\cdot\frac{\boldsymbol{i}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}V' = -\nabla\int_{\mathrm{S}}\frac{\boldsymbol{i}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\cdot\mathrm{d}\boldsymbol{S}'.$$

If we suppose an infinitely large sphere for S, $\boldsymbol{i}$ decreases to zero on S. Thus, the relationship $\nabla\times\boldsymbol{B} = \mu_0\boldsymbol{i}$, Eq. (6.27), is valid.

A2.4 Proof of Eq. (6.33)

The rotation of Eq. (6.33) is

$$\nabla\times\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int_{\mathrm{V}}\nabla\times\frac{\boldsymbol{i}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}V'.$$

Noting that the rotation is a derivative with respect to $\boldsymbol{r}$, Eq. (A1.42) leads to

$$\nabla\times\frac{\boldsymbol{i}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|} = -\boldsymbol{i}(\boldsymbol{r}')\times\nabla\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} = \boldsymbol{i}(\boldsymbol{r}')\times\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3}.$$

Thus, the above equation is written as

$$\nabla\times\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int_{\mathrm{V}}\frac{\boldsymbol{i}(\boldsymbol{r}')\times(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\mathrm{d}V'.$$

Since this agrees with the right side of Eq. (6.7), Eq. (6.33) is valid.

A2.5 Proof of Eq. (6.45)

For Eq. (6.45) to hold, only the following equation has to be satisfied:

$$\nabla\times\left(\frac{\boldsymbol{m}\times\boldsymbol{r}}{r^3}\right) = -\nabla\left(\frac{\boldsymbol{m}\cdot\boldsymbol{r}}{r^3}\right).$$

Here we put

$$\boldsymbol{a} = \frac{\boldsymbol{r}}{r^3} = -\nabla\frac{1}{r}.$$

Then, from Eq. (A1.43) we have

$$\nabla \times (\boldsymbol{m} \times \boldsymbol{a}) = (\boldsymbol{a}\cdot\nabla)\boldsymbol{m} - (\boldsymbol{m}\cdot\nabla)\boldsymbol{a} + \boldsymbol{m}(\nabla\cdot\boldsymbol{a}) - \boldsymbol{a}(\nabla\cdot\boldsymbol{m}).$$

The first and fourth differentiation terms of the constant vector $\boldsymbol{m}$ are zero. Substituting $\boldsymbol{a} = (1/r^2)\boldsymbol{i}_r$ leads to

$$\nabla\cdot\boldsymbol{a} = \frac{1}{r^2}\cdot\frac{\partial}{\partial r}\left(r^2\frac{1}{r^2}\right) = 0.$$

Thus, only the third term remains. On the other hand, Eq. (A1.39) gives

$$\nabla(\boldsymbol{m}\cdot\boldsymbol{a}) = (\boldsymbol{m}\cdot\nabla)\boldsymbol{a} + (\boldsymbol{a}\cdot\nabla)\boldsymbol{m} + \boldsymbol{m} \times (\nabla \times \boldsymbol{a}) + \boldsymbol{a} \times (\nabla \times \boldsymbol{m}).$$

The second and fourth differentiation terms of the constant vector $\boldsymbol{m}$ are zero. We can easily show that $\nabla \times \boldsymbol{a} = 0$ in the third term from Eq. (A1.45). Thus, only the first term remains. As a result, the target equation is valid and we prove Eq. (6.45).

A2.6 Proof of Eq. (9.5)

Using the relationship,

$$\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3} = -\nabla\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} = \nabla'\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|},$$

on the right side of Eq. (9.4), this equation is written as

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int_{\mathrm{V}} \boldsymbol{M}(\boldsymbol{r}') \times \nabla'\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\,\mathrm{d}V'.$$

If we put $\phi = 1/|\boldsymbol{r}-\boldsymbol{r}'|$ in Eq. (A1.42), this leads to

$$\boldsymbol{M} \times \nabla'\phi = \phi(\nabla' \times \boldsymbol{M}) - \nabla' \times (\phi\boldsymbol{M}).$$

Thus, Eq. (9.4) becomes

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int_{\mathrm{V}} \frac{\nabla' \times \boldsymbol{M}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\,\mathrm{d}V' - \frac{\mu_0}{4\pi}\int_{\mathrm{V}} \nabla' \times \frac{\boldsymbol{M}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\,\mathrm{d}V'.$$

Equation (A1.41) becomes

$$\nabla\cdot(\boldsymbol{a}\times\boldsymbol{b}) = \boldsymbol{b}\cdot(\nabla\times\boldsymbol{a}) - \boldsymbol{a}\cdot(\nabla\times\boldsymbol{b}) = \boldsymbol{b}\cdot(\nabla\times\boldsymbol{a})$$

for a constant vector $\boldsymbol{b}$. The volume integral on the left side leads to $\int_{\mathrm{V}}\nabla\cdot(\boldsymbol{a}\times\boldsymbol{b})\mathrm{d}V = \int_{\mathrm{S}}(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{n}\mathrm{d}S$, where $\boldsymbol{n}$ is the unit vector normal to the surface denoted by S. Since vector $\boldsymbol{b}$ is constant, the above surface integral reduces to $\int_{\mathrm{S}}\boldsymbol{b}\cdot(\boldsymbol{n}\times\boldsymbol{a})\mathrm{d}S = \boldsymbol{b}\cdot\int_{\mathrm{S}}\boldsymbol{n}\times\boldsymbol{a}\,\mathrm{d}S$. On the other hand, the volume integral on the right side leads to $\int_{\mathrm{V}}\boldsymbol{b}\cdot(\nabla\times\boldsymbol{a})\mathrm{d}V = \boldsymbol{b}\cdot\int_{\mathrm{V}}\nabla\times\boldsymbol{a}\,\mathrm{d}V$. Since $\boldsymbol{b}$ is an arbitrary vector, we obtain the general relationship,

$$\int_{\mathrm{V}}\nabla\times\boldsymbol{a}\,\mathrm{d}V = \int_{\mathrm{S}}\boldsymbol{n}\times\boldsymbol{a}\,\mathrm{d}S.$$

Using this relationship, the second integral of the vector potential leads to

$$-\frac{\mu_0}{4\pi}\int_{\mathrm{S}}\boldsymbol{n}'\times\frac{\boldsymbol{M}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\,\mathrm{d}S'.$$

If we assume an infinitely large sphere for S ($|\boldsymbol{r}'| = r' \to \infty$), $|\boldsymbol{M}|$ and $|\boldsymbol{r}-\boldsymbol{r}'|^{-1}$ are of the orders of $1/r'^2$ and $1/r'$, and the surface integral is of the order of r'^2 in magnitude. As a result, this surface integral is of the order of $1/r'$, and we can neglect it. Thus, the vector potential is

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int_{\mathrm{V}}\frac{\nabla'\times\boldsymbol{M}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\,\mathrm{d}V',$$

and we derive Eq. (9.5).

A3 Superconductivity

A3.1 Phenomenological Electromagnetism

Here we introduce the phenomenological London theory that describes the electromagnetic phenomenon associated with the Meissner–Ochsenfeld effect in superconductors. We denote the mass, electric charge and velocity of a superconducting electron by m^*, $-e^*$ and $\boldsymbol{v}_{\mathrm{s}}$, respectively. The theory assumes the equation of motion of the superconducting electron:

$$m^*\frac{\mathrm{d}\boldsymbol{v}_{\mathrm{s}}}{\mathrm{d}t} = -e^*\boldsymbol{E}.$$

The right side is the Coulomb force. The viscous force in Eq. (5.21) does not act on superconducting electrons. The above equation requires a superconducting current to flow without decaying in a steady state where there is no electric field. It is known that the superconducting electron is a pair of two electrons, thus $e^* = 2e$. If we denote the density of superconducting electrons by n_s, from Eq. (5.24) the superconducting current density is

$$\boldsymbol{i} = -e^* n_\mathrm{s} \boldsymbol{v}_\mathrm{s}.$$

Eliminating $\boldsymbol{v}_\mathrm{s}$ using this equation yields

$$\boldsymbol{E} = \frac{m^*}{n_\mathrm{s} e^{*2}} \cdot \frac{\mathrm{d}\boldsymbol{i}}{\mathrm{d}t}. \tag{A3.1}$$

Taking a rotation of this equation and using Eqs. (10.39) and (6.27) for $\nabla \times \boldsymbol{E}$ and $\boldsymbol{i}$, we have

$$\frac{\partial}{\partial t}\left(\boldsymbol{B} + \frac{m^*}{\mu_0 n_\mathrm{s} e^{*2}} \nabla \times \nabla \times \boldsymbol{B}\right) = 0.$$

Hence, we find that the quantity in the parentheses is a constant value. When it is zero, the Meissner–Ochsenfeld effect can be explained. Namely, it leads to

$$\boldsymbol{B} + \lambda^2 \nabla \times \nabla \times \boldsymbol{B} = 0, \tag{A3.2}$$

where

$$\lambda = \left(\frac{m^*}{\mu_0 n_\mathrm{s} e^{*2}}\right)^{1/2}$$

is a quantity with the dimension of length called the **penetration depth** of the magnetic field. Equations (A3.1) and (A3.2) are called the **London equations**. Since $\nabla \cdot \boldsymbol{B} = 0$, using Eq. (A1.46) reduces Eq. (A3.2) to

$$\nabla^2 \boldsymbol{B} - \frac{1}{\lambda^2} \boldsymbol{B} = 0. \tag{A3.3}$$

Here we show that Eq. (A3.3) describes the Meissner–Ochsenfeld effect. Suppose a semi-infinite superconductor that occupies the region $x \geq 0$ with the surface at $x = 0$. We apply an external magnetic field of magnetic flux density B_0 parallel to the z-axis. In this case it is reasonable to assume that the internal magnetic flux density has only the z-component and is uniform in the y-z plane. Thus, Eq. (A3.3) reduces to

$$\frac{\mathrm{d}^2 B}{\mathrm{d}x^2} - \frac{1}{\lambda^2} B = 0.$$

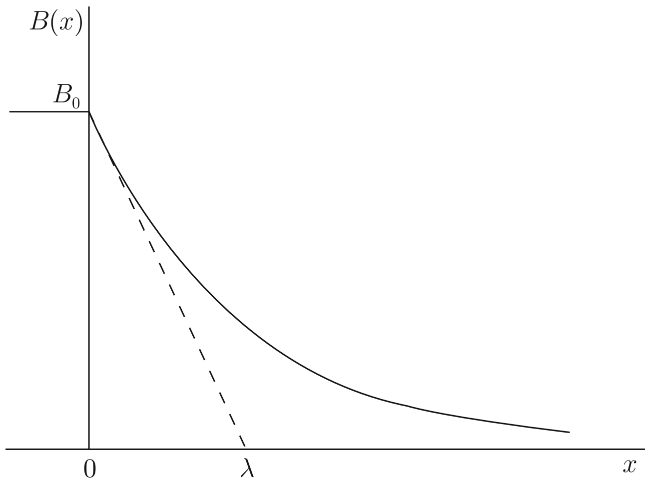

Fig. A3.1 Distribution of magnetic flux density in the vicinity of the superconductor surface

From the boundary conditions that $B(0) = B_0$ and B should be finite at infinity ($x \to \infty$), we have

$$B(x) = B_0 \exp\left(-\frac{x}{\lambda}\right).$$

This shows that the magnetic flux penetrates from the surface to a depth of about λ (see Fig. A3.1). For this reason λ is called the penetration depth. Usually λ takes a value of the order of tens of nm and can be neglected in comparison with the specimen size, and hence, the Meissner–Ochsenfeld effect can be explained. From Eq. (6.27) a current of density

$$i(x) = \frac{B_0}{\mu_0 \lambda} \exp\left(-\frac{x}{\lambda}\right).$$

flows along the y-axis. That is, the diamagnetism in the superconductor is caused by the current flowing on the surface, and this current is called the **Meissner current**. If we regard this as a real surface current, the current that flows within a unit width along the z-axis is given by

$$\tau = \int_0^\infty i(x)\mathrm{d}x = \frac{B_0}{\mu_0},$$

which satisfies Eq. (7.8).

As was just stated, the London equation is assumed to explain the Meissner–Ochsenfeld effect. It is easy to show that Eqs. (A3.1) and (A3.2) are derived from the following equation:

$$\boldsymbol{i} = -\frac{n_s e^{*2}}{m^*}\boldsymbol{A}. \tag{A3.4}$$

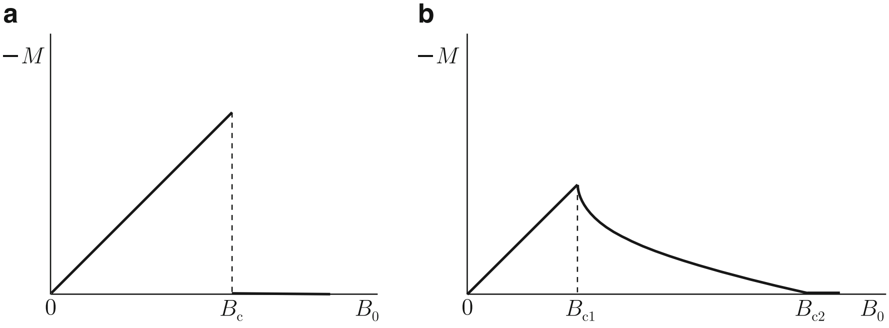

Fig. A3.2 Magnetization of (**a**) type 1 and (**b**) type 2 superconductors

Namely, we obtain Eq. (A3.1) by differentiating this equation with respect to time using the fact that there is no electrostatic field ($-\nabla\phi = 0$) in the superconductor. The rotation of Eq. (A3.4) directly derives Eq. (A3.2). Equation (A3.4) is derived from the rigorous Ginzburg–Landau theory, and hence, we are justified in assuming the London equations.

We can also derive the London equation, (A3.3), by minimizing a suitable energy. This is given by

$$\frac{1}{2\mu_0}\boldsymbol{B}^2 + \frac{\lambda^2}{2\mu_0}(\nabla \times \boldsymbol{B})^2. \tag{A3.5}$$

The first term is the magnetic energy and the second term is the kinetic energy of superconducting electrons.

A3.2 Mixed State

There are two kinds of superconductors, i.e., type 1 and type 2 superconductors. When there is no geometrical effect as in the case of a long slab superconductor in a parallel magnetic flux density, the magnetizations of these superconductors are like those shown in Fig. A3.2a, b. Most superconducting elements are classified into type 1 in (a). In this case when the external magnetic flux density B_0 is small, the superconductor is in the Meissner state, a perfect diamagnetic state. When B_0 exceeds the critical value B_c, the superconductivity disappears with a jump in magnetization to zero, and the superconductor enters in the normal state with an electric resistivity. We call B_c the critical magnetic flux density, or $B_c/\mu_0 = H_c$ is called the critical magnetic field.

Alloy or compound superconductors, including practical superconductors, are classified into type 2 shown in Fig. A3.2b. When B_0 is below B_{c1} the superconductor is in the Meissner state, but when B_0 exceeds B_{c1} the superconductor enters an

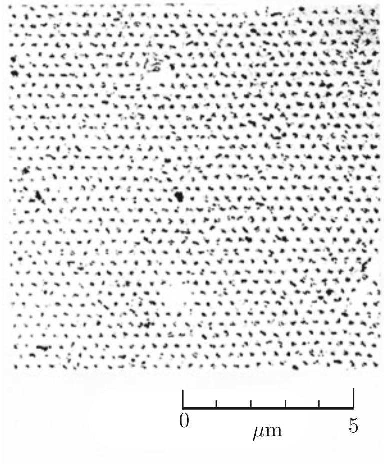

Fig. A3.3 Micrograph of quantized magnetic fluxes in superconducting Pb–Tl (courtesy of Dr. B. Obst at the Research Center in Karlsruhe). *Black dots* are ferromagnetic particles attached to the central part of each quantized magnetic flux

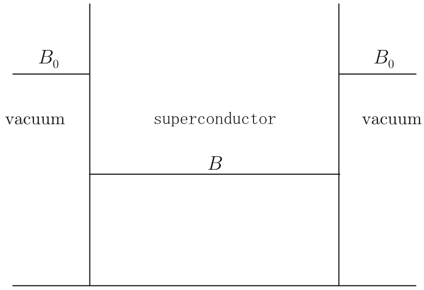

Fig. A3.4 Magnetic flux distribution in type 2 superconductor in the mixed state

imperfectly diamagnetic state called the **mixed state** with a penetration of magnetic flux. When B_0 exceeds B_{c2} the superconductor enters the normal state with zero magnetization. The quantities B_{c1} and B_{c2} are called the lower and upper critical magnetic flux density.

In the mixed state the magnetic flux is quantized as shown in Fig. A3.3, and each has a magnetic quantum of

$$\phi_0 = \frac{h_P}{2e} = 2.0678 \times 10^{-15}\ \mathrm{Wb},$$

where h_P is Planck's constant. The central part of each quantized magnetic flux is in the normal state and the magnetic flux is concentrated in the region about λ from the center. Hence, the circular current flows stably around the center, and the quantized magnetic flux is also called a vortex.

The structure of each quantized magnetic flux is much smaller than the size of a superconductor specimen, and the internal magnetic flux density can be regarded as uniform, as schematically shown in Fig. A3.4. If this magnetic flux density is B, the magnetization of the superconductor is given by Eq. (7.38).

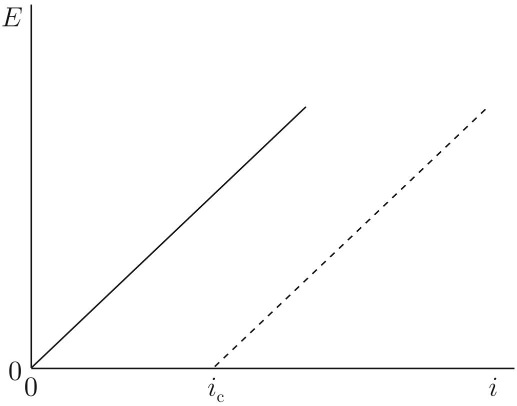

Fig. A3.5 Relationship between current density and induced electric field in superconductor. The *solid* and *dashed lines* show the characteristics for the superconductor without and with pinning centers

A3.3 Motion of Quantized Magnetic Flux

In operating conditions of superconducting equipment, the superconductors are generally in the mixed state with penetration of quantized magnetic fluxes. Thus, the Lorentz force,

$$\boldsymbol{F}' = \boldsymbol{i} \times \boldsymbol{B}$$

is exerted on quantized magnetic fluxes in a unit volume under the transport current. In this case the current is not localized only near the surface but flows uniformly inside the superconductor. When quantized magnetic fluxes are driven to move with velocity $\boldsymbol{V}$ by the Lorentz force, the electric field given by Eq. (10.21) is induced:

$$\boldsymbol{E} = \boldsymbol{B} \times \boldsymbol{V}. \tag{A3.6}$$

This is Josephson's relation. Since the condition is steady without a change in the magnetic flux density with time, this induced electric field satisfies Eq. (1.28).

The relationship between the current density and electric field in the superconductor in the mixed state is similar to Ohm's law for a usual metal, as shown by the solid line in Fig. A3.5. That is, when the current density increases, the Lorentz force increases and the velocity of quantized magnetic flux increases, resulting in an increase in the electric field. In such a resistive state heat is generated because of the energy dissipation. This occurs because the central part of each quantized magnetic flux is in the normal state. The induced electric field drives normal electrons, resulting in energy dissipation similar to that in a normal metal. Thus, when the magnetic flux density increases, the number density of quantized magnetic flux also increases and the electric resistance increases. When the magnetic flux density reaches B_{c2}, all the area in the superconductor reaches the normal state and the electric resistance reaches the normal value.

To transport a current without appearance of electric resistance, the motion of quantized magnetic fluxes needs to stop ($\boldsymbol{V} = 0$) even under the Lorentz force. This action is called flux pinning, and defects such as normal precipitates or grain boundaries are known to act effectively. These defects are called pinning centers. Practical superconductors contain such pinning centers dispersed with a high concentration. The condition of the force equilibrium on quantized magnetic fluxes is given by

$$\boldsymbol{i} \times \boldsymbol{B} + \boldsymbol{F}_{\mathrm{p}} = 0, \tag{A3.7}$$

where $\boldsymbol{F}_{\mathrm{p}}$ is the pinning force density. The corresponding relationship between the current density and electric field under the influence of flux pinning is shown by the dashed line in Fig. A3.5. The current density i_{c} at which the electric field starts to appear is called the critical current density. In this condition Eq. (A3.7) gives

$$i_{\mathrm{c}} = \frac{F_{\mathrm{p}}}{B}. \tag{A3.8}$$

To transport a current of high density without appearance of electric resistance, the strength of the pinning force needs to be enhanced.

A3.4 Electromagnetism and Superconductivity

Here we carefully look at the fundamental factors that construct electromagnetism. The independent principles are

(a) the Coulomb force (with Coulomb's law),
(b) the Lorentz force (with the Biot–Savart law),
(c) the law of electromagnetic induction,
(d) the displacement current.

(a) gives Eq. (11.9), (b) gives Eq. (11.10) and a part of Eq. (11.8), (c) gives Eq. (11.7), and (d) gives a part of Eq. (11.8). Thus, the above four principles are arranged into Maxwell's equations. From the comprehensive Maxwell theory based on these equations, the Coulomb force and Lorentz force are derived in terms of the Maxwell stress tensor. In fact, the Lorentz force is derived from a theoretical investigation of the energy in Exercise 11.9.

However, it should be noted that these principles are not enough to describe electromagnetic phenomena completely. That is, we need empirical Ohm's law for a system in which current flows. This law is not derived theoretically. Hence, electromagnetic theory is not complete in this sense.

