

Undergraduate Texts in Mathematics

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Undergraduate Texts in Mathematics are generally aimed at third- and fourth-year undergraduate mathematics students at North American universities. These texts strive to provide students and teachers with new perspectives and novel approaches. The books include motivation that guides the reader to an appreciation of interrelations among different aspects of the subject. They feature examples that illustrate key concepts as well as exercises that strengthen understanding.

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Introduction to Partial Differential Equations

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To the memory of my father, Frank W.J. Olver (1924-2013) and mother, Grace E. Olver (née Smith, 1927-1980), whose love, patience, and guidance formed the heart of it all.

Preface

The momentous revolution in science precipitated by Isaac Newton's calculus soon revealed the central role of partial differential equations throughout mathematics and its manifold applications. Notable examples of fundamental physical phenomena modeled by partial differential equations, most of which are named after their discoverers or early proponents, include quantum mechanics (Schrödinger, Dirac), relativity (Einstein), electromagnetism (Maxwell), optics (eikonal, Maxwell–Bloch, nonlinear Schrödinger), fluid mechanics (Euler, Navier–Stokes, Korteweg–de Vries, Kadomstev–Petviashvili), superconductivity (Ginzburg–Landau), plasmas (Vlasov), magneto-hydrodynamics (Navier–Stokes + Maxwell), elasticity (Lamé, von Karman), thermodynamics (heat), chemical reactions (Kolmogorov–Petrovsky–Piskounov), finance (Black–Scholes), neuroscience (FitzHugh–Nagumo), and many, many more. The challenge is that, while their derivation as physical models — classical, quantum, and relativistic — is, for the most part, well established, [57, 69], most of the resulting partial differential equations are notoriously difficult to solve, and only a small handful can be deemed to be completely understood. In many cases, the only means of calculating and understanding their solutions is through the design of sophisticated numerical approximation schemes, an important and active subject in its own right. However, one cannot make serious progress on their numerical aspects without a deep understanding of the underlying analytical properties, and thus the analytical and numerical approaches to the subject are inextricably intertwined.

This textbook is designed for a one-year course covering the fundamentals of partial differential equations, geared towards advanced undergraduates and beginning graduate students in mathematics, science, and engineering. No previous experience with the subject is assumed, while the mathematical prerequisites for embarking on this course of study will be listed below. For many years, I have been teaching such a course to students from mathematics, physics, engineering, statistics, chemistry, and, more recently, biology, finance, economics, and elsewhere. Over time, I realized that there is a genuine need for a well-written, systematic, modern introduction to the basic theory, solution techniques, qualitative properties, and numerical approximation schemes for the principal varieties of partial differential equations that one encounters in both mathematics and applications. It is my hope that this book will fill this need, and thus help to educate and inspire the next generation of students, researchers, and practitioners.

While the classical topics of separation of variables, Fourier analysis, Green's functions, and special functions continue to form the core of an introductory course, the inclusion of nonlinear equations, shock wave dynamics, dispersion, symmetry and similarity methods, the Maximum Principle, Huygens' Principle, quantum mechanics and the Schrödinger equation, and mathematical finance makes this book more in tune with recent developments and trends. Numerical approximation schemes should also play an essential role in an introductory course, and this text covers the two most basic approaches: finite differences and finite elements.

On the other hand, modeling and the derivation of equations from physical phenomena and principles, while not entirely absent, has been downplayed, not because it is unimportant, but because time constraints limit what one can reasonably cover in an academic year's course. My own belief is that the primary purpose of a course in partial differential equations is to learn the principal solution techniques and to understand the underlying mathematical analysis. Thus, time devoted to modeling effectively lessens what can be adequately covered in the remainder of the course. For this reason, modeling is better left to a separate course that covers a wider range of mathematics, albeit at a more cursory level. (Modeling texts worth consulting include [57, 69].) Nevertheless, this book continually makes contact with the physical applications that spawn the partial differential equations under consideration, and appeals to physical intuition and familiar phenomena to motivate, predict, and understand their mathematical properties, solutions, and applications. Nor do I attempt to cover stochastic differential equations — see [83] for this increasingly important area — although I do work through one important by-product: the Black–Scholes equation, which underlies the modern financial industry. I have tried throughout to balance rigor and intuition, thus giving the instructor flexibility with their relative emphasis and time to devote to solution techniques versus theoretical developments.

The course material has now been developed, tested, and revised over the past six years here at the University of Minnesota, and has also been used by several other universities in both the United States and abroad. It consists of twelve chapters along with two appendices that review basic complex numbers and some essential linear algebra. See below for further details on chapter contents and dependencies, and suggestions for possible semester and year-long courses that can be taught from the book.

