

Solutions

Solutions, answers, hints or relevant remarks to selected exercises are provided. Unexplained notation can be found in the text.

Chapter 1

1.1 The only non-empty open and closed subset of \mathbb{R}^n is \mathbb{R}^n . The union of open sets is open and the intersection of closed sets is closed.

- (a) Interior of a solid ellipse, open and bounded, not compact.
- (b) Surface of a cone, closed, not bounded since $(n, 0, n)$ lies in the surface for all n , not compact.
- (c) First octant in \mathbb{R}^3 —like the first quadrant in \mathbb{R}^2 . Closed, not bounded since $(n, 0, 0)$ lies in the set for all n . Hence not compact.
- (d) $x^2 + y^2 + (z - 1)^2 = 1$ is the surface of a sphere with centre $(0, 0, 1)$ and radius 1. Closed, bounded and hence compact.
- (e) Intersection of sphere of radius 2 and centre $(0, 0, 2)$ and cylinder parallel to z -axis based on circle in (x, y) -plane with centre $(0, 0)$ and radius 2. Intersection is circle in the plane $z = 2$ of radius 2 with centre $(0, 0, 2)$, compact.

1.2 (a) $\frac{\partial f}{\partial x} = 2x \log(1 + x^2 y^2) + \frac{2(z^2 + x^2)xy^2}{1 + x^2 y^2}$, $\frac{\partial f}{\partial y} = \frac{2(z^2 + x^2)x^2 y}{1 + x^2 y^2}$, $\frac{\partial f}{\partial z} = 2z \log(1 + x^2 y^2)$.

(b) $\frac{\partial g}{\partial x} = y \tan^{-1}(xz) + \frac{xyz}{1 + x^2 z^2}$, $\frac{\partial g}{\partial y} = x \tan^{-1}(xz)$, $\frac{\partial g}{\partial z} = \frac{x^2 y}{1 + x^2 z^2}$.

$$1.3 \quad F'(x, y, z, w) = \begin{pmatrix} 2x & -2y & 0 & 0 \\ 2y & 2x & 0 & 0 \\ z & 0 & x & 0 \\ 2xz^2w^2 & 0 & 2zw^2x^2 & 2z^2wx^2 \end{pmatrix},$$

$$D_{\mathbf{v}}F(1, 2, -1, -2) = \begin{pmatrix} 2 & -4 & 0 & 0 \\ 4 & 2 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 8 & 0 & -8 & -4 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 10 \\ -4 \\ 36 \end{pmatrix}.$$

1.4 $\nabla f(x, y, z) = (2x - y, -x + z^3, 3yz^2 - 6)$. $\nabla f(x, y, z) = (0, 0, 0) \Leftrightarrow 2x - y = 0, -x + z^3 = 0, 3yz^2 - 6 = 0 \Leftrightarrow y = 2x, z^3 = x, yz^2 = 2 \Leftrightarrow y = 2x, z^3 = x, xz^2 = 1 \Leftrightarrow y = 2x, z^3 = x, z^5 = 1 \Leftrightarrow z = 1, x = 1, y = 2$. Solution $(1, 2, 1)$.

1.5 $\nabla f = (2xe^y, x^2e^y, 0)$, $\nabla g = (zy^2e^{xz}, 2ye^{xz}, xy^2e^{xz})$, $fg = x^2y^2e^{y+xz}$. $\nabla(fg) = (2xy^2e^{y+xz} + x^2y^2ze^{y+xz}, 2x^2ye^{y+xz} + x^2y^2e^{y+xz}, x^3y^2e^{y+xz}) = x^2e^y(zy^2e^{xz}, 2ye^{xz}, xy^2e^{xz}) + y^2e^{xz}(2xe^y, x^2e^y, 0) = f\nabla g + g\nabla f$.

1.6 If $P(t) = (x_1(t), x_2(t), \dots, x_n(t))$ then $\|P(t)\|^2 = x_1^2(t) + \dots + x_n^2(t)$ and $\frac{d}{dt}(\|P(t)\|^2) = 2x_1(t)x_1'(t) + \dots + 2x_n(t)x_n'(t) = 2 \sum_{i=1}^n x_i(t)x_i'(t) = 2P(t) \circ P'(t) = 2\langle P(t), P'(t) \rangle$ where we supposed for convenience that $P(t)$ and $P'(t)$ are in the same space. If $\|P(t)\|$ does not depend on t then $\frac{d}{dt}(\|P(t)\|^2) = 0$ and $\langle P(t), P'(t) \rangle = 0$. The inner product of two vectors is zero if and only if they are perpendicular. In this exercise, which is extensively used in Chaps. 5–8 and 16–18, we used the inner product notation. If we use matrix notation then $P(t)$ is a $1 \times n$ matrix, $P'(t)$ is an $n \times 1$ matrix and $P(t) \circ P'(t)$ is a scalar.

$$1.7 \quad F'(X) = \begin{pmatrix} 2x & 0 & 0 \\ 0 & 2y & 2z \\ yz & xz & xy \end{pmatrix}, \quad G'(X) = \begin{pmatrix} e^x & 0 & 0 \\ 0 & 2y & -2z \\ yz & xz & xy \end{pmatrix}$$

$$H(x, y, z) = x^2e^x + y^4 - z^4 + x^2y^2z^2,$$

$$\nabla H(x, y, z) = (2xe^x + x^2e^x + 2xy^2z^2, 4y^3 + 2x^2yz^2, -4z^3 + 2x^2y^2z).$$

Note $F: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ hence F' is a 3×3 matrix, $G: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ hence G' is a 3×3 matrix and $\langle F, G \rangle: \mathbb{R}^3 \rightarrow \mathbb{R}^1$ hence $\nabla(\langle F, G \rangle)$ is a 1×3 matrix. This means $G \circ F' + F \circ G'$ is a 1×3 matrix and $F' \circ G + F \circ G'$ would *not* give the correct answer since it is not possible to add a 1×3 matrix and a 3×1 matrix.

$$1.8 \quad \frac{\partial}{\partial x} ((x^2 + y^2 + z^2)^{-1/2}) = -\frac{1}{2}(x^2 + y^2 + z^2)^{-3/2} \cdot 2x = \frac{-x}{(x^2 + y^2 + z^2)^{3/2}}.$$

1.9 Let $F = (f_1, \dots, f_m)$. Then $\|F\| = (f_1^2 + \dots + f_m^2)^{1/2}$ and $\frac{\partial}{\partial x_i}(\|F\|) = \frac{1}{2}(f_1^2 + f_2^2 + \dots + f_m^2)^{-1/2} \cdot \frac{\partial}{\partial x_i}(\sum_{j=1}^m f_j^2) = \frac{1}{\|F\|} \cdot \sum_{j=1}^m f_j \frac{\partial f_j}{\partial x_i} = \langle F, \frac{\partial F}{\partial x_i} \rangle / \|F\|$, $\nabla_{\mathbf{v}}(\|F\|) = \sum_{i=1}^n v_i \frac{\partial}{\partial x_i}(\|F\|) = \frac{1}{\|F\|} \sum_{i=1}^n v_i \langle F, \frac{\partial F}{\partial x_i} \rangle = \frac{1}{\|F\|} \langle F, \sum_{i=1}^n v_i \frac{\partial F}{\partial x_i} \rangle = \frac{\langle F, D_{\mathbf{v}}F \rangle}{\|F\|}$. We require $F(P) \neq 0$ since we divide by $\|F\|$. The result is not true otherwise, e.g. $f(x) = |x|$, $x \in \mathbb{R}$, is not differentiable at the origin. Note the use of the notation $\nabla_{\mathbf{v}}$ in place of $D_{\mathbf{v}}$ for scalar-valued functions.

1.10 Let $F(x_1, \dots, x_n) = (x_1, \dots, x_n)$ in Exercise 1.9. Then $D_{\mathbf{e}_i} F = \mathbf{e}_i$ and $D_{\mathbf{e}_i}(1/\|X\|) = -\frac{1}{\|X\|^2} D_{\mathbf{e}_i}(\|X\|) = -\frac{1}{\|X\|^2} D_{\mathbf{e}_i}(\|F\|) = -\frac{\langle F, D_{\mathbf{e}_i} F \rangle}{\|X\|^2 \|F\|} = -\frac{\langle \mathbf{e}_i, F \rangle}{\|X\|^3} = -x_i/\|X\|^3$. Hence $\nabla(1/\|X\|) = -X/\|X\|^3$.

1.11 $H(x, y, z) = ((xyz)^2 + (x^2 + y^2)^2, (xyz)^2 - (x^2 + y^2)^2, (x^2 - y^2)^2 - z^4, (x^2 - y^2)^2 + z^4)$ and $H_2(x, y, z) = (xyz)^2 - (x^2 + y^2)^2$. Hence $\frac{\partial H_2}{\partial x} = 2xy^2z^2 - 4x(x^2 + y^2)$.

$$G' \circ F' = \begin{pmatrix} 2u & 2v & 0 & 0 \\ 2u & -2v & 0 & 0 \\ 0 & 0 & 2w & -2t \\ 0 & 0 & 2w & 2t \end{pmatrix} \begin{pmatrix} yz & xz & xy \\ 2x & 2y & 0 \\ 2x & -2y & 0 \\ 0 & 0 & 2z \end{pmatrix} = \begin{pmatrix} - & - & - \\ \frac{\partial H_2}{\partial x} & - & - \\ - & - & - \\ - & - & - \end{pmatrix}$$

$$\frac{\partial H_2}{\partial x} = 2uyz - 2v \cdot 2x = 2xyz \cdot yz - 2(x^2 + y^2)2x = 2xy^2z^2 - 4x(x^2 + y^2).$$

1.12 h is the composition

$$\begin{aligned} (x_1, \dots, x_n) &\rightarrow (\mathbf{e}^{x_1}, \dots, \mathbf{e}^{x_n}) \\ &\parallel \\ (y_1, \dots, y_n) &\rightarrow f(y_1, \dots, y_n) \end{aligned}$$

$$\frac{\partial h}{\partial x_i} = \mathbf{e}^{x_i} \frac{\partial f}{\partial y_i}, \quad \frac{\partial^2 h}{\partial x_i^2} = \mathbf{e}^{x_i} \frac{\partial f}{\partial y_i} + \mathbf{e}^{x_i} \frac{\partial}{\partial y_i} \left(\frac{\partial f}{\partial y_i} \right) = \mathbf{e}^{x_i} \frac{\partial f}{\partial y_i} + \mathbf{e}^{2x_i} \frac{\partial^2 f}{\partial y_i^2} = y_i \frac{\partial f}{\partial y_i} + y_i^2 \frac{\partial^2 f}{\partial y_i^2}. \text{ Hence } \sum_{i=1}^n \frac{\partial^2 h}{\partial x_i^2} = \sum_{i=1}^n y_i \frac{\partial f}{\partial y_i} + \sum_{i=1}^n y_i^2 \frac{\partial^2 f}{\partial y_i^2} = 0.$$

1.14 $P = (1, 1, 1)$, $\Delta X = (0.1, 0.05, -0.05)$, $f(P) = 3$, $f(P + \Delta X) = 3.42628125$, $\nabla f(P) = (3, 5, 3)$, $f(P) + \nabla f(P) \cdot \Delta X = 3.40$, $\text{Error} = 0.02628125$, $\text{Error} \times 100 / f(P) = 0.87604166\%$.

1.15 $F^{-1}(C)$ is the intersection of the cone $z^2 - x^2 - y^2 = 1$ and the plane $2x - y = 2$. Solving these equations yields $y = 2x - 2$ and $z^2 = 1 + x^2 + 4(x - 1)^2 = 1 + 5x^2 - 8x + 4 = (9/5) + 5(x - 4/5)^2$. This shows that the level set is a hyperbola.

