

Appendix A

Complex Numbers

The purpose of this short appendix is to review the basics of complex numbers and complex arithmetic, which are used throughout much of the text.

A *complex number* is an expression of the form $z = x + iy$, where $x, y \in \mathbb{R}$ are real and $i = \sqrt{-1}$ is the imaginary unit. The set of all complex numbers is denoted by \mathbb{C} . We call $x = \operatorname{Re} z$ the *real part* of z and $y = \operatorname{Im} z$ the *imaginary part* of $z = x + iy$. (Note: The imaginary part is the real number y , not iy .) A real number x is merely a complex number with zero imaginary part, $\operatorname{Im} z = 0$, and so we may regard $\mathbb{R} \subset \mathbb{C}$. Complex addition and multiplication are based on simple adaptations of the rules of real arithmetic to include the identity $i^2 = -1$, and so

$$\begin{aligned}(x + iy) + (u + iv) &= (x + u) + i(y + v), \\ (x + iy)(u + iv) &= (xu - yv) + i(xv + yu).\end{aligned}\tag{A.1}$$

Complex numbers enjoy all the usual laws of real addition and multiplication, *including commutativity*: $zw = wz$.

We can identify a complex number $x + iy$ with a vector $(x, y) \in \mathbb{R}^2$ in the real, two-dimensional plane. For this reason, \mathbb{C} is sometimes referred to as the *complex plane*. (Although keep in mind that, as a complex vector space, \mathbb{C} is only one-dimensional.) Based on this identification, we shall employ the standard terminology of planar vector calculus — domain, curve, etc. — without alteration. Complex addition (A.1) corresponds to vector addition, but the vector interpretation of complex multiplication is more obscure.

The *complex conjugate* of $z = x + iy$ is $\bar{z} = x - iy$. Note that $\operatorname{Re} \bar{z} = \operatorname{Re} z$, while $\operatorname{Im} \bar{z} = -\operatorname{Im} z$. Geometrically, the complex conjugate of z is obtained by reflecting the corresponding vector through the real axis, as illustrated in [Figure A.1](#). In particular, $\bar{\bar{z}} = z$ if and only if z is real. In general,

$$\operatorname{Re} z = \frac{z + \bar{z}}{2}, \quad \operatorname{Im} z = \frac{z - \bar{z}}{2i}.\tag{A.2}$$

Complex conjugation is compatible with complex arithmetic:

$$\overline{z + w} = \bar{z} + \bar{w}, \quad \overline{zw} = \bar{z}\bar{w}.$$

In particular, the product of a complex number and its conjugate,

$$z\bar{z} = (x + iy)(x - iy) = x^2 + y^2,\tag{A.3}$$

is real and nonnegative. Its square root is known as the *modulus* or *norm* of the complex number $z = x + iy$, and written

$$|z| = \sqrt{x^2 + y^2}.\tag{A.4}$$

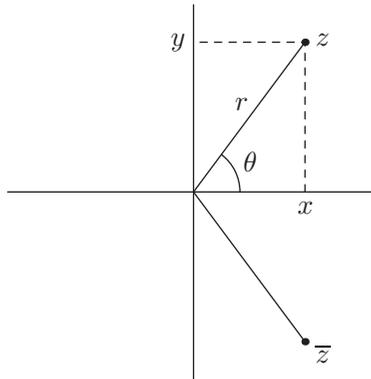


Figure A.1. Complex numbers.

Note that $|z| \geq 0$, with $|z| = 0$ if and only if $z = 0$. The modulus $|z|$ generalizes the absolute value of a real number and coincides with the standard Euclidean norm in the (x, y) -plane. This implies the validity of the triangle inequality

$$|z + w| \leq |z| + |w|. \quad (\text{A.5})$$

Equation (A.3) can be rewritten in terms of the modulus as

$$z \bar{z} = |z|^2. \quad (\text{A.6})$$

Rearranging the factors, we deduce the formula for the reciprocal of a nonzero complex number:

$$\frac{1}{z} = \frac{\bar{z}}{|z|^2}, \quad z \neq 0, \quad \text{or, equivalently,} \quad \frac{1}{x + iy} = \frac{x - iy}{x^2 + y^2}. \quad (\text{A.7})$$

The general formula for complex division,

$$\frac{w}{z} = \frac{w \bar{z}}{|z|^2} \quad \text{or} \quad \frac{u + iv}{x + iy} = \frac{(xu + yv) + i(xv - yu)}{x^2 + y^2}, \quad (\text{A.8})$$

is an immediate consequence.

The modulus of a complex number,

$$r = |z| = \sqrt{x^2 + y^2},$$

is one component of its polar coordinate representation

$$x = r \cos \theta, \quad y = r \sin \theta \quad \text{or} \quad z = r(\cos \theta + i \sin \theta). \quad (\text{A.9})$$

The polar angle θ , which measures the angle that the line connecting z to the origin makes with the horizontal axis, is known as the *phase*, and written

$$\theta = \text{ph } z. \quad (\text{A.10})$$

As such, the phase is defined only up to an integer multiple of 2π . The unique *principal value* of the phase is restricted to $-\pi < \text{ph } z \leq \pi$. A more common term for the polar

angle is the *argument* of z , written $\arg z = \text{ph } z$. However, in conformity with [85, 86], we prefer to use “phase” here, in part to avoid confusion with the argument z of a function $f(z)$.

Euler’s celebrated formula for the complex exponential,

$$e^{i\theta} = \cos \theta + i \sin \theta, \quad (\text{A.11})$$

can be used to compactly rewrite the polar form (A.9) of a complex number as

$$z = r e^{i\theta}, \quad \text{where} \quad r = |z|, \quad \theta = \text{ph } z. \quad (\text{A.12})$$

Consequently, the complex logarithm has the form

$$\log z = \log(r e^{i\theta}) = \log r + \log e^{i\theta} = \log r + i\theta = \log |z| + i \text{ph } z. \quad (\text{A.13})$$

More generally, the complex exponential is given by

$$e^z = e^x \cos y + i e^x \sin y, \quad \text{for} \quad z = x + iy. \quad (\text{A.14})$$

We note that the modulus and phase of a product of complex numbers can be readily computed:

$$|zw| = |z| |w|, \quad \text{ph}(zw) = \text{ph } z + \text{ph } w, \quad (\text{A.15})$$

the latter formula requiring that we allow multiply valued phases; the formula does *not* hold as stated for all z, w when the principal value of the phase is used. Similarly, the modulus and phase of the reciprocal of a nonzero complex number are

$$\left| \frac{1}{z} \right| = \frac{1}{|z|}, \quad \text{ph} \left(\frac{1}{z} \right) = -\text{ph } z. \quad (\text{A.16})$$

On the other hand, complex conjugation preserves the modulus, but negates the phase:

$$|\bar{z}| = |z|, \quad \text{ph } \bar{z} = -\text{ph } z. \quad (\text{A.17})$$

The latter formula is not valid for the principal value of the phase when z lies on the negative real axis.

Appendix B

Linear Algebra

In this appendix, we collect basic results and definitions from linear algebra that are used in our study of partial differential equations. The reader is referred to [89] for the proofs and further details.

B.1 Vector Spaces and Subspaces

Vector spaces and their ancillary structures provide the common language of linear algebra. The basic definition is modeled on the prototypical finite-dimensional example: the *Euclidean space* \mathbb{R}^n , which is the set of all real (column) vectors with n entries, equipped with the operations of vector addition and scalar multiplication. More generally:

Definition B.1. A (real) *vector space* is a set V equipped with two operations:

- (i) *Addition*: adding any pair of elements $\mathbf{v}, \mathbf{w} \in V$ produces another vector $\mathbf{v} + \mathbf{w} \in V$.
- (ii) *Scalar Multiplication*: multiplying an element $\mathbf{v} \in V$ by a scalar $c \in \mathbb{R}$ produces a vector $c\mathbf{v} \in V$.