Prerequisites

The initial prerequisite is a reasonable level of mathematical sophistication, which includes the ability to assimilate abstract constructions and apply them in concrete situations. Some physical insight and familiarity with basic mechanics, continuum physics, elementary thermodynamics, and, occasionally, quantum mechanics is also very helpful, but not essential.

Since partial differential equations involve the partial derivatives of functions, the most fundamental prerequisite is calculus — both univariate and multivariate. Fluency in the basics of differentiation, integration, and vector analysis is absolutely essential. Thus, the student should be at ease with limits, including one-sided limits, continuity, differentiation, integration, and the Fundamental Theorem. Key techniques include the chain rule, product rule, and quotient rule for differentiation, integration by parts, and change of variables in integrals. In addition, I assume some basic understanding of the convergence of sequences and series, including the standard tests — ratio, root, integral — along with Taylor's theorem and elementary properties of power series. (On the other hand, Fourier series will be developed from scratch.)

When dealing with several space dimensions, some familiarity with the key constructions and results from two- and three-dimensional vector calculus is helpful: rectangular (Cartesian), polar, cylindrical, and spherical coordinates; dot and cross products; partial derivatives; the multivariate chain rule; gradient, divergence, and curl; parametrized curves and surfaces; double and triple integrals; line and surface integrals, culminating in Green's Theorem and the Divergence Theorem — as well as very basic point set topology: notions of

open, closed, bounded, and compact subsets of Euclidean space; the boundary of a domain and its normal direction; etc. However, all the required concepts and results will be quickly reviewed in the text at the appropriate juncture: Section 6.3 covers the two-dimensional material, while Section 12.1 deals with the three-dimensional counterpart.

Many solution techniques for partial differential equations, e.g., separation of variables and symmetry methods, rely on reducing them to one or more ordinary differential equations. In order to make progress, the student should therefore already know how to find the general solution to first-order linear equations, both homogeneous and inhomogeneous, along with separable nonlinear first-order equations, linear constant-coefficient equations, particularly those of second order, and first-order linear systems with constant-coefficient matrices, in particular the role of eigenvalues and the construction of a basis of solutions. The student should also be familiar with initial value problems, including statements of the basic existence and uniqueness theorems, but not necessarily their proofs. Basic references include [18, 20, 23], while more advanced topics can be found in [52, 54, 59]. On the other hand, while boundary value problems for ordinary differential equations play a central role in the analysis of partial differential equations, the book does not assume any prior experience, and will develop solution techniques from the beginning.

Students should also be familiar with the basics of complex numbers, including real and imaginary parts; modulus and phase (or argument); and complex exponentials and Euler's formula. These are reviewed in Appendix A. In the numerical chapters, some familiarity with basic computer arithmetic, i.e., floating-point and round-off errors, is assumed. Also, on occasion, basic numerical root finding algorithms, e.g., Newton's Method; numerical linear algebra, e.g., Gaussian Elimination and basic iterative methods; and numerical solution schemes for ordinary differential equations, e.g., Runge–Kutta Methods, are mentioned. Students who have forgotten the details can consult a basic numerical analysis textbook, e.g., [24, 60], or reference volume, e.g., [94].

Finally, knowledge of the basic results and conceptual framework provided by modern linear algebra will be essential throughout the text. Students should already be on familiar terms with the fundamental concepts of vector space, both finite- and infinite-dimensional, linear independence, span, and basis, inner products, orthogonality, norms, and Cauchy–Schwarz and triangle inequalities, eigenvalues and eigenvectors, determinants, and linear systems. These are all covered in Appendix B; a more comprehensive and recommended reference is my previous textbook, [89], coauthored with my wife, Cheri Shakiban, which provides a firm grounding in the key ideas, results, and methods of modern applied linear algebra. Indeed, Chapter 9 here can be viewed as the next stage in the general linear algebraic framework that has proven to be so indispensable for the modern analysis and numerics of not just linear partial differential equations but, indeed, all of contemporary pure and applied mathematics.

While applications and solution techniques are paramount, the text does not shy away from precise statements of theorems and their proofs, especially when these help shed light on the applications and development of the subject. On the other hand, the more advanced results that require analytical sophistication beyond what can be reasonably assumed at this level are deferred to a subsequent, graduate-level course. In particular, the book does *not* assume that the student has taken a course in real analysis, and hence, while the basic ideas underlying Hilbert space are explained in the context of Fourier analysis, knowledge of measure theory and Lebesgue integration is neither assumed nor used. Consequently, the precise definitions of Hilbert space and generalized functions (distributions) are necessarily left somewhat vague, with the level of detail being similar

to that found in a basic physics course on quantum mechanics. Indeed, one of the goals of the course is to inspire mathematics students (and others) to take a rigorous real analysis course, because it is so indispensable to the more advanced theory and applications of partial differential equations that build on the material presented here.