1.19 The level set can be rewritten as $y^6 + (x - y)^2 + (xy - 4z)^2 = 51$, hence $y^6 < 64$ and $|y| < 2$, $(x - y)^2 < 64$ implies $|x| < |y| + 8 < 10$. Hence $|xy| < 20$ and $(xy - 4z)^2 < 64$ implies $|4z| < |xy| + 8 < 28$. Hence $|z| < 7$. This shows that the level set is bounded. It is also closed.

1.21 $AX = \lambda X$, $AY = \mu Y$, $\lambda \neq \mu \Rightarrow \lambda \langle X, Y \rangle = \langle \lambda X, Y \rangle = \langle AX, Y \rangle = \langle X, AY \rangle = \langle X, \mu Y \rangle = \mu \langle X, Y \rangle \Rightarrow (\lambda - \mu) \langle X, Y \rangle = 0 \Rightarrow X \perp Y$ since $\lambda \neq \mu$.

Chapter 2

- 2.1 $F'_1(X) = \nabla F_1(X) = (2x_1, -2x_2, 0, 0)$. F_1 has full rank except when $x_1 = x_2 = 0$. $x_1 \neq 0 \Rightarrow \{x_2, x_3, x_4\}$ can be taken as the independent variables, $x_2 \neq 0 \Rightarrow \{x_1, x_3, x_4\}$ can be taken as the independent variables. $F_1(P_1) = -3$. If $x_1 \neq 0$ let $\phi_1(x_2, x_3, x_4) = (x_2^2 - 3)^{1/2}$ near $(x_2, x_3, x_4) = (2, 0, -1)$

$$F'_2(X) = \begin{pmatrix} 2x_1 & -2x_2 & 0 & 0 \\ 0 & 0 & 2x_3 & -2x_4 \end{pmatrix},$$

full rank $\Leftrightarrow x_1$ or $x_2 \neq 0$ and x_3 or $x_4 \neq 0$. Pairs of independent variables (x_1, x_3) , (x_1, x_4) , (x_2, x_3) , (x_2, x_4) . $F_2(P_2) = (1, 3)$, $\phi_2(x_2, x_3) = ((1 + x_2^2)^{1/2}, -(x_3^2 - 3)^{1/2})$ near $(x_2, x_3) = (0, 2)$, $F'_3(X) = \begin{pmatrix} 2x_1 & -2x_2 & 0 & 0 \\ 0 & 0 & 2x_3 & -2x_4 \\ -2x_1 & 0 & 0 & 2x_4 \end{pmatrix}$, full rank \Leftrightarrow any three of the variables x_1, x_2, x_3, x_4 are non-zero. $F_3(P_3) = (-3, -7, 15)$, $\phi_3(x_1) = (\sqrt{x_1^2 + 3}, \sqrt{x_1^2 + 8}, \sqrt{x_1^2 + 15})$ near $x_1 = 1$. Other solutions also exist.

- 2.2 (i) $x^2 + y^2 = u^2 \cos^2 v + u^2 \sin^2 v = u^2 \Rightarrow u = \pm\sqrt{x^2 + y^2} \Rightarrow \frac{\partial u}{\partial x} = \pm x/\sqrt{x^2 + y^2}$, $y/x = u \sin v / u \cos v = \tan v \Rightarrow v = \tan^{-1}(y/x) \Rightarrow \frac{\partial v}{\partial x} = -y/(x^2 + y^2)$.
- (ii) $\frac{\partial u}{\partial x} \cos v - u \sin v \frac{\partial v}{\partial x} = 1$, $\frac{\partial u}{\partial x} \sin v + u \cos v \frac{\partial v}{\partial x} = 0$. Solving these two linear equations for $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial x}$ gives us $\frac{\partial u}{\partial x} = \cos v$ and $\frac{\partial v}{\partial x} = -\sin v / u$ which agree with (i).

$$2.3 \quad F'(X) = \begin{pmatrix} 2x_1x_2^2 & 2x_1^2x_2 & 0 & 0 \\ x_2x_3 & x_1x_3 & x_1x_2 & 0 \\ 0 & 0 & 0 & 2x_1 \end{pmatrix}, F'(1, 2, 3, 4) = \begin{pmatrix} 8 & 4 & 0 & 0 \\ 6 & 3 & 2 & 0 \\ 0 & 0 & 0 & 8 \end{pmatrix},$$

hence $8x_1 + 4x_2 = 0$, $6x_1 + 3x_2 + 2x_3 = 0$, $8x_4 = 0$. Solution set $\{(x_1, -2x_1, 0, 0) : x_1 \in \mathbb{R}\}$, Basis = $\{(1, -2, 0, 0)\}$, Tangent line = $\{(1 + t, 1 - 2t, 3, 4) : t \in \mathbb{R}\}$.

- 2.4 (a) Let $f(x, y, z) = xe^y - z$. Surface = $f^{-1}(0)$, $\nabla f(x, y, z) = (e^y, xe^y, -1)$, $\nabla f(1, 0, 1) = (1, 1, -1)$, Normal line = $\{(1 + t, t, 1 - t) : t \in \mathbb{R}\}$, Tangent plane = $\{(x, y, z) : (x - 1) \cdot 1 + y \cdot 1 + (z - 1) \cdot (-1) = 0\} = \{(x, y, z) : x + y - z = 0\}$.

- (b) Let $F(x, y, z) = (x^2 + y^2 - z^2, x + y + z)$. Γ is the set $F^{-1}(1, 5)$.
 $F'(x, y, z) = \begin{pmatrix} 2x & 2y & -2z \\ 1 & 1 & 1 \end{pmatrix}$, $F'(1, 2, 2) = \begin{pmatrix} 2 & 4 & -4 \\ 1 & 1 & 1 \end{pmatrix}$

Tangent line = $(1, 2, 2) + \{(x, y, z) : 2x + 4y - 4z = 0, x + y + z = 0\} = (1, 2, 2) + \{(x, y, z) : y = 3z, x = -4z\} = \{(1 - 4t, 2 + 3t, 2 + t) : t \in \mathbb{R}\}$.

- 2.5 Equation of plane is $ax + by + cz = d$. Using (1, 2, 3) and (4, 5, 6) we obtain $a + 2b + 3c = d$ and $4a + 5b + 6c = d$. Since the plane is perpendicular to the plane $7x + 8y + 9z = 10$ it follows that $(a, b, c) \cdot (7, 8, 9) = 0$, i.e. $7a + 8b + 9c = 0$. Solving for a, b, c and d gives $b = -2a, c = a$ and $d = 0$. Solution is $\{(x, y, z) : x - 2y + z = 0\}$.
- 2.6 Let $f(x, y, z) = \sqrt{x} + \sqrt{y} + \sqrt{z}$, $\nabla f(1, 4, 1) = (1/2, 1/4, 1/2)$. Tangent plane to $f^{-1}(4)$ at (1, 4, 1) is $\{(x, y, z) : (x - 1) \cdot (1/2) + (y - 4) \cdot (1/4) + (z - 1) \cdot (1/2) = 0\} = \{(x, y, z) : 2x + y + 2z = 8\}$.
- 2.7 Let $f(x, y, z) = x^2 + 4y^2 + 4z^2$, $\nabla f(x, y, z) = (2x, 8y, 8z)$. Tangent planes to $f^{-1}(1)$ at $(1/\sqrt{2}, 1/4, 1/4)$ and $(\sqrt{3}/2, 0, 1/4)$ are $\sqrt{2}x + 2y + 2z = 2$ and $\sqrt{3}x + 2z = 2$. Line of intersection = $\{(t, (\sqrt{3} - \sqrt{2})t/2, 1 - t\sqrt{3}/2) : t \in \mathbb{R}\}$. $k =$ distance squared of line to the origin = $(9 - 2\sqrt{6})/(12 - 2\sqrt{6}) = (14 - \sqrt{6})/20$.
- 2.8 Substitute $x^2 = 1 + y^2$ into $x^2 + 2y^2 = 4$ to get $1 + 3y^2 = 4$. Hence $3y^2 = 3$ and $y = \pm 1$, $x = \pm\sqrt{2}$. Four points are $(\pm\sqrt{2}, \pm 1)$, $(a, b) = (\sqrt{2}, 1)$. Tangent line to hyperbola at $(\sqrt{2}, 1)$, $\sqrt{2}x - y = 1$ has slope $\sqrt{2}$. Normal line to ellipse points in direction $(2\sqrt{2}, 4)$ and hence has slope $4/2\sqrt{2} = \sqrt{2}$. Both lines pass through $(\sqrt{2}, 1)$ and hence coincide. Tangent lines meet at $(0, \pm 1)$ and at $(\pm 1/\sqrt{2}, 0)$. Area = $\sqrt{2}$.
- 2.9 Direction of normal line to paraboloid at (1, 1, 4) is (2, 2, -1). Tangent plane at (1, 1, 4) is $\{(x, y, z) : 2x + 2y - z = 0\}$. Normal line through (1, 1, 4) is $\{(1 + 2t, 1 + 2t, 4 - t) : t \in \mathbb{R}\}$, $t = -9/8$ gives the point $(-5/4, -5/4, 41/8)$ on paraboloid and on normal. Normal line through $(-5/4, -5/4, 41/8)$ has direction $(-5/2, -5/2, -1)$, $\cos \theta = \frac{(-5/2, -5/2, -1) \cdot (2, 2, -1)}{(25/4 + 25/4 + 1)^{1/2} (4 + 4 + 1)^{1/2}} = \frac{-9}{(27.9/2)^{1/2}} = -\sqrt{\frac{2}{3}}$.
- 2.10 Let $f(x, y, z) = \log(x^2 + y^2) - 2z$, $S = f^{-1}(0) = \text{Graph}(g)$, where $g(x, y) = \frac{1}{2} \log(x^2 + y^2)$, $\nabla f(1, -1, \frac{1}{2} \log 2) = (1, -1, -2)$. Tangent plane = $\{(x, y, z) : x - y - 2z = 2 - \log 2\}$, Normal line = $\{(1 + t, -1 - t, \frac{1}{2} \log 2 - t) : t \in \mathbb{R}\}$.
- 2.11 At points of contact normals coincide. Hence $(f'(x), -1, 0) = \lambda(2z, 1, 2z + 2x) \Rightarrow \lambda = -1 \Rightarrow z = -x$ and $f'(x) = (-1)(-2x) \Rightarrow f(x) = x^2$.