These are subject to the following axioms: for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and all scalars $c, d \in \mathbb{R}$,

- (a) *Commutativity of Addition*: $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$.
- (b) *Associativity of Addition*: $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.
- (c) *Additive Identity*: There is a zero element $\mathbf{0} \in V$ satisfying $\mathbf{v} + \mathbf{0} = \mathbf{v} = \mathbf{0} + \mathbf{v}$.
- (d) *Additive Inverse*: For each $\mathbf{v} \in V$ there is an element $-\mathbf{v} \in V$ such that
$$\mathbf{v} + (-\mathbf{v}) = \mathbf{0} = (-\mathbf{v}) + \mathbf{v}.$$
- (e) *Distributivity*: $(c + d)\mathbf{v} = (c\mathbf{v}) + (d\mathbf{v})$, and $c(\mathbf{v} + \mathbf{w}) = (c\mathbf{v}) + (c\mathbf{w})$.
- (f) *Associativity of Scalar Multiplication*: $c(d\mathbf{v}) = (cd)\mathbf{v}$.
- (g) *Unit for Scalar Multiplication*: the scalar $1 \in \mathbb{R}$ satisfies $1\mathbf{v} = \mathbf{v}$.

Complex vector spaces are defined in an identical manner, the only difference being that the scalars are allowed to be complex numbers. In this case, the prototype is the space \mathbb{C}^n consisting of column vectors with n complex entries.

While finite-dimensional vector spaces play a significant role in the study of partial differential equations, particularly in the design of numerical solution schemes, for us the more important examples are infinite-dimensional vector spaces whose elements (“vectors”) are functions. The main example is the following:

Example B.2. Let $I \subset \mathbb{R}$ be an interval. The *function space* $\mathcal{F} = \mathcal{F}(I)$, whose elements are all real-valued functions $f(x)$ defined for $x \in I$, has the structure of a vector space. Addition of functions in \mathcal{F} is defined in the usual manner: $(f + g)(x) = f(x) + g(x)$ for all $x \in I$. Multiplication by scalars $c \in \mathbb{R}$ is the same as multiplication by constants, $(cf)(x) = cf(x)$. The zero element is the constant function that is identically 0 for all $x \in I$. With these operations, all the vector space axioms listed in Definition B.1 are valid, and hence $\mathcal{F}(I)$ is a real vector space.

More generally, if $\Omega \subset \mathbb{R}^n$ is any subset of n -dimensional Euclidean space, the function space $\mathcal{F}(\Omega)$ is defined as the set of all real-valued functions $f(x_1, \dots, x_n)$ defined for all $x = (x_1, \dots, x_n) \in \Omega$. Addition and scalar (constant) multiplication of functions are defined in the same manner.

A *subspace* of a vector space V is a subset $W \subset V$ that is a vector space in its own right. In particular, a subspace W *must* contain the zero element of V .

Proposition B.3. A nonempty subset $W \subset V$ of a vector space is a subspace if and only if

- (a) for every $\mathbf{v}, \mathbf{w} \in W$, the sum $\mathbf{v} + \mathbf{w} \in W$, and
- (b) for every $\mathbf{v} \in W$ and every $c \in \mathbb{R}$, the scalar product $c\mathbf{v} \in W$.

For example, a complete list of subspaces of $V = \mathbb{R}^3$ is (i) the origin $\{\mathbf{0}\}$; (ii) every line through the origin; (iii) every plane through the origin; (iv) all of \mathbb{R}^3 .

Example B.4. Here are some examples of subspaces of the function space $\mathcal{F}(I)$.

- (a) The space $\mathcal{P}^{(n)}$ of polynomials of degree $\leq n$.
- (b) The space $C^0(I)$ of all continuous functions on the interval I .
- (c) The space $C^n(I)$ consisting of all functions $f(x)$ that have n continuous derivatives $f'(x), f''(x), \dots, f^{(n)}(x)$ on[†] I .
- (d) The space $C^\infty(I) = \bigcap_{n \geq 0} C^n(I)$ of infinitely differentiable, or *smooth*, functions is also a subspace.
- (e) The space $\mathcal{A}(I)$ of analytic functions. Recall that a function $f(x)$ is called *analytic* at a point a if it is smooth, and, moreover, its Taylor series

$$f(a) + f'(a)(x - a) + \frac{1}{2}f''(a)(x - a)^2 + \dots = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n \quad (\text{B.1})$$

converges to $f(x)$ for all x sufficiently close to a . (The series is not required to converge on the entire interval I .) Not every smooth function is analytic, and so $\mathcal{A}(I) \subsetneq C^\infty(I)$; see Exercise 11.3.21 for an explicit example.

B.2 Bases and Dimension

Definition B.5. Let $\mathbf{v}_1, \dots, \mathbf{v}_k$ belong to a vector space V . A sum of the form

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_k\mathbf{v}_k = \sum_{i=1}^k c_i\mathbf{v}_i, \quad (\text{B.2})$$

[†] We use one-sided derivatives at any endpoint that belongs to the interval.

where the coefficients c_1, c_2, \dots, c_k are any scalars, is known as a *linear combination* of the elements $\mathbf{v}_1, \dots, \mathbf{v}_k$. Their *span* is the subspace $W = \text{span} \{\mathbf{v}_1, \dots, \mathbf{v}_k\} \subset V$ consisting of all possible linear combinations.

Definition B.6. The elements $\mathbf{v}_1, \dots, \mathbf{v}_k \in V$ are called *linearly dependent* if there exist scalars c_1, \dots, c_k , *not all zero*, such that

$$c_1 \mathbf{v}_1 + \dots + c_k \mathbf{v}_k = \mathbf{0}. \quad (\text{B.3})$$

Elements that are not linearly dependent are called *linearly independent*.

In particular, a collection of functions $f_1(x), \dots, f_n(x)$ is linearly dependent if and only if there exist constants c_1, \dots, c_n , *not all zero*, such that the linear combination

$$c_1 f_1(x) + \dots + c_n f_n(x) \equiv 0 \quad (\text{B.4})$$

is identically zero. Conversely, if the only choice of constants for which (B.4) holds is $c_1 = \dots = c_n = 0$, then the functions are linearly independent.

Definition B.7. A *basis* of a vector space V is a finite collection of elements $\mathbf{v}_1, \dots, \mathbf{v}_n \in V$ that (a) spans V , and (b) is linearly independent.

The simplest example is the *standard basis* of \mathbb{R}^n , consisting of the n vectors

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad \dots, \quad \mathbf{e}_n = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \quad (\text{B.5})$$

so that \mathbf{e}_i is the vector with 1 in the i^{th} slot and 0's elsewhere. However, there are many other bases of \mathbb{R}^n ; indeed, any n linearly independent vectors $\mathbf{v}_1, \dots, \mathbf{v}_n \in \mathbb{R}^n$ form a basis.

Lemma B.8. *The elements $\mathbf{v}_1, \dots, \mathbf{v}_n$ form a basis of V if and only if every $\mathbf{v} \in V$ can be written uniquely as a linear combination of the basis elements:*

$$\mathbf{v} = c_1 \mathbf{v}_1 + \dots + c_n \mathbf{v}_n = \sum_{i=1}^n c_i \mathbf{v}_i. \quad (\text{B.6})$$

The coefficients (c_1, \dots, c_n) are called the *coordinates of the vector \mathbf{v} with respect to the given basis*.