Outline of Chapters

The first chapter is brief and serves to set the stage, introducing some basic notation and describing what is meant by a partial differential equation and a (classical) solution thereof. It then describes the basic structure and properties of linear problems in a general sense, appealing to the underlying framework of linear algebra that is summarized in Appendix B. In particular, the fundamental superposition principles for both homogeneous and inhomogeneous linear equations and systems are employed throughout.

The first three sections of Chapter 2 are devoted to first-order partial differential equations in two variables — time and a single space coordinate — starting with simple linear cases. Constant-coefficient equations are easily solved, leading to the important concepts of characteristic and traveling wave. The method of characteristics is then extended, initially to linear first-order equations with variable coefficients, and then to the nonlinear case, where most solutions break down into discontinuous shock waves, whose subsequent dynamics relies on the underlying physics. The material on shocks may be at a slightly higher level of difficulty than the instructor wishes to deal with this early in the course, and hence may be downplayed or even omitted, perhaps returned to at a later stage, e.g., when studying Burgers' equation in Section 8.4, or when the concept of weak solution is introduced in Chapter 10. The final section of Chapter 2 is essential, and shows how the second-order wave equation can be reduced to a pair of first-order partial differential equations, thereby producing the celebrated solution formula of d'Alembert.

Chapter 3 covers the essentials of Fourier series, which is *the* most important tool in our analytical arsenal. After motivating the subject by adapting the eigenvalue method for solving linear systems of ordinary differential equations to the heat equation, the remainder of the chapter develops basic Fourier series analysis, in both real and complex forms. The final section investigates the various modes of convergence of Fourier series: pointwise, uniform, in norm. Along the way, Hilbert space and completeness are introduced, at an appropriate level of rigor. Although more theoretical than most of the material, this section is nevertheless strongly recommended, even for applications-oriented students, and can serve as a launching pad for higher-level analysis.

Chapter 4 immediately delves into the application of Fourier techniques to construct solutions to the three paradigmatic second-order partial differential equations in two independent variables — the heat, wave, and Laplace/Poisson equations — via the method of separation of variables. For dynamical problems, the separation of variables approach reinforces the importance of eigenfunctions. In the case of the Laplace equation, separation is performed in both rectangular and polar coordinates, thereby establishing the averaging property of solutions and, consequently, the Maximum Principle as important by-products. The chapter concludes with a short discussion of the classification of second-order partial differential equations, in two independent variables, into parabolic, hyperbolic, and elliptic categories, emphasizing their disparate natures and the role of characteristics.

Chapter 5 is the first devoted to numerical approximation techniques for partial differential equations. Here the emphasis is on finite difference methods. All of the

preceding cases are discussed: heat equation, transport equations, wave equation, and Laplace/Poisson equation. The student learns that, in contrast to the field of ordinary differential equations, numerical methods must be specially adapted to the particularities of the partial differential equation under investigation, and may well not converge unless certain stability constraints are satisfied.

Chapter 6 introduces a second important solution method, founded on the notion of a Green's function. Our development relies on the use of distributions (generalized functions), concentrating on the extremely useful "delta function", which is characterized both as an unconventional limit of ordinary functions and, more rigorously but more abstractly, by duality in function space. While, as with Hilbert space, we do not assume familiarity with the analysis tools required to develop the fully rigorous theory of such generalized functions, the aim is for the student to assimilate the basic ideas and comfortably work with them in the context of practical examples. With this in hand, the Green's function approach is then first developed in the context of boundary value problems for ordinary differential equations, followed by consideration of elliptic boundary value problems for the Poisson equation in the plane.

Chapter 7 returns to Fourier analysis, now over the entire real line, resulting in the Fourier transform. Applications to boundary value problems are followed by a further development of Hilbert space and its role in modern quantum mechanics. Our discussion culminates with the Heisenberg Uncertainty Principle, which is viewed as a mathematical property of the Fourier transform. Space and time considerations persuaded me not to press on to develop the Laplace transform, which is a special case of the Fourier transform, although it can be profitably employed to study initial value problems for both ordinary and partial differential equations.