Chapter 3

- 3.1 Let $g(x, y, z) = xy + yz$, $f_1(x, y, z) = x^2 + y^2 - 1$, $f_2(x, y, z) = x - yz$.
 $\nabla g = \lambda_1 \nabla f_1 + \lambda_2 \nabla f_2 \Rightarrow (y, x + z, y) = \lambda_1(2x, 2y, 0) + \lambda_2(1, -z, -y)$,
 $y = -\lambda_2 y \Rightarrow y = 0$ or $\lambda_2 = -1$, $y = 0 \Rightarrow x = 0$ which contradicts
 $x^2 + y^2 = 1$. $\lambda_2 = -1 \Rightarrow x + z = \lambda_1 2y + z \Rightarrow x = \lambda_1 2y = yz \Rightarrow z = 2\lambda_1$,
 $y = 2\lambda_1 x - 1 = zx - 1 = (x^2/y) - 1 \Rightarrow y^2 = x^2 - y = 1 - y^2 - y \Rightarrow$
 $2y^2 + y - 1 = 0$. Hence $y = -1$ or $y = 1/2$, $y = -1 \Rightarrow x = 0$ and $z = 0$.
 Solution $(0, -1, 0)$ and $g(0, -1, 0) = 0$. If $y = 1/2$, $x = \pm\sqrt{3}/2$, $z = \pm\sqrt{3}$.
 Maximum $= 3\sqrt{3}/4$, minimum $= -3\sqrt{3}/4$.
- 3.2 Maximize z on $\{x^2 + y^2 = 1\} \cap \{x + y + z = 1\}$, $(0, 0, 1) = \lambda_1(2x, 2y, 0) +$
 $\lambda_2(1, 1, 1) \Rightarrow \lambda_2 = 1$, $2x\lambda_1 = -1 = 2y\lambda_1 \Rightarrow x = y = \pm 1/\sqrt{2}$, $z =$
 $1 \mp \sqrt{2}$. Maximum $1 + \sqrt{2}$, minimum $1 - \sqrt{2}$.
- 3.3 Maximize $ab + (1/2)bd$ subject to $b + 2a + 2c = P$, $b^2/4 + d^2 = c^2$. $(b, a +$
 $(1/2)d, 0, (1/2)b) = \lambda_1(2, 1, 2, 0) + \lambda_2(0, b/2, -2c, 2d) \Rightarrow c = 2d$, $b =$
 $\sqrt{12}d$, $a = (\sqrt{3} + 1)d \Rightarrow P = (4\sqrt{3} + 6)d$. Maximum $= (2 - \sqrt{3})P^2/4$.
- 3.4 $(2x, 2y, 2z) = \lambda_1(1, 1, -1) + \lambda_2(1, 3, 1) \Rightarrow 2x = \lambda_1 + \lambda_2$, $2y = \lambda_1 +$
 $3\lambda_2$, $2z = -\lambda_1 + \lambda_2 \Rightarrow 2x - y + z = 0$. Together with constraints this gives
 solution $(0, 1/2, 1/2)$. Minimum $1/4$. The two constraints are planes which
 intersect in a line. Question asks to find minimum and maximum distance
 squared from line to the origin. Line contains points which tend to infinity so
 no maximum.
- 3.5 The two constraints are planes which intersect in a line. This line contains
 points which tend to infinity. f consists of positive terms added together and
 tends to infinity as either x or y or z goes to infinity. Since f is always positive it
 must have an absolute minimum. $(2x, 2y, 4z) = \lambda_1(1, 1, 1) + \lambda_2(1, -1, 3) \Rightarrow$
 $\dots \Rightarrow x = 17/14$, $y = 16/14$, $z = 9/14$. Minimum $= 707/196$. $z =$
 $3 - x - y = \phi(x, y)$, $f(x, y, \phi(x, y)) = x^2 + y^2 + 2(3 - x - y)^2$ and con-
 straint becomes $2x + 4y = 7$.
- $\phi_1(x) = (7 - 2x)/4$, $\phi_2(x) = (1/4)(5 - 2x)$, $f(x, \phi_1(x), \phi_2(x)) = x^2 +$
 $(7 - 2x)^2/16 + (5 - 2x)^2/8$.
- 3.6 $(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}) = \lambda(\frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2}) \Rightarrow \lambda \neq 0$, $\frac{1}{a} = \frac{2\lambda x}{a^2} \Rightarrow 1 = \frac{2\lambda x}{a} = \frac{2\lambda y}{b} =$
 $\frac{2\lambda z}{c} \Rightarrow \frac{x}{a} = \frac{y}{b} = \frac{z}{c} \Rightarrow x = \pm \frac{a}{\sqrt{3}}$, $y = \pm \frac{b}{\sqrt{3}}$, $z = \pm \frac{c}{\sqrt{3}}$.
- 3.7 $(yz, xz, xy) = \lambda(-1/x^2, -1/y^2, -1/z^2) \Rightarrow \lambda \neq 0$, $x^2 yz = -\lambda = xy^2 z =$
 $xyz^2 \Rightarrow x = y = z$ and $3/x = 1 \Rightarrow x = y = z = 3$. Minimum 27.
- 3.8 $V(x, y, z) = xyz$, $(yz, xz, xy) = \lambda(1/a, 1/b, 1/c) \Rightarrow \dots \Rightarrow x/a = y/b =$
 z/c . Maximum $= abc/27$.
- 3.9 Join the vertices to the centre and let x , y , z be the angles at the centre. Products
 of lengths of sides $= 8R^3 \sin \frac{x}{2} \sin \frac{y}{2} \sin \frac{z}{2} = f(x, y, z)$. Sum of squares of
 lengths $= 4R^2(\sin^2 \frac{x}{2} + \sin^2 \frac{y}{2} + \sin^2 \frac{z}{2}) = g(x, y, z)$. R is the radius of the
 circle. $h(x, y, z) = x + y + z - 2\pi$.

- (a) $\nabla f = \lambda \nabla h \dots \Rightarrow \cos(x/2) = \cos(y/2) = \cos(z/2) \Rightarrow x = y = z.$
 (b) $\nabla g = \lambda \nabla h \dots \Rightarrow \sin(x/2) = \sin(y/2) = \sin(z/2) \Rightarrow x = y = z.$
- 3.10 Since $x^2 \geq 0$, $3x^2 - y^5 = 0 \Rightarrow y^5 \geq 0 \Rightarrow y \geq 0$. Hence $f(x, y) = 2y \geq 0$ and $f(0, 0) = 0$ implies f has minimum value 0 at $(0, 0)$. The method of Lagrange multipliers does not work since the surface $3x^2 - y^5 = 0$ does not have full rank at $(0, 0)$.
- 3.11 Minimize $g(\mathbf{v}) = \langle \nabla f(P), \mathbf{v} \rangle$ over $\langle \mathbf{v}, \mathbf{v} \rangle = h(\mathbf{v}) = 1$. $\nabla g(\mathbf{v}) = \lambda \nabla h(\mathbf{v}) \Rightarrow \nabla f(P) = \lambda \mathbf{v}$ and $\|\mathbf{v}\| = 1 \Rightarrow \lambda = \pm \|\nabla f(P)\|/2$. Hence $g(\mathbf{v}) = \dots = \pm \|\nabla f(P)\|$. Maximum increase in direction $\nabla f(P)/\|\nabla f(P)\|$.
- 3.12 Let $y_i = x_i/i$. Maximize $n!y_1 \cdots y_n$ on $\sum_{i=1}^n y_i^2 = 1$. Use the method in Example 3.2 to get $y_1 = y_2 = \dots = y_n = 1/\sqrt{n}$ at maximum.
- 3.13 If x, y and z are perpendicular distances to the sides, of length a, b and c , then it is necessary to minimize d , where $d^2 = x^2 + y^2 + z^2$, subject to constraint Area = $A = (1/2)(ax + by + cz)$. Minimum = $4A^2/(a^2 + b^2 + c^2)$.
- 3.14 Nearest point on line, $(11, 2, -4)$. Distance = $\sqrt{6}$.
- 3.15 Since $-1 \leq \cos \theta \leq +1$, the result in Exercise 2.12 is equivalent to the Cauchy-Schwarz inequality. We get the equality case by considering when $\cos \theta = \pm 1$.
- 3.16 The constraints form a compact subset of \mathbb{R}^4 so the function has a maximum and a minimum. If $(x, y, u, v) = (a, 0, b, 0)$ then $(xv - yu)^2 = 0$ and the minimum is 0. Let $c = xv - yu$. It suffices to consider $c \neq 0$. If $(x, y, u, v) = (a, 0, 0, b)$ then $(xv - yu)^2 = a^2b^2$ and the maximum is positive. $\nabla((xv - yu)^2) = 2c(v, -u, -y, x) = \lambda_1(2x, 2y, 0, 0) + \lambda_2(0, 0, 2u, 2v)$, $cv = \lambda_1x$, $c(-u) = \lambda_1y$, $c(-y) = \lambda_2u$, $cx = \lambda_2v$. $c^2v^2 + c^2u^2 = \lambda_1^2x^2 + \lambda_1^2y^2 \Rightarrow c^2b^2 = \lambda_1^2a^2$, $c^2 = cvx - cuy = c(vx - uy) = \lambda_1(x^2 + y^2) = \lambda_1a^2$. Hence $\lambda_1 = b^2$ and $c^2 = a^2b^2 = \text{Maximum}$.

Alternatively, let $x = a \cos \theta$, $y = a \sin \theta$, $u = b \cos \phi$, $v = b \sin \phi$ then $(xv - yu)^2 = a^2b^2 \sin^2(\theta - \phi)$ and as $\sin^2(\theta - \phi)$ ranges between 0 and 1 we get the above result. Also this exercise is a special case of the Cauchy-Schwarz inequality.

Chapter 4

- 4.1 (a) $\nabla f(x, y) = (2x + y + 2, x + 2) = (0, 0) \Rightarrow (-2, 2)$ is the only critical point. $H_{f(x,y)} = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$, $\det(H_{f(-2,2)}) = -1 < 0 \Rightarrow f$ has a saddle point at $(-2, 2)$.
- (b) local minimum at $(1, 1)$, saddle point at $(0, 0)$. To show the local minimum is an absolute minimum over $\{(x, y); x > 1/2, y > 1/2\}$ note that $\det(H_{f(x,y)}) = 36xy - 9 > 0$ when $x > 1/2$ and $y > 1/2$.
- (c) $\nabla f(x, y, z) = (3x^2z - 192, 2y - z, x^3 - y)$,

$$f''(x, y, z) = H_{f(x,y,z)} = \begin{pmatrix} 6xz & 0 & 3x^2 \\ 0 & 2 & -1 \\ 3x^2 & -1 & 0 \end{pmatrix}.$$

Critical point $(2, 8, 16)$.

$$H_{f(2,8,16)} = \begin{pmatrix} 192 & 0 & 12 \\ 0 & 2 & -1 \\ 12 & -1 & 0 \end{pmatrix}$$

saddle point.

- (d) saddle points at $(2, 4)$, $(-1, 4)$, $(2, 1)$, local maximum at $(1, 3)$ (the change of variables $u = 2 - x$, $v = 4 - y$ simplifies the calculations).
- (e) local maxima at $(\pm 1, \pm 1, 1)$, $(\pm 1, 1, \pm 1)$, $(1, \pm 1, \pm 1)$.
- (f) critical points at $(\pm 1/\sqrt{2}, \pm 1/\sqrt{2}, \pm 1/\sqrt{2})$, local maxima if even number of negative signs otherwise local minima. Using one variable calculus, max of xe^{-x^2} over \mathbb{R} is $1/\sqrt{2}e^{1/2}$ and min is $-1/\sqrt{2}e^{1/2}$. Hence max of $xyz e^{-x^2-y^2-z^2}$ over \mathbb{R}^3 is $1/2\sqrt{2}e^{3/2}$ and min is $-1/2\sqrt{2}e^{3/2}$.
- (g) saddle point at $(1, 1, 1/2)$.
- (h) saddle point at $(2\sqrt{2}, 2, -2)$.
- (i) saddle point at $(-1, 1/2, 1/2)$.
- (j) saddle point at $(1, 2, 1)$.
- 4.2 We always have $f(x, y, z) \geq 0$. Since $f(0, 0, 0) = 0$, f has an absolute minimum at $(0, 0, 0)$. Critical points satisfy $2ax = 2x(ax^2 + by^2 + cz^2)$, $2by = 2y(ax^2 + by^2 + cz^2)$, $2cz = 2z(ax^2 + by^2 + cz^2)$. Critical points are $(0, 0, 0)$, $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, $(0, 0, \pm 1)$ since if (x, y, z) is a critical point, $x \neq 0$ and $y \neq 0$ then $2ax/2by = 2x/2y \Rightarrow axy = bxy \Rightarrow a = b$ contradiction. Similarly all critical points can have only one non-zero component and this must be 1. Local minimum at $(0, 0, 0)$, local maxima at $(\pm 1, 0, 0)$, saddle points at $(0, \pm 1, 0)$ and $(0, 0, \pm 1)$. Since $2e^{x^2} > x^4$, $x^2e^{-x^2} \rightarrow 0$ as $x \rightarrow \pm\infty$ and hence $f(x, y, z) \rightarrow 0$ if any one of $x, y, z \rightarrow \pm\infty$. Use the method in Example 3.3 to show that f has an absolute maximum over \mathbb{R}^3 at a critical point. Absolute maximum a/e .