Theorem B.9. *Suppose the vector space V has a basis $\mathbf{v}_1, \dots, \mathbf{v}_n$. Then every other basis of V has the same number of elements in it. This number is called the *dimension* of V , and written $\dim V = n$.*

On the other hand, if the vector space contains infinitely many linearly independent elements, then it does not have a basis in the sense of Definition B.7, and is thus *infinite-dimensional*. All of the function spaces and subspaces listed above are infinite-dimensional vector spaces. An example of a finite-dimensional function space is the space $\mathcal{P}^{(n)} \subset \mathcal{F}(\mathbb{R})$ consisting of all polynomials $p(x) = a_0 + a_1 x + \dots + a_n x^n$ of degree $\leq n$. The monomials $1, x, x^2, \dots, x^n$ form a basis, and hence $\mathcal{P}^{(n)}$ has dimension $n + 1$. (On the other hand, the vector space containing *all* polynomials is infinite-dimensional.)

B.3 Inner Products and Norms

The dot product on Euclidean space \mathbb{R}^n plays an essential role in geometry, analysis, and mechanics. Its basic properties inspire the general definition of an inner product on a vector space.

Definition B.10. An *inner product* on the real vector space V is a pairing that takes two elements $\mathbf{v}, \mathbf{w} \in V$ and produces a real number $\langle \mathbf{v}, \mathbf{w} \rangle \in \mathbb{R}$, subject to the following three axioms for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, and scalars $c, d \in \mathbb{R}$.

(i) *Bilinearity:*

$$\begin{aligned}\langle c\mathbf{u} + d\mathbf{v}, \mathbf{w} \rangle &= c\langle \mathbf{u}, \mathbf{w} \rangle + d\langle \mathbf{v}, \mathbf{w} \rangle, \\ \langle \mathbf{u}, c\mathbf{v} + d\mathbf{w} \rangle &= c\langle \mathbf{u}, \mathbf{v} \rangle + d\langle \mathbf{u}, \mathbf{w} \rangle.\end{aligned}\tag{B.7}$$

(ii) *Symmetry:*

$$\langle \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{w}, \mathbf{v} \rangle.\tag{B.8}$$

(iii) *Positivity:*

$$\langle \mathbf{v}, \mathbf{v} \rangle > 0 \quad \text{whenever} \quad \mathbf{v} \neq \mathbf{0}, \quad \text{while} \quad \langle \mathbf{0}, \mathbf{0} \rangle = 0.\tag{B.9}$$

Given an inner product, the associated *norm* of an element $\mathbf{v} \in V$ is defined as the positive square root of its inner product with itself:

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}.\tag{B.10}$$

Bilinearity of the inner product implies that

$$\|c\mathbf{v}\| = |c| \|\mathbf{v}\|$$

for any scalar c . The positivity axiom implies that $\|\mathbf{v}\| \geq 0$ is real and nonnegative, and equals 0 if and only if $\mathbf{v} = \mathbf{0}$ is the zero element. A vector space norm induces a notion of *distance* between elements $\mathbf{v}, \mathbf{w} \in V$, with $\text{dist}(\mathbf{v}, \mathbf{w}) = \|\mathbf{v} - \mathbf{w}\|$. In particular, $\text{dist}(\mathbf{v}, \mathbf{w}) = 0$ if and only if $\mathbf{v} = \mathbf{w}$.

Example B.11. The most familiar example of an inner product is the *dot product*[†]

$$\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v} \cdot \mathbf{w} = \mathbf{v}^T \mathbf{w} = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n\tag{B.11}$$

on the Euclidean space \mathbb{R}^n . The associated *Euclidean norm*

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}\tag{B.12}$$

conforms to our usual notion of distance between points in Euclidean space.

To find the most general inner product on \mathbb{R}^n , we need to introduce the important class of positive definite matrices.

Definition B.12. An $n \times n$ matrix C is called *positive definite* if it satisfies the positivity condition

$$\mathbf{v}^T C \mathbf{v} > 0 \quad \text{for all} \quad \mathbf{0} \neq \mathbf{v} \in \mathbb{R}^n.\tag{B.13}$$

We will sometimes write $C > 0$ to mean that C is a positive definite matrix.

[†] The elements $\mathbf{v} \in \mathbb{R}^n$ are to be regarded as column vectors, while the transpose, written \mathbf{v}^T , is the corresponding row vector.

Warning: The condition $C > 0$ does *not* mean that all the entries of C are positive. For example, $\begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix}$ is positive definite, whereas $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ is not.

Many authors, including [89], require that a positive definite matrix also be symmetric. We will *not* impose this condition here a priori. However, most of the positive definite matrices we will encounter in applications will be symmetric (or, more generally, self-adjoint — as in Example 9.15). For a symmetric matrix, the most useful test for positive definiteness is to perform Gaussian Elimination on the matrix C , which is positive definite if and only if no row interchanges are needed, and all the pivots are positive, [89].

Proposition B.13. *Every inner product on \mathbb{R}^n is given by*

$$\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v}^T C \mathbf{w} \quad \text{for} \quad \mathbf{v}, \mathbf{w} \in \mathbb{R}^n, \tag{B.14}$$

where $C > 0$ is a symmetric positive definite matrix.

The next example is of particular significance in Fourier analysis and partial differential equations.

Example B.14. Let $[a, b] \subset \mathbb{R}$ be a bounded closed interval. The integral

$$\langle f, g \rangle = \int_a^b f(x) g(x) dx \tag{B.15}$$

defines an inner product on the space $C^0[a, b]$ of continuous functions. The associated norm

$$\|f\| = \sqrt{\int_a^b f(x)^2 dx} \tag{B.16}$$

is known as the L^2 norm of the function f over the interval $[a, b]$. The positivity of the norm: $\|f\| > 0$ for $f \neq 0$, follows from the fact that the only continuous nonnegative function $g(x) \geq 0$ that satisfies $\int_a^b g(x) dx = 0$ is the zero function $g(x) \equiv 0$. Extending this construction to spaces containing discontinuous functions is trickier, since there are discontinuous functions that are not identically zero, but nevertheless have zero norm integral. An example is a function that is zero except at a single point. Further discussion can be found in Section 3.5.

The two most important inequalities in mathematical analysis apply to any inner product space.

Theorem B.15. *Every inner product satisfies the Cauchy–Schwarz and triangle inequalities*

$$|\langle \mathbf{v}, \mathbf{w} \rangle| \leq \|\mathbf{v}\| \|\mathbf{w}\|, \quad \|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|, \quad \text{for all} \quad \mathbf{v}, \mathbf{w} \in V. \tag{B.17}$$

Equality holds if and only if \mathbf{v} and \mathbf{w} are parallel, i.e., scalar multiples of each other.