Chapter 8 integrates and further develops several different themes that arise in the analysis of dynamical evolution equations, both linear and nonlinear. The first section introduces the fundamental solution for the heat equation, and describes applications in mathematical finance through the celebrated Black–Scholes equation. The second section is a brief discussion of symmetry methods for partial differential equations, a favorite topic of the author and the subject of his graduate-level monograph [87]. Section 8.3 introduces the Maximum Principle for the heat equation, an important tool, inspired by physics, in the advanced analysis of parabolic problems. The last two sections study two basic higher-order nonlinear equations. Burgers' equation combines dissipative and nonlinear effects, and can be regarded as a simplified model of viscous fluid mechanics. Interestingly, Burgers' equation can be explicitly solved by transforming it into the linear heat equation. The convergence of its solutions to the shock-wave solutions of the limiting nonlinear transport equation underlies the modern analytic method of viscosity solutions. The final section treats basic third-order linear and nonlinear evolution equations arising, for example, in the modeling of surface waves. The linear equation serves to introduce the phenomenon of dispersion, in which different Fourier modes move at different velocities, producing common physical effects observed in, for instance, water waves. We also highlight the recently discovered and fascinating Talbot effect of dispersive quantization and fractalization on periodic domains. The nonlinear Korteweg–de Vries equation has many remarkable properties, including localized soliton solutions, first discovered in the 1960s, that result from its status as a completely integrable system.

Before proceeding further, Chapter 9 takes time to formulate a general abstract framework that underlies much of the more advanced analysis of linear partial differential equations. The material is at a slightly higher level of abstraction (although amply illustrated

by concrete examples), so the more computationally oriented reader may wish to skip ahead to the last two chapters, referring back to the relevant concepts and general results in particular contexts as needed. Nevertheless, I strongly recommend covering at least some of this chapter, both because the framework is important to understanding the commonalities among various concrete instantiations, and because it demonstrates the pervasive power of mathematical analysis, even for those whose ultimate goal is applications. The development commences with the adjoint of a linear operator between inner product spaces — a powerful and far-ranging generalization of the matrix transpose — which naturally leads to consideration of self-adjoint and positive definite operators, all illustrated by finite-dimensional linear algebraic systems and boundary value problems governed by ordinary and partial differential equations. A particularly important construction, forming the foundation of the finite element numerical method, is the characterization of solutions to positive definite boundary value problems via minimization principles. Next, general results concerning eigenvalues and eigenfunctions of self-adjoint and positive definite operators are established, which serve to explain the key features of reality, orthogonality, and completeness that underlie Fourier and more general eigenfunction series expansions. A general characterization of complete eigenfunction systems based on properties of the Green's function nicely ties together two of the principal themes of the text.

Chapter 10 returns to the numerical analysis of partial differential equations, introducing the powerful finite element method. After outlining the general construction based on the preceding abstract minimization principle, we present its practical implementation, first for one-dimensional boundary value problems governed by ordinary differential equations and then for elliptic boundary value problems governed by the Laplace and Poisson equations in the plane. The final section develops an alternative approach, based on the idea of a weak solution to a partial differential equation, a concept of independent interest. Indeed, the nonclassical shock-wave solutions encountered in Section 2.3 are properly characterized as weak solutions.

The final two Chapters, 11 and 12, survey the analysis of partial differential equations in, respectively, two and three space dimensions, concentrating, as before, on the Laplace, heat, and wave equations. Much of the analysis relies on separation of variables, which, in curvilinear coordinates, leads to new classes of special functions that arise as solutions to certain linear second-order non-constant-coefficient ordinary differential equations. Since we are not assuming familiarity with this subject, the method of power series solutions to ordinary differential equations is developed in some detail. We also present the methods of Green's functions and fundamental solutions, including their qualitative properties and various applications. The material has been arranged according to spatial dimension rather than equation type; thus Chapter 11 deals with the planar heat and wave equations (the planar Laplace and Poisson equations having been treated earlier, in Chapters 4 and 6), while Chapter 12 covers all their three-dimensional counterparts. This arrangement allows a more orderly treatment of the required classes of special functions; thus, Bessel functions play the leading role in Chapter 11, while spherical harmonics, Legendre/Ferrers functions, and Laguerre polynomials star in Chapter 12. The last chapter also presents the Kirchhoff formula that solves the wave equation in three-dimensional space, an important consequence being the validity of Huygens' Principle concerning the localization of disturbances in space, which, surprisingly, does not hold in a two-dimensional universe. The book culminates with an analysis of the Schrödinger equation for the hydrogen atom, whose bound states are the atomic energy levels underlying the periodic table, atomic spectroscopy, and molecular chemistry.

