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Multivariate Calculus and Geometry

Third Edition

Springer

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To my four godchildren
Anne-Marie Dineen, Donal Coffey,
Kevin Timoney, Eoghan Wallace

Preface to First Edition

The importance assigned to accuracy in basic mathematics courses has, initially, a useful disciplinary purpose but can, unintentionally, hinder progress if it fosters the belief that exactness is all that makes mathematics what it is. *Multivariate calculus* occupies a pivotal position in undergraduate mathematics programmes in providing students with the opportunity to outgrow this narrow viewpoint and to develop a flexible, intuitive and independent vision of mathematics. This possibility arises from the extensive nature of the subject.

Multivariate calculus links together in a non-trivial way, perhaps for the first time in a student's experience, four important subject areas: *analysis*, *linear algebra*, *geometry* and *differential calculus*. Important features of the subject are reflected in the variety of alternative titles we could have chosen, e.g. "Advanced Calculus", "Vector Calculus", "Multivariate Calculus", "Vector Geometry", "Curves and Surfaces" and "Introduction to Differential Geometry". Each of these titles partially reflects our interest but it is more illuminating to say that here we study differentiable functions, i.e.

functions which enjoy a good local approximation by linear functions.

The main emphasis of our presentation is on understanding the underlying fundamental principles. These are discussed at length, carefully examined in simple familiar situations and tested in technically demanding examples. This leads to a structured and systematic approach of manageable proportions which gives shape and coherence to the subject and results in a comprehensive and unified exposition.

We now discuss the four underlying topics and the background we expect—bearing in mind that the subject can be approached with different levels of mathematical maturity. Results from *analysis* are required to justify much of this book, yet many students have little or no background in analysis when they approach multivariate calculus. This is not surprising as differential calculus preceded and indeed motivated the development of analysis. We do not list analysis as a prerequisite, but hope that our presentation shows its importance and motivates the reader to study it further.

Since linear approximations appear in the definition of differentiable functions, it is not surprising that *linear algebra* plays a part in this book. Several-variable

calculus and linear algebra developed, to a certain extent, side by side to their mutual benefit. The primary role of linear algebra, in our study, is to provide a suitable notation and framework in which we can clearly and compactly introduce concepts and present and prove results. This is more important than it appears since to quote T. C. Chaundy, *"notation biases analysis as language biases thought"*. An elementary knowledge of matrices and determinants is assumed and particular results from linear algebra are introduced as required.

We discuss the role of *geometry* in multivariate calculus throughout the text and confine ourselves here to a brief comment. The natural setting for *functions* which enjoy a good local approximation by *linear functions* are *sets* which enjoy a good local approximation to *linear spaces*. In one and two dimensions this leads to *curves* and *surfaces*, respectively, and in higher dimensions to *differentiable manifolds*.

We assume the reader has acquired a reasonable knowledge of *one-variable differential and integral calculus* before approaching this book. Although not assumed, some experience with partial derivatives allows the reader to proceed rapidly through routine calculations and to concentrate on important concepts. A reader with no such experience should definitely read Chapter 1 a few times before proceeding and may even wish to consult the author's *Functions of Two Variables* (Chapman and Hall 1995).

We now turn to the contents of this book. Our general approach is holistic and we hope that the reader will be equally interested in all parts of this book. Nevertheless, it is possible to group certain chapters thematically.

Differential Calculus on Open Sets and Surfaces (Chapters 1–4).

We discuss extremal values of real-valued functions on surfaces and open sets. The important principle here is the *Implicit Function Theorem*, which links linear approximations with systems of linear equations and sets up a relationship between graphs and surfaces.

Integration Theory (Chapters 6, 9, 11–15).