Proof: We begin with the Cauchy–Schwarz inequality: $|\langle \mathbf{v}, \mathbf{w} \rangle| \leq \|\mathbf{v}\| \|\mathbf{w}\|$. The case $\mathbf{w} = \mathbf{0}$ is trivial, and so we assume $\mathbf{w} \neq \mathbf{0}$. Let $t \in \mathbb{R}$ be an arbitrary scalar. Using the three inner product axioms, we have

$$0 \leq \|\mathbf{v} + t\mathbf{w}\|^2 = \langle \mathbf{v} + t\mathbf{w}, \mathbf{v} + t\mathbf{w} \rangle = \|\mathbf{v}\|^2 + 2t \langle \mathbf{v}, \mathbf{w} \rangle + t^2 \|\mathbf{w}\|^2, \tag{B.18}$$

with equality holding if and only if $\mathbf{v} = -t\mathbf{w}$, which requires \mathbf{v} and \mathbf{w} to be parallel vectors. We fix \mathbf{v} and \mathbf{w} , and consider the right-hand side of (B.18) as a quadratic function of t . Its minimum value occurs when $t = \|\mathbf{w}\|^{-2} \langle \mathbf{v}, \mathbf{w} \rangle$. Substituting this value into (B.18), we obtain

$$0 \leq \|\mathbf{v}\|^2 - 2 \frac{\langle \mathbf{v}, \mathbf{w} \rangle^2}{\|\mathbf{w}\|^2} + \frac{\langle \mathbf{v}, \mathbf{w} \rangle^2}{\|\mathbf{w}\|^2} = \|\mathbf{v}\|^2 - \frac{\langle \mathbf{v}, \mathbf{w} \rangle^2}{\|\mathbf{w}\|^2},$$

and hence $\langle \mathbf{v}, \mathbf{w} \rangle^2 \leq \|\mathbf{v}\|^2 \|\mathbf{w}\|^2$, which, upon taking the square root, establishes the Cauchy–Schwarz inequality. Again, as noted above, equality holds if and only if \mathbf{v} and \mathbf{w} are parallel.

To establish the triangle inequality, we compute

$$\begin{aligned} \|\mathbf{v} + \mathbf{w}\|^2 &= \langle \mathbf{v} + \mathbf{w}, \mathbf{v} + \mathbf{w} \rangle = \|\mathbf{v}\|^2 + 2\langle \mathbf{v}, \mathbf{w} \rangle + \|\mathbf{w}\|^2 \\ &\leq \|\mathbf{v}\|^2 + 2\|\mathbf{v}\| \|\mathbf{w}\| + \|\mathbf{w}\|^2 = (\|\mathbf{v}\| + \|\mathbf{w}\|)^2, \end{aligned}$$

where the middle inequality follows from the Cauchy–Schwarz inequality (which clearly also holds if the absolute value is removed.) Taking square roots of both sides completes the proof. *Q.E.D.*

We will also have occasion to use inner products on complex vector spaces. To ensure that the associated norm remains positive, the real definition must be modified. The complex conjugate of a complex scalar $c = a + ib$, with $a, b \in \mathbb{R}$, will be indicated by an overbar: $\bar{c} = a - ib$. When dealing with a complex inner product space, one must pay careful attention to complex conjugation.

Definition B.16. An *inner product* on the complex vector space V is a pairing that takes two vectors $\mathbf{v}, \mathbf{w} \in V$ and produces a complex number $\langle \mathbf{v}, \mathbf{w} \rangle \in \mathbb{C}$, subject to the following requirements, for $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, and $c, d \in \mathbb{C}$:

(i) *Sesquilinearity:*

$$\begin{aligned} \langle c\mathbf{u} + d\mathbf{v}, \mathbf{w} \rangle &= c\langle \mathbf{u}, \mathbf{w} \rangle + d\langle \mathbf{v}, \mathbf{w} \rangle, \\ \langle \mathbf{u}, c\mathbf{v} + d\mathbf{w} \rangle &= \bar{c}\langle \mathbf{u}, \mathbf{v} \rangle + \bar{d}\langle \mathbf{u}, \mathbf{w} \rangle. \end{aligned} \tag{B.19}$$

(ii) *Conjugate Symmetry:*

$$\langle \mathbf{v}, \mathbf{w} \rangle = \overline{\langle \mathbf{w}, \mathbf{v} \rangle}. \tag{B.20}$$

(iii) *Positivity:*

$$\|\mathbf{v}\|^2 = \langle \mathbf{v}, \mathbf{v} \rangle \geq 0, \quad \text{and} \quad \langle \mathbf{v}, \mathbf{v} \rangle = 0 \quad \text{if and only if} \quad \mathbf{v} = \mathbf{0}. \tag{B.21}$$

Example B.17. The simplest example is the *Hermitian dot product*

$$\mathbf{z} \cdot \mathbf{w} = \mathbf{z}^T \bar{\mathbf{w}} = z_1 \bar{w}_1 + z_2 \bar{w}_2 + \cdots + z_n \bar{w}_n, \quad \text{for} \quad \mathbf{z} = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}, \tag{B.22}$$

between complex vectors $\mathbf{v}, \mathbf{w} \in \mathbb{C}^n$.

Example B.18. Let $C^0[-\pi, \pi]$ denote the complex vector space consisting of all complex-valued continuous functions $f(x) = u(x) + iv(x)$ depending on the *real* variable $-\pi \leq x \leq \pi$. The L^2 *Hermitian inner product* on $C^0[-\pi, \pi]$ is defined as

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x) \overline{g(x)} dx, \tag{B.23}$$

i.e., the integral of f times the complex conjugate of g , with corresponding norm

$$\|f\| = \sqrt{\int_{-\pi}^{\pi} |f(x)|^2 dx} = \sqrt{\int_{-\pi}^{\pi} [u(x)^2 + v(x)^2] dx}. \quad (\text{B.24})$$

Inner products on complex vector spaces also satisfy the Cauchy–Schwarz and triangle inequalities (B.17). The proof is left as an exercise for the reader; see [89; Exercise 3.6.46].

B.4 Orthogonality

Definition B.19. Two elements $\mathbf{v}, \mathbf{w} \in V$ of an inner product space V are called *orthogonal* if their inner product vanishes: $\langle \mathbf{v}, \mathbf{w} \rangle = 0$.

For ordinary Euclidean space equipped with the dot product, two vectors are orthogonal if and only if they are perpendicular, i.e., meet at a right angle.

Definition B.20. A basis $\mathbf{u}_1, \dots, \mathbf{u}_n$ of an inner product space V is called *orthogonal* if $\langle \mathbf{u}_i, \mathbf{u}_j \rangle = 0$ for all $i \neq j$. The basis is called *orthonormal* if, in addition, each vector has unit length: $\|\mathbf{u}_i\| = 1$, for all $i = 1, \dots, n$.

For example, the standard basis vectors (B.5) form an orthonormal basis of \mathbb{R}^n with respect to the dot product, but they are not orthonormal for any other inner product thereon.

Theorem B.21. If $\mathbf{v}_1, \dots, \mathbf{v}_n$ form an orthogonal basis, then the corresponding coordinates of a vector

$$\mathbf{v} = a_1 \mathbf{v}_1 + \dots + a_n \mathbf{v}_n \quad \text{are given by} \quad a_i = \frac{\langle \mathbf{v}, \mathbf{v}_i \rangle}{\|\mathbf{v}_i\|^2}. \quad (\text{B.25})$$

Moreover, the vector's norm can be computed using the formula

$$\|\mathbf{v}\|^2 = \sum_{i=1}^n a_i^2 \|\mathbf{v}_i\|^2 = \sum_{i=1}^n \left(\frac{\langle \mathbf{v}, \mathbf{v}_i \rangle}{\|\mathbf{v}_i\|} \right)^2. \quad (\text{B.26})$$

Proof: We compute the inner product of (B.25) with one of the basis vectors. By orthogonality,

$$\langle \mathbf{v}, \mathbf{v}_i \rangle = \left\langle \sum_{j=1}^n a_j \mathbf{v}_j, \mathbf{v}_i \right\rangle = \sum_{j=1}^n a_j \langle \mathbf{v}_j, \mathbf{v}_i \rangle = a_i \|\mathbf{v}_i\|^2.$$