Course Outlines and Chapter Dependencies

With sufficient planning and a suitably prepared and engaged class, most of the material in the text can be covered in a year. The typical single-semester course will finish with Chapter 6. Some pedagogical suggestions:

Chapter 1: Go through quickly, the main take-away being linearity and superposition.

Chapter 2: Most is worth covering and needed later, although Section 2.3, on shock waves, is optional, or can be deferred until later in the course.

Chapter 3: Students that have already taken a basic course in Fourier analysis can move directly ahead to the next chapter. The last section, on convergence, is important, but could be shortened or omitted in a more applied course.

Chapter 4: The heart of the first semester's course. Some of the material at the end of Section 4.1 — Robin boundary conditions and the root cellar problem — is optional, as is the very last subsection, on characteristics.

Chapter 5: A course that includes numerics (as I strongly recommend) should start with Section 5.1 and then cover at least a couple of the following sections, the selection depending upon the interests of the students and instructor.

Chapter 6: The material on distributions and the delta function is important for a student's general mathematical education, both pure and applied, and, in particular, for their role in the design of Green's functions. The proof of Green's representation formula (6.107) might be heavy going for some, and can be omitted by just covering the preceding less-rigorous justification of the logarithmic formula for the free-space Green's function.

Chapter 7: Sections 7.1 and 7.2 are essential, and convolution in Section 7.3 is also important. Section 7.4, on Hilbert space and quantum mechanics, can easily be omitted.

Chapter 8: All five sections are more or less independent of each other and, except for the fundamental solution and maximum principle for the heat equation, not used subsequently. Thus, the instructor can pick and choose according to interest and time allotted.

Chapter 9: This chapter is at a more abstract level than the bulk of the text, and can be skipped entirely (referring back when required), although if one intends to cover the finite element method, the material in the first three sections leading to minimization principles is required. Chapters 11 and 12 can, if desired, be launched into straight after Chapter 8, or even Chapter 7 plus the material on the heat equation in Chapter 8.

Chapter 10: Again, for a course that includes numerics, finite elements is extremely important and well worth covering. The final Section 10.4, on weak solutions, is optional, particularly the revisiting of shock waves, although if this was skipped in the early part of the course, now might be a good time to revisit Section 2.3.

Chapters 11 and 12: These constitute another essential component of the classical partial differential equations course. The detour into series solutions of ordinary

differential equations is worth following, unless this is done elsewhere in the curriculum. I recommend trying to cover as much as possible, although one may well run out of time before reaching the end, in which case, consider omitting the end of Section 11.6, on Chladni figures and nodal curves, Section 12.6, on Kirchhoff's formula and Huygens' Principle, and Section 12.7, on the hydrogen atom. Of course, if Chapter 6, on Green's functions, and Section 8.1, on fundamental solutions, were omitted, those aspects will also presumably be omitted here; even if they were covered, there is not a compelling reason to revisit these topics in higher dimensions, and one may prefer to jump ahead to the more novel material appearing in the final sections.

Exercises and Software

Exercises appear at the end of almost every subsection, and come in a variety of genres. Most sets start with some straightforward computational problems to develop and reinforce the principal new techniques and ideas. Ability to solve these basic problems is a minimal requirement for successfully assimilating the material. More advanced exercises appear later on. Some are routine, but others involve challenging computations, computer-based projects, additional practical and theoretical developments, etc. Some will challenge even the most advanced reader. A number of straightforward technical proofs, as well as interesting and useful extensions of the material, particularly in the later chapters, have been relegated to the exercises to help maintain continuity of the narrative.

Don't be afraid to assign only a few parts of a multi-part exercise. I have found the True/False exercises to be particularly useful for testing of a student's level of understanding. A full answer is not merely a T or F, but must include a detailed explanation of the reason, e.g., a proof or a counterexample, or a reference to a result in the text. Many computer projects are included, particularly in the numerical chapters, where they are essential for learning the practical techniques. However, computer-based exercises are not tied to any specific choice of language or software; in my own course, MATLAB is the preferred programming platform. Some exercises could be streamlined or enhanced by the use of computer algebra systems, such as MATHEMATICA and MAPLE, but, in general, I have avoided assuming access to any symbolic software.

As a rough guide, some of the exercises are marked with special signs:

- ◇ indicates an exercise that is referred to in the body of the text, or is important for further development or applications of the subject. These include theoretical details, omitted proofs, or new directions of importance.
- ♡ indicates a project — usually a longer exercise with multiple interdependent parts.
- ♠ indicates an exercise that requires (or at least strongly recommends) use of a computer. The student could be asked either to write their own computer code in, say, MATLAB, MAPLE, or MATHEMATICA, or to make use of pre-existing packages.
- ♣ = ♠ + ♡ indicates a more extensive computer project.