Here we discuss electromagnetic phenomena in superconductors. These phenomena are independent of Ohm's law, and the mechanism that determines the current is obtained by minimizing the free energy. Hence, we can say that electromagnetic theory is complete for superconductors including the case where current flows. In

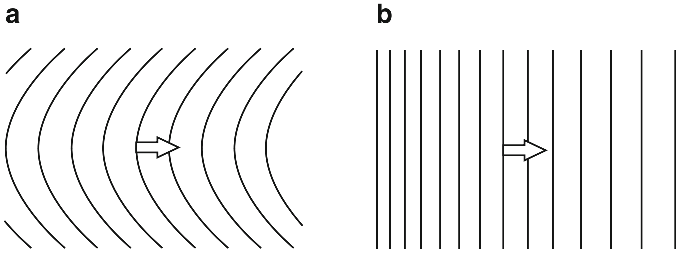

Fig. A3.6 Distortions of quantized magnetic fluxes: (**a**) bending of magnetic fluxes and (**b**) gradient of magnetic flux density. The Lorentz forces shown by the *arrows* to reduce the distortion are the line tension and magnetic pressure, respectively

addition, if we include the cases of pressurization or films, more than half of the elements become superconducting at low temperatures, and most metallic compounds and some organic compounds are superconductors. That is, superconductors are fairly common substances. This textbook shows that superconductor has its own place in the $\boldsymbol{E}$–$\boldsymbol{B}$ analogy. In principle it was even possible to predict the existence of superconductors in the 19th Century.

Second, we discuss electromagnetic phenomena in a superconductor with pinning centers in the mixed state. In many cases we can neglect the kinetic energy in Eq. (A3.4), and the suitable energy density to be minimized is

$$\frac{1}{2\mu_0}\boldsymbol{B}^2 + U_\mathrm{p}, \tag{A3.9}$$

where U_p is the pinning energy. Minimizing this energy with respect to the displacement of quantized magnetic fluxes leads to Eq. (A3.7) for an isolated superconductor.(1) That is, the variation in the magnetic energy due to the deformation of magnetic structure brings about the Lorentz force, and the variation in the pinning energy gives the pinning force density. We can extend this relationship to a non-isolated case and then to a general irreversible case. Thus, we obtain the force balance equation that describes practical electromagnetic phenomena in superconductors. The Lorentz force given by the first term in Eq. (A3.7) is transformed to

$$\boldsymbol{i} \times \boldsymbol{B} = \frac{1}{\mu_0}\left[(\boldsymbol{B}\cdot\nabla)\boldsymbol{B} - \frac{1}{2}\nabla\boldsymbol{B}^2\right]. \tag{A3.10}$$

Each term on the right side is expressed as an elastic restoring force against the distortion of quantized magnetic fluxes lines. The first term gives the line tension for the bent magnetic fluxes in Fig. A3.6a, and the second term gives the magnetic pressure to make the magnetic flux density uniform in Fig. A3.6b. The Lorentz force derived in Exercise 11.9 is the magnetic pressure.

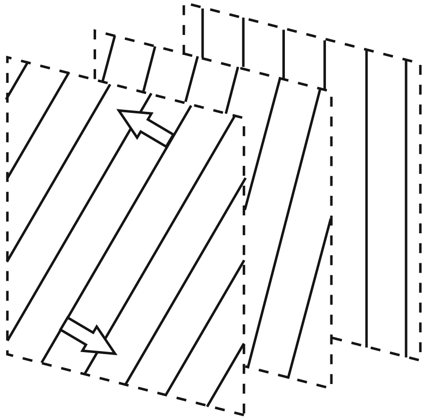

Fig. A3.7 Distortion of magnetic flux lines in the force free-state. Current flows parallel to the magnetic flux lines. Restoring torque is predicted to work on the flux lines as shown by the *arrows*

It is known that various peculiar electromagnetic phenomena called longitudinal magnetic field effects are observed when we apply a current to a long superconducting wire or slab in a parallel magnetic field.(2) In this condition the current and magnetic flux density are parallel to each other, and the Lorentz force on the quantized magnetic flux is zero ($\boldsymbol{i} \times \boldsymbol{B} = 0$). This state is called the force-free state. However, the magnetic structure contains a twisted distortion produced by the current, as shown in Fig. A3.7. We can expect that some restoring torque works to reduce the distortion, as shown by the arrows in the figure. In fact, we can derive the **force-free torque** using a similar method in Exercise 11.9.(3) We can explain that the rotational motion of quantized magnetic flux driven by the restoring torque causes the peculiar electromagnetic phenomena of the longitudinal magnetic field effects.(2)

Such a torque in a static condition is not known in electromagnetism. We can easily show that $\nabla \times \boldsymbol{J} \neq 0$ when a current flows as in Fig. A3.7. Hence, this situation cannot be realized in normal conductors (see Exercise 7.11). To say this in more detail, the helicity given by $\boldsymbol{J} \cdot \boldsymbol{B}$ or $\boldsymbol{A} \cdot \boldsymbol{B}$ is not zero in this condition.

In this textbook we showed that a superconductor can be considered a general material in electromagnetism. Here we showed that a superconductor is a more purely physical material described by a complete theory. We can even expect that superconductors will open the door to electromagnetic phenomena that people have never yet experienced.

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Answers to Exercises

Chapter 1

1.1. We presume the electric charge in a small region between x and $x + \mathrm{d}x$ from point A, $\mathrm{d}Q = (Q/L)\mathrm{d}x$, to be a point charge. Then, the Coulomb force on point charge q by this charge is $\mathrm{d}F = qQ\mathrm{d}x/[4\pi\epsilon_0 L(L + d - x)^2]$ and all the forces from each position point in the same direction. Thus, the total force is

$$F = \int_0^L \frac{qQ\mathrm{d}x}{4\pi\epsilon_0 L(L + d - x)^2} = \frac{qQ}{4\pi\epsilon_0 d(L + d)}.$$

1.2. The electric field strength due to the electric charge $\lambda\mathrm{d}y$ in the region y to $y + \mathrm{d}y$ from the lower edge of the bar is $\mathrm{d}E = \lambda\mathrm{d}y/[4\pi\epsilon_0(y^2 + b^2)]$. We define angle θ as shown in Fig. B1.1. The x- and y-components of the electric field are $\mathrm{d}E\cos\theta$ and $-\mathrm{d}E\sin\theta$, respectively, and $y = b\tan\theta$ with $\theta_a = \tan^{-1}(a/b)$. Thus, we obtain

$$E_x = \frac{\lambda}{4\pi\epsilon_0 b}\int_0^{\theta_a}\cos\theta\mathrm{d}\theta = \frac{\lambda a}{4\pi\epsilon_0 b(a^2 + b^2)^{1/2}},$$

$$E_y = -\frac{\lambda}{4\pi\epsilon_0 b}\int_0^{\theta_a}\sin\theta\mathrm{d}\theta = -\frac{\lambda}{4\pi\epsilon_0 b}\left[1 - \frac{b}{(a^2 + b^2)^{1/2}}\right].$$

1.3. The distance from one side to point P is $r = [(a^2/4) + z^2]^{1/2}$ and the electric field strength due to the electric charge on one side is $E' = \lambda a/\{4\pi\epsilon_0 r[(a^2/4) + z^2]^{1/2}\}$. From symmetry only the vertical component of the electric field remains (see Fig. B1.2), and we obtain the electric field by summing the contributions from the four sides as

$$E = 4E'\sin\beta = \frac{\lambda az}{\pi\epsilon_0[(a^2/4) + z^2][(a^2/2) + z^2]^{1/2}}.$$

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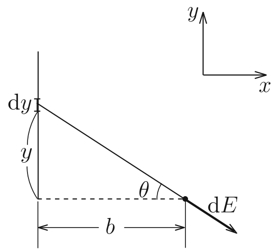


Fig. B1.1 Definition of angle θ

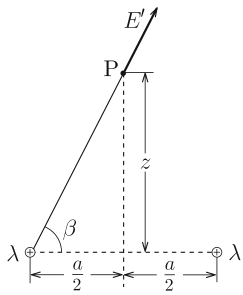


Fig. B1.2 Electric field produced by electric charge on one side

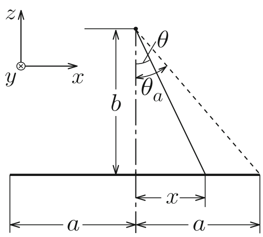


Fig. B1.3 Electric field produced by line charge in a part of slab

1.4. We define the coordinates as shown in Fig. B1.3. Although we cannot directly apply Gauss' law, it can be used to estimate the electric field produced by the line charge of density $\sigma \mathrm{d}x$ in a thin region between x and $x + \mathrm{d}x$. The electric field strength at point A due to this line charge is $\mathrm{d}E'_{\mathrm{A}} = \sigma \mathrm{d}x/[2\pi\epsilon_0(x^2 + b^2)^{1/2}]$. From symmetry only the z-component, $\mathrm{d}E_{\mathrm{A}} = \mathrm{d}E'_{\mathrm{A}} \cos\theta$, remains. Integration yields the electric field at A;

$$E_{\mathrm{A}} = \int_{-a}^{a} \frac{\sigma \cos\theta \mathrm{d}x}{2\pi\epsilon_0(x^2 + b^2)^{1/2}} = \frac{\sigma}{2\pi\epsilon_0} \int_{-\theta_a}^{\theta_a} \mathrm{d}\theta = \frac{\sigma\theta_a}{\pi\epsilon_0},$$

where we have transformed as $x = b\tan\theta$ with $\theta_a = \tan^{-1}(a/b)$.

The electric field at point B produced by the line charge in a thin region between x and $x + \mathrm{d}x$ has a strength of $\mathrm{d}E_{\mathrm{B}} = \sigma \mathrm{d}x/[2\pi\epsilon_0(d - x)]$ and is directed along the x-axis. Thus, a simple summation yields

$$E_{\mathrm{B}} = \frac{\sigma}{2\pi\epsilon_0} \int_{-a}^{a} \frac{\mathrm{d}x}{d - x} = \frac{\sigma}{2\pi\epsilon_0} \log\frac{d + a}{d - a}.$$

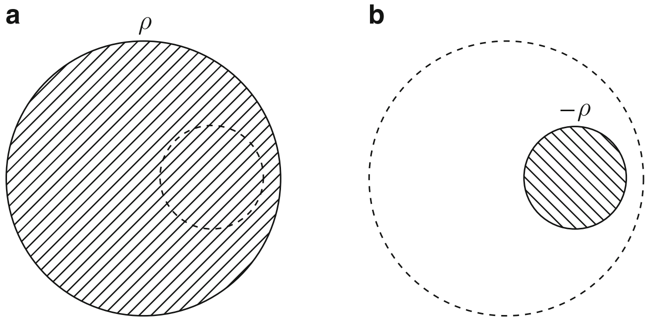


Fig. B1.4 Superposition of (**a**) uniformly distributed electric charge with density ρ over the whole sphere and (**b**) uniformly distributed electric charge with density $-\rho$ in the vacancy

1.5. Although we cannot directly obtain the electric field using Gauss' law, we can solve this problem using a superposition of two solvable cases. The given condition can be realized by superposing the situation where the electric charge is uniformly distributed with density ρ in the whole region of the sphere as shown in Fig. B1.4a, and that where the electric charge is uniformly distributed with density $-\rho$ in the vacant region as shown in Fig. B1.4b. We can calculate the electric field in each case using Gauss' law. First we determine the electric field at the center A of the vacancy. In (a) the total electric charge is $Q' = (4\pi/3)d^3\rho$ and the electric field is $E_1 = Q'/(4\pi\epsilon_0 d^2) = \rho d/(3\epsilon_0)$. We similarly determine the electric field in (b) to be $E_2 = \rho r/(3\epsilon_0) \to 0$ in the limit $r \to 0$. Thus, we have

$$E_A = E_1 + E_2 = \frac{\rho d}{3\epsilon_0}.$$

Second, we determine the electric field at point B. The contributions from (a) and (b) are $E_3 = \rho a^3/(3\epsilon_0 r^2)$ and $E_4 = -\rho b^3/[3\epsilon_0(r-d)^2]$, respectively. Thus, we have

$$E_B = E_3 + E_4 = \frac{\rho}{3\epsilon_0}\left[\frac{a^3}{r^2} - \frac{b^3}{(r-d)^2}\right].$$

1.6. From Eq. (1.25) we obtain the electric potential due to the electric charge $\lambda \mathrm{d}y$ in a small region between y and $y + \mathrm{d}y$ from origin O as $\mathrm{d}\phi = \lambda \mathrm{d}y/[4\pi\epsilon_0(y^2 + b^2)^{1/2}]$. Integrating this with respect to y from $-a$ to a yields

$$\phi = \int_{-a}^{a} \frac{\lambda \mathrm{d}y}{4\pi\epsilon_0(y^2+b^2)^{1/2}} = \frac{\lambda}{2\pi\epsilon_0}\int_0^{\theta_a}\frac{\mathrm{d}\theta}{\cos\theta} = \frac{\lambda}{2\pi\epsilon_0}\log\frac{(a^2+b^2)^{1/2}+a}{b},$$

where we have transformed as $y = b\tan\theta$ with $\theta_a = \tan^{-1}(a/b)$.

1.7. We easily obtain the electric potential from Eq. (1.27) in this case. Since all the electric charge is located at distance a from the center, the electric potential is

$$\phi = \frac{1}{4\pi\epsilon_0 a}\int_{\mathrm{V}} \rho(\boldsymbol{r}')\mathrm{d}V' = \frac{Q}{4\pi\epsilon_0 a} = \frac{\sigma a}{\epsilon_0},$$

where $Q = 4\pi a^2\sigma$ is the total electric charge. We can also calculate the electric potential from the electric field [see Eq. (2.9b)].

1.8. Since all the electric charge $Q = 2\pi a\lambda$ is located at the same distance $(z^2 + a^2)^{1/2}$ from point P, we calculate the electric potential as

$$\phi = \frac{Q}{4\pi\epsilon_0(z^2 + a^2)^{1/2}} = \frac{a\lambda}{2\epsilon_0(z^2 + a^2)^{1/2}}.$$

From symmetry the electric field at point P is directed vertically and its strength is

$$E = -\frac{\partial\phi}{\partial z} = \frac{az\lambda}{2\epsilon_0(z^2 + a^2)^{3/2}}.$$

1.9. We denote the distance from the sphere center by r. Then, the electric potential outside the sphere $(r > a)$ is $\phi(r) = Q/(4\pi\epsilon_0 r)$. Thus, from Eq. (1.33) the work is determined to be

$$W = q[\phi(r_{\mathrm{B}}) - \phi(r_{\mathrm{A}})] = \frac{qQ}{4\pi\epsilon_0}\left(\frac{1}{r_{\mathrm{B}}} - \frac{1}{r_{\mathrm{A}}}\right).$$

Chapter 2

2.1. Electric charge Q_1 is distributed uniformly on the surface of the inner sphere $(r = a)$ and the electric charge $-Q_1$ induced by the electrostatic induction is distributed uniformly on the inner surface of the outer sphere $(r = b)$. Thus, the electric charge $Q_1 + Q_2$ appears on the outer surface of the outer sphere $(r = c)$, following the principle of conservation of charge. The electric field is directed radially and its strength is

$$\begin{aligned} E_r &= 0; && 0 \le r < a, \\ &= \frac{Q_1}{4\pi\epsilon_0 r^2}; && a < r < b, \\ &= 0; && b < r < c, \\ &= \frac{Q_1 + Q_2}{4\pi\epsilon_0 r^2}; && r > c. \end{aligned}$$

The electric potential is determined to be

$$\begin{aligned}
\phi &= \frac{Q_1 + Q_2}{4\pi\epsilon_0 r}; && r > c, \\
&= \frac{Q_1 + Q_2}{4\pi\epsilon_0 c}; && b < r < c, \\
&= \frac{Q_1}{4\pi\epsilon_0}\left(\frac{1}{r} - \frac{1}{b} + \frac{1}{c}\right) + \frac{Q_2}{4\pi\epsilon_0 c}; && a < r < b, \\
&= \frac{Q_1}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b} + \frac{1}{c}\right) + \frac{Q_2}{4\pi\epsilon_0 c}; && 0 \le r < a.
\end{aligned}$$

2.2. We denote by Q_0 the electric charge induced on the inner surface ($r = a$). Using Gauss' law, the electric charge on the inner surface of the outer sphere ($r = b$) is determined to be $-Q_0$. Hence, the electric charge on the outer surface ($r = c$) is $Q + Q_0$. If we define the electric potential to be zero at infinity, the electric potential of the outer sphere is

$$\phi = \frac{Q + Q_0}{4\pi\epsilon_0 c}.$$

On the other hand, the electric field in the region $a < r < b$ is $E = Q_0/(4\pi\epsilon_0 r^2)$, and the electric potential there is $\phi = Q_0/(4\pi\epsilon_0 r) + C$ with C denoting a constant. From the condition that $\phi = 0$ at $r = a$ because of grounding, we have $C = -Q_0/(4\pi\epsilon_0 a)$. Thus, the electric potential of the outer sphere is

$$\phi = -\frac{Q_0}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right).$$

The requirement that this is equal to the electric potential determined from infinity yields

$$Q_0 = -\left(\frac{1}{a} - \frac{1}{b} + \frac{1}{c}\right)^{-1} \frac{Q}{c}.$$

2.3. We denote the electric charge on the surface at $x = b$ by Q_b. Then, the electric charge on the surface at $x = a$ is $-Q_b$. So that the electric field does not penetrate the conductor of $a < x < b$, the electric charge at $x = b$ must be the same as the total electric charge in the region $x < a$, i.e., $Q - Q_b$. Thus, we have $Q_b = Q/2$ and the electric charge at $x = a$ is $-Q/2$. We similarly obtain the electric charges on the surfaces of the left conductor. As a result, the electric charges at $x = -b, -a, a$ and b are $Q/2, Q/2, -Q/2$ and $Q/2$, respectively.

The electric field is directed along the x-axis and its strength is

$$
\begin{aligned}
E &= -\frac{Q}{2\epsilon_0 S}; && x < -b, \\
&= 0; && -b \le x \le -a,\ a \le x \le b, \\
&= \frac{Q}{2\epsilon_0 S}; && -a < x < a, \\
&= \frac{Q}{2\epsilon_0 S}; && x > b.
\end{aligned}
$$

We obtain the electric potential from $E = -\partial\phi/\partial x$ as

$$
\begin{aligned}
\phi &= \frac{Q(x + 2a + b)}{2\epsilon_0 S}; && x < -b, \\
&= \frac{Qa}{\epsilon_0 S}; && -b \le x \le -a, \\
&= -\frac{Q(x - a)}{2\epsilon_0 S}; && -a < x < a, \\
&= 0; && a \le x \le b, \\
&= \frac{Q(-x + b)}{2\epsilon_0 S}; && x > b,
\end{aligned}
$$

where ϕ of the right conductor with no electric charge is arbitrarily defined to be zero.

2.4. The reason why the electric field produced by the electric charge distributed on the conductor surface is doubled is that there are other electric field contributions from electric charges in other areas. For the same reason the electric field inside the conductor cancels to zero. Examples are found in the case where an electric charges of different kind is distributed on the surface of the opposite electrode of a capacitor, as shown in Fig. B2.1a, or in the case where an electric charge of the same kind stays on the opposite surface of the conductor, as shown in Fig. B2.1b. The situation in Example 1.4 corresponds to the thin limit of the conductor in Fig. B2.1b.

2.5. We define two-dimensional polar coordinates (R, φ) on the conductor surface with the origin on the point at which the vertical line from point charge q meets the surface. We consider a thin ring of radius R to $R + \mathrm{d}R$ and presume the electric charge in a small part of the azimuthal angle φ to $\varphi + \mathrm{d}\varphi$, $\mathrm{d}Q = -qaR\mathrm{d}R\mathrm{d}\varphi/[2\pi(R^2 + a^2)^{3/2}]$, as a point electric charge. The Coulomb force on q caused by this point charge is $\mathrm{d}F = q\mathrm{d}Q/[4\pi\epsilon_0(R^2 + a^2)]$, and only its vertical component, $\mathrm{d}F_z = [a/(R^2 + a^2)^{1/2}]\mathrm{d}F$, remains from symmetry. Integrating this over the surface, we have

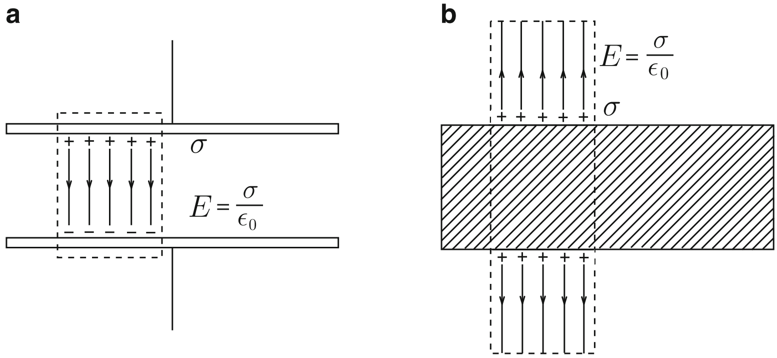


Fig. B2.1 Examples of doubled electric field strength: (**a**) parallel-plate capacitor and (**b**) distribution of electric charge of the same kind on the opposite surface of the conductor

$$F_z = -\frac{q^2a^2}{8\pi^2\epsilon_0}\int_0^\infty \frac{2\pi R\mathrm{d}R}{(R^2+a^2)^3} = -\frac{q^2}{16\pi\epsilon_0 a^2}.$$

This agrees with the image force, Eq. (2.15).