To prove formula (B.26), we similarly expand

$$\|\mathbf{v}\|^2 = \langle \mathbf{v}, \mathbf{v} \rangle = \sum_{i,j=1}^n a_i a_j \langle \mathbf{v}_i, \mathbf{v}_j \rangle = \sum_{i=1}^n a_i^2 \|\mathbf{v}_i\|^2. \quad Q.E.D.$$

In the case of an orthonormal basis, the formulas (B.25–26) simplify to

$$\mathbf{v} = c_1 \mathbf{u}_1 + \dots + c_n \mathbf{u}_n, \quad \text{where} \quad c_i = \langle \mathbf{v}, \mathbf{u}_i \rangle, \quad \|\mathbf{v}\| = c_1^2 + \dots + c_n^2. \quad (\text{B.27})$$

Example B.22. A particularly important orthogonal basis is provided by the following vectors lying in \mathbb{C}^n :

$$\begin{aligned} \boldsymbol{\omega}_k &= (1, \zeta^k, \zeta^{2k}, \zeta^{3k}, \dots, \zeta^{(n-1)k})^T \\ &= (1, e^{2k\pi i/n}, e^{4k\pi i/n}, \dots, e^{2(n-1)k\pi i/n})^T, \end{aligned} \quad k = 0, \dots, n-1, \quad (\text{B.28})$$

where

$$\zeta = e^{2\pi i/n}. \quad (\text{B.29})$$

Orthogonality relies on the fact that its powers, $\zeta^k = e^{2k\pi i/n}$, $k = 0, \dots, n-1$, are the complex roots of the elementary polynomial

$$z^n - 1 = (z - 1)(1 + z + z^2 + \dots + z^{n-1}), \quad (\text{B.30})$$

while

$$\bar{\zeta} = e^{-2\pi i/n} = \zeta^{-1}.$$

Since when $0 < k \leq n-1$, the complex number $\zeta^k \neq 1$ is a root of the polynomial (B.30), it must also be a root of the second factor. This implies that

$$1 + \zeta^k + \zeta^{2k} + \zeta^{3k} + \dots + \zeta^{(n-1)k} = \begin{cases} n, & k \equiv 0 \pmod n, \\ 0, & k \not\equiv 0 \pmod n, \end{cases}$$

where the former case $k \equiv 0 \pmod n$ follows by direct substitution of $\zeta^k = 1$. Thus, the Hermitian inner products of the vectors (B.28) equal

$$\langle \boldsymbol{\omega}_k, \boldsymbol{\omega}_l \rangle = \sum_{j=0}^{n-1} \zeta^{jk} \bar{\zeta}^{jl} = \sum_{j=0}^{n-1} \zeta^{j(k-l)} = \begin{cases} n, & k = l, \\ 0, & k \neq l, \end{cases} \quad (\text{B.31})$$

provided $0 \leq k, l \leq n-1$, thereby establishing orthogonality. These vectors are the discrete analogues of the orthogonal complex exponential functions that are used to construct complex Fourier series. They are the basis of the discrete Fourier transform, [89; §5.7], and their orthogonality is the key to modern signal processing.

B.5 Eigenvalues and Eigenvectors

The eigenvalues and eigenvectors of a matrix first appear when solving linear systems of ordinary differential equations. But their essential importance extends across all of mathematics and its manifold applications. Extensions of the eigenvalue method to linear operators on function spaces are critical to the analysis of partial differential equations.

Definition B.23. Let A be an $n \times n$ matrix. A scalar λ is called an *eigenvalue* of A if there is a *nonzero* vector $\mathbf{v} \neq \mathbf{0}$, called an associated *eigenvector*, such that

$$A\mathbf{v} = \lambda\mathbf{v}. \quad (\text{B.32})$$

In particular, a matrix has $\lambda = 0$ as an eigenvalue if and only if it has a *null eigenvector* $\mathbf{v} \neq \mathbf{0}$, satisfying $A\mathbf{v} = \mathbf{0}$, and hence is a singular (non-invertible) matrix, with vanishing determinant: $\det A = 0$. An eigenvalue is called *simple* if it admits only one linearly independent eigenvalue; more generally, the *multiplicity* of an eigenvalue is defined as the

dimension of the *eigenspace* consisting of all solutions to the eigenequation (B.32), including $\mathbf{0}$. Thus, a simple eigenvalue has multiplicity 1.

Even if A is a real matrix, we must allow the possibility of complex eigenvectors. Matrices with a “complete” set of eigenvectors are the most common, and also the easiest to deal with.

Definition B.24. An $n \times n$ real or complex matrix A is called *complete* if there exists a basis of \mathbb{C}^n consisting of its (complex) eigenvectors.

It is not hard to show that eigenvectors corresponding to different eigenvalues are necessarily linearly independent. This means that matrices with all distinct (and hence simple) eigenvalues are necessarily complete:

Proposition B.25. Any $n \times n$ matrix with n distinct eigenvalues is complete.

Unfortunately, not all matrices with repeated eigenvalues are complete. For instance, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is complete, since, for instance, $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ form an eigenvector basis of \mathbb{C}^2 , whereas $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is not, since it has only one independent eigenvector, namely $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Incomplete matrices are much more challenging to deal with, both theoretically and numerically. Fortunately, we can safely ignore the incomplete cases in this text.

The most common way for orthogonal bases to arise is as eigenvector bases of symmetric matrices. (Orthogonality is with respect to the standard dot product on \mathbb{R}^n .) The extension of this result to “self-adjoint” operators on function space forms the foundation of Fourier analysis and its generalizations.

Theorem B.26. Let $A = A^T$ be a real symmetric $n \times n$ matrix. Then

- (a) All the eigenvalues of A are real.
- (b) Eigenvectors corresponding to distinct eigenvalues are orthogonal.
- (c) There is an orthonormal basis of \mathbb{R}^n consisting of n eigenvectors of A .

Let us demonstrate orthogonality, leaving the remaining steps in the proof to [89; Theorem 8.20]. If

$$A\mathbf{v} = \lambda\mathbf{v}, \quad A\mathbf{w} = \mu\mathbf{w},$$

where $\lambda \neq \mu$ are distinct real eigenvalues, then, by symmetry of A ,

$$\lambda\mathbf{v} \cdot \mathbf{w} = (A\mathbf{v}) \cdot \mathbf{w} = (A\mathbf{v})^T \mathbf{w} = \mathbf{v}^T A\mathbf{w} = \mathbf{v} \cdot (A\mathbf{w}) = \mathbf{v} \cdot (\mu\mathbf{w}) = \mu\mathbf{v} \cdot \mathbf{w},$$

and hence

$$(\lambda - \mu)\mathbf{v} \cdot \mathbf{w} = 0.$$

Since $\lambda \neq \mu$, this implies that the eigenvectors \mathbf{v}, \mathbf{w} are necessarily orthogonal.

B.6 Linear Iteration

For numerical applications, we will require some basic results on iteration of linear systems. Consider first a *homogeneous linear iterative system* of the form

$$\mathbf{u}^{(k+1)} = A\mathbf{u}^{(k)}, \quad \mathbf{u}^{(0)} = \mathbf{u}_0, \quad (\text{B.33})$$

in which A is an $n \times n$ matrix and $\mathbf{u}_0 \in \mathbb{R}^n$ or \mathbb{C}^n . The solution to such a system is evidently obtained by repeatedly multiplying the initial vector \mathbf{u}_0 by the matrix A , and so

$$\mathbf{u}^{(k)} = A^k \mathbf{u}_0. \tag{B.34}$$

Definition B.27. A matrix A is called *convergent* if every solution to the homogeneous linear iterative system (B.33) tends to zero in the limit: $\mathbf{u}^{(k)} \rightarrow \mathbf{0}$ as $k \rightarrow \infty$. Equivalently, A is convergent if and only if its powers converge to the zero matrix: $A^k \rightarrow \mathbf{0}$ as $k \rightarrow \infty$.