Movies

In the course of writing this book, I have made a number of movies to illustrate the dynamical behavior of solutions and their numerical approximations. I have found that

they are an extremely effective pedagogical tool and strongly recommend showing them in the classroom with appropriate commentary and discussion. They are an ideal medium for fostering a student's deep understanding and insight into the phenomena exhibited by the at times indigestible analytical formulas — much better than the individual snapshots that appear in the figures in the printed book.

While it is clearly impossible to include the movies directly in the printed text, the electronic e-book version will contain direct links. In addition, I have posted all the movies on my own web site, along with the MATHEMATICA code used to generate them:

<http://www.math.umn.edu/~olver/mov.html>

When a movie is available, the sign $\boxed{+}$ appears in the figure caption.

Conventions and Notation

A complete list of symbols employed can be found in the Symbol Index that appears at the end of the book.

Equations are numbered consecutively within chapters, so that, for example, (3.12) refers to the 12th equation in Chapter 3, irrespective of which section it appears in.

Theorems, lemmas, propositions, definitions, and examples are also numbered consecutively within each chapter, using a single scheme. Thus, in Chapter 1, Definition 1.2 follows Example 1.1, and precedes Proposition 1.3 and Theorem 1.4. I find this numbering system to be the most helpful for speedy navigation through the book.

References (books, papers, etc.) are listed alphabetically at the end of the text, and are referred to by number. Thus, [89] is the 89th listed reference, namely my *Applied Linear Algebra* text.

Q.E.D. signifies the end of a proof, an acronym for “quod erat demonstrandum”, which is Latin for “which was to be demonstrated”.

The variables that appear throughout will be subject to consistent notational conventions. Thus t always denotes time, while x, y, z represent (Cartesian) space coordinates. Polar coordinates r, θ , cylindrical coordinates r, θ, z , and spherical coordinates r, θ, φ , will also be used when needed, and our conventions appear at the appropriate places in the exposition; be especially careful with the last case, since the angular variables θ, φ are subject to two contradictory conventions in the literature. The above are almost always independent variables in the partial differential equations under study; the dependent variables or unknowns will mostly be denoted by u, v, w , while f, g, h and F, G, H represent known functions, appearing as forcing terms or in boundary data. See Chapter 4 for our convention, used in differential geometry, used to denote functions in different coordinate systems, i.e., $u(x, y)$ versus $u(r, \theta)$.

In accordance with standard contemporary mathematical notation, the “blackboard bold” letter \mathbb{R} denotes the real number line, \mathbb{C} denotes the field of complex numbers, \mathbb{Z} denotes the set of integers, both positive and negative, while \mathbb{N} denotes the natural numbers, i.e., the nonnegative integers, including 0. Similarly, \mathbb{R}^n and \mathbb{C}^n denote the corresponding n -dimensional real and complex vector spaces consisting of n -tuples of elements of \mathbb{R} and \mathbb{C} , respectively. The zero vector in each is denoted by $\mathbf{0}$.

Boldface lowercase letters, e.g., $\mathbf{v}, \mathbf{x}, \mathbf{a}$, usually denote vectors (almost always column vectors), whose entries are indicated by subscripts: v_1, x_i , etc. Matrices are denoted by ordinary capital letters, e.g., A, C, K, M — but not all such letters refer to matrices; for

instance, V often refers to a vector space, while F is typically a forcing function. The entries of a matrix, say A , are indicated by the corresponding subscripted lowercase letters: a_{ij} , with i the row index and j the column index.

Angles are *always* measured in radians, although occasionally degrees will be mentioned in descriptive sentences. All trigonometric functions are evaluated on radian angles. Following the conventions advocated in [85, 86], we use $\text{ph } z$ to denote the *phase* of a complex number $z \in \mathbb{C}$, which is more commonly called the *argument* and denoted by $\arg z$. Among the many reasons to prefer “phase” are to avoid potential confusion with the argument x of a function $f(x)$, as well as to be in accordance with the “Method of Stationary Phase” mentioned in Chapter 8.