2.6. We denote two conductor surfaces that are perpendicular to each other by the x-y and y-z planes, as shown in Fig. B2.2. Assume the given electric charge is located on the plane $y = 0$. We virtually remove the conductor and place three electric charges, $-Q$, $-Q$ and Q, at $(a, 0, -b)$, $(-a, 0, b)$ and $(-a, 0, -b)$, respectively. Then, the electric potential in the vacuum region $(x > 0, z > 0)$ is

$$\phi(x, y, z) = \frac{Q}{4\pi\epsilon_0}\left\{\frac{1}{[(x-a)^2+y^2+(z-b)^2]^{1/2}} - \frac{1}{[(x-a)^2+y^2+(z+b)^2]^{1/2}}\right.$$
$$\left. - \frac{1}{[(x+a)^2+y^2+(z-b)^2]^{1/2}} + \frac{1}{[(x+a)^2+y^2+(z+b)^2]^{1/2}}\right\}.$$

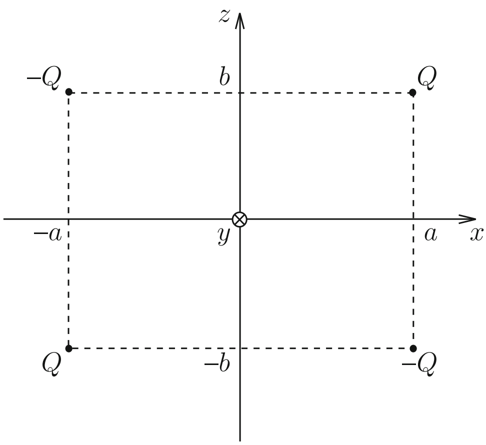


Fig. B2.2 True electric charge Q and three mirror charges

This satisfies $\phi = 0$ on the surfaces $x = 0$ and $z = 0$, and hence, this gives the correct electric potential. We determine the electric charge density on the x-y and y-z planes to be

$$\sigma(x,y,0) = -\epsilon_0\left(\frac{\partial\phi}{\partial z}\right)_{z=0} = -\frac{Qb}{2\pi}\left\{\frac{1}{[(x-a)^2+y^2+b^2]^{3/2}} - \frac{1}{[(x+a)^2+y^2+b^2]^{3/2}}\right\},$$

$$\sigma(0,y,z) = -\epsilon_0\left(\frac{\partial\phi}{\partial x}\right)_{x=0} = -\frac{Qa}{2\pi}\left\{\frac{1}{[y^2+(z-b)^2+a^2]^{3/2}} - \frac{1}{[y^2+(z+b)^2+a^2]^{3/2}}\right\}.$$

2.7. A simple calculation gives

$$E_r = -\frac{\partial\phi}{\partial r} = \frac{q}{4\pi\epsilon_0}\left\{\frac{r-d\cos\theta}{(r^2+d^2-2rd\cos\theta)^{3/2}} - \frac{a}{d}\cdot\frac{r-(a^2/d)\cos\theta}{[r^2+(a^2/d)^2-(2a^2r/d)\cos\theta]^{3/2}}\right\},$$

$$E_\theta = -\frac{1}{r}\cdot\frac{\partial\phi}{\partial\theta} = \frac{q\sin\theta}{4\pi\epsilon_0}\left\{\frac{d}{(r^2+d^2-2rd\cos\theta)^{3/2}} - \frac{a^3}{d^2}\cdot\frac{1}{[r^2+(a^2/d)^2-(2a^2r/d)\cos\theta]^{3/2}}\right\},$$

$$E_\varphi = -\frac{1}{r\sin\theta}\cdot\frac{\partial\phi}{\partial\varphi} = 0.$$

2.8. We virtually remove the conductor and place a line electric charge of density λ' at the line located at distance h from the center O. As shown in Example 1.8, the electric potential at point P on the surface of cylindrical conductor is

$$\phi = \frac{\lambda}{2\pi\epsilon_0}\log\frac{R_0}{(a^2+d^2-2ad\cos\varphi)^{1/2}} + \frac{\lambda'}{2\pi\epsilon_0}\log\frac{R_0'}{(a^2+h^2-2ah\cos\varphi)^{1/2}},$$

where φ is the angle defined in Fig. B2.3 and R_0 and R_0' are distances from O to reference points of the electric potential. So that the electric potential does not depend on φ, the conditions $\lambda' = -\lambda$ and $h = a^2/d$ must be fulfilled. In addition, $R_0' = (a/d)R_0$ is requited so that the electric potential of the conductor is zero. Thus, the electric potential outside the conductor is

$$\phi(R,\varphi) = \frac{\lambda}{2\pi\epsilon_0}\log\frac{d[R^2+(a^2/d)^2-2(a^2R/d)\cos\varphi]^{1/2}}{a(R^2+d^2-2Rd\cos\varphi)^{1/2}}.$$

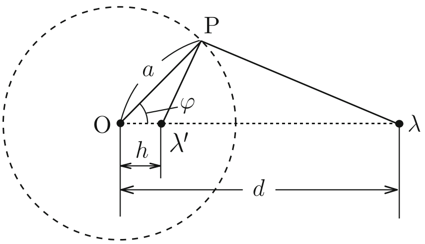


Fig. B2.3 Definition of angle φ

The electric charge density on the conductor surface is

$$\sigma = -\epsilon_0 \left(\frac{\partial \phi(R, \varphi)}{\partial R} \right)_{R=a} = -\frac{\lambda(d^2 - a^2)}{2\pi a(a^2 + d^2 - 2ad \cos\varphi)}.$$

This gives the electric charge in a unit length;

$$\int_0^{2\pi} \sigma a \, \mathrm{d}\varphi = -\frac{\lambda(d^2 - a^2)}{\pi} \int_0^{\pi} \frac{\mathrm{d}\varphi}{a^2 + d^2 - 2ad \cos\varphi} = -\lambda,$$

where we have used Eq. (7.26).

2.9. We assume that the electric field produced by the electric charge on the cylindrical conductor surface is the same as that produced by the line charge of density λ placed at distance h from the center of the cylinder after virtually removing the cylinder (see Fig. B2.4). If we place an image line charge of density $-\lambda$ in the infinite conductor at distance $l-h$ from its surface after virtually removing the infinite conductor, the infinite conductor surface is equipotential. Hence, if the distance $2l-h$ between the image charge $-\lambda$ and the cylinder center corresponds to d in Exercise 2.8, the cylindrical conductor surface is also equipotential, and all the required conditions are satisfied. The result in Exercise 2.8 gives $h = a^2/d$. From the above conditions we have

$$d = l + \sqrt{l^2 - a^2}, \qquad h = l - \sqrt{l^2 - a^2}.$$

Substituting these into the result in Exercise 2.8 yields the electric potential outside the conductors;

$$\phi(R, \varphi) = -\frac{\lambda}{4\pi\epsilon_0} \log \frac{R^2 + (l - \sqrt{l^2 - a^2})^2 - 2R(l - \sqrt{l^2 - a^2}) \cos\varphi}{R^2 + (l + \sqrt{l^2 - a^2})^2 - 2R(l + \sqrt{l^2 - a^2}) \cos\varphi}.$$

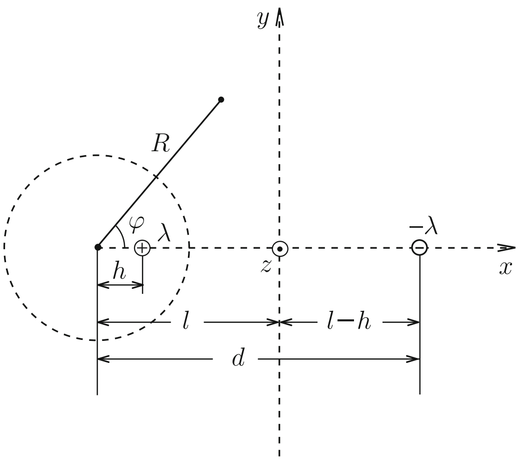


Fig. B2.4 Image line charges placed in two conductors

We find that the electric potential on the surface, $\phi(a,\varphi) = -[\lambda/(2\pi\epsilon_0)]\log[(l - \sqrt{l^2 - a^2})/a]$, is constant. The electric charge density on the cylindrical surface is

$$\sigma = -\epsilon_0 \left(\frac{\partial \phi}{\partial x}\right)_{R=a} = \frac{\lambda}{2\pi a} \cdot \frac{\sqrt{l^2 - a^2}}{l - a\cos\varphi}.$$

Next we define Cartesian coordinates with the y-z plane ($x = 0$) on the infinite conductor surface and the central axis of the cylindrical conductor at $y = 0$. From the relationships $R\cos\varphi = x + l$ and $R\sin\varphi = y$, the electric potential is also expressed as

$$\phi(x, y) = -\frac{\lambda}{4\pi\epsilon_0} \log \frac{(x + \sqrt{l^2 - a^2})^2 + y^2}{(x - \sqrt{l^2 - a^2})^2 + y^2}.$$

Thus, we can easily confirm that $\phi(x = 0) = 0$ is satisfied. The electric charge density on the infinite conductor surface is

$$\sigma = \epsilon_0 \left(\frac{\partial \phi}{\partial x}\right)_{x=0} = -\frac{\lambda\sqrt{l^2 - a^2}}{\pi(y^2 + l^2 - a^2)}.$$

It should be noted that the sign is opposite, since the normal vector on the conductor surface is directed along the negative x-axis.

2.10. We remove the conductor and place an image electric charge, q, on a line extending from the center and the electric charge Q (see Fig. B2.5). We denote the distance of this point from the center by d. The electric potential at point P on the inner surface of the conductor is

$$\phi(a, \theta) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{(a^2 + d^2 - 2ad\cos\theta)^{1/2}} + \frac{Q}{(a^2 + h^2 - 2ah\cos\theta)^{1/2}}\right].$$

The conditions that satisfy $\phi(a, \theta) = 0$ are

$$d = \frac{a^2}{h}, \qquad q = -\frac{dQ}{a} = -\frac{aQ}{h}.$$

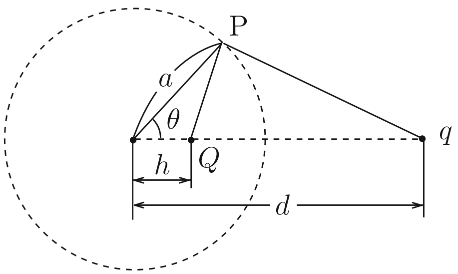


Fig. B2.5 Image electric charge q

Thus, the electric potential in the hollow is given by

$$\phi(r,\theta) = \frac{Q}{4\pi\epsilon_0}\left\{-\frac{a}{h[r^2+(a^2/h)^2-2(a^2r/h)\cos\theta]^{1/2}} + \frac{1}{(r^2+h^2-2rh\cos\theta)^{1/2}}\right\}.$$

The electric charge density on the inner surface is

$$\sigma(\theta) = -\epsilon_0 E_r(r=a) = \epsilon_0\left(\frac{\partial\phi}{\partial r}\right)_{r=a} = -\frac{Q(a^2-h^2)}{4\pi a(a^2+h^2-2ah\cos\theta)^{3/2}}.$$

Chapter 3

3.1. First we determine the coefficients. Assuming $Q_1 = 1$ and $Q_2 = 0$, we have

$$\phi_1 = p_{11} = \frac{1}{4\pi\epsilon_0 a}, \qquad \phi_2 = p_{21} = \frac{1}{4\pi\epsilon_0 d} = p_{12}.$$

When $Q_1 = 0$ and $Q_2 = q$, the electric potential of the spherical conductor is

$$\phi_1 = p_{11}Q_1 + p_{12}Q_2 = \frac{q}{4\pi\epsilon_0 d}.$$

We easily find this agrees with the result, $\phi(a,\theta)$, in Example 2.3.

3.2. We denote the cylindrical conductor and a thin linear conductor placed at the position of the line charge as conductors 1 and 2, respectively. We give a unit electric charge to conductor 1 of a unit length ($\lambda_1 = 1$) and no electric charge to conductor 2 ($\lambda_2 = 0$). Then, the electric potentials of conductors 1 and 2 are

$$\phi_1 = p'_{11} = \frac{1}{2\pi\epsilon_0}\log\frac{R_0}{a}, \qquad \phi_2 = p'_{21} = \frac{1}{2\pi\epsilon_0}\log\frac{R_0}{d}.$$

Thus, we obtain the coefficients of electrostatic potential. In a general case where $\lambda_1 = \Lambda$ and $\lambda_2 = \lambda$, the electric potential of conductor 1 is

$$\phi_1 = p'_{11}\Lambda + p'_{12}\lambda.$$

When conductor 1 is grounded, $\phi_1 = 0$. This with $p'_{12} = p'_{21}$ yields

$$\Lambda = -\frac{p'_{21}}{p'_{11}}\lambda = -\frac{\log R_0 - \log a}{\log R_0 - \log d}\,\lambda.$$

If the reference point is infinity ($R_0 \to \infty$), this reduces to

$$\Lambda = -\lambda.$$

3.3. We denote the inner and outer conductors as conductors 1 and 2, respectively. In a general case where conductor 1 is not grounded, the assumptions $Q_1 = 1$ and $Q_2 = 0$ give

$$\phi_1 = p_{11} = \frac{1}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b} + \frac{1}{c}\right), \qquad \phi_2 = p_{21} = p_{12} = \frac{1}{4\pi\epsilon_0 c}.$$

For $Q_1 = q$ and $Q_2 = Q$, the electric potential of conductor 1 is

$$\phi_1 = p_{11}q + p_{12}Q = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b} + \frac{1}{c}\right) + \frac{Q}{4\pi\epsilon_0 c}.$$

Hence, when conductor 1 is grounded ($\phi_1 = 0$), we have

$$q = -\left(\frac{1}{a} - \frac{1}{b} + \frac{1}{c}\right)^{-1}\frac{Q}{c}.$$

This agrees with the result obtained in Exercise 2.2.

3.4. The electric charge distributed in a unit length of the concentric conductor is λ on the surface of the inner conductor ($R = a$) and $-\lambda$ on the inner surface of the outer conductor ($R = b$). As a result, the electric field is $E = \lambda/(2\pi\epsilon_0 R)$ in the region $a < R < b$ and is zero in other regions. Hence, the electrostatic energy density in this region is $u_\mathrm{e} = \epsilon_0 E^2/2 = \lambda^2/(8\pi^2\epsilon_0 R^2)$ and the electrostatic energy in the conductor of a unit length is

$$U'_\mathrm{e} = \int_a^b \frac{\lambda^2}{8\pi^2\epsilon_0 R^2}\cdot 2\pi R\,\mathrm{d}R = \frac{\lambda^2}{4\pi\epsilon_0}\log\frac{b}{a}.$$

The electric potential of the outer conductor is zero and that of the inner conductor is

$$\phi = \frac{\lambda}{2\pi\epsilon_0}\log\frac{b}{a}.$$

Hence, we obtain the same electrostatic energy from Eq. (3.36) with $U'_\mathrm{e} = \lambda\phi/2$.

Using this result and $U'_\mathrm{e} = \lambda^2/(2C')$ corresponding to Eq. (3.38), the capacitance in a unit length is

$$C' = \frac{2\pi\epsilon_0}{\log(b/a)}.$$

3.5. (1) The electric field is $E(r) = Q/(4\pi\epsilon_0 r^2)$ in the region $a < r < b$ and is zero in other regions. The electrostatic energy density has a non-zero value, $u_e = Q^2/(32\pi^2\epsilon_0 r^4)$, only in the region $a < r < b$. We calculate the electrostatic energy as

$$U_e = \int_a^b u_e 4\pi r^2 \mathrm{d}r = \frac{(b-a)Q^2}{8\pi\epsilon_0 ab}.$$

(2) Using the electric field in (1), the electric potential of the outer conductor is zero and that of the inner conductor is

$$\phi = -\int_b^a \frac{Q}{4\pi\epsilon_0 r^2}\mathrm{d}r = \frac{(b-a)Q}{4\pi\epsilon_0 ab}.$$

The electrostatic energy is

$$U_e = \frac{1}{2}Q\phi = \frac{(b-a)Q^2}{8\pi\epsilon_0 ab}.$$

(3) We denote the inner and outer conductors as conductors 1 and 2, respectively. The coefficients of electric potential are

$$p_{11} = \frac{1}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b} + \frac{1}{c}\right), \qquad p_{12} = p_{21} = p_{22} = \frac{1}{4\pi\epsilon_0 c}.$$

The electric charges are $Q_1 = Q$ and $Q_2 = -Q$. Thus, the electrostatic energy is

$$U_e = \frac{1}{2}p_{11}Q^2 - p_{12}Q^2 + \frac{1}{2}p_{22}Q^2 = \frac{(b-a)Q^2}{8\pi\epsilon_0 ab}.$$

3.6. Suppose that electric charges $\pm\lambda$ are given to each conductor in a unit length. We define the x-axis normal to these conductors in such a way that it passes through the centers of these conductors. We denote the positions of the centers of the conductors with negative and positive electric charges by $x = 0$ and $x = d$, respectively. Since the diameter of these conductors is much smaller than the interval, d, we can approximate the electric charges as being uniformly distributed on each surface. Hence, the electric field at position x is

$$E = -\frac{\lambda}{2\pi\epsilon_0}\left(\frac{1}{x} + \frac{1}{d-x}\right)$$

under the definition of positive electric field directed along the positive x-axis. The electric potential difference between the two conductors is

$$V = -\int_a^{d-a} E\mathrm{d}x = \frac{\lambda}{\pi\epsilon_0}\log\left(\frac{d}{a} - 1\right).$$

The capacitance in a unit length is

$$C' = \frac{\lambda}{V} = \frac{\pi\epsilon_0}{\log[(d/a) - 1]}.$$

3.7. The electric field in the space where the conductor is not inserted is $E = V/d$ and the densities of electric charges on the electrode surfaces in this region are $\pm\sigma_1 = \pm\epsilon_0 E = \pm\epsilon_0 V/d$. On the other hand, the electric field is concentrated only in the vacuum in other region and its strength is $E = V/(d-t)$. Thus, the densities of electric charges on the electrode surfaces in this region are $\pm\sigma_2 = \pm\epsilon_0 V/(d-t)$. Hence, when the depth of insertion changes from x to $x + \Delta x$, the change in the electric charge in the electrode is $\Delta Q = \epsilon_0 btV\Delta x/[d(d-t)]$. The electrostatic energy of the capacitor is

$$U_\mathrm{e} = \frac{1}{2}[(a-x)\sigma_1 + x\sigma_2]bV = \frac{\epsilon_0 bV^2}{2}\left(\frac{a-x}{d} + \frac{x}{d-t}\right).$$

The variation in the electrostatic energy when x increases by Δx is $\Delta U_\mathrm{e} = \epsilon_0 btV^2\Delta x/[2d(d-t)]$.

If we denote the force on the conductor by F, the work done by the conductor is $F\Delta x$. The input energy from the electric power source to the system is $V\Delta Q$. Hence, from the relationship $\Delta U_\mathrm{e} = -F\Delta x + V\Delta Q$, we obtain the force as

$$F = \lim_{\Delta x \to 0}\frac{V\Delta Q - \Delta U_\mathrm{e}}{\Delta x} = \frac{\epsilon_0 btV^2}{2d(d-t)}.$$

Thus, the force is positive for increasing x and is attractive.

3.8. In the solution of Exercise 3.7, the electric charge,

$$Q = (a-x)b\sigma_1 + xb\sigma_2 = \epsilon_0 bV\left(\frac{a-x}{d} + \frac{x}{d-t}\right),$$

is kept constant and there is no energy flow from the electric power source. Thus, the electrostatic energy is given by

$$U_\mathrm{e} = \frac{1}{2}QV = \frac{Q^2}{2\epsilon_0 b}\left(\frac{a-x}{d} + \frac{x}{d-t}\right)^{-1}.$$

We obtain the force on the conductor as

$$F = -\frac{\partial U_\mathrm{e}}{\partial x} = \frac{dt(d-t)Q^2}{2\epsilon_0 b[a(d-t) + tx]^2}.$$

Confirm that this force is identical with that in Exercise 3.7.

Chapter 4

4.1. Assume that electric charges Q and $-Q$ appear on the inner and outer electrodes, respectively, when we apply potential difference V between the two electrodes. The electric flux density is directed radially between the two electrodes and its values are $D_1 = D_2 = Q/(4\pi r^2)$ in each region of different dielectric materials. Hence, the electric fields in each region are $E_1 = Q/(4\pi\epsilon_1 r^2)$ and $E_2 = Q/(4\pi\epsilon_2 r^2)$. The electric potential difference between the two electrodes is

$$V = \int_a^b E_1 \mathrm{d}r + \int_b^c E_2 \mathrm{d}r = \frac{Q}{4\pi\epsilon_1}\left(\frac{1}{a} - \frac{1}{b}\right) + \frac{Q}{4\pi\epsilon_2}\left(\frac{1}{b} - \frac{1}{c}\right).$$

We obtain the capacitance as

$$C = \frac{Q}{V} = \frac{4\pi\epsilon_1\epsilon_2 abc}{\epsilon_1 a(c-b) + \epsilon_2 c(b-a)}.$$

4.2. Assume that electric charges of density σ_1 and σ_2 appear on the inner electrode surface regions ($r = a$) faced to dielectric materials of ϵ_1 and ϵ_2, respectively, when we apply potential difference V between the two electrodes. The electric flux density is directed radially between the two electrodes, and its values in dielectric materials 1 and 2 are $D_1 = a^2\sigma_1/r^2$ and $D_2 = a^2\sigma_2/r^2$. The electric fields in respective regions are $E_1 = a^2\sigma_1/(\epsilon_1 r^2)$ and $E_2 = a^2\sigma_2/(\epsilon_2 r^2)$. Since the integration of these electric fields between the two electrodes is V, we have

$$\frac{a^2\sigma_1}{\epsilon_1}\left(\frac{1}{a} - \frac{1}{b}\right) = \frac{a^2\sigma_2}{\epsilon_2}\left(\frac{1}{a} - \frac{1}{b}\right) = V.$$

Thus, we determine the surface charge densities to be

$$\sigma_1 = \frac{b\epsilon_1 V}{a(b-a)}, \qquad \sigma_2 = \frac{b\epsilon_2 V}{a(b-a)}.$$

This yields the total electric charge on the internal electrode,

$$Q = 2\pi a^2(\sigma_1 + \sigma_2) = \frac{2\pi ab(\epsilon_1 + \epsilon_2)V}{b-a}.$$

We obtain the capacitance as

$$C = \frac{Q}{V} = \frac{2\pi ab(\epsilon_1 + \epsilon_2)}{b-a}.$$

4.3. We denote the plane determined by the normal vector $\boldsymbol{n}$ on the interface and the electric field $\boldsymbol{E}_1$ in dielectric material 1 as S. Assume that the electric field $\boldsymbol{E}_2$ in dielectric material 2 does not lie on this plane. We consider a plane, S′, normal to both the interface and S and define a small rectangle on S′ that includes the interface. The two sides of the rectangle are parallel to the interface. When we integrate the electric field along this rectangle, the integral in dielectric material 2 is not zero, while that in dielectric material 1 is zero. This is contradictory, since Eq. (1.30) is not satisfied under this assumption. Thus, we prove that the electric field $\boldsymbol{E}_2$ also lies on plane S.