The solution formula (B.34), while elementary, is not particularly enlightening. An alternative approach is to recognize that if λ_j is an eigenvalue of A and \mathbf{v}_j a corresponding eigenvector, then

$$\mathbf{u}_j^{(k)} = \lambda_j^k \mathbf{v}_j \tag{B.35}$$

is a solution, since

$$A \mathbf{u}_j^{(k)} = \lambda_j^k A \mathbf{v}_j = \lambda_j^{k+1} \mathbf{v}_j = \mathbf{u}_j^{(k+1)}.$$

Moreover, linear combinations of such *eigensolutions* are also solutions. In particular, if A is complete, then we can write down the general solution to (B.33) as a linear combination of the independent eigensolutions:

$$\mathbf{u}^{(k)} = c_1 \lambda_1^k \mathbf{v}_1 + c_2 \lambda_2^k \mathbf{v}_2 + \cdots + c_n \lambda_n^k \mathbf{v}_n, \tag{B.36}$$

where $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is the eigenvector basis. The coefficients c_1, \dots, c_n are uniquely determined by the initial conditions,

$$\mathbf{u}^{(0)} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \cdots + c_n \mathbf{v}_n = \mathbf{u}_0,$$

which relies on the fact that the eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ form a basis. Now, A is convergent if and only if all solutions $\mathbf{u}^{(k)} \rightarrow \mathbf{0}$. The individual eigensolution (B.35) goes to zero if and only if its associated eigenvalue is strictly less than 1 in modulus: $|\lambda_j| < 1$. This proves the following result for complete matrices. The proof in the incomplete case relies on the Jordan canonical form, [89; Chapter 10].

Theorem B.28. *The matrix A is convergent if and only if all its eigenvalues satisfy $|\lambda| < 1$.*

Definition B.29. The *spectral radius* of a matrix A is defined as the maximal modulus of all of its real and complex eigenvalues: $\rho(A) = \max \{ |\lambda_1|, \dots, |\lambda_k| \}$.

Corollary B.30. *The matrix A is convergent if and only if $\rho(A) < 1$.*

Indeed, the spectral radius essentially governs the rate of convergence of the iterative system — the closer it is to 0, the faster the convergence rate.

Next, consider the *inhomogeneous linear iterative system*

$$\mathbf{v}^{(k+1)} = A \mathbf{v}^{(k)} + \mathbf{b}, \quad \mathbf{v}^{(0)} = \mathbf{v}_0, \tag{B.37}$$

where \mathbf{b} a fixed vector. A *fixed point* is a vector \mathbf{v}^* that satisfies

$$\mathbf{v}^* = A \mathbf{v}^* + \mathbf{b}, \quad \text{or, equivalently,} \quad (\mathbf{I} - A) \mathbf{v}^* = \mathbf{b}, \tag{B.38}$$

where \mathbf{I} is the *identity matrix* of the same size as A . Thus, if 1 is not an eigenvalue of A (which cannot happen when A is convergent), then $\mathbf{I} - A$ is nonsingular, and so the iterative system has a unique fixed point.

Theorem B.31. Assume that 1 is not an eigenvalue of A . Then all solutions to (B.37) converge to the fixed point, $\mathbf{v}^{(k)} \rightarrow \mathbf{v}^*$ as $k \rightarrow \infty$ if and only if A is a convergent matrix.

Proof: Let $\mathbf{u}^{(k)} = \mathbf{v}^{(k)} - \mathbf{v}^*$, so that $\mathbf{v}^{(k)} \rightarrow \mathbf{v}^*$ if and only if $\mathbf{u}^{(k)} \rightarrow \mathbf{0}$. Now,

$$\mathbf{u}^{(k+1)} = \mathbf{v}^{(k+1)} - \mathbf{v}^* = (A\mathbf{v}^{(k)} + \mathbf{b}) - (A\mathbf{v}^* + \mathbf{b}) = A(\mathbf{v}^{(k)} - \mathbf{v}^*) = A\mathbf{u}^{(k)},$$

and hence $\mathbf{u}^{(k)}$ solves the homogeneous version (B.33). Thus, the result is an immediate consequence of Definition B.27. *Q.E.D.*

B.7 Linear Functions and Systems

The most basic structural features of linear differential equations, both ordinary and partial, linear boundary value problems, etc., are founded on the concept of a linear function between vector spaces.

Definition B.32. Let U and V be real vector spaces. A function $L: U \rightarrow V$ is called *linear* if it obeys two basic rules:

$$L[\mathbf{u} + \mathbf{v}] = L[\mathbf{u}] + L[\mathbf{v}], \quad L[c\mathbf{u}] = cL[\mathbf{u}], \quad (\text{B.39})$$

for all $\mathbf{u}, \mathbf{v} \in U$ and all scalars c .

We will refer to U as the *domain space* of the function L , and V as the *target space*. The latter is to emphasize the fact that the *range* of L , namely

$$\text{rng } L = \{ \mathbf{v} \in V \mid \mathbf{v} = L[\mathbf{u}] \text{ for some } \mathbf{u} \in U \}, \quad (\text{B.40})$$

may very well be a proper subspace of the target space V .

Theorem B.33. Every linear function $L: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is given by matrix multiplication, $L[\mathbf{v}] = A\mathbf{v}$, where A is an $m \times n$ matrix.

Proving that matrix multiplication satisfies the linearity conditions (B.39) is easy. The converse is established by seeing what the linear function does to the basis vectors of \mathbb{R}^n ; see [89; Theorem 7.5].

Corollary B.34. Every linear function $L: \mathbb{R}^n \rightarrow \mathbb{R}$ is given by taking the dot product with a fixed vector $\mathbf{a} \in \mathbb{R}^n$:

$$L[\mathbf{v}] = \mathbf{a} \cdot \mathbf{v}. \quad (\text{B.41})$$

When U is a function space, a linear function is also referred to as a *linear operator* in order to avoid confusion with the elements of U . If the target space $V = \mathbb{R}$, then the term *linear functional* is also often used for $L: U \rightarrow \mathbb{R}$.

Here are some representative examples that arise in applications.

Example B.35. (a) Evaluation of a function at a point, namely $L[f] = f(x_0)$, defines a linear operator $L: C^0[a, b] \rightarrow \mathbb{R}$.

(b) Integration,

$$I[f] = \int_a^b f(x) dx, \quad (\text{B.42})$$

also defines a linear functional $I: C^0[a, b] \rightarrow \mathbb{R}$.

(c) The operation $M_a[f(x)] = a(x)f(x)$ of multiplication by a continuous function a defines a linear operator $M_a: C^0[a, b] \rightarrow C^0[a, b]$.

(d) Differentiation of functions, $D[f] = f'$, serves to define a linear operator $D: C^1[a, b] \rightarrow C^0[a, b]$.

(e) A general *linear ordinary differential operator* of order n ,

$$L = a_n(x)D^n + a_{n-1}(x)D^{n-1} + \cdots + a_1(x)D + a_0(x), \quad (\text{B.43})$$

is obtained by summing such operators. If the coefficient functions $a_0(x), \dots, a_n(x)$ are continuous, then

$$L[u] = a_n(x) \frac{d^n u}{dx^n} + a_{n-1}(x) \frac{d^{n-1} u}{dx^{n-1}} + \cdots + a_1(x) \frac{du}{dx} + a_0(x)u \quad (\text{B.44})$$

defines a linear operator from $C^n[a, b]$ to $C^0[a, b]$.