We use $\{f|C\}$ to denote a set, where f gives the formula for the members of the set and C is a (possibly empty) list of conditions. For example, $\{x \mid 0 \leq x \leq 1\}$ means the closed unit interval from 0 to 1, also written $[0, 1]$, while $\{ax^2 + bx + c \mid a, b, c \in \mathbb{R}\}$ is the set of real quadratic polynomials, and $\{0\}$ is the set consisting only of the number 0. We use $x \in S$ to indicate that x is an element of the set S , while $y \notin S$ says that y is not an element. Set theoretic union and intersection are denoted by $S \cup T$ and $S \cap T$, respectively. The subset sign $S \subset U$ includes the possibility that the sets S and U might be equal, although for emphasis we sometimes write $S \subseteq U$. On the other hand, $S \subsetneq U$ specifically implies that the two sets are not equal. We use $U \setminus S = \{x \mid x \in U, x \notin S\}$ to denote the set-theoretic difference, meaning all elements of U that do not belong to S . We use the abbreviations \max and \min to denote the maximum and minimum elements of a set of real numbers, or of a real-valued function.

The symbol \equiv is used to emphasize when two functions are identically equal, so $f(x) \equiv 1$ means that f is the constant function, equal to 1 at all values of x . It is also occasionally used in modular arithmetic, whereby $i \equiv j \pmod n$ means $i - j$ is divisible by n . The symbol $:=$ will define a quantity, e.g., $f(x) := x^2 - 1$. An arrow is used in two senses: first, to indicate convergence of a sequence, e.g., $x_n \rightarrow x^*$ as $n \rightarrow \infty$, or, alternatively, to indicate a function, so $f: X \rightarrow Y$ means that the function f maps the domain set X to the image or target set Y , with formula $y = f(x)$. Composition of functions is denoted by $f \circ g$, while f^{-1} indicates the inverse function. Similarly, A^{-1} denotes the inverse of a matrix A .

By an *elementary function* we mean a combination of rational, algebraic, trigonometric, exponential, logarithmic, and hyperbolic functions. Familiarity with their basic properties is assumed. We always use $\log x$ for the natural (base e) logarithm — avoiding the ugly modern notation $\ln x$. On the other hand, the required properties of the various special functions — the error and complementary error functions, the gamma function, Airy functions, Bessel and spherical Bessel functions, Legendre and Ferrers functions, Laguerre functions, spherical harmonics, etc. — will be developed as needed.

Summation notation is used throughout, so $\sum_{i=1}^n a_i$ denotes the finite sum $a_1 + a_2 + \dots + a_n$ or, if the upper limit is $n = \infty$, an infinite series. Of course, the lower limit need not be 1; if it is $-\infty$ and the upper limit is $+\infty$, the result is a doubly infinite series, e.g., the complex Fourier series in Chapter 3. We use $\lim_{n \rightarrow \infty} a_n$ to denote the usual limit of a sequence a_n . Similarly, $\lim_{x \rightarrow a} f(x)$ denotes the limit of the function $f(x)$ at a point a , while $f(x^-) = \lim_{x \rightarrow a^-} f(x)$ and $f(x^+) = \lim_{x \rightarrow a^+} f(x)$ are the one-sided (left- and right-hand, respectively) limits, which agree if and only if $\lim_{x \rightarrow a} f(x)$ exists.

We will employ a variety of standard notations for derivatives. In the case of ordinary

derivatives, the most basic is the Leibniz notation $\frac{du}{dx}$ for the derivative of u with respect to x . As for partial derivatives, both the full Leibniz notation $\frac{\partial u}{\partial t}$, $\frac{\partial u}{\partial x}$, $\frac{\partial^2 u}{\partial x^2}$, $\frac{\partial^3 u}{\partial t \partial x^2}$, and the more compact subscript notation u_t , u_x , u_{xx} , u_{txx} , etc. will be interchangeably employed throughout; see also Chapter 1. Unless specifically mentioned, all functions are assumed to be sufficiently smooth that any indicated derivatives exist and the relevant mixed partial derivatives are equal. Ordinary derivatives can also be indicated by the Newtonian notation u' instead of $\frac{du}{dx}$ and u'' for $\frac{d^2 u}{dx^2}$, while $u^{(n)}$ denotes the n^{th} order derivative $\frac{d^n u}{dx^n}$. If the variable is time, t , instead of space, x , then we may employ dots, \dot{u} , \ddot{u} , instead of primes.

Definite integrals are denoted by $\int_a^b f(x) dx$, while $\int f(x) dx$ is the corresponding indefinite integral or anti-derivative. We assume familiarity only with the Riemann theory of integration, although students who have learned Lebesgue integration may wish to take advantage of that on occasion, e.g., during the discussion of Hilbert space.