4.4. Since the parallel component of the electric field is continuous across the wide interface, the electric field inside the slit is also E_0 and the electric flux density is $D = \epsilon_0 E_0$.

4.5. Since the normal component of the electric flux density is continuous across the wide interface, the electric flux density inside the slit is also $D = \epsilon E_0$ and the electric field is $E = D/\epsilon_0 = (\epsilon/\epsilon_0)E_0$.

4.6. Applying Gauss' law to a closed surface including the dielectric material surface, we determine the electric flux density in the dielectric material to be $D = \epsilon_0 E_0$. Hence, the electric field in the dielectric material is $E = (\epsilon_0/\epsilon)E_0$. The surface polarization charge density σ_p is equal to the electric polarization and we obtain

$$\sigma_\mathrm{p} = P = (\epsilon - \epsilon_0)E = \frac{\epsilon_0(\epsilon - \epsilon_0)}{\epsilon} E_0.$$

4.7. The electric field E is given by the sum of E_0 and the electric field produced by the polarization charge of surface density, $\sigma_\mathrm{p}(\theta) = [3\epsilon_0(\epsilon - \epsilon_0)/(\epsilon + 2\epsilon_0)]E_0 \cos\theta$, with θ denoting the zenithal angle. Since the electric charge of surface density $\sigma = 3\epsilon_0 E_0 \cos\theta$ in Eq. (2.29) produces the uniform electric field $-E_0$ inside the sphere, the above polarization charge produces the uniform electric field $-(\epsilon - \epsilon_0)E_0/(\epsilon + 2\epsilon_0)$. Thus, we have

$$E = E_0 - \frac{(\epsilon - \epsilon_0)E_0}{\epsilon + 2\epsilon_0} = \frac{3\epsilon_0}{\epsilon + 2\epsilon_0} E_0.$$

This agrees with the result in Example 4.4.

4.8. We define cylindrical coordinates with the z-axis at the central axis of the dielectric cylinder and the azimuthal angle measured from the direction of the applied electric field. We assume that the electric field outside the dielectric cylinder ($R > a$) produced by the polarized charge is given by the linear electric dipole of moment $\hat{p}$ in a unit length placed at the central axis after virtually removing the dielectric cylinder. The direction of the dipole moment is the same as that of the applied electric field. We assume that the electric field inside the dielectric cylinder ($R < a$) has a uniform strength E and is directed parallel to the applied electric field ($\varphi = 0$). The continuity conditions for the parallel (azimuthal) component of the electric field and the normal (radial) component of the electric flux density give

$$\hat{p} = \frac{\epsilon - \epsilon_0}{\epsilon + \epsilon_0} \cdot 2\pi\epsilon_0 a^2 E_0, \qquad E = \frac{2\epsilon_0}{\epsilon + \epsilon_0} E_0.$$

The electric field is

$$E_R = \frac{D_R}{\epsilon_0} = \left(1 + \frac{\epsilon - \epsilon_0}{\epsilon + \epsilon_0} \cdot \frac{a^2}{R^2}\right) E_0 \cos\varphi, \qquad E_\varphi = \frac{D_\varphi}{\epsilon_0} = -\left(1 - \frac{\epsilon - \epsilon_0}{\epsilon + \epsilon_0} \cdot \frac{a^2}{R^2}\right) E_0 \sin\varphi,$$

outside the dielectric cylinder ($R > a$) and

$$E_R = \frac{D_R}{\epsilon} = \frac{2\epsilon_0}{\epsilon + \epsilon_0} E_0 \cos\varphi, \qquad E_\varphi = \frac{D_\varphi}{\epsilon} = -\frac{2\epsilon_0}{\epsilon + \epsilon_0} E_0 \sin\varphi.$$

inside the dielectric cylinder ($0 \le R < a$). The electric polarization inside the dielectric cylinder is

$$P = (\epsilon - \epsilon_0)E = \frac{2\epsilon_0(\epsilon - \epsilon_0)}{\epsilon + \epsilon_0} E_0.$$

Here we apply Eq. (4.9) to a small shell that includes the surface of the dielectric cylinder, as shown in Fig. 4.17. Since there is no true electric charge on the surface, the surface polarization charge density is given by the difference in the normal component of the electric field on the surface multiplied by ϵ_0;

$$\sigma_p(\varphi) = \frac{2\epsilon_0(\epsilon - \epsilon_0)}{\epsilon + \epsilon_0} E_0 \cos\varphi = P \cos\varphi.$$

4.9. We define the x-y plane ($z = 0$) on the dielectric material surface and the position of the line current as $x = 0$. To determine the electric potential in the vacuum region ($z > 0$), we assume that all the space is vacuum and the electric potential is produced by both the line charge of linear density λ and a virtual line charge of linear density λ' located at the symmetric line with respect to the dielectric material surface;

$$\phi_v(x, z) = \frac{1}{2\pi\epsilon_0} \left\{ \lambda \log \frac{R_0}{[x^2 + (z - a)^2]^{1/2}} + \lambda' \log \frac{R_0}{[x^2 + (z + a)^2]^{1/2}} \right\}.$$

In the above R_0 is the distance of the reference point from the line at x=0 on the surface. To determine the electric potential inside the dielectric material ($z < 0$), we assume that all the space is occupied by the dielectric material and the electric potential is given by a line charge of linear density λ'' placed at the original position. Hence, the electric potential at (x, z) inside the dielectric material is

$$\phi_d(x, z) = \frac{1}{2\pi\epsilon} \lambda'' \log \frac{R_0}{[x^2 + (z - a)^2]^{1/2}}.$$

The continuity condition of the parallel component of the electric field, Eq. (4.24), gives $\phi_{\rm v}(z=0) = \phi_{\rm d}(z=0)$. This yields

$$\frac{\lambda + \lambda'}{\epsilon_0} = \frac{\lambda''}{\epsilon}.$$

Since there is no true electric charge on the surface, the normal component of the electric flux density is continuous. Then, $\epsilon_0(\partial\phi_{\rm v}/\partial z)_{z=0} = \epsilon(\partial\phi_{\rm d}/\partial z)_{z=0}$ given by Eq. (4.20) yields

$$\lambda - \lambda' = \lambda''.$$

From these conditions we obtain the linear electric charge densities as

$$\lambda' = -\frac{\epsilon - \epsilon_0}{\epsilon + \epsilon_0}\lambda, \qquad \lambda'' = \frac{2\epsilon}{\epsilon + \epsilon_0}\lambda.$$

The electric potential is

$$\begin{aligned} \phi &= \frac{\lambda}{2\pi\epsilon_0}\left\{\log\frac{R_0}{[x^2 + (z-a)^2]^{1/2}} - \frac{\epsilon - \epsilon_0}{\epsilon + \epsilon_0}\log\frac{R_0}{[x^2 + (z+a)^2]^{1/2}}\right\}; \qquad z > 0, \\ &= \frac{\lambda}{\pi(\epsilon + \epsilon_0)}\log\frac{R_0}{[x^2 + (z-a)^2]^{1/2}}; \qquad z < 0. \end{aligned}$$

Chapter 5

5.1. We apply voltage V between the two edges. The electric field along the circle of radius R from the center is $E(R) = 2V/(\pi R)$ (see Fig. B5.1). Hence, the current density at this point is $i(R) = 2V/(\pi\rho_{\rm r}R)$. Here we define the angle θ as in the

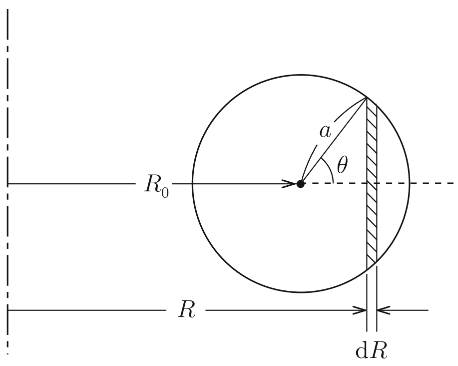


Fig. B5.1 Part in the region R to $R + {\rm d}R$ from the center figure. Then, $R = R_0 + a\cos\theta$. The current flowing in the region between R and $R + \mathrm{d}R$ is $i(R)2a\sin\theta\mathrm{d}R = 4Va^2\sin^2\theta\mathrm{d}\theta/[\pi\rho_\mathrm{r}(R_0 + a\cos\theta)]$. Hence, the total current is

$$I = \int_0^\pi \frac{4Va^2\sin^2\theta\mathrm{d}\theta}{\pi\rho_\mathrm{r}(R_0 + a\cos\theta)}.$$

We transform the integrand as

$$\frac{\sin^2\theta}{R_0 + a\cos\theta} = \frac{R_0}{a^2} - \frac{1}{a}\cos\theta - \left(\frac{R_0^2}{a^2} - 1\right)\frac{1}{R_0 + a\cos\theta}.$$

For integration of the third term, we use Eq. (7.26) with $\theta = \pi - \varphi$. A simple calculation gives

$$I = \frac{4V}{\rho_\mathrm{r}}\,[R_0 - (R_0^2 - a^2)^{1/2}].$$

Then, we obtain the electric resistance as

$$R_\mathrm{r} = \frac{\rho_\mathrm{r}}{4[R_0 - (R_0^2 - a^2)^{1/2}]}.$$

5.2. The cross-sectional area at height x from the bottom is $S(x) = a[b-(b-c)x/h]$, and the current density there is $i(x) = I/\{a[b-(b-c)x/h]\}$, when we apply current I. Since the electric field is $E(x) = \rho_\mathrm{r} i(x)$, the voltage between the two edges is

$$V = \int_0^h \frac{\rho_\mathrm{r} I}{a[b-(b-c)x/h]}\mathrm{d}x = \frac{h\rho_\mathrm{r} I}{a(b-c)}\log\frac{b}{c}.$$

The electric resistance is

$$R_\mathrm{r} = \frac{h\rho_\mathrm{r}}{a(b-c)}\log\frac{b}{c}.$$

5.3. When we apply current I, the current density at distance R from the central axis is

$$i(R) = \frac{I}{2\pi l R}.$$

The electric field is $E(R) = \rho_{\mathrm{r}1} i(R)$ for $a < R < b$ and $E(R) = \rho_{\mathrm{r}2} i(R)$ for $b < R < c$. Hence, the voltage between the two electrodes is

$$V = \int_a^b \frac{\rho_{r1} I}{2\pi l R} dR + \int_b^c \frac{\rho_{r2} I}{2\pi l R} dR = \frac{I}{2\pi l} \left(\rho_{r1} \log \frac{b}{a} + \rho_{r2} \log \frac{c}{b} \right).$$

The resistance is

$$R_r = \frac{V}{I} = \frac{1}{2\pi l} \left(\rho_{r1} \log \frac{b}{a} + \rho_{r2} \log \frac{c}{b} \right).$$

5.4. We use I_1 and I_2 to denote the currents flowing in the respective regions with the electric resistivities ρ_{r1} and ρ_{r2} when we apply voltage V between the electrodes. Then, the current densities at positions at distance R ($a \le R \le b$) in the respective regions are $i_1(R) = I_1/(\pi R l)$ and $i_2(R) = I_2/(\pi R l)$, and the electric fields are $E_1(R) = \rho_{r1} I_1/(\pi R l)$ and $E_2(R) = \rho_{r2} I_2/(\pi R l)$. From the conditions that the integrations of the electric fields from $R = a$ to $R = b$ are V, we have

$$I_1 = \frac{\pi l V}{\rho_{r1} \log(b/a)}, \qquad I_2 = \frac{\pi l V}{\rho_{r2} \log(b/a)}.$$

Since the total current is $I = I_1 + I_2$, we obtain the electric resistance as

$$R_r = \frac{\rho_{r1} \rho_{r2} \log(b/a)}{\pi l (\rho_{r1} + \rho_{r2})}.$$

5.5. When we apply voltage V between the two edges, the electric field at a point of radius R is $E(R) = 2V/(\pi R)$. Hence, the electric power density is $p(R) = 4V^2/(\pi^2 \rho_r R^2)$. The electric power in the region R to $R + dR$ is

$$dP = \frac{\pi w R dR p(R)}{2} = \frac{2wV^2}{\pi \rho_r} \frac{dR}{R}.$$

Thus, the total dissipated electric power is

$$P = \frac{2wV^2}{\pi \rho_r} \int_{R_0 - d/2}^{R_0 + d/2} \frac{dR}{R} = \frac{2wV^2}{\pi \rho_r} \log \frac{R_0 + d/2}{R_0 - d/2}.$$

This is equal to IV.

5.6. When we apply voltage V between the electrodes of the capacitor with a dielectric material of dielectric constant ϵ in the space, the electric field is $E = V/d$ and the electric flux density is $D = \epsilon E = \epsilon V/d$. The surface charge density on the electrode is $\sigma = D = \epsilon V/d$ and the total electric charge is $Q = \sigma S = \epsilon S V/d$. Thus, the capacitance of the capacitor is $C = Q/V = \epsilon S/d$.

When we apply voltage V between the electrodes of the resistor with a substance of electric conductivity σ_c in the space, the electric field is $E = V/d$ and the current density is $i = \sigma_c E = \sigma_c V/d$. The total current is $I = iS = \sigma_c S V/d$. Thus, the electric resistance of the resistor is $R_r = V/I = d/(\sigma_c S)$.

From the above results we obtain the same result as Eq. (5.38):

$$CR_{\mathrm{r}} = \frac{\epsilon}{\sigma_{\mathrm{c}}}.$$

5.7. We can use the answer to Exercise 4.5, if we convert the electric flux density $\boldsymbol{D}$ to the current density $\boldsymbol{i}$ with conversion of the dielectric constants ϵ_0 and ϵ to the electric conductivities $\sigma_{\mathrm{c}0}$ and σ_{c}. The uniform electric field E_0 corresponds to $\sigma_{\mathrm{c}0} i_0$. We define cylindrical coordinates with the z-axis at the central axis of the cylinder and azimuthal angle φ measured from the direction of the applied uniform current. The current density outside the cylinder ($R > a$) is

$$i_R = \left(1 + \frac{\sigma_{\mathrm{c}} - \sigma_{\mathrm{c}0}}{\sigma_{\mathrm{c}} + \sigma_{\mathrm{c}0}} \cdot \frac{a^2}{R^2}\right) i_0 \cos\varphi, \qquad i_\varphi = -\left(1 - \frac{\sigma_{\mathrm{c}} - \sigma_{\mathrm{c}0}}{\sigma_{\mathrm{c}} + \sigma_{\mathrm{c}0}} \cdot \frac{a^2}{R^2}\right) i_0 \sin\varphi,$$

and that inside the cylinder ($R < a$) is

$$i_R = \frac{2\sigma_{\mathrm{c}}}{\sigma_{\mathrm{c}} + \sigma_{\mathrm{c}0}} i_0 \cos\varphi, \qquad i_\varphi = -\frac{2\sigma_{\mathrm{c}}}{\sigma_{\mathrm{c}} + \sigma_{\mathrm{c}0}} i_0 \sin\varphi.$$

5.8. Suppose we replace a substance of electric conductivity σ_{c} with a dielectric material of dielectric constant ϵ and place virtual line charges $\pm\lambda$ at the positions shown in Fig. B5.2. Then, the electric potential at point P on the surface of the left conductor is

$$\phi(a, \theta) = \frac{\lambda}{2\pi\epsilon} \left\{ \log \frac{R_0}{(a^2 + h^2 - 2ah\cos\theta)^{1/2}} - \log \frac{R_0'}{[a^2 + (d-h)^2 - 2a(d-h)\cos\theta]^{1/2}} \right\},$$

where R_0 and R_0' are constants. So that this electric potential is constant and independent of angle θ, the following condition should be satisfied;

$$\frac{ah}{a^2 + h^2} = \frac{a(d-h)}{a^2 + (d-h)^2},$$

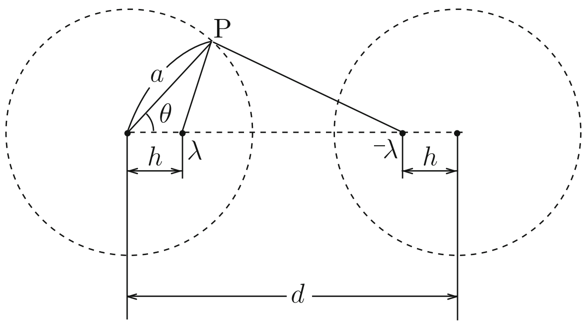


Fig. B5.2 Virtual line charges

which reduces to $h(d-h)=a^2$. This condition for h is simply solved as

$$h=\frac{d-\sqrt{d^2-4a^2}}{2}.$$

The symmetry condition of $\phi=0$ on the central plane between the two conductors gives $R_0=R_0'$. Then, the electric potential of the left conductor is

$$\phi_+=\phi(a,\theta)=\frac{\lambda}{4\pi\epsilon}\log\frac{d+\sqrt{d^2-4a^2}}{d-\sqrt{d^2-4a^2}}=\frac{\lambda}{2\pi\epsilon}\log\frac{d+\sqrt{d^2-4a^2}}{2a}.$$

The electric potential of the right conductor is $\phi_-=-\phi(a,\theta)$. Hence, the capacitance in a unit length is

$$C'=\frac{\lambda}{2\phi(a,\theta)}=\frac{\pi\epsilon}{\log[(d+\sqrt{d^2+4a^2})/2a]}.$$

Using Eq. (5.38), we obtain the electric resistance in a unit length as

$$R_{\mathrm{r}}'=\frac{1}{\pi\sigma_{\mathrm{c}}}\log\frac{d+\sqrt{d^2+4a^2}}{2a}.$$

Chapter 6

6.1. All contributions to the magnetic flux density at the center O produced by elementary currents at respective regions point normal backward. The angle θ in Eq. (6.5) is zero on any point on the left straight section and is π on any point on the right straight section. Thus, there is no contribution to the resultant magnetic flux density from these sections. The angle θ is $\pi/2$ and $r=a$ on the semicircle. The contribution from the elementary current in this section is $\mathrm{d}B=\mu_0 I\,\mathrm{d}s/(4\pi a^2)$. Integrating this over the semicircle yields

$$B=\frac{\mu_0 I}{4\pi a^2}\cdot\pi a=\frac{\mu_0 I}{4a}.$$

6.2. The distance between one side and point P is $l=[(a^2/4)+b^2]^{1/2}$. Using the same method as in Example 6.2, we calculate the magnetic flux density produced by the current on one side as

$$B'=\frac{\mu_0 I}{4\pi l}\int_{\theta_1}^{\pi-\theta_1}\sin\theta\,\mathrm{d}\theta=\frac{\mu_0 I}{2\pi l}\cos\theta_1=\frac{\mu_0 I a}{4\pi l[(a^2/4)+l^2]^{1/2}},$$

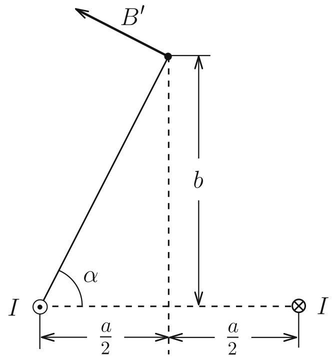


Fig. B6.1 Magnetic flux density produced at point P by current on one side

where θ_1 is the angle of point P measured from the edge of the side. Figure B6.1 schematically shows the magnetic flux density produced by the current on one side. Only the vertical component remains from symmetry, and we obtain as

$$B = 4B' \cos\alpha = \frac{\mu_0 I a^2}{2\pi l^2[(a^2/4) + l^2]^{1/2}} = \frac{\mu_0 I a^2}{2\pi[(a^2/4) + b^2][(a^2/2) + b^2]^{1/2}}.$$

6.3. The current density is expressed as $i = nqv$ in terms of the velocity v of an electric charge. The force of $F = -qvB$ acts on the charge along the y-axis, resulting in the condition that the charges are accumulated on the side of the negative y-axis. Such an accumulation causes the electric field E along the y-axis and the electric force $F' = qE$ works on the charge. In the steady state we attain the balanced condition given by $F + F' = 0$. Hence, we obtain the electric field as

$$E = vB = \frac{iB}{nq}.$$

This is called the **Hall electric field**. The sign of the Hall electric field is determined by the sign of electric charge. That is, the sign of the Hall electric field clarifies whether the current-carrying charges are electrons ($q = -e$) or holes ($q = e$). This phenomenon, i.e., the induction of the electric field in the direction normal to the current and magnetic flux density is called the **Hall effect**.