- 4.3 Local minimum at $(1/4, 1/4, 1/4)$, degenerate critical points at $(x, 0, 0)$, $(0, y, 0)$, $(0, 0, z)$, $(0, y, 1 - y)$, $(x, 1 - x, 0)$, $(x, 0, 1 - x)$, $x, y, z \in \mathbb{R}$.
- 4.4 The function is only defined when $x \neq 0$, $y \neq 0$ and $z \neq 0$. At a critical point $2x^3 = y^3 + z^3 \Rightarrow 3x^3 = x^3 + y^3 + z^3$. Similarly $2y^3 = x^3 + z^3$ and $2z^3 = x^3 + y^3 \Rightarrow x^3 = y^3 = z^3 \Rightarrow x = y = z$ and $(*)\{x, x, x : x \neq 0\}$ is a set of critical points.

$$H_{f(x,x,x)} = \frac{1}{x^6} \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$$

and $\det(H_{f(x,x,x)}) = 0$.

- 4.5 Minimize $d^2 = (x + 1)^2 + (y - 1)^2 + (xy - 1)^2$. Critical points satisfy $x + 1 + (xy - 1)y = 0$ and $y - 1 + (xy - 1)x = 0$. Adding we obtain $x^2y + y^2x = xy(x + y) = 0$. If $x = 0$ then $y = 1$ and if $y = 0$ then $x = -1$. If $y = -x \Rightarrow x^3 + 2x + 1 = 0 \Rightarrow x < 0$ and $d^2 = 2(x + 1)^2 + (x^2 + 1)^2 = x^4 + 4(x + \frac{1}{2})^2 + 2 > 2$. Minimum = $\sqrt{2}$. Lagrange multipliers can also be used for this problem.
- 4.7 $\nabla(\sum_{i=1}^m \|X - Y_i\|^2) = 2(mX - \sum_{i=1}^m Y_i)$.
- 4.8 $2x + 6\phi\phi_x - 2y - 2y\phi_x = 0$, $4y + 6\phi\phi_y - 2x - 2\phi - 2y\phi_y = 0$ and $\phi_x = \phi_y = 0 \Rightarrow x = y = \phi \Rightarrow x = \pm 1$. Critical points of ϕ at $\pm(1, 1)$, $\det(H_{\phi(1,1)}) = \det(H_{\phi(-1,-1)}) = 1/4$. $\phi_{xx}(1, 1) < 0$, local maximum at $(1, 1)$, $\phi_{xx}(-1, -1) > 0$, local minimum at $(-1, -1)$. The equation of the level set yields two solutions; $\phi_1(x, y) = (y + \sqrt{6 - 3x^2 - 5y^2 + 6xy})/3$, $\phi_2(x, y) = (y - \sqrt{6 - 3x^2 - 5y^2 + 6xy})/3$. Both have a local maximum at $(1, 1)$ and both have a local minimum $(-1, -1)$.
- 4.10 $|x| \leq 2$ and $|y| \leq 2$ since $x^2 + y^2 = 2$ and $|z| = |1 - x| \leq 1 + |x| \leq 3$ and $F^{-1}(2, 1)$ is bounded and hence compact since it is clearly closed.

Chapter 5

5.1 $P'(t) = (6 \sinh 3t, -6 \cosh 3t, 6)$, $\|P'(t)\| = 6\sqrt{2} \cosh 3t$, length $= 2 \sinh 15$.

5.2 (c) On $[0, 1]$, $(\cos^{-1}(s))' = -(1 - s^2)^{-1/2}$, $P'(s) = (-(1 - s^2)^{1/2}, -s, 0)$.

5.3 $s(t) = t\sqrt{r^2 + h^2}$, $P(s^{-1}(t)) = \left(r \cos\left(\frac{t}{\sqrt{r^2+h^2}}\right), r \sin\left(\frac{t}{\sqrt{r^2+h^2}}\right), \frac{ht}{\sqrt{r^2+h^2}} \right)$.

5.4 (a) $s(t) = \sqrt{3}(e^t - 1)$, $s^{-1}(t) = \log\left(\frac{t}{\sqrt{3}} + 1\right)$. Unit speed parametrization $t \rightarrow$

$$\left(\frac{t}{\sqrt{3}} + 1\right) \left(\cos\left(\log\left(\frac{t}{\sqrt{3}} + 1\right)\right), \sin\left(\log\left(\frac{t}{\sqrt{3}} + 1\right)\right), 1\right) \text{ for } 0 \leq t \leq \sqrt{3}(e - 1).$$

(b) $s^{-1}(t) = \sinh^{-1}(t/\sqrt{2})$. Unit speed parametrization

$$t \rightarrow \left(\left(1 + \frac{t^2}{2}\right)^{1/2}, \frac{t}{\sqrt{2}}, \sinh^{-1}\left(\frac{t}{\sqrt{2}}\right) \right)$$

where $0 \leq t \leq \sqrt{2} \sinh(1)$.

5.5 Let $P(t) = \left(\sqrt{t^2 - \frac{t^4}{16}}, 4 - \frac{t^2}{4}, t\right)$ for $0 \leq t \leq 4$.

5.6 We need $\phi: [0, 1] \rightarrow [0, 1]$, $\phi'(t) > 0$ for $0 < t < 1$, $\phi(0) = 0$, $\phi(1) = 1$, $\phi'(0) = 0$, $\phi'(1) = 0$. Take $\phi(t) = 3t^2 - 2t^3$.

$$P(t) = \begin{cases} (1, 2) + (3t^2 - 2t^3)(-2, -4), & 0 \leq t \leq 1 \\ (-1, -2) + (3(t-1)^2 - 2(t-1)^3)(5, 2), & 1 \leq t \leq 2 \\ (4, 0) + (3(t-2)^2 - 2(t-2)^3)(-3, 2), & 2 \leq t \leq 3. \end{cases}$$

5.8 This exercise shows that the rate of change of f at X_0 along two curves, which pass through X_0 , depends only on the tangents to the curves at X_0 .

5.9 To show $\langle TX, TY \rangle = \langle X, Y \rangle$ expand $\|T(X + Y)\|^2$ and $\|T(X - Y)\|^2$ and consider the difference. Use exercise 2.12 to show that angles are preserved. To show that area is preserved it suffices (?) to show that the area of rectangles is preserved. (Hint: think of Riemann sums.)

Chapter 6

- 6.1 (a) $F(P(t)) \cdot P'(t) = t$, $2\pi^2$; (b) $F(P(t)) \cdot P'(t) = t^2 + 2t^7 + 3t^6$, $85/84$; (c) $F(P(t)) \cdot P'(t) = (\cos e^t, e, e^t) \cdot (0, 1, e^t) = e + e^{2t}$, $(e^8 + 8e - 1)/2$.
- 6.2 (a) $x^2 + yz + e^{xz}$, (b) does not have a potential, (c) $xy + z^2 + \sin(xyz)$, (d) $\sin(x^2 + yz)$.
- 6.4 (a) Use Exercise 1.10, $(-2y+z, 2x-3z, -x+3y)/\|X\|^3$; (b) $(xz^2 - xy^2, yx^2 - yz^2, zy^2 - zx^2)/\|X\|^3$; (c) $(0, 0, 0)$.
- 6.6 This exercise shows that *div* and *curl* operate like derivatives. In (b) the minus sign is, perhaps, unexpected. A very careful application of the definitions is needed to verify these formulae.
- 6.9 Use Exercise 6.7, $\nabla(f(\|X\|)) = \nabla(f(\sqrt{x^2 + y^2 + z^2})) = f'(\|X\|)X/\|X\|$ and

$$\begin{aligned} \nabla^2(f(\|X\|)) &= \frac{\partial}{\partial x} \left(f'(\|X\|) \cdot \frac{x}{\|X\|} \right) + \frac{\partial}{\partial y} \left(f'(\|X\|) \cdot \frac{y}{\|X\|} \right) \\ &\quad + \frac{\partial}{\partial z} \left(f'(\|X\|) \cdot \frac{z}{\|X\|} \right) \\ &= f''(\|X\|) \frac{x^2 + y^2 + z^2}{\|X\|} + f'(\|X\|) \cdot \frac{2(x^2 + y^2 + z^2)}{\|X\|^3}. \end{aligned}$$

Then g harmonic $\Leftrightarrow \nabla^2(f(\|X\|)) = 0 \Leftrightarrow 2f'(r)/r + f''(r) = 0 \Leftrightarrow r^2 f''(r) + 2rf'(r) = (r^2 f'(r))' = 0 \Leftrightarrow f'(r) = C/r^2 \Leftrightarrow f(r) = B + (A/r)$.

Chapter 7

7.1 $P(t) = (3 \cos t, 9 \sin t), 0 \leq t \leq 2\pi$, ellipse, Curvature $27(9 \sin^2 t + 81 \cos^2 t)^{-3/2}$, maximum curvature at $(0, \pm 9)$.

7.2 $|\kappa| = \frac{\left| \frac{2}{a^2} \left(\frac{2y}{b^2}\right)^2 + \frac{2}{b^2} \left(\frac{2x}{a^2}\right)^2 \right|}{\left(\left(\frac{2x}{a^2}\right)^2 + \left(\frac{2y}{b^2}\right)^2 \right)^{3/2}} = \frac{ab}{\left(\frac{x^2 b^2}{a^2} + \frac{a^2 y^2}{b^2} \right)^{3/2}}$. If the ellipse has an anti-clockwise orientation then $|\kappa| = \kappa$, otherwise $\kappa = -|\kappa|$.

7.4 Let $P(t) = (x(t), y(t), z(t))$ be unit speed. $P'(t) = (x'(t), y'(t), z'(t))$ is independent of $t \Leftrightarrow$ all tangents are parallel $\Leftrightarrow P(t) = (a + bt, c + dt, e + ft)$.

7.5 (a) $T(t) = ((1 + t)^{1/2}, -(1 - t)^{1/2}, \sqrt{2})/2, N(t) = ((1 - t)^{1/2}, (1 + t)^{1/2}, 0)/\sqrt{2}, B(t) = (-(1 + t)^{1/2}, (1 - t)^{1/2}, \sqrt{2})/2, \kappa(t) = \tau(t) = \sqrt{2}/4\sqrt{1 - t^2}$. (b) $T(t) = (-(1 - t^2)^{1/2}, -t, 0), N(t) = (t, -(1 - t^2)^{1/2}, 0), B(t) = (0, 0, 1), \kappa(t) = 1/\sqrt{1 - t^2}, \tau(t) = 0$. (c) $T(t) = (t(1 + t^2)^{-1/2}, 2, (1 + t^2)^{-1/2})/\sqrt{5}, N(t) = ((1 + t^2)^{-1/2}, 0, -t(1 + t^2)^{-1/2}), B(t) = (-2t(1 + t^2)^{-1/2}, 1, -2(1 + t^2)^{-1/2})/\sqrt{5}, \kappa(t) = 1/\sqrt{5}(1 + t^2), \tau(t) = 2/\sqrt{5}(1 + t^2)$. (The point of this exercise was to calculate the Frenet-Serret apparatus directly. The curve in (b) lies in the xy -plane. By Proposition 8.1, $\tau(t) = 0$ and $B(t) = \pm(0, 0, 1)$. It is necessary to calculate $B(t)$ in order to check which sign to take but the above advance information is useful in order to know what to expect.)

7.6 $\|P'(t)\| = a \sec \alpha$. Let $ab = \cos \alpha$. Then $Q(t) := (a \cos bt, a \sin bt, t \sin \alpha)$ is unit speed. $T'(t) = (-ab^2 \cos bt, -ab^2 \sin bt, 0), \kappa(t) = ab^2, N(t) = (-\cos bt, -\sin bt, 0), N(0) = (-1, 0, 0)$, and normal points into cylinder, hence centre of curvature = $Q(t) + \kappa(t)N(t) = Q(t) + T'(t) = ((a - ab^2) \cos bt, (a - ab^2) \sin bt, t \sin \alpha)$, a helix, on the cylinder $x^2 + y^2 = a^2(1 - b^2)^2$.