Historical Matters

Mathematics is both a historical and a social activity, and many notable algorithms, theorems, and formulas are named after famous (and, on occasion, not-so-famous) mathematicians, scientists, and engineers — usually, but not necessarily, the discoverer(s). The text includes a succinct description of many of the named contributors. Readers who are interested in more extensive historical details, complete biographies, and, when available, portraits or photos, are urged to consult the informative University of St. Andrews Mac-tutor web site:

<http://www-history.mcs.st-andrews.ac.uk/history/index.html>

Early prominent contributors to the subject include the Bernoulli family, Euler, d'Alembert, Lagrange, Laplace, and, particularly, Fourier, whose remarkable methods in part sparked the nineteenth century's rigorization of mathematical analysis and then mathematics in general, as pursued by Cauchy, Riemann, Cantor, Weierstrass, and Hilbert. In the twentieth century, the subject of partial differential equations reached maturity, producing an ever-increasing number of research papers, both theoretical and applied. Nevertheless, it remains one of the most challenging and active areas of mathematical research, and, in some sense, we have only scratched the surface of this deep and fascinating subject.

Textbooks devoted to partial differential equations began to appear long ago. Of particular note, Courant and Hilbert's monumental two-volume treatise, [34, 35], played a central role in the development of applied mathematics in general, and partial differential equations in particular. Indeed, it is not an exaggeration to state that all modern treatments, including this one, as well as large swaths of research, have been directly influenced by this magnificent text. Modern undergraduate textbooks worth consulting include [50, 91, 92, 114, 120], which are more or less at the same mathematical level but have a variety of points of view and selection of topics. The graduate-level texts [38, 44, 61, 70, 99] are recommended starting points for the more advanced reader and beginning researcher. More specialized monographs and papers will be referred to at the appropriate junctures.

This book began life in 1999 as a part of a planned comprehensive introduction to applied math, inspired in large part by Gilbert Strang's wonderful text, [112]. After some

time and much effort, it was realized that the original vision was much too ambitious a goal, so my wife, Cheri Shakiban, and I recast the first part as our applied linear algebra textbook, [89]. I later decided that a large fraction of the remainder could be reworked into an introduction to partial differential equations, which, after some time and classroom testing, resulted in the book you are now reading.

Some Final Remarks

To the student: You are about to delve into the vast and important field of partial differential equations. I hope you enjoy the experience and profit from it in your future studies and career, wherever they may take you. Please send me your comments. Did you find the explanations helpful or confusing? Were enough examples included? Were the exercises of sufficient variety and appropriate level to enable you to learn the material? Do you have suggestions for improvements to be incorporated into a new edition?

To the instructor: Thank you for adopting this text! I hope you enjoy teaching from it as much as I enjoyed writing it. Whatever your experience, I want to hear from you. Let me know which parts you liked and which you didn't. Which sections worked and which were less successful. Which parts your students enjoyed, which parts they struggled with, and which parts they disliked. How can it be improved?

To all readers: Like every author, I sincerely hope that I have eliminated all errors in the text. But, more realistically, I know that no matter how many times one proofreads, mistakes still manage to squeeze through (or, worse, be generated during the editing process). Please email me your questions, typos, mathematical errors, comments, suggestions, and so on. The book's dedicated web site

<http://www.math.umn.edu/~olver/pde.html>

will actively maintain a comprehensive list of known corrections, commentary, feedback, and resources, as well as links to the movies and MATHEMATICA code mentioned above.

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I would like to thank Achi Dosanjh for encouraging me to publish this book with Springer and for her enthusiastic encouragement and help during the production process. I am grateful to David Kramer for his thorough job copyediting the manuscript. While I did not always follow his suggested changes (and, sometimes, chose to deliberately go against certain grammatical and stylistic conventions in the interests of clarity), they were all seriously considered and the result is a much-improved exposition.

And last, but far from least, my mathematical family — my wife, Cheri Shakiban, my father, Frank W.J. Olver, and my son, Sheehan Olver — had a profound impact with their many comments, help, and advice over the years. Sadly, my father passed away at age 88 on April 23, 2013, and so never got to see the final printed version. I am dedicating this book to him and to my mother, Grace, who died in 1980, for their amazing influence on my life.

Peter J. Olver
September 2013

Corrected Printing

This is the corrected printing of the book that fixes all currently known typos and errors. Some minor improvements of the exposition, typesetting, and figures have also been incorporated. Again, I thank the staff at Springer for all their help and understanding during the production process. I would also like to thank to Adrian Fellhauer, Samuel Fleischer, Ulrich Gerlach, Christopher Grant, Joost Hulshof, Qunli Ji, Ted Kroon, Christoph Leuenberger, Artem Novozhilov, Ercüment Ortaccgil, Paul Georg Papatzacos, and Mikhail Shvartsman for their helpful comments on and corrections to the first printing.

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