6.4. Closed circuit C is projected on a plane normal to the current, as shown in Fig. B6.2. We denote the projected closed trajectory and elementary line vector d$\boldsymbol{s}$ as C′ and d$\boldsymbol{s}'$, respectively. Since the magnetic flux density $\boldsymbol{B}$ stays in a plane normal to the current, we have $\boldsymbol{B} \cdot \mathrm{d}\boldsymbol{s} = \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{s}'$. That is, the following relationship holds;

$$\int_{\mathrm{C}} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{s} = \int_{\mathrm{C}'} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{s}'.$$

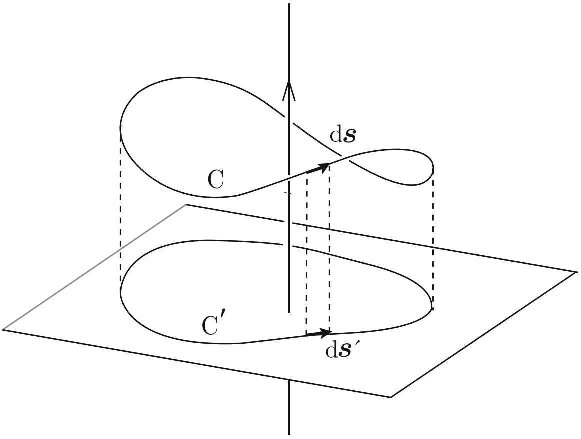


Fig. B6.2 Closed line C and its projection C′ on a plane perpendicular to straight current

Hence, Eq. (6.22) holds for an arbitrary closed line. We can similarly prove Eq. (6.23) when the current does not penetrate the closed line.

6.5. The magnetic flux density is directed along the z-axis and its value is

$$\begin{aligned} B_z(x) &= 0; && x < -b, \\ &= \mu_0 i(x+b); && -b < x < -a, \\ &= \mu_0 i(b-a): && -a < x < a, \\ &= \mu_0 i(b-x); && a < x < b, \\ &= 0; && x > b. \end{aligned}$$

The vector potential has only the y-component, A_y, and the from the relationship $B_z = \partial A_y/\partial x$, we have

$$\begin{aligned} A_y(x) &= -\frac{\mu_0 i}{2}(b^2-a^2); && x < -b, \\ &= \frac{\mu_0 i}{2}(x^2+2bx+a^2); && -b < x < -a, \\ &= \mu_0 i(b-a)x: && -a < x < a, \\ &= \frac{\mu_0 i}{2}(-x^2+2bx-a^2); && a < x < b, \\ &= \frac{\mu_0 i}{2}(b^2-a^2); && x > b. \end{aligned}$$

6.6. We define the x- and y-axes along the slab width and current, respectively, with $x = 0$ at the center of the slab. We presume the current $\mathrm{d}I = I\mathrm{d}x/w$ flowing in a thin region x to $x + dx$ as a line current. The magnetic flux density at point P produced by this line current is directed along the negative z-axis, and its value is $dB_z = -\mu_0 dI/[2\pi(d - x)] = -\mu_0 I dx/[2\pi w(d - x)]$. The magnetic flux density at point P is

$$B_z(d) = -\frac{\mu_0 I}{2\pi w}\int_{-w/2}^{w/2}\frac{dx}{d - x} = -\frac{\mu_0 I}{2\pi w}\log\frac{d + w/2}{d - w/2}.$$

The vector potential at $x > w/2$ is directed along the y-axis, and from $B_z(x) = \partial A_y/\partial x$ we have

$$A_y(x) = \int B_z(x)dx = -\frac{\mu_0 I}{2\pi w}\left[\left(x + \frac{w}{2}\right)\log\left(x + \frac{w}{2}\right) - \left(x - \frac{w}{2}\right)\log\left(x - \frac{w}{2}\right) + C\right],$$

where C is a constant determined by the position of reference point and $A_y(d)$ is the value on point P.

6.7. The current density is $i = I/[\pi(a^2 - b^2)]$. We can solve this problem by superposing case (a) in which the current flows uniformly with density i in the whole cross-section (see Fig. B6.3a) and case (b) in which the current flows uniformly with density i along the opposite direction inside the vacancy (see Fig. B6.3b). The contribution to the magnetic flux density at the vacancy center A from case (a) is $B_1 = \mu_0 \pi d^2 i/(2\pi d) = \mu_0 I d/[2\pi(a^2 - b^2)]$ and that from case (b) is $B_2 = 0$. Hence, the magnetic flux density at A is directed upward and its strength is

$$B_A = B_1 = \frac{\mu_0 I d}{2\pi(a^2 - b^2)}.$$

The contribution to the magnetic flux density at point B from (a) is $B_3 = \mu_0 a^2 i/(2R) = \mu_0 I a^2/[2\pi(a^2 - b^2)R]$ and that from (b) is $B_4 = -\mu_0 b^2 i/[2(R -$

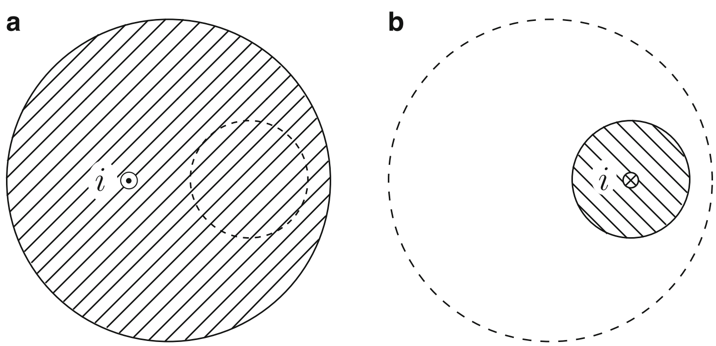


Fig. B6.3 Superposition of (**a**) current flowing uniformly with density i and (**b**) current flowing uniformly with density i along the opposite direction inside the vacancy

$d)] = -\mu_0 I b^2/[2\pi(a^2-b^2)(R-d)]$. These point in the same direction. Thus, the magnetic flux density at B is directed upward and its strength is

$$B_{\rm B} = B_3 + B_4 = \frac{\mu_0 I}{2\pi(a^2-b^2)}\left(\frac{a^2}{R} - \frac{b^2}{R-d}\right).$$

6.8. The divergence of Eq. (6.33) is

$$\nabla\cdot\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int_{\rm V}\nabla\cdot\frac{\boldsymbol{i}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\,{\rm d}V'.$$

Since ∇ is the differential operator with respect to $\boldsymbol{r}$, from Eq. (A1.40) we have

$$\nabla\cdot\frac{\boldsymbol{i}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|} = \boldsymbol{i}(\boldsymbol{r}')\cdot\nabla\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}.$$

If we use the differential operator ∇' with respect to $\boldsymbol{r}'$, $\nabla|\boldsymbol{r}-\boldsymbol{r}'|^{-1} = -\nabla'|\boldsymbol{r}-\boldsymbol{r}'|^{-1}$. Thus, using Eq. (A1.40) again, the integrand is written as

$$-\boldsymbol{i}(\boldsymbol{r}')\cdot\nabla'\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} = \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\nabla'\cdot\boldsymbol{i}(\boldsymbol{r}') - \nabla'\cdot\frac{\boldsymbol{i}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}.$$

Since $\nabla'\cdot\boldsymbol{i}(\boldsymbol{r}') = 0$, applying Gauss' law yields

$$-\int_{\rm V}\nabla'\cdot\frac{\boldsymbol{i}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\,{\rm d}V' = -\int_{\rm S}\frac{\boldsymbol{i}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\cdot{\rm d}\boldsymbol{S}'.$$

If we assume the surface S of region V at infinity, the surface integral reduces to zero, and we prove Eq. (6.30).

6.9. The vector potential has only the azimuthal component, A_φ, since the current flows only along this direction. This is related to the axial magnetic flux density, B_z, through $(1/R)(\partial R A_\varphi/\partial R) = B_z$. Outside the coil ($R > a$), substituting $B_z = 0$ yields $A_\varphi = C_1/R$ with C_1 being a constant. Inside the coil ($0 \le R < a$), substituting $B_z = \mu_0 n I$ yields

$$A_\varphi = \frac{1}{2}\mu_0 n I R + \frac{C_2}{R}$$

with C_2 being a constant. Since the value of A_φ must be finite at $R = 0$, we find that $C_2 = 0$. The continuity at $R = a$ gives $C_1 = \mu_0 n I a^2/2$. Thus, the vector potential is

$$A_\varphi(R) = \frac{\mu_0 n I R}{2}; \qquad 0 \le R < a,$$

$$= \frac{\mu_0 n I a^2}{2R}; \qquad R > a.$$

This agrees with the result in Example 6.7.

6.10. We use the coordinates in Fig. 6.21 and place magnetic charges $\pm q_{\rm m}$ at points $(0, 0, \pm d/2)$. Then, we obtain the magnetic potential due to positive and negative magnetic charges similarly to the calculation in Sect. 1.6 as $\phi_{\rm m\pm}(r) = \pm[\mu_0 q_{\rm m}/(4\pi r^2)][r \pm (d/2)\cos\theta]$. The magnetic potential due to the magnetic charge pair is

$$\phi_{\rm m}(r) = \phi_{\rm m+}(r) + \phi_{\rm m-}(r) = \frac{\mu_0 q_{\rm m} d}{4\pi r^2}\cos\theta = \frac{\mu_0 m}{4\pi r^2}\cos\theta.$$

The magnetic flux density, Eq. (6.44c), is derived using Eq. (6.49).

Chapter 7

7.1. The current I_1 flows uniformly on the surface of the inner superconductor ($R = a$) and the induced current $-I_1$ flows uniformly on the inner surface of the outer superconductor ($R = b$). The current $I_1 + I_2$ flows on the outer surface of the outer superconductor ($R = c$), following the conservation law of current. The resultant magnetic flux density has the azimuthal component and its value is

$$\begin{aligned} B_\varphi &= 0; & 0 \le R < a, \\ &= \frac{\mu_0 I_1}{2\pi R}; & a < R < b, \\ &= 0; & b < R < c, \\ &= \frac{\mu_0 (I_1 + I_2)}{2\pi R}; & R > c. \end{aligned}$$

The vector potential has the z-component and its value is

$$\begin{aligned} A_z &= \frac{\mu_0 (I_1 + I_2)}{2\pi}\log\frac{R_0}{R}; & R > c, \\ &= \frac{\mu_0 (I_1 + I_2)}{2\pi}\log\frac{R_0}{c}; & b < R < c, \\ &= \frac{\mu_0 I_1}{2\pi}\log\frac{bR_0}{cR} + \frac{\mu_0 I_2}{2\pi}\log\frac{R_0}{c}; & a < R < b, \\ &= \frac{\mu_0 I_1}{2\pi}\log\frac{bR_0}{ac} + \frac{\mu_0 I_2}{2\pi}\log\frac{R_0}{c}; & 0 \le R < a, \end{aligned}$$

where $R = R_0 (> c)$ is the position of the reference point.

7.2. We denote the current flowing on the surface at $x = b$ by I_b. Then, the current at $x = a$ is $-I_b$. So that the magnetic flux density does not penetrate the superconductor of $a < x < b$, the current at $x = b$ must be the same as the total current in the region $x < a$, i.e., $I - I_b$. Thus, we have $I_b = I/2$ and the current at $x = a$ is $-I/2$. We similarly obtain the currents on the surfaces of the left superconductor. As a result, the currents at $x = -b, -a, a$ and b are $I/2, I/2, -I/2$ and $I/2$, respectively.

The magnetic flux density is directed along the z-axis and its value is

$$
\begin{aligned}
B_z &= -\frac{\mu_0 I}{2l}; && x < -b, \\
&= 0; && -b < x < -a,\ a < x < b, \\
&= \frac{\mu_0 I}{2l}; && -a < x < a, \\
&= \frac{\mu_0 I}{2l}; && x > b.
\end{aligned}
$$

The vector potential has only the y-component and we obtain from $B_z = \partial A_y/\partial x$ as

$$
\begin{aligned}
A_y &= -\frac{\mu_0 I(x + 2a + b)}{2l}; && x < -b, \\
&= -\frac{\mu_0 I a}{l}; && -b < x < -a, \\
&= \frac{\mu_0 I(x - a)}{2l}; && -a < x < a, \\
&= 0; && a < x < b, \\
&= \frac{\mu_0 I(x - b)}{2l}; && x > b,
\end{aligned}
$$

where the vector potential of the right superconductor without electric current is defined to be zero.

7.3. The reason why the magnetic flux density produced by the current flowing on the superconductor surface is doubled is that there is other magnetic flux density contributions from currents flowing in other areas. For the same reason the magnetic flux density inside the superconductor cancels to zero. Examples are found in the case where a current flows along the opposite direction on the surface of the opposite plate of a superconducting transmission line, as shown in Fig. B7.1a, or in the case where a current flows along the same direction on the opposite surface of the superconductor, as shown in Fig. B7.1b. The situation in Example 6.5 corresponds to the thin limit of the superconductor in Fig. B7.1b.

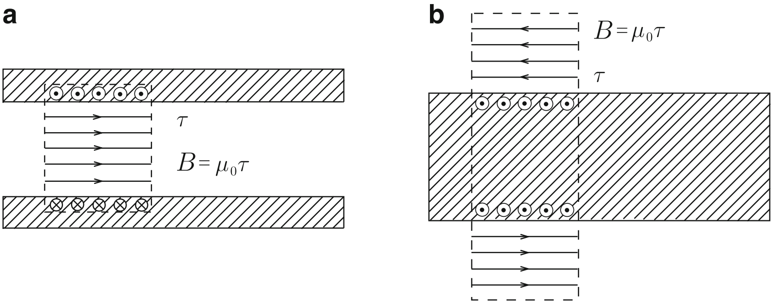


Fig. B7.1 Examples of doubled magnetic flux density: (**a**) superconducting transmission line with opposite currents and (**b**) superconductor with same currents in the both sides

7.4. We define the x-axis on the superconductor surface along the direction normal to the current and the position of the current to be $x = 0$. The density of the current induced on the surface is given by Eq. (7.16). The force on I in a unit length caused by the current $\mathrm{d}I = \tau(x)\mathrm{d}x$ flowing in a thin region x to $x + \mathrm{d}x$ is $\mathrm{d}F' = \mu_0 I^2 a\mathrm{d}x/[2\pi^2(x^2+a^2)^{3/2}]$. From symmetry only the component normal to the surface remains: $\mathrm{d}F'_z = [a/(x^2+a^2)^{1/2}]\mathrm{d}F'$. Thus, the total force in a unit length is

$$F'_z = \frac{\mu_0 I^2 a^2}{\pi^2}\int_0^\infty \frac{\mathrm{d}x}{(x^2+a^2)^2} = \frac{\mu_0 I^2}{\pi^2 a}\int_0^{\pi/2}\cos^2\theta\mathrm{d}\theta = \frac{\mu_0 I^2}{4\pi a}.$$

This agrees with the image force, Eq. (7.18).

7.5. We denote two superconductor surfaces that are perpendicular to each other by the x-y and y-z planes, as shown in Fig. B7.2. Assume the given current I is located at (a, b) on the x-z plane. We virtually remove the superconductor and place three image currents, $-I$, $-I$ and I, at $(a, -b)$, $(-a, b)$ and $(-a, -b)$, respectively. Then,

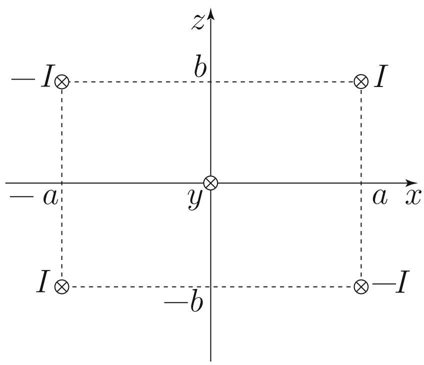


Fig. B7.2 Current I and three image currents the vector potential in the vacuum region ($x > 0, z > 0$) is

$$A_y(x,z) = \frac{\mu_0 I}{4\pi} \log \frac{[(x-a)^2 + (z+b)^2][(x+a)^2 + (z-b)^2]}{[(x-a)^2 + (z-b)^2][(x+a)^2 + (z+b)^2]},$$

neglecting a constant term associated with a choice of the reference point. This satisfies $A_y = 0$ on the surfaces $x = 0$ and $z = 0$ and hence, this gives the correct vector potential. We determine the current density on the x-y and y-z planes to be

$$\tau(x,y,0) = -\frac{1}{\mu_0}\left(\frac{\partial A_y}{\partial z}\right)_{z=0} = -\frac{4Iabx}{\pi[(x-a)^2 + b^2][(x+a)^2 + b^2]},$$

$$\tau(0,y,z) = \frac{1}{\mu_0}\left(\frac{\partial A_y}{\partial x}\right)_{x=0} = -\frac{4Iabz}{\pi[(z-b)^2 + a^2][(z+b)^2 + a^2]}.$$

7.6. A simple calculation gives

$$B_R = \frac{1}{R}\cdot\frac{\partial A_z}{\partial\varphi} = \frac{\mu_0 I \sin\varphi}{2\pi}\left[\frac{a^2/d}{R^2 + (a^2/d)^2 - 2(a^2R/d)\cos\varphi} - \frac{d}{R^2 + d^2 - 2dR\cos\varphi}\right],$$

$$B_\varphi = -\frac{\partial A_z}{\partial R} = -\frac{\mu_0 I}{2\pi}\left[\frac{R - (a^2/d)\cos\varphi}{R^2 + (a^2/d)^2 - 2(a^2R/d)\cos\varphi} - \frac{R - d\cos\varphi}{R^2 + d^2 - 2dR\cos\varphi}\right],$$

$$B_z = 0.$$

7.7. From Eq. (7.13) the equivector-potential surface is given by

$$\frac{x^2 + (z+a)^2}{x^2 + (z-a)^2} = K,$$

with K denoting a constant. This is transformed to

$$x^2 + \left(z - \frac{K+1}{K-1}\,a\right)^2 = \frac{4a^2K}{(K-1)^2}.$$

This expresses a cylindrical surface parallel to the y-axis. Thus, we can see that the vector potential lines on the surface.

7.8. We assume that the magnetic flux density in the vacuum region is the same as that when we place an image current I in the superconducting cylinder at distance h from the center of the cylinder and an image current $-I$ in the infinite superconductor at distance $l - h$ from its surface after virtually removing the two superconductors (see Fig. B7.3), similarly to the answer to Exercise 2.9. In this case the boundary condition on the infinite superconductor surface is satisfied. If the distance $2l - h$ between the image current $-I$ and the cylinder center corresponds to d in Fig. 7.10, the boundary condition on the cylinder surface is also satisfied. From the above relationship and Eq. (7.22) we obtain

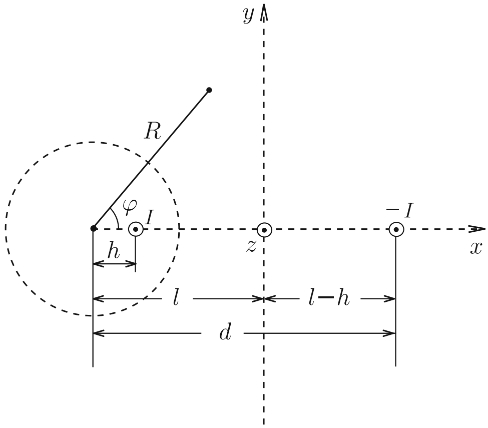


Fig. B7.3 Image currents placed in two superconductors

$$d = l + \sqrt{l^2 - a^2}, \qquad h = l - \sqrt{l^2 - a^2}.$$

Substituting these into Eq. (7.23) yields the vector potential outside the superconductors;

$$A_z(R, \varphi) = -\frac{\mu_0 I}{4\pi} \log \frac{R^2 + (l - \sqrt{l^2 - a^2})^2 - 2R(l - \sqrt{l^2 - a^2}) \cos\varphi}{R^2 + (l + \sqrt{l^2 - a^2})^2 - 2R(l + \sqrt{l^2 - a^2}) \cos\varphi}.$$

We find that the vector potential on the surface, $A_z(a, \varphi) = -[\mu_0 I/(2\pi)] \log[(l - \sqrt{l^2 - a^2})/a]$, is constant. The current density on the cylinder surface is

$$\tau = -\frac{1}{\mu_0} \left(\frac{\partial A_z}{\partial R} \right)_{R=a} = \frac{I}{2\pi a} \cdot \frac{\sqrt{l^2 - a^2}}{l - a \cos\varphi}.$$

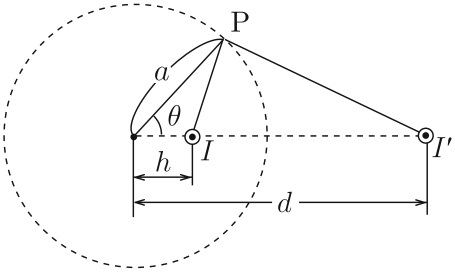
Next we define Cartesian coordinates with the y-z plane ($x = 0$) on the infinite superconductor surface and the central axis of the cylindrical superconductor at $y = 0$. From the relationships $R \cos\varphi = x + l$ and $R \sin\varphi = y$, the vector potential is also expressed as

$$A_z(x, y) = -\frac{\mu_0 I}{4\pi} \log \frac{(x + \sqrt{l^2 - a^2})^2 + y^2}{(x - \sqrt{l^2 - a^2})^2 + y^2}.$$

Thus, we can easily confirm that $A_z(x = 0) = 0$ is satisfied. The density of the current (along the z-axis) on the infinite superconductor surface, which is equal to $-B_y(x=0)/\mu_0$, is

$$\tau = \frac{1}{\mu_0} \left(\frac{\partial A_z}{\partial x} \right)_{x=0} = -\frac{I\sqrt{l^2 - a^2}}{\pi(y^2 + l^2 - a^2)}.$$

Fig. B7.4 Image current I'



7.9. We virtually remove the superconductor and place an image current I' parallel to the current I on a plane including the central axis and the current I (see Fig. B7.4). We denote the distance between the central axis and the image current by d. The vector potential at point P on the inner surface of the superconductor is

$$A_z(a,\varphi) = \frac{\mu_0 I'}{2\pi}\log\frac{R_0'}{(a^2+d^2-2ad\cos\varphi)^{1/2}} + \frac{\mu_0 I}{2\pi}\log\frac{R_0}{(a^2+h^2-2ah\cos\varphi)^{1/2}}.$$

The conditions that satisfy $A_z(a,\varphi)$=const give

$$I' = -I, \qquad d = \frac{a^2}{h}.$$

Since the total current is zero ($I + I' = 0$), we can choose the infinity as the reference point of the vector potential and we have $R_0 = R_0'$. The vector potential in the hollow is

$$A_z(R,\varphi) = \frac{\mu_0 I}{2\pi}\log\frac{[R^2+(a^2/h)^2-2(a^2R/h)\cos\varphi]^{1/2}}{(R^2+h^2-2Rh\cos\varphi)^{1/2}}.$$

The current density on the inner surface is

$$\tau(\varphi) = \frac{B_\varphi(R=a)}{\mu_0} = -\frac{1}{\mu_0}\left(\frac{\partial A_z}{\partial R}\right)_{R=a} = -\frac{I(a^2-h^2)}{2\pi a(a^2+h^2-2ah\cos\varphi)}.$$

7.10. The radial and zenithal components of the applied magnetic flux density outside the spherical superconductor are $B_0\cos\theta$ and $-B_0\sin\theta$, respectively. The radial and zenithal components due to the magnetic moment m at a point at distance r from the origin are $\mu_0 m\cos\theta/(2\pi r^3)$ and $\mu_0 m\sin\theta/(4\pi r^3)$, respectively. The condition that the radial component of the magnetic flux density just outside the surface is zero is written as

$$B_0 \cos\theta + \frac{\mu_0 m \cos\theta}{2\pi a^3} = 0,$$

which gives

$$m = -\frac{2\pi a^3}{\mu_0} B_0.$$

The zenithal component of the magnetic flux density just outside the surface is equal to the surface current density τ multiplied by μ_0. Thus, we have

$$\tau = \frac{1}{\mu_0}\left(-B_0 \sin\theta + \frac{\mu_0 m \sin\theta}{4\pi a^3}\right) = -\frac{3}{2\mu_0} B_0 \sin\theta.$$

These results agree with Eqs. (7.30) and (7.33).