7.7 $Q(t) = (a \cos(\frac{b}{a} \sinh^{-1}(\frac{t}{b})), a \sin(\frac{b}{a} \sinh^{-1}(\frac{t}{b})), \sqrt{b^2 + t^2})$ is a unit speed parametrization. Osculating plane at $Q(t) := (q_1(t), q_2(t), q_3(t))$ is perpendicular to $Q'(t) \times Q''(t)$ and tangent plane at $Q(t)$ is perpendicular to $\tilde{Q}(t) := (q_1(t), q_2(t), 0)$. It suffices to show $[Q'(t) \times Q''(t) \cdot \tilde{Q}(t)] / \|Q'(t)\| \cdot \|\tilde{Q}(t)\| = \alpha(t)$ is constant. $\|\tilde{Q}(t)\| = a, Q'(t) \times Q''(t) \cdot \tilde{Q}(t) = ab/(b^2 + t^2), \|Q''(t)\| = b\sqrt{a^2 + b^2}/a(t^2 + b^2)$ and $\alpha(t) = a/\sqrt{a^2 + b^2}$. Alternatively, if $P(t) = (a \cos t, a \sin t, b \cosh \frac{at}{b})$ and $\bar{P}(t) = (a \cos t, a \sin t, 0)$ then results in Chap. 8 show it suffices to prove $[P'(t) \times P''(t) \cdot \bar{P}(t)] \|P'(t) \times P''(t)\| \cdot \|\bar{P}(t)\|$ is constant. $\|\bar{P}(t)\| = a, P'(t) \times P''(t) \cdot \bar{P}(t) = [a^4 \cosh(at/b)]/b$, and

$$\|P'(t) \times P''(t)\| = [a^2(a^2 + b^2)^{1/2} \cosh(at/b)]/b.$$

7.11 $Q'(t) = B(t)$ implies Q is unit speed and $Q' = T_{\bar{\Gamma}} = B, Q'' = T'_{\bar{\Gamma}} = B' = -\tau N = \kappa_{\bar{\Gamma}} N_{\bar{\Gamma}}$. Hence $\kappa_{\bar{\Gamma}} = \tau$ (since $\tau \geq 0$ and $\kappa_{\bar{\Gamma}}$ is positive) and $N_{\bar{\Gamma}} = -N. B_{\bar{\Gamma}} = T_{\bar{\Gamma}} \times N_{\bar{\Gamma}} = B \times (-N) = T, B'_{\bar{\Gamma}} = -\tau_{\bar{\Gamma}} N_{\bar{\Gamma}} = T' = \kappa N$. Hence $\tau_{\bar{\Gamma}} = \kappa$.

7.12 $X = \tau T + \kappa B$.

Chapter 8

- 8.1 By inspection one can take $\mathbf{u} = (0, 0, 1)$ in (a) and (b) and $\mathbf{u} = (0, 1, 0)$ in (c).
 8.2 $T(t) = (1, -t^{-2}, -t^{-2} - 1)/(2 + 2t^{-2} + 2t^{-4})^{1/2}$, $N(t) = (1 + 2t^{-2}, 2 + t^{-2}, 1 - t^{-2})/(6 + 6t^{-2} + 6t^{-4})^{1/2}$, $B(t) = (1, -1, 1)/\sqrt{3}$, $\kappa(t) = 2t^{-3}\sqrt{3}(2 + 2t^{-2} + 2t^{-4})^{-3/2}$, $\tau(t) = 0$.

Note By inspection $P(t) \cdot (1, -1, 1) = -1$ so Γ lies in the plane $x - y + z = -1$. By Proposition 8.1, $\tau(t) = 0$ and $B(t) = \pm(1, -1, 1)/\sqrt{3}$. To know which sign to take it was necessary to do the above calculations.

- 8.3 $P'(\theta) = (-\tan \theta, \cot \theta, \sqrt{2})$, $\|P'(\theta)\| = \frac{2}{\sin 2\theta}$, $P''(\theta) = (-\sec^2 \theta, -\operatorname{cosec}^2 \theta, 0)$, $\|P'(\theta) \times P''(\theta)\| = 4\sqrt{2}/(\sin^2 2\theta)$, $\kappa(\theta) = \sin 2\theta/\sqrt{2}$.
 8.4 $P'(t) = 3(2t, 1 - t^2, 1 + t^2)$, $\|P'(t)\| = 3\sqrt{2}(1 + t^2)$, $T(t) = \left(\frac{\sqrt{2}t}{1+t^2}, \frac{1-t^2}{\sqrt{2}(1+t^2)}, \frac{1}{\sqrt{2}}\right)$, $T(t) \cdot (0, 0, \pm 1) = \pm \frac{1}{\sqrt{2}}$, $P'(t) \times P''(t) = 18(2t, 1 - t^2, -1 - t^2)$, $\|P'(t) \times P''(t)\| = 18\sqrt{2}(1 + t^2)$, $\kappa(t) = 1/3(1 + t^2)^2$, $P'''(t) = 6(0, -1, 1)$, $\langle P'''(t), P'(t) \times P''(t) \rangle = -18 \cdot 6 \cdot 2$, $\tau(t) = -1/3(1 + t^2)^2$.
 8.5 $\kappa(t) = (\sqrt{2}/3)e^{-t}$, $\tau(t) = (1/3)e^{-t}$.
 8.6 The normal plane at $P(t)$ is $\{X : \langle P(t) - X, P'(t) \rangle = 0\}$. If X_0 lies in every normal plane then $\langle P(t) - X_0, P'(t) \rangle = 0$ for all t . Hence $\frac{d}{dt} \langle P(t) - X_0, P'(t) - X_0 \rangle = 0$ and $\|P(t) - X_0\|$ is independent of t , and the curve lies on a sphere. $\langle P(\theta), P'(\theta) \rangle = -2 \sin 2\theta$, $\langle (a, b, c), P'(\theta) \rangle = 2a \sin 2\theta + 2b \sin \theta + 2c \cos 2\theta$. Hence $a = -1$, $b = 0$ and $c = 0$ imply $\langle P(\theta) - (a, b, c), P'(\theta) \rangle = 0$. Centre $(-1, 0, 0)$, radius 2.
 8.7 $P'(t) = (a, 2bt, 3t^2)$, $\|P'(t)\|^2 = a^2 + 4b^2t^2 + 9t^4$, $P''(t) = (0, 2b, 6t)$, $P'(t) \times P''(t) = (6bt^2, -6at, 2ab)$, $\|P'(t) \times P''(t)\|^2 = 36b^2t^4 + 36a^2t^2 + 4a^2b^2$, $P'''(t) = (0, 0, 6)$, $P'''(t) \cdot P'(t) \times P''(t) = 12ab$,

$$\begin{aligned} \tau(t)/\kappa(t) &= \|P'(t)\|^3 (P'''(t) \cdot P'(t) \times P''(t)) / \|P'(t) \times P''(t)\|^3 \\ &= 12ab(a^2 + 4b^2t^2 + 9t^4)^{3/2} / (36b^2t^4 + 36a^2t^2 + 4a^2b^2)^{3/2}. \end{aligned}$$

P parametrizes a generalised helix $\Leftrightarrow a^2 + 4b^2t^2 + 9t^4 = \alpha(36b^2t^4 + 36a^2t^2 + 4a^2b^2)$ for some $\alpha \in \mathbb{R} \Leftrightarrow \alpha = 1/4b^2$ and $4b^2 = \alpha 36a^2 \Leftrightarrow 4b^4 = 9a^2$.

Chapter 9

- 9.1 $\int_0^1 x^2 dx \cdot \int_0^{\pi/4} \sin^2 y dy = [x^3/3]_0^1 [x/2 - (\sin 2x)/4]_0^{\pi/4} = (\pi - 2)/24$.
- 9.2 (a) $\int_0^\pi \left\{ \int_0^x x \cos(x+y) dy \right\} dx = \frac{\pi}{2}$,
 (b) $\int_{-1}^{+1} \left\{ \int_{1-\sqrt{1-x^2}}^{1+\sqrt{1-x^2}} (x^2 + y^2) dy \right\} dx = 8/3 \int_0^1 (1-x^2)^{1/2} (2+x^2) dx = 3\pi/2$; or alternatively, let $x = r \cos \theta$, $y = 1 + r \sin \theta$, $dx dy = r dr d\theta$
 and $\int_0^1 \int_0^{2\pi} (r^2 + 1 + 2r \sin \theta) r dr d\theta = 3\pi/2$,
 (c) $\int_1^2 \left\{ \int_{1/y}^y \frac{y^2}{x^2} dx \right\} dy = \int_1^2 (-y + y^3) dy = 9/4$.
- 9.3 $\int_0^2 \left\{ \int_{y^2}^{4y-y^2} dx \right\} dy = 8/3$.
- 9.4 Let $\mathbf{F} = (Q, -P)$. If $t \rightarrow (x(t), y(t))$ is a unit speed parametrization then $\mathbf{n} = (y', -x')$. Now apply Green's Theorem.
- 9.5 (a) $\int_0^2 \left\{ \int_0^{3y} e^{y^2} dx \right\} dy = 3(e^4 - 1)/2$,
 (b) $\int_0^2 \left\{ \int_{x^2}^{2x} e^{y/x} dy \right\} dx = e^2 - 1$.
- 9.6 $\iint_{\substack{x^2+y^2 \leq 1 \\ x \geq 0, y \geq 0}} (1-xy) dx dy = \int_0^1 \left\{ \int_0^{\sqrt{1-x^2}} (1-xy) dy \right\} dx = (2\pi - 1)/8$. The calculations are easier using polar coordinates.
- 9.7 20π .
- 9.8 (a) Once Green's theorem has been used, symmetry implies that the answer is 0.
 (b) By Green's theorem $\int_\Gamma = \iint_{(x-2)^2+y^2 < 4} 2x dx dy$. This can be evaluated in the usual fashion but some geometry avoids all the calculations:
- $$\int_\Gamma = \iint_{(x-2)^2+y^2 < 4} 2(x-2) dx dy + 4 \iint_{(x-2)^2+y^2 < 4} dx dy = 0 + 4\pi \cdot 4 = 16\pi.$$
- The first integral is zero since $x - 2$ has average value 0 on the disc and the second integral is $4 \times$ (area of disc).
- (c) $\int_\Gamma = \iint_\Omega (-3y - 4x^2 y) dx dy = \int_3^5 dx \cdot \int_1^4 -3y dy - \int_3^5 4x^2 dx \cdot \int_1^4 y dy = -1025$.

Chapter 10

- 10.1 (a) $(u, v) \rightarrow (a \cosh u \cos v, b \cosh u \sin v, c \sinh u)$, $-\infty < u < +\infty$, $0 < v < 2\pi$.
 (b) $(u, v) \rightarrow (a \cosh u, b \sinh u \cos v, c \sinh u \sin v)$, $-\infty < u < +\infty$, $0 < v < 2\pi$;
 10.2 (a) P_1 is easily seen to be injective. For the remainder see Example 13.5.
 (b) $(P_2)_x = (\cos \theta, 0, 1)$, $(P_2)_\theta = (-x \sin \theta, y \cos \theta, 0)$ and

$$(P_2)_x \times (P_2)_\theta = (-y \cos, -x \sin, y \cos^2 \theta).$$

If $(P_2)_x \times (P_2)_\theta = 0$ then $\cos \theta = \sin \theta = 0$. This is impossible since $\cos^2 \theta + \sin^2 \theta = 1$. If $P_2(x, \theta) = P_2(x_1, \theta_1)$ then $x + a = x_1 + a$ and $\cos \theta = \cos \theta_1$, $\sin \theta = \sin \theta_1$. Hence $x = x_1$ and $\theta = \theta_1$.