Linear partial differential equations are based on linear partial differential operators, which are discussed in Chapter 1. They are particular examples of the general concept of a linear system.

Definition B.36. A *linear system* is an equation of the form

$$L[\mathbf{u}] = \mathbf{f}, \quad (\text{B.45})$$

in which $L: U \rightarrow V$ is a linear function, $\mathbf{f} \in V$, while the desired solution $\mathbf{u} \in U$. The system is *homogeneous* if $\mathbf{f} = \mathbf{0}$; otherwise, it is called *inhomogeneous*.

Note that, by the definition (B.40) of the range of L , the linear system (B.45) will have a solution if and only if $\mathbf{f} \in \text{rng } L$. In particular, a homogeneous linear system always has a solution, namely $\mathbf{u} = \mathbf{0}$. However, it may possibly admit other, nonzero, solutions.

Theorem B.37. If $\mathbf{z}_1, \dots, \mathbf{z}_k$ are all solutions to the same homogeneous linear system

$$L[\mathbf{z}] = \mathbf{0}, \quad (\text{B.46})$$

then any linear combination $c_1 \mathbf{z}_1 + \cdots + c_k \mathbf{z}_k$, for any scalars c_1, \dots, c_k , is also a solution.

In other words, the set of solutions to a homogeneous linear system (B.46) forms a subspace of the domain space U , known as the *kernel* of the linear function L :

$$\ker L = \{ \mathbf{z} \in U \mid L[\mathbf{z}] = \mathbf{0} \}. \quad (\text{B.47})$$

Theorem B.38. If the inhomogeneous linear system $L[\mathbf{u}] = \mathbf{f}$ has a particular solution \mathbf{u}^* , which requires $\mathbf{f} \in \text{rng } L$, then the general solution is $\mathbf{u} = \mathbf{u}^* + \mathbf{z}$, where $\mathbf{z} \in \ker L$ is any solution to the corresponding homogeneous system $L[\mathbf{z}] = \mathbf{0}$.

The *Superposition Principle* for inhomogeneous linear systems allows us to combine solutions corresponding to different right-hand sides.

Theorem B.39. Suppose that for each $i = 1, \dots, k$, we know a particular solution \mathbf{u}_i^* to the inhomogeneous linear system $L[\mathbf{u}] = \mathbf{f}_i$ for some $\mathbf{f}_i \in \text{rng } L$. Then, given scalars c_1, \dots, c_k , a particular solution to the combined inhomogeneous system

$$L[\mathbf{u}] = c_1 \mathbf{f}_1 + \cdots + c_k \mathbf{f}_k \quad (\text{B.48})$$

is the corresponding linear combination

$$\mathbf{u}^* = c_1 \mathbf{u}_1^* + \cdots + c_k \mathbf{u}_k^* \quad (\text{B.49})$$

of particular solutions. The general solution to the inhomogeneous system (B.48) is

$$\mathbf{u} = \mathbf{u}^* + \mathbf{z} = c_1 \mathbf{u}_1^* + \cdots + c_k \mathbf{u}_k^* + \mathbf{z}, \quad (\text{B.50})$$

where $\mathbf{z} \in \ker L$ is an arbitrary solution to the associated homogeneous system $L[\mathbf{z}] = \mathbf{0}$.

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Symbol Index

Symbol	Meaning	Page(s)
$c + d$	addition of scalars	575
$z + w$	complex addition	571
$A + B$	addition of matrices	575
$\mathbf{v} + \mathbf{w}$	addition of vectors	575
$f + g$	addition of functions	575
zw	complex multiplication	571
z/w	complex division	572
$c\mathbf{v}$, cA , cf	scalar multiplication	575
\bar{z}	complex conjugate	571
$\bar{\Omega}$	closure of subset or domain	243
$\mathbf{0}$	zero vector	xvi, 575
> 0	positive definite	355, 578
≥ 0	positive semi-definite	355
f^{-1}	inverse function	xvi
A^{-1}	inverse matrix	xvi
$f(x^+)$, $f(x^-)$	one-sided limits	xvi
$n!$	factorial	163, 453
$\binom{n}{k}$	binomial coefficient	163
$ \cdot $	absolute value, modulus	94, 225, 571
$\ \cdot\ $	norm	73, 89, 106, 284, 356, 578, 579, 581
$\ \cdot\ $	double norm	380
$\ \ \cdot\ \ $	norm	356
$\mathbf{v} \cdot \mathbf{w}$	dot product	578
$\mathbf{z} \cdot \mathbf{w}$	Hermitian dot product	580
$\langle \cdot \rangle$	expected value	287
$\langle \cdot, \cdot \rangle$	inner product	73, 89, 107, 285, 341, 578, 579, 581
$\langle\langle \cdot, \cdot \rangle\rangle$	inner product	341
$[0, 1]$	closed interval	xvi
$\{f \mid C\}$	set	xvi
\in	element of	xvi

\notin	not element of	xvi
\subset, \subsetneq	subset	xvi
\cup	union	xvi
\cap	intersection	xvi
\setminus	set theoretic difference	xvi
$:=$	definition of symbol	xvi
\equiv	identical equality of functions	xvi
\equiv	equivalence in modular arithmetic	xvi
\circ	composition	xvi
$*$	convolution	95, 281
L^*	adjoint operator	341
\sim	Fourier series representation	74
\sim	asymptotic equality	300
$f: X \rightarrow Y$	function	xvi
$x_n \rightarrow x$	convergent sequence	xvi
$f_n \rightarrow f$	weak convergence	230
$f(x^+), f(x^-)$	one-sided limits	41, 79
u', u'', \dots	space derivatives	xvii
\dot{u}, \ddot{u}, \dots	time derivatives	xvii
$u_x, u_{xx}, u_{tx}, \dots$	partial derivatives	xvii, 1
$\frac{du}{dx}, \frac{d^2u}{dx^2}, \dots$	ordinary derivatives	xvii, 1
∂	partial derivative	xvii, 1
∂	boundary of domain	5, 152, 504
$\frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial t \partial x}, \dots$	partial derivatives	xvii, 1
$\partial_x, \frac{\partial}{\partial x}$	partial derivative operator	2
$\frac{\partial}{\partial \mathbf{n}}$	normal derivative	153, 244, 504
∇	gradient	150, 242, 345, 505
$\nabla \cdot$	divergence	242, 347, 505
$\nabla \times$	curl	242
∇^2	Laplacian	243
\square	wave operator	50
$\sum_{i=1}^n$	summation	xvi
$\int f(x) dx$	indefinite integral	xvii
$\int_a^b f(x) dx$	definite integral	xvii

$\int_{-\infty}^{\infty} f(x) dx$	principal value integral	283
$\iint_{\Omega} f(x, y) dx dy$	double integral	243
$\iiint_{\Omega} f(x, y, z) dx dy dz$	triple integral	505
$\int_C f(s) ds$	line integral with respect to arc length	244
$\int_C \mathbf{v} d\mathbf{x}$	line integral	243
$\oint_C \mathbf{v} d\mathbf{x}$	line integral around closed curve	243
$\iint_{\partial\Omega} f dS$	surface integral	505
a	Bohr radius	567
\mathcal{A}	space of analytic functions	576
a_k	Fourier coefficient	74, 89
Ai	Airy function	327, 460
arg	argument (see phase)	xvi, 573
\mathbf{b}	finite element vector	401
\mathbf{B}	magnetic field	551
b_k	Fourier coefficient	74, 89
Bi	Airy function of the second kind	462
c	wave speed	19, 24, 50, 486, 546
\mathbf{c}	finite element coefficient vector	401
\mathbb{C}	complex numbers	xv, 571
c_g	group velocity	331
c_k	complex Fourier coefficient	89
c_k	eigenfunction series coefficient	378
c_p	phase velocity	330
C^0	space of continuous functions	108, 576
C^n	space of differentiable functions	5, 576
C^∞	space of smooth functions	576
\mathbb{C}^n	n -dimensional complex space	xv, 575
coker	cokernel	350
cos	cosine	6, 89
cosh	hyperbolic cosine	88
coth	hyperbolic cotangent	91, 317
csc	cosecant	230
curl	curl (see also $\nabla \times$)	242
d	ordinary derivative	xvii, 1
D	derivative operator	342, 585
D	domain	5