7.11. For simplicity we consider the case where we apply a current to a wide slab material. Since the internal magnetic flux density must be zero, the current must flow on the surface. If Ohm's law holds, the electric field $\boldsymbol{E}$ has a non-zero value only on the surface, while it is zero inside the material. Then, we can easily show that $\nabla \times \boldsymbol{E}$ is not zero in violation of the fundamental relationship of Eq. (1.28).

Chapter 8

8.1. When we apply current I to the left line, the magnetic flux that penetrates the coil is

$$\Phi_{\mathrm{l}} = \int_c^{b+c} \frac{\mu_0 I}{2\pi x} w(x) \mathrm{d}x,$$

where $w(x) = (a/b)(b+c-x)$ is the width of the triangle at distance x from the line. A simple calculation gives

$$\Phi_{\mathrm{l}} = \frac{\mu_0 I a}{2\pi}\left(\frac{b+c}{b} \log \frac{b+c}{c} - 1\right).$$

The magnetic flux produced by the current on the right line is similarly given by

$$\Phi_{\mathrm{r}} = \frac{\mu_0 I a}{2\pi}\left(1 - \frac{d-b-c}{b} \log \frac{d-c}{d-b-c}\right).$$

Thus, we obtain the mutual inductance as

$$M = \frac{\Phi_l + \Phi_r}{I} = \frac{\mu_0 a}{2\pi b}\left[(b+c)\log\frac{b+c}{c} - (d-b-c)\log\frac{d-c}{d-b-c}\right].$$

8.2. The magnetic flux stays only in the region between the two superconductors and the density is $B = \mu_0 I/(2\pi R)$. Hence, the magnetic flux in a unit length is

$$\Phi' = \int_a^b \frac{\mu_0 I}{2\pi R}\,\mathrm{d}R = \frac{\mu_0 I}{2\pi}\log\frac{b}{a}.$$

We obtain the self-inductance in a unit length as

$$L' = \frac{\Phi'}{I} = \frac{\mu_0}{2\pi}\log\frac{b}{a}.$$

This agrees with the result calculated from the magnetic energy.

8.3. In the case of conductor, the current flows uniformly inside the conductor, and the magnetic flux densities in the regions $0 \le R < a$ and $b < R < c$ are respectively given by

$$\begin{aligned} B(R) &= \frac{\mu_0 I R}{2\pi a^2}; && 0 \le R < a, \\ &= \frac{\mu_0 I}{2\pi(c^2-b^2)}\left(\frac{c^2}{R} - R\right); && b < R < c. \end{aligned}$$

Hence, in comparison with the case of superconductor, the magnetic energy increases by

$$\begin{aligned} \Delta U'_{\mathrm{m}} &= \frac{1}{2\mu_0}\int_0^a \left(\frac{\mu_0 I R}{2\pi a^2}\right)^2 \cdot 2\pi R\mathrm{d}R + \frac{1}{2\mu_0}\int_b^c \left[\frac{\mu_0 I}{2\pi(c^2-b^2)}\right]^2 \left(\frac{c^2}{R} - R\right)^2 \cdot 2\pi R\mathrm{d}R \\ &= \frac{\mu_0 c^2 I^2}{8\pi(c^2-b^2)}\left(\frac{2c^2}{c^2-b^2}\log\frac{c}{b} - 1\right). \end{aligned}$$

Adding this contribution to the result in Exercise 8.5, we obtain the self-inductance in a unit length as

$$L' = \frac{\mu_0}{2\pi}\log\frac{b}{a} + \frac{\mu_0 c^2}{4\pi(c^2-b^2)}\left(\frac{2c^2}{c^2-b^2}\log\frac{c}{b} - 1\right).$$

8.4. When current I flows in the parallel-wire transmission line as shown in Fig. B8.1, the magnetic flux that penetrates upward the coil by the right current is

$$\phi = w\int_b^{(a^2+b^2)^{1/2}} \frac{\mu_0 I}{2\pi r}\mathrm{d}r = \frac{\mu_0 I w}{2\pi}\log\frac{(a^2+b^2)^{1/2}}{b}.$$

The magnetic flux produced by the left current is the same and the total magnetic flux is $\Phi = 2\phi$. The mutual inductance is

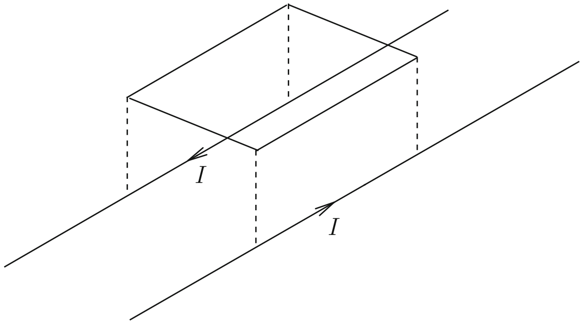


Fig. B8.1 Current in parallel-wire transmission line

$$M = \frac{\Phi}{I} = \frac{\mu_0 w}{\pi} \log \frac{(a^2 + b^2)^{1/2}}{b}.$$

8.5. When we apply current I to the outer coil, the magnetic flux density produced in the inner coil is $B = \mu_0 n_\mathrm{b} I$. Hence, the magnetic flux that penetrates one turn of the inner coil is $\phi = \pi a^2 B = \pi \mu_0 n_\mathrm{b} a^2 I$. The magnetic flux penetrating a unit length of this coil is

$$\Phi' = n_\mathrm{a} \phi = \pi \mu_0 n_\mathrm{a} n_\mathrm{b} a^2 I.$$

The mutual inductance in a unit length is

$$M' = \frac{\Phi'}{I} = \pi \mu_0 n_\mathrm{a} n_\mathrm{b} a^2.$$

8.6. We apply Ampere's law to circle C of radius R from the central axis (see Fig. B8.2). The magnetic flux density at this position is $B = \mu_0 NI/(2\pi R)$. If we define the two-dimensional polar coordinates as in the figure, we have $R = d + r\cos\theta$. The magnetic energy is

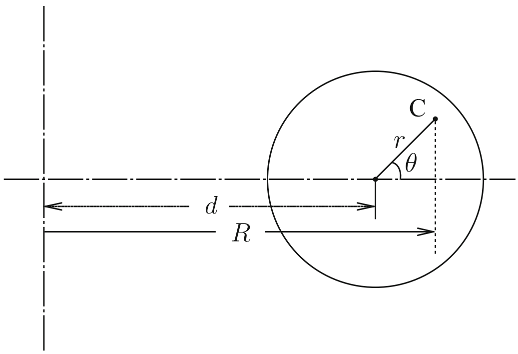


Fig. B8.2 Cross-section of toroidal coil

$$U_{\rm m} = 2\int_0^{\pi} {\rm d}\theta \int_0^a \frac{\mu_0 N^2 I^2}{8\pi^2(d + r\cos\theta)^2}\cdot 2\pi(d + r\cos\theta) r{\rm d}r$$
$$= \frac{\mu_0 N^2 I^2}{2\pi}\int_0^{\pi} {\rm d}\theta \int_0^a \frac{r{\rm d}r}{d + r\cos\theta}.$$

Using Eq. (7.26) for the integral with respect to θ, the magnetic energy leads to

$$U_{\rm m} = \frac{\mu_0 N^2 I^2}{2}\int_0^a \frac{r{\rm d}r}{(d^2 - r^2)^{1/2}} = \frac{\mu_0 N^2 I^2}{2}[d - (d^2 - a^2)^{1/2}].$$

The self-inductance is

$$L = \frac{2U_{\rm m}}{I^2} = \mu_0 N^2[d - (d^2 - a^2)^{1/2}].$$

8.7. (a) The inductance coefficients are

$$L_{11} = \frac{\mu_0}{2\pi}\log\frac{R_1 R_3 R_\infty}{R_0 R_2 R_4},$$
$$L_{21} = L_{12} = L_{22} = \frac{\mu_0}{2\pi}\log\frac{R_3 R_\infty}{R_2 R_4},$$
$$L_{31} = L_{32} = L_{13} = L_{23} = L_{33} = \frac{\mu_0}{2\pi}\log\frac{R_\infty}{R_4}.$$

(b) From Eq. (8.35) we calculate the magnetic energy as

$$U_{\rm m} = \frac{1}{2}L_{11}I_1^2 + \frac{1}{2}L_{22}I_2^2 + \frac{1}{2}L_{33}I_3^2 + L_{12}I_1I_2 + L_{23}I_2I_3 + L_{31}I_3I_1$$
$$= \frac{\mu_0}{4\pi}\left[I_1^2\log\frac{R_1R_3R_\infty}{R_0R_2R_4} + (I_2^2 + 2I_1I_2)\log\frac{R_3R_\infty}{R_2R_4} + (I_3^2 + 2I_2I_3 + 2I_3I_1)\log\frac{R_\infty}{R_4}\right]$$
$$= \frac{\mu_0}{4\pi}\left[(I_1 + I_2 + I_3)^2\log\frac{R_\infty}{R_4} + (I_1 + I_2)^2\log\frac{R_3}{R_2} + I_1^2\log\frac{R_1}{R_0}\right].$$

This result can also be obtained from Eq. (8.40).

8.8. The magnetic flux density in the vacuum region where the superconducting rod is not inserted is $B_1 = \mu_0 I'$. Thus, the magnetic flux that penetrates the superconducting hollow cylinder is $\Phi = \pi b^2 B_1 = \pi\mu_0 b^2 I'$. The magnetic flux is the same in the space of the region where the superconducting rod is inserted, and the magnetic flux density there is $B_2 = b^2 B_1/(b^2 - a^2) = \mu_0 b^2 I'/(b^2 - a^2)$. The current in a unit length flowing on the inner surface of the superconducting hollow cylinder is

$$I_2' = \frac{B_2}{\mu_0} = \frac{b^2 I'}{b^2 - a^2}.$$

On the surface of the inserted superconducting rod the current of the same surface density flows along the opposite direction. Thus, the total magnetic energy is

$$U_{\mathrm{m}} = \frac{1}{2}\Phi I'(l-x) + \frac{1}{2}\Phi I_2'x = \frac{\pi\mu_0 b^2 I'^2}{2}\left(l - x + \frac{b^2 x}{b^2 - a^2}\right),$$

where l is the length of the superconducting hollow cylinder. We obtain the same result from Eq. (8.40). The force on the cylindrical rod is

$$F = -\frac{\partial U_{\mathrm{m}}}{\partial x} = -\frac{\pi\mu_0 a^2 b^2 I'^2}{2(b^2 - a^2)},$$

indicating a repulsive force, since it is negative for increasing x.

8.9. When the distance between the two coils, x, changes to $x + \Delta x$, we assume that I_1 and I_2 change to $I_1 + \Delta I_1$ and $I_2 + \Delta I_2$, respectively. If we neglect small terms of the second order, the conditions that the magnetic fluxes do not change in each coil are given by

$$\Delta\Phi_1 = L_{11}\Delta I_1 + \Delta L_{21} I_2 + L_{21}\Delta I_2 = 0,$$
$$\Delta\Phi_2 = \Delta L_{21} I_1 + L_{21}\Delta I_1 + L_{22}\Delta I_2 = 0,$$

where L_{11}, L_{22} and L_{21} are inductance coefficients and ΔL_{21} is the change in the mutual inductance coefficient. The corresponding change in the magnetic energy is

$$\Delta U_{\mathrm{m}} = L_{11} I_1 \Delta I_1 + L_{21}(I_1\Delta I_2 + I_2\Delta I_1) + \Delta L_{21} I_1 I_2 + L_{22} I_2 \Delta I_2.$$

Using the above two conditions, this reduces to $\Delta U_{\mathrm{m}} = -\Delta L_{21} I_1 I_2$. The mutual inductance coefficient L_{21} is given by

$$L_{21} = -\frac{\mu_0 l}{2\pi}\log\frac{(x+a)(x+b)}{x(x+a+b)}.$$

and the change in L_{21} due to the change in x is $\Delta L_{21} = (\partial L_{21}/\partial x)\Delta x$. Thus, we calculate the magnetic force as

$$F = -\frac{\partial U_{\mathrm{m}}}{\partial x} = \frac{\partial L_{21}}{\partial x} I_1 I_2$$
$$= \frac{\mu_0 l}{2\pi}\left(\frac{1}{x} - \frac{1}{x+a} - \frac{1}{x+b} + \frac{1}{x+a+b}\right) I_1 I_2.$$

We can easily confirm that this agrees with the Lorentz force between the two circuits.

Chapter 9

9.1. The magnetic flux density and magnetic field are parallel to the slab. We denote these values in magnetic materials 1 and 2 by B_1, H_1, B_2 and H_2, respectively. Ampere's law derives $H_1 = H_2 = I/w$, and these satisfy the continuity of the parallel component of the magnetic field on the boundary. These yield $B_1 = \mu_1 I/w$ and $B_2 = \mu_2 I/w$. The magnetic flux in a unit length is $\Phi' = d(B_1 + B_2)/2$ and the self-inductance in a unit length is

$$L' = \frac{(\mu_1 + \mu_2)d}{2w}.$$

9.2. We denote the distance from the center by R. When we apply current I to the parallel-wire transmission line, the magnetic field is $H(R) = I/(2\pi R)$ in the region $a < R < c$ and zero in other regions. Hence, the magnetic flux densities in magnetic materials 1 and 2 are $B_1 = \mu_1 I/w$ and $B_2 = \mu_2 I/w$, respectively. The magnetic flux in a unit length is

$$\Phi' = \int_a^b \frac{\mu_1 I}{2\pi R}\,\mathrm{d}R + \int_b^c \frac{\mu_2 I}{2\pi R}\,\mathrm{d}R = \frac{I}{2\pi}\left(\mu_1 \log\frac{b}{a} + \mu_2 \log\frac{c}{b}\right).$$

The self-inductance in a unit length is

$$L' = \frac{1}{2\pi}\left(\mu_1 \log\frac{b}{a} + \mu_2 \log\frac{c}{b}\right).$$

9.3. We denote the plane determined by the normal vector $\boldsymbol{n}$ on the interface and the magnetic field $\boldsymbol{H}_1$ in magnetic material 1 as S. Assume that the magnetic field $\boldsymbol{H}_2$ in magnetic material 2 does not lie on this plane. We consider a plane, S′, normal to both the interface and S and define a small rectangle on S′ that includes the interface. The two sides of the rectangle are parallel to the interface. When we integrate the magnetic field along this rectangle, the integral in magnetic material 2 is not zero, while that in magnetic material 1 is zero. The circular integral of the magnetic field should be zero, since the planar current $\boldsymbol{\tau}$ flows on plane S′. Hence, the above assumption is contradictory, and we prove that the magnetic field $\boldsymbol{H}_2$ also lies on plane S.

9.4. Since the parallel component of the magnetic field is continuous across the interface, the magnetic field inside the slit is also B_0/μ and the magnetic flux density is $B = (\mu_0/\mu)B_0$.

9.5. Since the normal component of the magnetic flux density is continuous across the interface, the magnetic flux density inside the slit is also $B = B_0$ and the magnetic field is $H = B/\mu_0 = B_0/\mu_0$.

9.6. Applying Ampere's law to a closed line including the magnetic material surface, the magnetic field in the magnetic material is $H = B_0/\mu_0$. Hence, the magnetic flux density in the magnetic material is $B = (\mu/\mu_0)B_0$. The surface magnetizing current density τ_{m} is equal to the magnetization and we obtain

$$\tau_{\mathrm{m}} = M = \left(\frac{1}{\mu_0} - \frac{1}{\mu}\right) B = \frac{(\mu - \mu_0)}{\mu_0^2} B_0.$$

9.7. The magnetic flux density B is given by the sum of B_0 and the component produced by the magnetizing current of surface density, $\tau_{\mathrm{m}}(\theta) = 3(\mu - \mu_0)B_0 \sin\theta/[\mu_0(\mu + 2\mu_0)]$, where θ is the zenithal angle. Since the current of surface density $\tau = -3(B_0/2\mu_0)\sin\theta$ in Eq. (7.33) produces the uniform magnetic flux density $-B_0$ inside the sphere, the magnetizing current produces a uniform magnetic flux density $2(\mu - \mu_0)B_0/(\mu + 2\mu_0)$. Thus, we have

$$B = B_0 + \frac{2(\mu - \mu_0)B_0}{\mu + 2\mu_0} = \frac{3\mu}{\mu + 2\mu_0} B_0.$$

This agrees with Eq. (9.37) in Example 9.4.

9.8. We define cylindrical coordinates with the z-axis at the central axis of the cylindrical magnetic material and the azimuthal angle φ measured from the direction of the applied magnetic flux density. We assume that the magnetic flux density outside the magnetic material ($R > a$) due to its magnetization is given by the linear magnetic dipole of moment $\hat{m}$ in a unit length placed at the central axis after virtually removing the magnetic material. The magnetic flux density inside the magnetic material ($R < a$) B is assumed to be constant. The directions of the linear magnetic dipole and inner magnetic flux density are parallel to that of the applied magnetic flux density. The continuities of the normal (radial) component of the magnetic flux density and the parallel (azimuthal) component of the magnetic field at the surface ($R = a$) give

$$\hat{m} = \frac{\mu - \mu_0}{\mu + \mu_0} \cdot \frac{2\pi a^2 B_0}{\mu_0}, \qquad B = \frac{2\mu}{\mu + \mu_0} B_0.$$

Using these results, the magnetic flux density outside the magnetic material ($R > a$) is

$$B_R = \mu_0 H_R = \left(1 + \frac{\mu - \mu_0}{\mu + \mu_0} \cdot \frac{a^2}{R^2}\right) B_0 \cos\varphi,$$

$$B_\varphi = \mu_0 H_\varphi = -\left(1 - \frac{\mu - \mu_0}{\mu + \mu_0} \cdot \frac{a^2}{R^2}\right) B_0 \sin\varphi,$$

and that inside the magnetic material ($R < a$) is

$$B_R = \mu H_R = \frac{2\mu}{\mu + \mu_0} B_0 \cos\varphi, \qquad B_\varphi = \mu H_\varphi = -\frac{2\mu}{\mu + \mu_0} B_0 \sin\varphi.$$

The magnetization of the magnetic material is

$$M = \left(\frac{1}{\mu_0} - \frac{1}{\mu}\right) B = \frac{2(\mu - \mu_0)}{\mu_0(\mu + \mu_0)} B_0.$$

Here we apply the integral form of Eq. (9.10) to a small rectangle on a plane normal to the central axis that includes the surface of the magnetic material, as shown in Fig. 9.15. Since there is no true current on the surface, the surface magnetizing current density is given by the difference in the parallel component of the magnetic flux density on the surface divided by μ_0;

$$\tau_{\mathrm{m}}(\varphi) = \frac{2(\mu - \mu_0)}{\mu_0(\mu + \mu_0)} B_0 \sin\varphi = M \sin\varphi.$$

9.9. We use B to denote the uniform magnetic flux density inside the spherical superconductor. This is directed parallel to the applied magnetic flux density. The boundary conditions are

$$\left(B_0 + \frac{\mu_0 m}{2\pi a^3}\right)\cos\theta = B\cos\theta, \qquad \frac{1}{\mu_0}\left(-B_0 + \frac{\mu_0 m}{4\pi a^3}\right)\sin\theta = -\frac{B}{\mu_0}\sin\theta + \tau.$$

From the former equation we have $m = 2\pi a^3(B - B_0)/\mu_0$. The magnetic flux density on the superconductor surface is maximum on the equator ($\theta = \pi/2$), and its absolute value is $B_0 - \mu_0 m/(4\pi a^3)$. The critical condition is that this value is equal to the critical magnetic flux density B_{c}. Thus, we have $m = 4\pi a^3(B_0 - B_{\mathrm{c}})/\mu_0$ or

$$-M = -\frac{m}{(4/3)\pi a^3} = \frac{3}{\mu_0}(B_{\mathrm{c}} - B_0).$$

This characteristic shows the descending line in Fig. 7.18 in Column (2) in Chap. 7.