- (c) $(P_3)_u \times (P_3)_v = (u + v, v - u, -2) \neq (0, 0, 0)$, $P_3(u_1, v_1) = P_3(u_2, v_2) \Rightarrow u_1 + v_1 = u_2 + v_2$ and $u_1 - v_1 = u_2 - v_2 \Rightarrow u_1 - u_2 = v_2 - v_1 = -(v_2 - v_1) \Rightarrow v_1 = v_2$ and $u_1 = u_2$.

- 10.3 Parametrize the ellipsoid $((x/\sqrt{2tr})^2 + (y/\sqrt{2tr})^2 + (z/\sqrt{2th})^2 = 1$ using ellipsoidal polar coordinates and take the $\theta = \pi/4$ cross section.

- 10.4 Parametrization formula unchanged. Range $0 < \theta < \pi/2$, $0 < \psi < \pi/2$.

- 10.5 $\|(0, 0, 1) + t((u, v, 0) - (0, 0, 1))\|^2 = 1 \Rightarrow t^2 u^2 + t^2 v^2 + (1 - t)^2 = 1$. Hence $t = 0$ or $t = 2/(1 + u^2 + v^2)$, $\phi(u, v) = (2u/(1 + u^2 + v^2), 2v/(1 + u^2 + v^2), 1 - 2/(1 + u^2 + v^2))$. $\phi(u_1, v_1) = \phi(u_2, v_2) \Rightarrow u_1^2 + v_1^2 = u_2^2 + v_2^2$ (from third coordinate) $\Rightarrow u_1 = u_2$ and $v_1 = v_2$ from the first and second coordinates and ϕ is injective. $\phi_u \times \phi_v = (-2u(1 + u^2 + v^2)^{-2}, -2v(1 + u^2 + v^2)^{-2}, (1 - (u^2 + v^2)^{-2})(1 + u^2 + v^2)^{-2}) \neq (0, 0, 0)$ for all $(u, v) \in \mathbb{R}^2$. Also $\phi(\mathbb{R}^2) = S \setminus (0, 0, 1)$.

- 10.6 Let $P(u) = (u \cos u, u \sin u, u\sqrt{3})$, $0 \leq u \leq 2$, Length $= \int_0^2 (4 + u^2)^{1/2} du = 2(\sqrt{2} + \sinh^{-1}(1))$. (Use $u = 2 \sinh \theta$, $\sinh^{-1}(1) = \log(1 + \sqrt{2})$ and $\sinh(2 \sinh^{-1}(1)) = ((1 + \sqrt{2})^2 - (1 + \sqrt{2})^{-2}) / 2 = 2\sqrt{2}$.)

- 10.7 $\nabla f(x, y, z) = (1 + y, x + z, y) \neq (0, 0, 0)$ since either $y \neq -1$ or $y \neq 0$. $U_0 = \{(x, y, z) : y \neq 0\}$ and $\phi_0(x, y) = (x, y, \frac{1-x-xy}{y})$, $(x, y) \in \mathbb{R}^2 \setminus \{(x, y); y \neq 0\}$. $U_1 = \{(x, y, z) \in S, y \neq -1\}$ and $\phi_1(y, z) = (\frac{1-yz}{1+y}, y, z)$ for $(y, z) \in \mathbb{R}^2 \setminus \{(y, z); y \neq -1\}$. (U_0, ϕ_0) and (U_1, ϕ_1) are graphs and hence parametrized surfaces.

- 10.8 At time t , L is at units above the xy -plane. The (x, y) coordinates of a point u units along L are $(u \sin bt, u \cos bt)$ after time t .

Chapter 11

- 11.1 $P(r, \theta) = (r \cos \theta, r \sin \theta, r^2)$, $0 < r < 2$, $0 < \theta < 2\pi$, $EG - F^2 = r^2(1 + 4r^2)$, surface area $= \pi(17^{3/2} - 1)/6$.
- 11.2 (a) $P(x, \theta) = (x, x^3 \cos \theta, x^3 \sin \theta)$, $0 < x < 1$, $0 < \theta < 2\pi$, $EG - F^2 = x^6(1 + 9x^4)$. Use substitution $u = 1 + 9x^4$, surface area $= \pi(10^{3/2} - 1)/27$.
- (b) $P(x, \theta) = (x, x^2 \cos \theta, x^2 \sin \theta)$, $0 < x < 1$, $0 < \theta < 2\pi$, $EG - F^2 = x^4(1 + 4x^2)$. Surface area $= \int_0^{2\pi} \int_0^1 x^2 \sqrt{1 + 4x^2} dx = \frac{2\pi}{8} \int_0^{\sinh^{-1}(2)} \sinh^2(\theta) \cosh^2(\theta) d\theta = \frac{\pi}{32} \int_0^{\sinh^{-1}(2)} (\cosh(4\theta) - 1) d\theta = \pi(18\sqrt{5} - \log(2 + \sqrt{5}))/32$ (use substitution $2x = \sinh \theta$ and identity $\sinh^{-1}(x) = \log(x + \sqrt{x^2 + 1})$).
- 11.3 $E = 1$, $F = 0$, $G = r^2 + 1$, surface area $= \int_0^{2\pi} \int_0^1 \sqrt{1 + r^2} dr d\theta = 2\pi \int_0^{\sinh^{-1}(1)} \cosh^2 \phi d\phi = \pi(\sqrt{2} + \log(1 + \sqrt{2}))$. Substitution $r = \sinh \phi$.
- 11.4 If $P(x, y) = (x, y, f(x, y))$ then $E = 1 + f_x^2$, $F = f_x f_y$ and $G = 1 + f_y^2$. Hence $EG - F^2 = (1 + f_x^2)(1 + f_y^2) - f_x^2 f_y^2 = 1 + f_x^2 + f_y^2 = 1 + \|\nabla f\|^2$.
- 11.5 Use parametrization $P(r, \theta) = (r \cos \theta, r \sin \theta, r^2/3)$, $0 < \theta < 2\pi$, $0 < r < \sqrt{3}$. Surface area $= \int_0^{2\pi} \int_0^{\sqrt{3}} r \sqrt{1 + (4r^2/9)} dr d\theta = (7\sqrt{21} - 9)\pi/6$.
- 11.6 Using $P(x, y) = (x, y, xy)$. By Exercise 11.4, surface Area $= \iint_{x^2 + y^2 \leq a^2} (1 + x^2 + y^2)^{1/2} dx dy$. Change to polar coordinates, $(x, y) \rightarrow (r \cos \theta, r \sin \theta)$. Alternatively, let $P_1(r, \theta) = (r \cos \theta, r \sin \theta, \frac{r^2}{2} \sin 2\theta)$, $0 < \theta < 2\pi$, $0 < r < a$. Then $E = 1 + r^2 \sin^2 2\theta$, $F = \frac{r^3}{2} \sin 4\theta$, $G = r^2(1 + r^2 \cos^2 2\theta)$, $EG - F^2 = r^2 + r^4$. Surface area $= \int_{\theta=0}^{2\pi} \int_{r=0}^a r \sqrt{1 + r^2} dr d\theta = 2\pi((1 + a^2)^{3/2} - 1)/3$ (substitution $u = 1 + r^2$).
- 11.7 First octant $\Rightarrow 0 < \theta < \pi/2$, between the planes $\Rightarrow 0 < r < 8\sqrt{3}$, inside cylinder $\Rightarrow r^2 \cos^2 \theta < r^2/4 \Rightarrow \cos \theta < \frac{1}{2} \Rightarrow \pi/3 < \theta < \pi/2$. $E = (r^2/64) + 1$, $F = 0$, $G = r^2$. Surface area $= \int_{\pi/3}^{\pi/2} \int_0^{8\sqrt{3}} r \sqrt{1 + (r^2/64)} dr d\theta = 224\pi/9$.

Chapter 12

- 12.1 $S \subset \{(x, y, z) : 6x + 3y + 2z = 6\}$, $\mathbf{n}(p) = (6, 3, 2)/7$, $\mathbf{F} \cdot \mathbf{n} = \frac{18}{7}$, $\iint_S \mathbf{F} = \frac{18}{7}$
 $\iint_S \sqrt{EG - F^2} dx dy = \frac{18}{7}(\text{Area } S) = 9$. Area of triangle in \mathbb{R}^3 with vertices
at \mathbf{a} , \mathbf{b} and \mathbf{c} is $\frac{1}{2}\|(\mathbf{a} - \mathbf{b}) \times (\mathbf{a} - \mathbf{c})\|$. Alternatively, using projections onto
coordinate planes, $\iint_S \mathbf{F} = \iint_S (f, g, h) = \iint_{P_{yz}} f + \iint_{P_{zx}} g + \iint_{P_{xy}} h = 1 \cdot \frac{1}{2} \cdot 2 \cdot 3 +$
 $2 \cdot \frac{1}{2} \cdot 1 \cdot 3 + 3 \cdot \frac{1}{2} \cdot 1 \cdot 2 = 9$. Also, $P(x, y) = (x, y, (6 - 6x - 3y)/2)$, $(x, y) \in$
triangle Δ in xy -plane with vertices $(0, 0)$, $(1, 0)$ and $(0, 2)$ parameterizes S
and $P_x \times P_y = (3, 3/2, 1)$ has the correct orientation. $\iint_S \mathbf{F} = \iint_{\Delta} (\mathbf{F}, P_x \times$
 $P_y) dx dy = 9(\text{Area } \Delta) = 9$.
- 12.3 $\phi(u, v) = (u, v, uv^3)$, $\phi_u = (1, 0, v^3)$, $\phi_v = (1, 0, 3uv^2)$ and $\phi_u \times \phi_v =$
 $(-v^3, -3uv^2, 1)$. Hence $\mathbf{F}(\phi(u, v)) \cdot \phi_u(u, v) \times \phi_v(u, v) = 2 + 18u^2v^4 + 2v^9$.
Answer 467.
- 12.4 $P(\theta, \psi) = ((b + a \cos \theta) \cos \psi, (b + a \cos \theta) \sin \psi, a \sin \theta)$, $0 < \theta, \psi < 2\pi$.
This surface is a torus (see Table 11.1). $\mathbf{F}(P(\theta, \psi)) = (a \cos \theta \cos \psi, a \cos \theta$
 $\sin \psi, a \sin \theta)$. $\langle \mathbf{F}, P_\theta \times P_\psi \rangle = -a^2(b + a \cos \theta)$, $\iint_S \mathbf{F} = -4\pi^2 a^2 b$. Since
 $P_\theta \times P_\psi(0, 0) = -a(b + a)(1, 0, 0)$ points inwards, P is not consistent with
the orientation. Answer $= -(-4\pi^2 a^2 b)$.
- 12.5 Use spherical polar coordinates (see Table 11.1) with range $0 < \theta < \pi/2$,
 $0 < \psi < 2\pi$. $EG - F^2 = a^4 \sin^2 \theta$.
(a) $y^2 + z^2 = a^2 \sin^2 \theta \sin^2 \psi + a^2 \cos^2 \theta$, $4\pi a^4/3$
(b) $(EG - F^2)^{1/2} (x^2 + y^2 + (z + a)^2)^{-1/2} = a \sin \theta / (2 + 2 \cos \theta)^{1/2}$, $2\pi a(2 -$
 $\sqrt{2})$.
- 12.6 Truncated cone, $f(r, \theta) = (r \cos \theta, r \sin \theta, r)$, $0 < \theta < 2\pi$, $1 < r < 3$,
 $f_r \times f_\theta = (-r \cos \theta, -r \sin \theta, r)$, $\langle \mathbf{F}(f(r, \theta)), f_r \times f_\theta \rangle = 2r$, 16π .
- 12.7 $F_r \times F_\theta = (\sin \theta, -\cos \theta, r)$. $\langle \mathbf{G}(F(r, \theta)), F_r \times F_\theta \rangle = \dots = -r\theta / (r^2 +$
 $\theta^2)^{3/2}$, $\iint_S \mathbf{G} = \int_\pi^{2\pi} \{ \int_\pi^\theta (-r\theta / (r^2 + \theta^2)^{3/2}) dr \} d\theta = \pi(3 - \sqrt{10})/\sqrt{2}$.
- 12.8 Method (b). Use polar coordinates on each coordinate plane and $z = 0$ on the
 xy -plane. $\iint_{P_{yz}(S)} (y^2 + z^2) dy dz + \iint_{P_{zx}(S)} \tan^{-1}(x/z) dz dx +$
 $\iint_{P_{xy}(S)} 0e^{x^2+y^2} dx dy = \iint_{0 < r < a, 0 < \theta < \pi/2} r^2 r dr d\theta + \iint_{0 < r < a, 0 < \theta < \pi/2} \theta r dr dt =$
 $a^2 \pi(2a^2 + \pi)/16$.