\det	determinant	582
\dim	dimension	577
div	divergence (see also $\nabla \cdot$)	242
ds	arc length element	244
dS	surface area element	505
e	base of natural logarithm	xvi
E	energy	61, 132, 151
\mathbf{E}	electric field	551
e^x	exponential	5
e^z	complex exponential	573
\mathbf{e}_i	standard basis vector	216, 577
erf	error function	55
erfc	complementary error function	302
\tilde{f}	periodic extension	77
\mathcal{F}	function space	575
\mathcal{F}	Fourier transform	264
\mathcal{F}^{-1}	inverse Fourier transform	265
$F(t, x; \xi)$	fundamental solution	292, 387, 481, 543
$G(x; \xi), G_\xi(x)$	Green's function	234, 240, 248, 527
$G(t, x; \tau, \xi)$	general fundamental solution	297
h	step size	182
\hbar	Planck's constant	6, 287, 394
H_n	Hermite polynomial	311
H_n^m, \tilde{H}_n^m	harmonic polynomial	520
$i = \sqrt{-1}$	imaginary unit	571
\mathbf{I}	identity matrix	575
Im	imaginary part	571
J_m	Bessel function	468
k	frequency variable	264
k	wave number	330
K	finite element matrix	401
$K[u]$	right hand side of evolution equation	291
k_{ij}^ν	elemental stiffness	417
K_n^m, \tilde{K}_n^m	complementary harmonic function	523
\ker	kernel	350, 577
l	angular quantum number	568
L^2	Hilbert space	106, 284
L_k	Laguerre polynomial	566
L_k^j	generalized Laguerre polynomial	566
$L[u]$	linear function/operator	10, 64, 585
$\lim_{x \rightarrow a}, \lim_{n \rightarrow \infty}$	limits	xvi

$\lim_{x \rightarrow a^-}$, $\lim_{x \rightarrow a^+}$	one-sided limits	xvi
log	natural or complex logarithm	xvi, 573
m	mass	6
m	magnetic quantum number	568
M	electron mass	564
M_r , M_r^*	spherical mean	553
max	maximum	xvi
min	minimum	xvi
mod	modular arithmetic	xvi
n	principal quantum number	568
\mathbf{n}	unit normal	153, 244, 505
\mathbb{N}	natural numbers	xv
\mathbf{O}	zero matrix	575
$O(h)$	Big Oh notation	182
p	pressure	3
p	option exercise price	299
P	Péclet number	311
P_n	Legendre polynomial	511, 525
p_n^m	trigonometric Ferrers function	515
P_n^m	Ferrers (associated Legendre) function	513
$\mathcal{P}^{(n)}$	space of polynomials of degree $\leq n$	577
ph	phase (argument)	xvi, 572
$Q[u]$	quadratic function(al)	362
r	radial coordinate	xv, 3, 160, 572
r	cylindrical radius	xv, 3, 508
r	spherical radius	xv, 3, 508
r	interest rate	299
\mathbb{R}	real numbers	xv
\mathbb{R}^n	n -dimensional Euclidean space	xv, 575
$R[u]$	Rayleigh quotient	375
Re	real part	571
rng	range	576
s	arc length	244
S	surface area	505
S_m	spherical Bessel function	539
s_n	partial sum	75, 113
S_r , S_r^*	sphere of radius r	553, 555
sech	hyperbolic secant	334
sign	sign function	94, 225
sin	sine	6, 89
sinh	hyperbolic sine	13, 88

span	span	576
supp	support	407
t	time	xv, 3
T	conserved density	38, 256
A^T	transpose of matrix	341, 578
T_ν	finite element triangle	411
tan	tangent	1
tanh	hyperbolic tangent	135
u	dependent variable	xv, 3
u_x, u_{xx}, \dots	partial derivative	1
v	dependent variable	xv, 3
v	eigenvector/eigenfunction	371
\mathbf{v}	vector	xv, 575
\mathbf{v}	eigenvector	66, 582
\mathbf{v}	vector field	3, 242
V	vector space	575
V	potential function	6
\mathbf{v}^\perp	perpendicular vector	244
v_{lmn}	atomic eigenfunction	568
V_λ	eigenspace	371
w	dependent variable	xv, 3
w	heat flux	122
\mathbf{w}	heat flux vector	437
x	Cartesian space coordinate	xv, 3, 152, 504
x	real part of complex number	571
X	flux	38, 256
y	Cartesian space coordinate	xv, 3, 152, 504
y	imaginary part of complex number	571
Y	flux	256
Y_m	Bessel function of the second kind	470
Y_n^m, \tilde{Y}_n^m	spherical harmonic	517
\mathcal{Y}_n^m	complex spherical harmonic	519
z	Cartesian space coordinate	xv, 3, 504
z	cylindrical coordinate	xv, 3, 508
z	complex number	571
\mathbb{Z}	integers	xv
α	electron charge	564
β_l^n	radial wave function	568
γ	thermal diffusivity	124, 438, 535
γ	Euler–Mascheroni constant	471
Γ	gamma function	454

δ, δ_ξ	delta function	217, 219, 246, 247, 527
$\tilde{\delta}$	periodically extended delta function	229
δ', δ'_ξ	derivative of delta function	225, 226
Δ	Laplacian	4, 152, 161, 243, 504, 509
Δ	discriminant	172, 173
Δx	step size	186
Δx	variance	287
Δ_S	spherical Laplacian	509
ε	thermal energy density	122, 437
ϵ_0	permittivity constant	551
$\zeta_{m,n}$	Bessel root	474
η	characteristic variable	51
θ	polar angle	xv, 3, 160, 572
θ	cylindrical angle	xv, 3, 508
θ	azimuthal angle	xv, 3, 508
ζ	root of unity	582
κ	thermal conductivity	65, 123, 437
κ	stiffness or tension	49
λ	eigenvalue	66, 371, 573
λ	magnification factor	189
μ_0	permeability constant	551
ν	viscosity	3
ξ	characteristic variable	19, 25, 32, 51
π	area of unit circle	5
ρ	density	49, 122, 438
ρ	spectral radius	584
ρ, ρ_ξ	ramp function	91, 223
$\rho_n, \rho_{n,\xi}$	n^{th} order ramp function	95, 223
$\rho_{m,n}$	relative vibrational frequency	495
σ	shock position	41
σ	heat capacity	65, 122, 438
σ	volatility	299
σ, σ_ξ	unit step function	61, 80, 222
$\sigma_{m,n}$	spherical Bessel root	540
φ	zenith angle	xv, 3, 508
φ	wave function	286
φ_k	orthogonal or orthonormal system	109
φ_k	basis for finite element subspace	401
χ	specific heat capacity	122, 431
χ_D	characteristic function	485

ψ	time-dependent wave function	394, 564
ω	frequency	59, 330
Ω	domain	152, 242, 504

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