Using this result, we obtain B and τ as

$$B = 3B_0 - 2B_{\mathrm{c}}, \qquad \tau = -\frac{3}{\mu_0}(B_{\mathrm{c}} - B_0)\sin\theta.$$

We can see that the values of B and τ agree with those in the Meissner state given by Eqs. (7.34) and (7.33) at $B_0 = (2/3)B_{\mathrm{c}}$. The quantities M and τ decrease to zero at $B_0 = B_{\mathrm{c}}$, showing the change to the normal state.

9.10. We suppose that currents of surface densities τ_0 and τ flow in the regions of the superconductor facing to the vacuum and the magnetic material, respectively. The magnetic field in the gap region is parallel to the superconductors and its strength is $H_0 = \tau_0$ and $H = \tau$ in the vacuum and magnetic material, and the corresponding magnetic flux density is $B_0 = \mu_0\tau_0$ and $B = \mu\tau$. The boundary condition yields $\mu_0\tau_0 = \mu\tau$. Since the total current is $\tau_0(a-x) + \tau x = I$, we obtain the surface current densities as

$$\tau_0 = \frac{\mu I}{\mu a - (\mu - \mu_0)x}, \qquad \tau = \frac{\mu_0 I}{\mu a - (\mu - \mu_0)x}.$$

The magnetic flux density is

$$B_0 = B = \frac{\mu\mu_0 I}{\mu a - (\mu - \mu_0)x}.$$

Thus, we calculate the magnetic energy as

$$U_\mathrm{m} = bd\left[\frac{B_0^2}{2\mu_0}(a-x) + \frac{B^2}{2\mu}x\right] = \frac{\mu\mu_0 bdI^2}{2[\mu a - (\mu - \mu_0)x]}.$$

The force on the magnetic material is

$$F = -\frac{\partial U_\mathrm{m}}{\partial x} = -\frac{\mu\mu_0(\mu - \mu_0)bdI^2}{2[\mu a - (\mu - \mu_0)x]^2}.$$

Since F is negative ($\mu > \mu_0$), it is directed opposite to increasing x, i.e., repulsive. This agrees with the answer in Example 4.6, if we substitute $Q \to bI, S \to ab, t \to d, \epsilon_0 \to \mu_0^{-1}$ and $\epsilon \to \mu^{-1}$.

Chapter 10

10.1. The magnetic flux penetrating the coil is that staying in the region d to $(a^2 + d^2)^{1/2}$ from the straight line:

$$\Phi = \frac{\mu_0 bI}{2\pi}\int_d^{(a^2+d^2)^{1/2}} \frac{\mathrm{d}R}{R} = \frac{\mu_0 bI}{2\pi}\log\frac{(a^2+d^2)^{1/2}}{d}.$$

The induced electromotive force is

$$V_\mathrm{em} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t} = -\frac{\mu_0 b}{2\pi}\log\frac{(a^2+d^2)^{1/2}}{d}\cdot\frac{\mathrm{d}I(t)}{\mathrm{d}t} = -\frac{\mu_0 I_\mathrm{m} b\omega}{2\pi}\log\frac{(a^2+d^2)^{1/2}}{d}\cos\omega t.$$

10.2. First, we use the magnetic flux law to determine the induced electromotive force. The magnetic flux penetrating the closed circuit is $\Phi = -a(b+vt)$, when the magnetic flux produced by a current flowing along PQRS is defined as positive. The induced electromotive force is

$$V_{\mathrm{em}} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t} = avB.$$

Second, we use the motional law. The electromotive force is induced only on side PQ, and $\boldsymbol{v} \times \boldsymbol{B}$ has magnitude vB and is directed from P to Q. Hence, the induced electromotive force is avB and the result agrees with that from the magnetic flux law.

10.3. We define the origin at R and the x- and y-axes on sides RQ and RS, respectively. Under the given condition, the continuity equation leads to

$$\nabla \times (\boldsymbol{B} \times \boldsymbol{V}) = -\alpha \boldsymbol{i}_z.$$

The left side reduces to

$$B\left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y}\right)\boldsymbol{i}_z.$$

The symmetry condition allows us to assume $(\partial V_x/\partial x) = (\partial V_y/\partial y)$. We can also assume the zero point of $\boldsymbol{V}$ at any point. Under the condition that $\boldsymbol{V} = 0$ at (0, 0), we have $V_x = -\alpha x/(2B)$ and $V_y = -\alpha y/(2B)$. On line PQ($x = b + vt$), $V_x = -\alpha(b + vt)/(2B)$ and $v_x = v$ give $V'_x = -\alpha(b + vt)/(2B) - v$, and the integral of the induced electric field from P to Q is

$$-\int_0^a (\boldsymbol{B} \times \boldsymbol{V'})_y \mathrm{d}y = \frac{\alpha a(b+vt)}{2} + Bva$$

On line SP($y = a$), $V_y = -\alpha a/(2B)$ and $v_y = 0$ give $V'_y = -\alpha a/(2B)$, and the integral of the induced electric field from S to P is

$$\int_0^{b+vt} (\boldsymbol{B} \times \boldsymbol{V}')_x \mathrm{d}x = \frac{\alpha a(b + vt)}{2}.$$

There are no contributions from sides QR and RS. Thus, the induced electromotive force is

$$V_{\mathrm{em}} = \alpha a(b + vt) + Bva.$$

10.4. Since the electric field is induced along the direction parallel to the applied current, there is no contribution to the electromotive force from sides QR and SP. The magnetic flux density on side PQ is $B = \mu_0 I/\{2\pi[R_0^2 + (a+b)^2]^{1/2}\}$ and the induced electric field $\boldsymbol{v} \times \boldsymbol{B}$ has a magnitude

$$E_{\mathrm{PQ}} = \frac{\mu_0 I v(a + d)}{2\pi[R_0^2 + (a + d)^2]},$$

and is directed from P to Q. The induced electric field on side RS has a magnitude

$$E_{RS} = \frac{\mu_0 I v d}{2\pi(R_0^2 + d^2)},$$

and is directed from S to R. Thus, we obtain the induced electromotive force as

$$V_{em} = b(E_{PQ} - E_{RS}) = \frac{\mu_0 I vab[R_0^2 - (d_0 + vt)(a + d_0 + vt)]}{2\pi[R_0^2 + (d_0 + vt)^2][R_0^2 + (a + d_0 + vt)^2]}.$$

10.5. Using the distance $r = (d^2 + a^2 + 2ad\cos\theta)^{1/2}$ between side PQ and the straight line, the magnetic flux that penetrates the coil is $\Phi = -(\mu_0 I b/2\pi)\log(r/d)$. Hence, the induced electromotive force is

$$V_{em} = -\frac{d\Phi}{dt} = -\frac{\mu_0 I abd\omega \sin\omega t}{2\pi(d^2 + a^2 + 2ad\cos\omega t)}.$$

10.6. Since side RS does not move, this does not contribute to the induced electromotive force. Since $\boldsymbol{v} \times \boldsymbol{B}$ is parallel to sides QR and SP, there are no contributions from these sides. Using the distance $r = (d^2 + a^2 + 2ad\cos\theta)^{1/2}$ between side PQ and the straight line, the magnetic flux density on this side is $B = \mu_0 I/(2\pi r)$ and $v = a\omega$. We denote the angle between $\boldsymbol{v}$ and $\boldsymbol{B}$ and the angle from the line to side PQ by α and β (see Fig. B10.1), respectively. From relationships $a\sin\alpha = d\sin\beta$ and $a\sin\theta = r\sin\beta$, we have $\sin\alpha = (d/r)\sin\theta$. Hence, the magnitude of $\boldsymbol{v} \times \boldsymbol{B}$ is $vB\sin\alpha = \mu_0 adI\omega\sin\theta/(2\pi r^2)$, and this is directed from Q to P, i.e., opposite to the integration. Thus, the induced electromotive force is

$$V_{em} = -vBb\sin\alpha = -\frac{\mu_0 I abd\omega \sin\omega t}{2\pi(d^2 + a^2 + 2ad\cos\omega t)}.$$

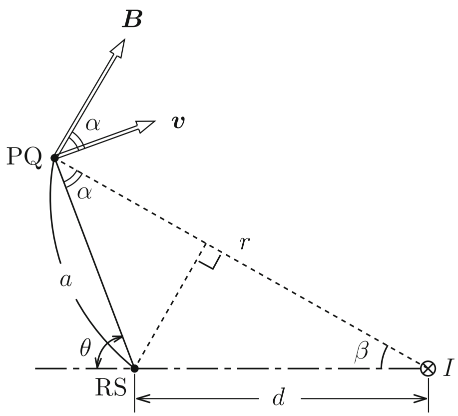


Fig. B10.1 Angles α and β

10.7. We denote the current flowing in the circuit by I. Since the electromotive force induced in the coil is $-L(\partial I/\partial t)$, the potential difference applied to the resistor is $V - L(\partial I/\partial t)$, which is equal to $R_\mathrm{r} I$. Thus, we have

$$V - L\frac{\partial I}{\partial t} = R_\mathrm{r} I.$$

The initial condition is $I(0) = 0$. The solution is

$$I(t) = \frac{V}{R_\mathrm{r}}\left[1 - \exp\left(-\frac{t}{\tau}\right)\right]$$

with $\tau = L/R_r$.

10.8. So that the current $\boldsymbol{I}$ flows in the conductor of electric resistance R'_r in a unit length, the electric field strength that the electric power source supplies is

$$\boldsymbol{E}' = R'_\mathrm{r}\boldsymbol{I} - \boldsymbol{v}\times\boldsymbol{B}.$$

The electric power inside the conductor is the sum of the component from the electric power source, $\boldsymbol{E}'\cdot\boldsymbol{I}$, and the component from the induced electric field, $\boldsymbol{I}\cdot(\boldsymbol{v}\times\boldsymbol{B})$. This leads to $R'_\mathrm{r}\boldsymbol{I}^2$, i.e., the electric power consumed in the conductor. The remaining electric power from the source contributes to the mechanical work on the outside as shown in Eq. (10.17).

Thus, the work done by the induced electric field is virtual and cannot really be measured. The induced electric field prevents all the electric energy from the electric power source from being consumed as Joule heat in the conductor but converts part of it to the mechanical work on the outside.

The Lorentz force does no mechanical work on moving electric charges (electrons). However, the charges driven by the Lorentz force do mechanical work on ions in a material through the Coulomb interaction. This is why the conductor moves. As a result, it looks as thought the Lorentz force does mechanical work. In this case the current decreases because of the law of conservation of energy, if the electric power from the source is not sufficient. The lost energy of the charges is the kinetic energy that is given by the electric power source in the initial state but not given by the Lorentz force.

10.9. We can assume that the derivatives with respect to y and z are zero from spatial symmetry and replace the time derivative by $\mathrm{i}\omega$. We can also assume that the inner electric field has only a z-component, E_z. Equation (10.39) leads to $\mathrm{d}E_z/\mathrm{d}x = \mathrm{i}\omega B_y$, showing that the magnetic flux density has only a y-component. Thus, Eq. (10.43) leads to $\mathrm{d}B_y/\mathrm{d}x = \mu\sigma_\mathrm{c} E_z$. The above two equations yield

$$\frac{\mathrm{d}^2 E_z}{\mathrm{d}x^2} - \mathrm{i}\omega\mu\sigma_\mathrm{c} E_z = 0.$$

We can easily solve this equation under the boundary condition $E_z(x=0) = E_0$. Taking the real part, we have

$$E_z(x,t) = E_0 e^{-x/\delta} \exp\left[i\left(\omega t - \frac{x}{\delta}\right)\right] \to E_0 e^{-x/\delta} \cos\left(\omega t - \frac{x}{\delta}\right).$$

Substituting the complex solution into the first equation yields

$$\begin{aligned} B_y(x,t) &= E_0 \left(\frac{\mu\sigma_c}{\omega}\right)^{1/2} e^{-x/\delta} \exp\left[i\left(\omega t - \frac{x}{\delta} + \frac{3\pi}{4}\right)\right] \\ &\to E_0 \left(\frac{\mu\sigma_c}{\omega}\right)^{1/2} e^{-x/\delta} E_0 \cos\left(\omega t - \frac{x}{\delta} + \frac{3\pi}{4}\right). \end{aligned}$$

10.10. We suppose that current I' flows uniformly on the thin conductor. When we carry a small current, $\Delta I'$, from the position $R = R_\infty$ to the conductor, an attractive force, $\mu_0 I' \Delta I'/(2\pi R)$, works on the small current of a unit length. Since the return current is uniformly distributed at R_∞, the force from the return current cancels. Hence, the work in a unit length necessary to carry the small current to the conductor is negative:

$$\Delta W_1 = \frac{\mu_0 I' \Delta I'}{2\pi} \int_{R_\infty}^{a} \frac{dR}{R} = -\frac{\mu_0 I' \Delta I'}{2\pi} \log \frac{R_\infty}{a}.$$

The electromotive force is induced to reduce the current in both the conductor circuit and the circuit composed of the small current. Hence, the electric power source in each circuit must supply an energy to maintain the current. For example, the magnetic flux penetrating a unit length of the circuit of the small current located at R is $\Phi' = (\mu_0 I'/2\pi)\log(R_\infty/R)$. The electromotive force induced in a unit length is $V_{em} = -d\Phi'/dt = [\mu_0 I'/(2\pi)] dR/dt$. Hence, the electric power necessary for the source to drive the current $\Delta I'$ continuously is $-V_{em}\Delta I'$, and the additional energy necessary to carry it from R_∞ to a is

$$\Delta W_2 = -\int V_{em} \Delta I' dt = -\frac{\mu_0 I' \Delta I'}{2\pi} \int_{R_\infty}^{a} \frac{dR}{R} = \frac{\mu_0 I' \Delta I'}{2\pi} \log \frac{R_\infty}{a}.$$

The same energy is also needed for the circuit composed of the conductor. Thus, the total energy needed to carry $\Delta I'$ is

$$\Delta W = \Delta W_1 + 2\Delta W_2 = \frac{\mu_0 I' \Delta I'}{2\pi} \log \frac{R_\infty}{a}.$$

The energy needed to carry the current I to the conductor is

$$W = \frac{\mu_0}{2\pi} \log \frac{R_\infty}{a} \int_0^I I' dI' = \frac{\mu_0 I^2}{4\pi} \log \frac{R_\infty}{a}.$$

Using the magnetic flux density $B(R) = \mu_0 I/(2\pi R)$ and Eq. (8.32), we can easily show that this is equal to the magnetic energy in the space of a unit length;

$$\int_a^{R_\infty} \frac{B^2(R)}{2\pi} 2\pi R \mathrm{d}R = \frac{\mu_0 I^2}{4\pi} \log \frac{R_\infty}{a}.$$

Thus, we can also derive the magnetic energy from the force between currents, if we correctly take into account the electromagnetic induction.

Chapter 11

11.1. The left side of Eq. (11.9) is $-\epsilon \Delta\phi$, and we obtain Poisson's equation for the electric potential,

$$\Delta\phi = -\frac{\rho}{\epsilon}.$$

The left side of Eq. (11.8) leads to

$$\frac{1}{\mu} \nabla\times(\nabla\times\boldsymbol{A}) = \frac{1}{\mu}[\nabla(\nabla\cdot\boldsymbol{A}) - \Delta\boldsymbol{A}] = -\frac{1}{\mu} \Delta\boldsymbol{A}.$$

The right side is the same as that shown in Exercise 11.3 and the equation for the vector potential is given by

$$\Delta\boldsymbol{A} - \epsilon\mu \frac{\partial^2 \boldsymbol{A}}{\partial t^2} - \epsilon\mu \nabla \frac{\partial\phi}{\partial t} = -\mu\boldsymbol{i}.$$

11.2. We use complex numbers and $\mathrm{e}^{\mathrm{i}\omega t}$ for the variation with time. We can assume that the internal electric field has only a z-component, E_z. Equation (10.39) leads to $\partial E_z/\partial x = \mathrm{i}\omega B_y$, showing that the magnetic flux density has only a y-component. Equation (11.4) leads to $\partial B_y/\partial x = \mathrm{i}\omega\mu\epsilon E_z$. Eliminating B_y yields

$$\frac{\partial^2 E_z}{\partial x^2} + \omega^2 \mu\epsilon E_z = 0.$$

The general solution including the time dependence is given by

$$E_z(x,t) = K_1 \exp[\mathrm{i}(\omega t + kx)] + K_2 \exp[\mathrm{i}(\omega t - kx)],$$

where $k = \omega(\mu\epsilon)^{1/2}$. The first and second terms show electromagnetic waves propagating along the negative and positive directions of the x-axis, respectively. From causality there is no wave propagating from infinity to the negative x-axis, and it is reasonable to assume $K_1 = 0$. Taking the real part the boundary condition $E_z(x = 0, t) = E_0 \cos\omega t$ gives $K_2 = E_0$. Thus, we have

$$E_z(x,t) = E_0 \cos(\omega t - kx), \qquad B_y(x,t) = -(\mu\epsilon)^{1/2} E_0 \cos(\omega t - kx).$$

11.3. From the answer to Exercise 11.2, the Poynting vector at depth x from the surface is

$$\boldsymbol{S}_{\mathrm{P}} = -\boldsymbol{i}_x \frac{E_z B_y}{\mu} = \frac{1}{(\mu\epsilon)^{1/2}} \cdot \epsilon E_0^2 \cos^2(\omega t - kx) \boldsymbol{i}_x.$$

It shows that the energy of density $\epsilon E_0^2 \cos^2(\omega t - kx)$ propagates with velocity $1/(\mu\epsilon)^{1/2}$ along the direction of the propagating electromagnetic wave. This does not decay with increasing x. This is because there is no energy dissipation due to electric resistivity.

11.4. When the electric charges on the electroplates are $\pm q(t)$, the electric field in the space between the electroplates is $E(t) = q(t)/\pi\epsilon_0 a^2$ and the displacement current there is $\partial D(t)/\partial t = (1/\pi a^2)[\partial q(t)/\partial t]$. The magnetic field on the surface of the space ($R = a$) is $H(t) = (1/2\pi a)[\partial q(t)/\partial t]$. Hence, the Poynting vector on the surface of the space is

$$S_{\mathrm{P}} = E(t)H(t) = \frac{q(t)}{2\pi^2 \epsilon_0 a^3} \cdot \frac{\partial q(t)}{\partial t}$$

and is directed inward the space. Integrating this with time gives

$$W = 2\pi a d \int S_{\mathrm{P}} \mathrm{d}t = \frac{d}{\pi\epsilon_0 a^2} \int_0^Q q \mathrm{d}q = \frac{dQ^2}{2\pi\epsilon_0 a^2} = \frac{Q^2}{2C},$$

where $C = \pi\epsilon_0 a^2/d$ is the capacitance of the capacitor. Thus, this energy is the electric energy stored in the capacitor.

11.5. Since the current density is $i = I/(\pi a^2)$, the electric field is $E = i/\sigma_{\mathrm{c}} = I/(\pi a^2 \sigma_{\mathrm{c}})$. The magnetic flux density on the surface is $B = \mu_0 I/(2\pi a)$. Thus, the Poynting vector on the surface has a magnitude

$$S_{\mathrm{P}} = \frac{EB}{\mu_0} = \frac{I^2}{2\pi^2 a^3 \sigma_{\mathrm{c}}}$$

and is directed normally inward the surface of the cylindrical conductor. The electric power penetrating into the conductor through a unit area is

$$P' = 2\pi a S_{\mathrm{P}} = \frac{I^2}{\pi a^2 \sigma_{\mathrm{c}}} = I^2 R'_{\mathrm{r}}$$

and is consumed in the conductor. In the above $R'_{\mathrm{r}} = 1/(\pi a^2 \sigma_{\mathrm{c}})$ is the electric resistance in a unit length of the cylindrical conductor.

11.6. The magnetic flux density produced in the coil when the current I' flows is $B' = \mu_0 I'/h$ and the electric field induced in the conducting plate is $E'_{\mathrm{i}} = -(\mu_0 a/2h)\mathrm{d}I'/\mathrm{d}t$. The electric field provided by the electric power source to keep the current constant is $E'_{\mathrm{s}} = -E'_{\mathrm{i}}$. Thus, the electric field inside the conductor is $E' = E'_{\mathrm{i}} - E'_{\mathrm{s}} = 0$. Hence, the Poynting vector on the conductor surface is zero, and there is no energy flow into the conductor. On the other hand, since the voltage between the gap at the terminal is $V = 2\pi a E'_{\mathrm{s}}$, the electric field there is

$$E' = \frac{V}{\delta} = \frac{\mu_0 \pi a^2}{h\delta} \cdot \frac{\mathrm{d}I'}{\mathrm{d}t}.$$

Hence, the Poynting vector at the terminal is directed inside the coil and the magnitude of the vector is

$$S_{\mathrm{P}} = \frac{B'E'}{\mu_0} = \frac{\mu_0 \pi a^2}{h^2 \delta} I' \frac{\mathrm{d}I'}{\mathrm{d}t}.$$

The energy supplied to the coil until the current reaches I is

$$U_{\mathrm{m}} = h\delta \int_0^I \frac{\mu_0 \pi a^2}{h^2 \delta} I' \mathrm{d}I' = \frac{\mu_0 \pi a^2}{2h} I^2 = \frac{B^2}{2\mu_0} \pi a^2 h,$$

where $B = \mu_0 I/h$ is the magnetic flux density in the final state and $\pi a^2 h$ is the volume of the space in which the magnetic flux is stored. Hence, we can see that all the energy fed by the energy source is stored in the coil as the magnetic energy.