Chapter 13

- 13.1 (a) $\text{curl}(z\mathbf{i} - x\mathbf{k}) = (0, 2, 0)$, $(3\pi - 8)2a^2/3$.
 (b) $\text{curl}(-y\mathbf{i} + x\mathbf{j}) = (0, 0, 2)$, πa^2 .
 (c) $\text{curl}(-z\mathbf{j} + y\mathbf{k}) = (2, 0, 0)$, $8a^2/3$.
- 13.2 $P_r \times P_\theta = \left(\frac{-r^2}{a} \sin \theta, \frac{-r^2}{a} \cos \theta, r\right)$, $\text{curl}(y, z, x) = (-1, -1, -1)$, $P_r \times P_\theta \cdot (-1, -1, -1) = \frac{r^2}{a}(\sin \theta + \cos \theta) - r$. $\int_\Gamma y dx + z dy + x dz = \int_0^b \left\{ \int_0^{2\pi} \left(\frac{r^2}{a}(\sin \theta + \cos \theta) - r\right) d\theta \right\} dr = -\pi b^2$.
- 13.3 The mapping $\theta \rightarrow (b + b \cos \theta, b - b \cos \theta, \sqrt{2b} \sin \theta)$, $0 \leq \theta \leq 2\pi$, parametrizes Γ a circle of radius $\sqrt{2b}$ containing the points $(2b, 0, 0)$, $(b, b, \sqrt{2b})$, $(0, 2b, 0)$ and $(b, b, -\sqrt{2b})$. Since $\text{curl}(y, z, x) = (-1, -1, -1)$ and $\mathbf{n} \cdot (-1, -1, -1) = -\sqrt{2}$, $\int_\Gamma y dx + z dy + x dz = -\sqrt{2}(\text{Area circle}) = -\sqrt{2} \pi (\sqrt{2b})^2 = -2\sqrt{2}\pi b^2$.
- 13.4 Γ coils around S n times. $\int_\Gamma -y dx + x dy = \int_0^{2\pi} (b + a \cos nt)^2 dt = \pi(2b^2 + a^2)$. $\text{Area} = \int_0^1 \int_0^{2\pi} r(b + a \cos nt)^2 dr dt = \pi(b^2 + \frac{a^2}{2})$.
- 13.5 If \mathbf{G} is the vector field in Example 6.6 then $\text{curl}(\mathbf{G}) = F$. For $d > 0$ let $\Gamma_d = \{(x, y, z) : x^2 + y^2 = 1 - d^2\}$. Then $\Gamma_b \cup \Gamma_c$ suitably oriented is the boundary of $S_{b,c}$ and $P(\theta) = (\sqrt{1-d^2} \cos \theta, \sqrt{1-d^2} \sin \theta, \sqrt{1-d^2})$, $0 \leq \theta \leq 2\pi$, parametrizes Γ_d so that the sphere above the plane $z = d$ is on the left as Γ_d is rotated. Hence $\mathbf{G}(P(\theta)) \cdot P'(\theta) = (\sin \theta, -\cos \theta, 0) \cdot \sqrt{1-d^2}(-\sin \theta, \cos \theta, 0) = -\sqrt{1-d^2}$ and $\int_{\Gamma_d} \mathbf{G} = (-\sqrt{1-d^2}) \cdot l(\Gamma_d) = -\sqrt{1-d^2} 2\pi \sqrt{1-d^2} = -2\pi(1-d^2)$. By Stokes' Theorem

$$\iint_{S_{b,c}} \mathbf{F} = \int_{\Gamma_c} \mathbf{G} - \int_{\Gamma_b} \mathbf{G} = 2\pi(c^2 - b^2).$$

- 13.6 $P(r, \theta) = (r \cos \theta, r \sin \theta, b(1 - \frac{r \cos \theta}{a}))$, $0 < r < a$, $0 < \theta < 2\pi$ parametrises the portion S of the plane inside the cylinder. $P_r \times P_\theta = (br/a, 0, r)$, $\text{curl}(y - z, z - x, x - y) = (-2, -2, -2)$, $\iint_S (\text{curl}(y - z, z - x, x - y), P_r \times P_\theta) dr d\theta = -2\pi a(a + b)$. Orientation inconsistent with positive answer. Choose opposite orientation. Answer $2\pi a(a + b)$.

Chapter 14

- 14.1 $P(r, \theta, z) = (r \cos \theta, a + r \sin \theta, z)$, $0 < r < a$, $0 < \theta < 2\pi$, $0 < z < (a^2 + r^2 + 2ar \sin \theta)/4a$, $|\det(P')| = r$.
- 14.2 $f(r, \theta, z) = (r \cos \theta, r \sin \theta, z)$, $0 < \theta < \pi$, $0 < z < r$, $0 < r < 2a \sin \theta$,
 volume = $\int_0^\pi \left\{ \int_0^{2a \sin \theta} \left\{ \int_0^r r dz \right\} dr \right\} d\theta = 8a^3/3 \int_0^\pi \sin \theta (1 - \cos^2 \theta) d\theta = 32a^3/9$. Note that the parametrization in Exercise 14.1 leads to an unsuitable integral. Use Fig. 14.13.
- 14.3 Use $(r, \theta, z) \rightarrow (r \cos \theta, r \sin \theta, z)$, $0 < r < 1$, $0 < \theta < 2\pi$, $\theta < z < 2\pi$.
- 14.4 Paraboloid and cone intersect when $x^2 + y^2 = z^2 = 2 - z$, $z^2 + z - 2 = 0$, $z = 1$, $z = -2$. Above plane $z = 0$, hence $z = 1$. Region projects onto the disc $\{(x, y, 1) : x^2 + y^2 \leq 1\}$ with paraboloid above and cone below. Parametrization $(r, \theta, z) \rightarrow (r \cos \theta, r \sin \theta, z)$, $0 < r < 1$, $0 < \theta < 2\pi$, $r < z < 2 - r^2$. Volume = $\int_0^{2\pi} \left\{ \int_0^1 \left\{ \int_r^{2-r^2} r dz \right\} dr \right\} d\theta = \frac{5\pi}{6}$.
- 14.5 (a) 8π ;
 (b) 8π ;
 (c) $\frac{81\pi}{4} - \frac{216}{5}$.
- 14.7 $2/3$.
- 14.8 $V = \int_0^1 \left\{ \int_0^1 \left\{ \int_0^{(1+x+y)^{1/2}} dx \right\} dy \right\} dx = 4(9\sqrt{3} - 8\sqrt{2} + 1)/15$. By symmetry the second volume is $8V$.
- 14.9 (a) Volume = $\int_0^{2a} A(z) dz = \int_0^{\sqrt{3}a} \pi(z^2/3) dz + \int_{\sqrt{3}a}^{2a} \pi(4a^2 - z^2) dz = (\pi a^3)/\sqrt{3} + \pi a^3(\frac{16}{3} - 3\sqrt{3}) = \frac{8\pi a^3}{3}(2 - \sqrt{3})$. (b) Use spherical polar coordinates $w : (r, \theta, \psi) \rightarrow (r \sin \theta \cos \psi, r \sin \theta \sin \psi, r \cos \theta)$, $0 < \theta < \pi/6$, $0 < r < 2a$, and $0 < \psi < 2\pi$. Since $\det(w') = r^2 \sin \theta$, Volume = $\int_0^{\pi/6} \sin \theta d\theta \cdot \int_0^{2a} r^2 dr \cdot \int_0^{2\pi} d\psi = \frac{8\pi a^3}{3}(2 - \sqrt{3})$. See also Example 15.3.
- 14.10 $\frac{7}{12}$.
- 14.11 $\det(F') = u^2 v$, $\iiint_V x dx dy dz = \int_0^1 u^3 du \int_0^1 v(1-v) dv \int_0^1 dw = 1/24$,
 $\iiint_V \frac{dx dy dz}{y+z} = \int_0^1 \int_0^1 \int_0^1 \frac{u^2 v}{uv} du dv dw = 1/2$.

Chapter 15

15.2 $\text{div}(\mathbf{F}) = y^2 + x^2$. Interior of $S = \{(r \cos \theta, r \sin \theta, z) : 0 < r < \sqrt{3}, \pi/2 < \theta < 3\pi/2, 0 < z < -r \cos \theta\}$, $\text{div}(\mathbf{F}) = r^2, 18\sqrt{3}/5$.

15.3 $\int_0^a (\int_0^{b(1-\frac{x}{a})} (\int_0^{c(1-\frac{x}{a}-\frac{y}{b})} x dz) dy) dx = a^2 cb/24$.

15.4 Order of integration is important, first x and then y to get $(e-2)/2e$.

15.5 $\text{div}(x^2, -y^2, 3xz) = 5x - 2y$,

$$\iiint_{\substack{x,y,x \geq 0 \\ x^2+y^2+x^2 \leq 4}} x dx dy dz = \iiint_{\substack{x,y,x \geq 0 \\ x^2+y^2+x^2 \leq 4}} y dx dy dz = \pi. \text{ Solution } 3\pi.$$

15.8 Volume = $\iint_D (ax^2 + by^2) dx dy$ where $D = \{(x, y) : (x/c)^2 + (y/d)^2 < 1\}$.

If $P(r, \theta) = (rc \cos \theta, rd \sin \theta)$, $0 < r < 1, 0 < \theta < 2\pi$ then $dx dy = rcd \cdot dr d\theta$. Volume = $\int_0^1 \int_0^{2\pi} (ac^2 r^2 \cos^2 \theta + bd^2 r^2 \sin^2 \theta) rcd \cdot dr d\theta = cd \int_0^1 r^3 dr \cdot \int_0^{2\pi} (ac^2 \cos^2 \theta + bd^2 \sin^2 \theta) d\theta = cd\pi(ac^2 + bd^2)/4$.