The key concepts are parameterizations (Chapters 5, 10 and 14) and oriented surfaces (Chapter 12). We build up our understanding and technical skill step by step, by discussing in turn line integrals (Chapter 6), integration over open subsets of $\mathbb{R}^2$ (Chapter 9), integration over simple surfaces without orientation (Chapter 11), integration over simple oriented surfaces (Chapter 12) and triple integrals over open subsets of $\mathbb{R}^3$(Chapter 14). At appropriate times we discuss generalizations of the fundamental theorem of calculus, i.e. *Green's Theorem* (Chapter 9), *Stokes' Theorem* (Chapter 13) and *the Divergence Theorem* (Chapter 15). Special attention is given to the parameterization of classical surfaces, the evaluation of surface integrals using projections, the change of variables formula and to the detailed examination of involved geometric examples.

Geometry of Curves and Surfaces (Chapters 5, 7–8, 10, 16–18).

We discuss signed curvature in $\mathbb{R}^2$ and use vector-valued differentiation to obtain the Frenet–Serret equations for curves in $\mathbb{R}^3$. The abstract geometric study of surfaces using Gaussian curvature is, regrettably, usually not covered in multivariate calculus courses. The fundamental concepts, parameterizations and plane curvature, are already in place (Chapters 5, 7 and 10) and examples from integration theory (Chapters 11–15) provide a concrete background and the required geometric insight. Using only curves in $\mathbb{R}^2$ and critical points of functions of two variables we develop the concept of Gaussian curvature. In addition, we discuss normal, geodesic and intrinsic curvature and establish a relationship between all three. In the final chapter we survey informally a number of interesting results from differential geometry.

This text is based on a course given by the author at University College, Dublin. The additions that emerged in providing details and arranging self-sufficiency suggest that it is suitable for a course of 30 lectures. Although the different topics come together to form a unified subject, with different chapters mutually supporting one another, we have structured this book so that each chapter is self-contained and devoted to a single theme.

This book can be used as a main text, as a supplementary text or for self-study. The groupings summarised above allow a selection of short courses at a slower pace. The exercises are extremely important as it is through them that a student can assess progress and understanding.

Our aim was to write a short book focusing on basic principles while acquiring technical skills. This precluded comments on the important applications of multivariate calculus which arise in physics, statistics, engineering, economics and, indeed, in most subjects with scientific aspirations.

It is a pleasure to acknowledge the help I received in bringing this project to fruition. Dana Nicolau displayed skill in preparing the text and great patience in accepting with a cheerful "OK, OK," the continuous stream of revisions, corrections and changes that flowed her way. Michael Mackey's diagrams speak for themselves. Brendan Quigley's geometric insight and Pauline Mellon's suggestions helped shape our outlook and the text. I would like to thank the Third Arts students at University College, Dublin, and especially Tim Cronin and Martin Brundin for their comments, reactions and corrections. Susan Hezlet of Springer provided instantaneous support, ample encouragement and helpful suggestions at all times. To all these and the community of mathematicians whose results and writings have influenced me, I say—thank you!

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Preface to Third Edition

Fifteen years have elapsed since the first edition was published and 5 years have gone by since I last taught a course on the topics in this book. It is nice to know that Springer still believes that new generations of teachers and students may still be interested in my approach and I am grateful to them for allowing me the opportunity to correct some errors, to revise some material, and to pass on to new readers comments of previous readers. I have, I hope, maintained the style, format, general approach and the results of previous editions. I have made changes in practically all chapters but the main changes occur in the final three chapters, which is an introduction to the differential geometry of surfaces in three-dimensional space. And now some important information which was not sufficiently stressed in earlier prefaces: as preparation to using this book readers should have completed a course in linear algebra and a first course on partial differentiation. Chapter 1 in this book is a summary of material that is presumed known and an introduction to notation that we use throughout the book.

It is a pleasure to thank Michael Mackey for his continued support and practical and mathematical help in preparing this edition. Joerg Sixt and Catherine Waite from Springer have been supportive and efficient throughout the period of preparation of this edition.

Dublin, Ireland

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