11.7. We denote the radius from the center by R. When the current applied to the coil is I', the magnetic flux density in the coil ($R < a$) is $B' = \mu_0 I'/h$. Because the conductor is sufficiently thin, we can assume that the current flows uniformly. Thus, the magnetic flux density in the conductor ($a \leq R \leq a + b$) is $B'(R) = \mu_0 (a + b - R) I'/(bh)$. The induced electric field has an azimuthal component, and from the relationship

$$(\nabla \times \boldsymbol{E})_z = \frac{E'_{\mathrm{i}}}{R} + \frac{\mathrm{d}E'_{\mathrm{i}}}{\mathrm{d}R} \simeq \frac{\mathrm{d}E'_{\mathrm{i}}}{\mathrm{d}R},$$

the induced electric field is given by

$$\begin{aligned} E'_{\mathrm{i}}(R) &= -\frac{\mu_0}{bh} \cdot \frac{\mathrm{d}I'}{\mathrm{d}t} \int_a^R (a + b - R)\mathrm{d}R + E'_{\mathrm{i}}(a) \\ &= -\frac{\mu_0 [ab + 2b(R-a) - (R-a)^2]}{2bh} \cdot \frac{\mathrm{d}I'}{\mathrm{d}t}, \end{aligned}$$

where we have used $E'_{\mathrm{i}}(a) = -(\mu_0 a/2h)\mathrm{d}I'/\mathrm{d}t$. Averaging this in the sufficiently thin conductor gives $\langle E'_{\mathrm{i}} \rangle = -[\mu_0 (3a + 2b)/(6h)](\mathrm{d}I'/\mathrm{d}t)$. Hence, so that the current I' flows in the conductor, the sum of the electrostatic field, E'_s, and $\langle E'_i \rangle$ should be equal to $\rho_r I'/(bh)$, and we have

$$E'_s = \frac{\rho_r I'}{bh} + \frac{\mu_0(3a+2b)}{6h} \cdot \frac{dI'}{dt}.$$

The electric field between the gap of the coil is $E'_0 = (2\pi a/\delta)E'_s$, and the energy that enters the coil while the current increases linearly from 0 to I within period T is

$$U = h\delta \int \frac{E'_0 B'}{\mu_0} dt = \frac{\mu_0 \pi a^2}{2h} I^2 + \frac{\mu_0 \pi ab}{3h} I^2 + \frac{2\pi a}{3bh} \rho_r I^2 T.$$

The first term is the magnetic energy stored in the space of the coil (see Exercise 11.6). As will be shown later, the second and third terms are the magnetic energy stored in the conductor and the dissipated energy. We can show that these energies penetrate from the inner surface of the coil using the Poynting vector.

The magnetic energy in the conductor is

$$2\pi ah \int_a^{a+b} \frac{1}{2\mu_0} B^2(R) dR = \frac{\pi \mu_0 ab}{3h} I^2.$$

Assuming $I' = (t/T)I$, the dissipated energy is

$$\frac{2\pi a \rho_r}{bh} \int_0^T \left(\frac{tI}{T}\right)^2 dt = \frac{2\pi a}{3bh} \rho_r I^2 T.$$

11.8. Exactly speaking, the conductors are not equipotential and hence, the electric field is not perpendicular to the conductor surfaces (see Fig. B11.1). Thus, the Poynting vector is not parallel to the surface and the dissipated energy enters the conductor.

11.9. The electric field induced along the y-axis while the magnetic flux density increases is

$$E_y(x) = -\int_0^x \frac{\partial B_z(x)}{\partial t} dx = -\frac{x^2}{2d} \frac{\partial b_0}{\partial t},$$

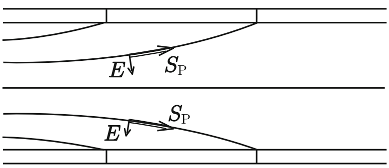


Fig. B11.1 Equipotential surface (*solid line*), electric field and the Poynting vector where we have used the symmetry condition, $E_y(0) = 0$. Thus, the Poynting vector at x directed along the positive x-axis is

$$S_\mathrm{P}(x) = -\frac{x^2}{2\mu_0 d}\left(B_0 + \frac{b_0 x}{d}\right)\frac{\partial b_0}{\partial t}.$$

The energy that penetrates into the region in unit time is

$$\Delta S_\mathrm{P} = -S_\mathrm{P}(d) + S_\mathrm{P}(d - \Delta x) = \frac{1}{\mu_0}\frac{\partial b_0}{\partial t}\left(B + \frac{3}{2}b_0\right)\Delta x.$$

Hence, the input energy while the magnetic flux density increases is

$$\Delta U = \int \Delta S_\mathrm{P}\mathrm{d}t = \frac{b_0}{\mu_0}\left(B_0 + \frac{3}{4}b_0\right)\Delta x.$$

On the other hand, the increase in the magnetic energy is

$$\Delta U_\mathrm{m} = \frac{1}{2\mu_0}\int_{d-\Delta x}^{d}\left[\left(B_0 + \frac{b_0}{d}x\right)^2 - {B_0}^2\right]\mathrm{d}x \simeq \frac{b_0}{\mu_0}\left(B_0 + \frac{b_0}{2}\right)\Delta x.$$

Hence, the work done by the expected restoring force to reduce the magnetic distortion is

$$\Delta W = \Delta U - \Delta U_\mathrm{m} = \frac{1}{4\mu_0}{b_0}^2\Delta x.$$

Here we determine the displacement of the flux lines, u. Integrating the continuity equation of magnetic flux with time gives $\nabla \times (\boldsymbol{B} \times \boldsymbol{u}) = -\boldsymbol{b}$. This leads to $\mathrm{d}u/\mathrm{d}x = -b_0 x/(B_0 d)$. Under the symmetry condition $u(0) = 0$, we obtain the displacement as

$$u(d) \simeq -\frac{b_0}{B_0 d}\int_0^d x\mathrm{d}x = -\frac{b_0 d}{2B_0}.$$

The work is written as $\Delta W = ({B_0}^2/\mu_0 d^2)u^2\Delta x$ in terms of the displacement. Hence, the force on this region is

$$f = \frac{\partial \Delta W}{\partial u} = -\frac{B_0 b_0}{\mu_0 d}\Delta x = -JB_0\Delta x,$$

where $J = b_0/\mu_0 d$ is the current density. Thus, we prove that the elastic restoring force is the Lorentz force. This force is directed along the negative x-axis to make the magnetic flux density uniform.

Thus, the Lorentz force is derived from the condition that the work on flux lines is equal to the difference between the input energy and stored energy. This should be equal to the dissipated energy. In this case we assume that the Lorentz force is counterbalanced with a virtual force to stably maintain such a state with a higher energy. In reality this virtual force is the pinning force (see Sect. A3.3 in the Appendix), and the work done by the Lorentz force is dissipated as the pinning loss.

Chapter 12

12.1. High frequency components of electromagnetic fields are completely shielded inside the conductor and the electric charge and current are consequently induced on the conductor surface. Hence, the fulfilled boundary conditions are only Eqs. (12.20) and (12.23). Corresponding Eqs. (12.35) and (12.38) are

$$\boldsymbol{n} \times (\boldsymbol{E}_0 + \boldsymbol{E}_0'') = 0, \qquad \boldsymbol{n} \cdot \left(\frac{1}{k}\boldsymbol{k} \times \boldsymbol{E}_0 + \frac{1}{k''}\boldsymbol{k}'' \times \boldsymbol{E}_0'' \right) = 0.$$

Since the electric field in the incident wave is normal to the plane of incidence (parallel to the y-axis in Fig. 12.3), the first equation leads to

$$E_0 + E_0'' = 0.$$

The second equation gives also the same result. In this case, taking the real part, the electric field in the vacuum region is

$$E_y = E_0 \cos(\omega t - \boldsymbol{k} \cdot \boldsymbol{r}) - E_0 \cos(\omega t - \boldsymbol{k}'' \cdot \boldsymbol{r}) = -2E_0 \sin(kz \cos\theta) \sin(\omega t - kx \sin\theta).$$

In the above $k = k''$ and we have used Eq. (12.33) and the following relations:

$$\boldsymbol{k} \cdot \boldsymbol{r} = kx \sin\theta - kz \cos\theta, \qquad \boldsymbol{k}'' \cdot \boldsymbol{r} = kx \sin\theta + kz \cos\theta.$$

In this configuration the electric charge does not appear on the surface since the electric field is parallel to the surface. The magnetic flux density is

$$\begin{aligned} B_x &= \frac{E_0}{c_0} \cos(\omega t - \boldsymbol{k} \cdot \boldsymbol{r}) \cos\theta - \frac{E_0''}{c_0} \cos(\omega t - \boldsymbol{k}'' \cdot \boldsymbol{r}) \cos\theta \\ &= \frac{2E_0}{c_0} \cos\theta \cos(kz \cos\theta) \cos(\omega t - kx \sin\theta), \\ B_z &= \frac{E_0}{c_0} \cos(\omega t - \boldsymbol{k} \cdot \boldsymbol{r}) \sin\theta + \frac{E_0''}{c_0} \cos(\omega t - \boldsymbol{k}'' \cdot \boldsymbol{r}) \sin\theta \\ &= -\frac{2E_0}{c_0} \sin\theta \sin(kz \cos\theta) \sin(\omega t - kx \sin\theta). \end{aligned}$$

The surface current density is given by

$$\tau_y(x) = \frac{B_x(z=0)}{\mu_0} = 2\left(\frac{\epsilon_0}{\mu_0}\right)^{1/2} E_0 \cos\theta \cos(\omega t - kx\sin\theta).$$

12.2. The same two equations as in Exercise 12.1 appear. Using the definition in Fig. 12.4, the first equation reduces to $(E_0 - E_0'')\cos\theta = 0$ and we obtain $E_0'' = E_0$. The second equation is fulfilled. Hence, it is sufficient if the above equation is satisfied. The magnetic flux density has only a y-component;

$$B_y = \frac{E_0}{c_0}[\cos(\omega t - \boldsymbol{k}\cdot\boldsymbol{r}) + \cos(\omega t - \boldsymbol{k}''\cdot\boldsymbol{r})] = \frac{2E_0}{c_0}\cos(kz\cos\theta)\cos(\omega t - kx\sin\theta).$$

Since the parallel component of the magnetic flux density is not zero on the surface, the surface current density is

$$\tau_x(x) = -\frac{B_y(z=0)}{\mu_0} = 2\left(\frac{\epsilon_0}{\mu_0}\right)^{1/2} E_0 \cos(\omega t - kx\sin\theta).$$

(Note the directions of the current and magnetic flux density.) The electric field is

$$\begin{aligned}
E_x &= -E_0\cos(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})\cos\theta + E_0''\cos(\omega t - \boldsymbol{k}''\cdot\boldsymbol{r})\cos\theta \\
&= 2E_0\cos\theta\sin(kz\cos\theta)\sin(\omega t - kx\sin\theta), \\
E_z &= -E_0\cos(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})\sin\theta - E_0''\cos(\omega t - \boldsymbol{k}''\cdot\boldsymbol{r})\sin\theta \\
&= -2E_0\sin\theta\cos(kz\cos\theta)\cos(\omega t - kx\sin\theta).
\end{aligned}$$

Since the normal component of the electric field is not zero on the surface, an electric charge appears on the surface and its density is

$$\sigma(x) = \epsilon_0 E_z(z=0) = -2E_0\epsilon_0\sin\theta\cos(\omega t - kx\sin\theta).$$

In this case we can see that the following relationship holds between the surface current and surface charge;

$$\nabla\cdot\boldsymbol{\tau} + \frac{\partial\sigma}{\partial t} = 0,$$

which corresponds to Eq. (5.10) for a three-dimensional case. It should be noted that $\nabla\cdot\boldsymbol{\tau} = 0$ in Exercise 12.1.

12.3. From Eqs. (12.24) and (12.25) we obtain the electric powers flowing from medium 1 to medium 2 through a unit area as the incident and reflected waves as

$$-\frac{1}{\mu_1}[\boldsymbol{E}(z=0)\times\boldsymbol{B}(z=0)]_z = \frac{E_0^2}{c_1\mu_1}\cos^2(\omega t - \boldsymbol{k}\cdot\boldsymbol{r}_0)\cos\theta,$$

$$-\frac{1}{\mu_1}[\boldsymbol{E}''(z=0)\times\boldsymbol{B}''(z=0)]_z = -\frac{E_0''^2}{c_1\mu_1}\cos^2(\omega t - \boldsymbol{k}''\cdot\boldsymbol{r}_0)\cos\theta'',$$

respectively. From Eq. (12.26) the electric power penetrating into medium 2 as the transmitted wave is

$$-\frac{1}{\mu_1}[\boldsymbol{E}'(z=0)\times\boldsymbol{B}'(z=0)]_z = \frac{E_0'^2}{c_2\mu_2}\cos^2(\omega t - \boldsymbol{k}'\cdot\boldsymbol{r}_0)\cos\theta'.$$

Because of Eq. (12.30) the factors dependent on time and space such as $\cos^2(\omega t - \boldsymbol{k}\cdot\boldsymbol{r}_0)$ are the same. Neglecting these factors, the rate of energy flow from medium 1 is

$$\frac{1}{\mu_1 c_1}(E_0^2 - E_0''^2)\cos\theta = \frac{4\alpha\cos^2\theta\cos\theta' E_0^2}{\mu_1 c_1(\cos\theta + \alpha\cos\theta')^2},$$

where $\alpha = (\epsilon_2\mu_1/\epsilon_1\mu_2)^{1/2}$ and we have used Eqs. (12.33) and (12.42b). On the other hand, Eq. (12.42a) yields the rate of energy penetration into medium 2;

$$\frac{1}{\mu_2 c_2}E_0'^2\cos\theta' = \frac{4\cos^2\theta\cos\theta' E_0^2}{\mu_2 c_2(\cos\theta + \alpha\cos\theta')^2}.$$

We can easily show that this is equal to the rate of energy flow from medium 1.

12.4. The x- and y-components of the Poynting vector are $-E_zB_y/\mu_0$ and E_zB_x/μ_0, respectively. From the condition of Eq. (12.56) these are zero on the surfaces of the wave guide, $(x = 0, a)$ and $(y = 0, b)$. Hence, there is no energy flow through these surfaces. Taking the real parts of the electric field and magnetic flux density, the z-component of the Poynting vector is

$$S_{\mathrm{P}z} = A^2\frac{\pi^2\epsilon_0\gamma\omega}{k^4}\left[\frac{m^2}{a^2}\cos^2\left(\frac{m\pi x}{a}\right)\sin^2\left(\frac{n\pi y}{b}\right) + \frac{n^2}{b^2}\sin^2\left(\frac{m\pi x}{a}\right)\cos^2\left(\frac{n\pi y}{b}\right)\right]\sin^2(\omega t - \gamma z).$$

Integrating this in the x-y plane, the electric power through a unit area along the z-axis is

$$P = A^2\frac{\pi^2\epsilon_0\gamma\omega(n^2a^2 + m^2b^2)}{4k^4ab}\sin^2(\omega t - \gamma z).$$

12.5. The boundary conditions on E_x, E_y, B_x and B_y are given by Eq. (12.56). The boundary conditions on B_z are: $\partial B_z/\partial x = 0$ at $x = 0$ and a from Eqs. (12.52b) and (12.52c), and $\partial B_z/\partial y = 0$ at $y = 0$ and b from Eqs. (12.52a) and (12.52d). Using these conditions, the general solution of Eq. (12.51b) is given by

$$B_z(x, y, z, t) = A' \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right).$$

Substituting this with $E_z = 0$ into Eqs. (12.52a)–(12.52d) yields

$$\begin{aligned} E_x &= \mathrm{i}A' \frac{n\pi\omega}{k^2 b} \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right), \\ E_y &= -\mathrm{i}A' \frac{m\pi\omega}{k^2 a} \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right), \\ B_x &= \mathrm{i}A' \frac{m\pi\gamma}{k^2 a} \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right), \\ B_y &= \mathrm{i}A' \frac{n\pi\gamma}{k^2 b} \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right). \end{aligned}$$

For simplicity, the factor $\exp[\mathrm{i}(\omega t - \gamma z)]$ is omitted. The Poynting vector along the z-axis is

$$\begin{aligned} S_{\mathrm{P}z} = A'^2 \frac{\pi^2\gamma\omega}{\mu_0 k^4} \Bigg[& \frac{n^2}{b^2} \cos^2\left(\frac{m\pi x}{a}\right) \sin^2\left(\frac{n\pi y}{b}\right) \\ & + \frac{m^2}{a^2} \sin^2\left(\frac{m\pi x}{a}\right) \cos^2\left(\frac{n\pi y}{b}\right) \Bigg] \sin^2(\omega t - \gamma z). \end{aligned}$$

Integrating this in the x-y plane yields the electric power through a unit area along the z-axis;

$$P = A'^2 \frac{\pi^2\gamma\omega(n^2a^2 + m^2b^2)}{4\mu_0 k^4 ab} \sin^2(\omega t - \gamma z).$$

12.6. Using the real parts of the electric field and magnetic flux density, the surface densities of electric charge and current on the plane $x = 0$ are respectively given by

$$\begin{aligned} \sigma(x = 0) &= \epsilon_0 E_x(x = 0) = \epsilon_0 A \frac{m\pi\gamma}{k^2 a} \sin\left(\frac{n\pi y}{b}\right) \sin(\omega t - \gamma z), \\ \tau_z(x = 0) &= \frac{1}{\mu_0} B_y(x = 0) = \epsilon_0 A \frac{m\pi\omega}{k^2 a} \sin\left(\frac{n\pi y}{b}\right) \sin(\omega t - \gamma z), \end{aligned}$$

where we have used $c_0 = 1/(\epsilon_0\mu_0)^{1/2}$. From the above results we have

$$\frac{\partial}{\partial z}\tau_z(x = 0) + \frac{\partial}{\partial t}\sigma(x = 0) = 0.$$

Thus, the continuity equation of current holds. Similar relationships are obtained for other surfaces.

12.7. The electric field in the plane normal to the conductors is similar to that in the case where the line charges $\pm\lambda$ are given at the image axes $(\pm l, 0)$ of the left and right conductors, respectively. The electric potential,

$$\phi(x, y) = \frac{\lambda}{4\pi\epsilon_0} \log \frac{(x-l)^2 + y^2}{(x+l)^2 + y^2},$$

gives

$$E_x = \frac{\lambda}{2\pi\epsilon_0}\left[\frac{x+l}{(x+l)^2+y^2} - \frac{x-l}{(x-l)^2+y^2}\right],$$
$$E_y = \frac{\lambda}{2\pi\epsilon_0}\left[\frac{y}{(x+l)^2+y^2} - \frac{y}{(x-l)^2+y^2}\right], \qquad \text{(A.1)}$$

where $l = (d/2) - h = [(d/2)^2 - a^2]^{1/2}$ (see Fig. B5.2). For the TEM wave λ is an arbitrary parameter associated with the electric field strength. The magnetic flux density is

$$B_x = \frac{E_y}{c_0}, \qquad B_y = -\frac{E_x}{c_0}.$$

Although a detailed calculation is not shown, the total electric charges that appear on the surface of each conductor of a unit length are equal to $\pm\lambda$, and the continuity equation of current holds with the surface charges.

12.8. Using $\phi(\boldsymbol{r}, t)$ in Eq. (12.78), the first term on the left side of Eq. (11.31) is

$$\Delta\phi(\boldsymbol{r}, t) = \frac{1}{4\pi\epsilon}\int_V \Delta\left[\frac{\rho(\boldsymbol{r}', t - R/c)}{R}\right] \mathrm{d}V',$$

where $R = |\boldsymbol{r} - \boldsymbol{r}'|$. Since Δ is the derivative with respect to $\boldsymbol{r}$, we have

$$\Delta\frac{\rho}{R} = \rho\Delta\frac{1}{R} + 2\nabla\rho\cdot\nabla\frac{1}{R} + \frac{\Delta\rho}{R}.$$

The volume integral of the first term on the right side including the abnormal point gives $-\rho/\epsilon$ in Eq. (11.31), as shown in Sect. A2.1. The second term is written as $-(2/R^2)(\partial\rho/\partial R)$, and $\Delta\rho$ in the third term is

$$\Delta\rho = \frac{1}{R}\cdot\frac{\partial^2}{\partial R^2}(R\rho) = \frac{\partial^2\rho}{\partial R^2} + \frac{2}{R}\cdot\frac{\partial\rho}{\partial R}.$$

Hence, only $(\partial^2\rho/\partial R^2)/R$ remains from the second and third terms. It is obvious that the following equation holds;

$$\frac{\partial^2\phi}{\partial t^2} = \frac{1}{4\pi\epsilon}\int_{\mathrm{V}}\frac{1}{R}\cdot\frac{\partial^2\rho}{\partial t^2}\mathrm{d}V'.$$

Hence, we find that Eq. (11.31) is proved, if the following equation holds;

$$\frac{\partial^2\rho}{\partial R^2} = \frac{1}{c^2}\cdot\frac{\partial^2\rho}{\partial t^2}.$$

The new definition $\xi = t - R/c$ gives $\partial\rho/\partial R = -(1/c)\partial\rho/\partial\xi$ and $\partial\rho/\partial t = \partial\rho/\partial\xi$. Thus, the following relations are resulted;

$$\frac{\partial^2\rho}{\partial R^2} = \frac{1}{c^2}\cdot\frac{\partial^2\rho}{\partial\xi^2}, \qquad \frac{\partial^2\rho}{\partial t^2} = \frac{\partial^2\rho}{\partial\xi^2}.$$

Hence, the above relation holds and we derive Eq. (11.31).

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