Chapter 16

- 16.1 $E = 1, F = 0, G = t^2 + b^2, l = 0, m = -b(t^2 + b^2)^{-1/2},$
 $K = -b^2 / (t^2 + b^2)^2.$
- 16.2 Since the surface of a cone always lies on one side of each tangent plane,
 $K \geq 0$, and since it contains a line, $K = 0$ and 0 is a principal curvature.
 $P(r, \theta) = (r \cos \theta, r \sin \theta, r), \mathbf{n} = (-\cos \theta, -\sin \theta, 1)/\sqrt{2}, E = 2, F =$
 $0, G = r^2, l = 0, m = 0, n = -r/\sqrt{2}.$ Principal curvatures $0, -1/r\sqrt{2}.$
- 16.3 Use (16.6), $E = 5, F = 4, G = 5, \mathbf{n} = (-2, -2, 1)/3, l = -2/3, m = 0,$
 $n = -2/3. \phi_u(1, 1) = (1, 0, 2), \phi_v(1, 1) = (0, 1, 2), v_1 = 2, v_2 = 1,$
 $k_p(\mathbf{v}) = 10/123.$
- 16.4 By (16.6), $\kappa_P(\mathbf{v}) = \frac{v_1^2 l + 2v_1 v_2 m + v_2^2 n}{v_1^2 E + 2v_1 v_2 F + v_2^2 G} = \alpha$ (constant) for all $(v_1, v_2) \Leftrightarrow v_1^2(l -$
 $\alpha E) + 2v_1 v_2(m - \alpha F) + v_2^2(n - \alpha G) = 0$ for all $(v_1, v_2) \Leftrightarrow l = \alpha E, m = \alpha F$
and $n = \alpha G.$
 $E = 1 + y^2, F = xy, G = 1 + x^2$ and $l = 0, m = (1 + x^2 + y^2)^{-1/2}, n = 0.$
At an umbilic point $l = \alpha E$ implies $\alpha = 0$ and $m = \alpha F$ implies $\alpha \neq 0.$
Hence there are no umbilics.
- 16.5 $K = 36uv \cdot (1 + 9u^4 + 9v^4)^{-2},$ elliptic points $uv > 0,$ hyperbolic points when
 $uv < 0.$
- 16.6 $m = n = 0 \Rightarrow K \equiv 0.$ This is the cylinder over the ellipse $x^2 + (y/2)^2 = 1.$
- 16.7 $u \rightarrow (av, -bv, 0) + u(a, b, v), v \rightarrow (au, bu, 0) + v(a, -b, u).$
- 16.8 $K = \frac{1}{a^2 b^2 c^2} \left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4} \right)^{-2}.$
- 16.9 It suffices to note that the coefficient of λ in (16.9) is $En + Gl - 2Fm.$
 $P(x, y) = (x, y, \log \cos y - \log \cos x), E = \sec^2 x, F = -\tan x \tan y, G =$
 $\sec^2 y, l = \sec^2 x / (1 + \tan^2 x + \tan^2 y)^{1/2}, m = 0, n = -\sec^2 y / (1 + \tan^2 x +$
 $\tan^2 y)^{1/2}.$

Chapter 17

- 17.1 Use toroidal polar coordinates, $L(xP_\theta + yP_\psi) = \frac{x}{a}P_\theta + \frac{y \cos \theta}{b+a \cos \theta}P_\psi$.
- 17.2 $\phi(u, v) = (u, v, u^2 - 2v^2)$, $\phi_u = (1, 0, 2u)$, $\phi_v = (0, 1, -4v)$, $\phi_{uu} = (0, 0, 2)$, $\phi_{uv} = (0, 0, 0)$, $\phi_{vv} = (0, 0, -4)$, $E = 1 + 4u^2$, $F = -8uv$, $G = 1 + 16v^2$, $\mathbf{n} = (-2u, 4v, 1)/(1 + 4u^2 + 16v^2)^{1/2}$, $l = 2/(1 + 4u^2 + 16v^2)^{1/2}$, $m = 0$, $n = -4/(1 + 4u^2 + 16v^2)^{1/2}$. $\kappa_P \left(\frac{v_1 \phi_u + v_2 \phi_v}{\|v_1 \phi_u + v_2 \phi_v\|} \right) = \frac{2v_1^2 - 4v_2^2}{(4u^2 + 16v^2 + 1)\|v_1 \phi_u + v_2 \phi_v\|^2}$. Let $P(t) = \phi(\sqrt{2}t, t) = (\sqrt{2}t, t, 0)$. By the chain rule $P'(t) = \sqrt{2}\phi_u + \phi_v$, and $k_{P(t)}(P'(t)) = C(2(\sqrt{2})^2 - 4) = 0$. The straight line $t \rightarrow P(t)$ lies in \mathbf{S} .
- 17.3 Suppose $\kappa \neq 0$ and \mathbf{n}_1 and \mathbf{n}_2 , the normals to \mathbf{S}_1 and \mathbf{S}_2 , are linearly independent. If $\mathbf{n}_1 \cdot \mathbf{n}_2 = \cos \theta$, then $\|\mathbf{n}_2 - \langle \mathbf{n}_1, \mathbf{n}_2 \rangle \mathbf{n}_1\|^2 = \sin^2 \theta$ and $\sin \theta \neq 0$. Let P denote a unit speed parametrization of Γ with $P(0) = p$ and $P'(t) = T(t)$. If N is the normal to Γ in \mathbb{R}^3 then $T \perp \mathbf{n}_1$, \mathbf{n}_2 and $N \in \text{span}(\mathbf{n}_1, \mathbf{n}_2)$. Then $D_{\mathbf{T}}\mathbf{T} = P''(t) = \kappa\mathbf{N}$ and $\lambda_i = -\langle D_{\mathbf{T}}\mathbf{n}_i, \mathbf{T} \rangle = \langle \mathbf{n}_i, D_{\mathbf{T}}\mathbf{T} \rangle = \langle \mathbf{n}_i, \kappa\mathbf{N} \rangle$. Hence

$$\kappa N = \langle \mathbf{n}_i, \kappa\mathbf{N} \rangle \mathbf{n}_i + \frac{\langle \mathbf{n}_2 - \mathbf{n}_1 \cos \theta, \kappa\mathbf{N} \rangle}{\sin \theta} \frac{(\mathbf{n}_2 - \mathbf{n}_1 \cos \theta)}{\sin \theta}$$

and $\kappa^2 = \lambda_1^2 + \frac{\lambda_2^2 - 2\lambda_1\lambda_2 \cos \theta + \lambda_1^2 \cos^2 \theta}{\sin^2 \theta}$, $\kappa^2 \sin^2 \theta = \lambda_1^2 \sin^2 \theta + \lambda_2^2 - 2\lambda_1\lambda_2 \cos \theta + \lambda_1^2 \cos^2 \theta = \lambda_1^2 + \lambda_2^2 - 2\lambda_1\lambda_2 \cos \theta$, $\kappa = 0 \Rightarrow \lambda_1 = \lambda_2 = 0$. \mathbf{n}_1 and \mathbf{n}_2 are not linearly independent $\Rightarrow \mathbf{n}_1 = \pm \mathbf{n}_2$ and $\lambda_1 = \pm \lambda_2$.

- 17.4 $P(1) = (1, 1, 2)$ and $P'(1)/\|P'(1)\| = (2, 1, 6)/\sqrt{41} = \mathbf{v}$, $S = g^{-1}(0)$ where $g(u, v, w) = u^2 + v^2 - w$, $\nabla g / \|\nabla g\| = \frac{(2u, 2v, -1)}{(4u^2 + 4v^2 + 1)^{1/2}}$, $\frac{\nabla g}{\|\nabla g\|}$ coincides with \mathbf{n} on S and is defined on an open set containing S . Calculate $D(\nabla g / \|\nabla g\|)$ and $-D_{\mathbf{v}}(\nabla g / \|\nabla g\|) = -D(\nabla g / \|\nabla g\|) \cdot \mathbf{v}$.

$$D\left(\frac{\nabla g}{\|\nabla g\|}\right)(1, 1, 2) = \frac{1}{27} \begin{pmatrix} 10 & -8 & 0 \\ -8 & 10 & 0 \\ 4 & 4 & 0 \end{pmatrix}, D_{\mathbf{v}}\left(\frac{\nabla g}{\|\nabla g\|}\right)(1, 1, 2) = \frac{1}{9\sqrt{41}}(4, -2, 4),$$

$$\kappa_{\mathbf{v}}(p) = \frac{1}{9\sqrt{41}}(4, -2, 4) \cdot (2, 1, 6)/\sqrt{41} = 10/123.$$

- 17.5 Use Euler's formula.

Chapter 18

- 18.1 Cylinder = $f^{-1}(0)$, $f(x, y, z) = x^2 + y^2 - 1$, $\nabla f = (2x, 2y, 0)$. Hence $\mathbf{n}(\phi(t)) \parallel (2 \cos(at + b), 2 \sin(at + b), 0)$, $\phi''(t) = (a^2 \cos(at + b), a^2 \sin(at + b), 0)$.
- 18.2 Unit speed parametrization $\phi(t) = (x(t), y(t)) \Rightarrow x''x' + y''y' = 0$. Note that φ is also unit speed. $\mathbf{n}(\varphi(t)) \parallel (y'(t), -x'(t) \cos \theta, -x'(t) \sin \theta)$, while $\varphi''(t) = (x''(t), y''(t) \cos \theta, y''(t) \sin \theta) \Rightarrow \varphi''(t) = \alpha(t)\mathbf{n}(\varphi(t))$ where $\alpha(t) = x''(t)/y'(t)$ if $y'(t) \neq 0$ and $\alpha(t) = y''(t)/x'(t)$ if $x'(t) \neq 0$. Since φ is a unit speed geodesic (18.5) implies $\varphi''(t) = T'(t) = \kappa_n(t)\mathbf{n}(\varphi(t))$, and the normal curvature at $P(t, \theta_0)$ is $x''(t)/y'(t)$, if $y'(t) \neq 0$, and $y''(t)/x'(t)$, if $x'(t) \neq 0$.
- 18.3 If $P(t) = \mathbf{a} + t\mathbf{b}$ parametrizes the straight line then P has constant speed and, since $P''(t) = T'(t) = 0$, (18.4) implies the geodesic curvature is zero and, by Proposition 18.2, straight lines are geodesics. $(x_0, y_0, z_0) + t(a, b, c)$ lies in the surface $z = x^2 - y^2 \Leftrightarrow (x_0 + ta)^2 - (y_0 + tb)^2 = z_0 + tc$ for all $t \Leftrightarrow (2x_0a - 2y_0b - c)t + (a^2 - b^2)t^2 = 0$ for all t . Letting $a = 1$, $b = 1$, $2x_0 - 2y_0 = c$ and $a = 1$, $b = -1$, $2x_0 + 2y_0 = c$ gives two lines on the surface.
- 18.4 Since P is unit speed $T(t) = P'(t)$ and $\kappa_n(t) = k_{P(t)}(T(t))$. By (18.3), $\tau_g = 0 \Leftrightarrow \mathbf{n}' = -\kappa_n T \Leftrightarrow L_{P(t)}(T(t)) = \kappa_n(t)T(t) \Leftrightarrow T(t)$ is a principal curvature for all $t \Leftrightarrow \Gamma$ is a line of curvature. Γ a geodesic $\implies \kappa_g = 0 \implies T' = \kappa N = \kappa_n \mathbf{n}$. Hence $N = \pm \mathbf{n}$, $B = T \times N = T \times (\pm \mathbf{n}) = \mp \mathbf{n}_s$ and $-\tau N = B' = \mp (\mathbf{n}_s)' = \mp (\tau_g \mathbf{n}) = \mp \tau_g (\pm N) = -\tau_g N$ and $\tau = \tau_g$.
- 18.5 By Proposition 8.1, Γ lies in a plane. Using the proof in Example 8.2, we see that the non-zero curvature is constant. An application of the result in Example 8.4 completes the proof.
- 18.6 By (18.5), since the tangent space to \mathbf{S} at $P(t)$ is 2 dimensional and $T(t) \perp N(t)$, we have $k_{P(t)}(T(t)) = 0 \iff N(t) \parallel \mathbf{n}_S(t) \iff N(P(t)) \in T_S(P(t)) \iff T_S(P(t)) = \text{span} \{T(t), N(t)\} \iff \text{tangent plane} = \text{osculating plane}$.
- 18.7 Assume sphere has centre at the origin and radius r . Let P be a unit speed parametrization of the curve. By (16.5) the normal curvature is $\pm 1/r$ and, by (18.5), $\kappa_g(t)$ constant $\iff \kappa(t)$ constant. By Example 8.2, $\kappa(t) \neq 0$ and $\tau(t)\langle P(t) - c, B(t) \rangle = -(\frac{1}{\kappa(t)})' = 0$. This implies $\tau(t) = 0$ and Exercise 18.5 implies the required result.

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