A
Differential Equations

PDEs are frequently solved by reducing them to one or more ODEs. This appendix contains a brief review of how to solve some of the basic ODEs encountered in this book.

For notation, we let $y = y(x)$ be the unknown function. Derivatives will be denoted by primes, i.e., $y' = y'(x)$, $y'' = y''(x)$. Sometimes we use the differential notation $y' = \frac{dy}{dx}$. If f is a function, an **antiderivative** is defined as a function F whose derivative is f, i.e., $F'(x) = f(x)$. Antiderivatives are unique only up to an additive constant, and they are often denoted by the usual indefinite integral sign:

$$F(x) = \int f(x)dx + C.$$

An arbitrary constant of integration C is added to the right side. However, in this last expression, it is often impossible to evaluate the antiderivative F at a particular value of x because the indefinite integral cannot be found in terms of familiar functions. $\Big($Take, for example, $f(x) = e^{-x^2}$.$\Big)$ In this case we must use the *general form of the antiderivative* as a definite integral with a variable upper limit,

$$F(x) = \int_a^x f(s)ds + C,$$

where a is any constant (observe that a and C are not independent, since changing one changes the other). If f is continuous at x, by the fundamental theorem of calculus, $F'(x) = f(x)$.

J. D. Logan, *Applied Partial Differential Equations,* 3rd edition.
Undergraduate Texts in Mathematics, DOI 10.1007/978-3-319-12493-3

First-Order Equations

An ODE of the first order is an equation of the form

$$G\Big(x, y, y'\Big) = 0 \quad \text{or} \quad y' = F(x, y).$$

There are three types of equations that occur regularly in PDEs: separable, linear, and Bernoulli. The general solution involves an arbitrary constant C that can be determined by an initial condition of the form $y(x_0) = y_0$.

Separable Equations. A first-order equation is separable if it can be written in the form

$$\frac{dy}{dx} = f(x)g(y).$$

In this case we separate variables and write

$$\frac{dy}{g(y)} = f(x)dx.$$

Integrating both sides gives

$$\int \frac{dy}{g(y)} = \int f(x)dx + C,$$

which defines the solution implicitly. As noted above, sometimes the antiderivatives must be written as definite integrals with a variable upper limit of integration.

Linear Equations. A first-order linear equation is one of the form

$$y' + p(x)y = q(x).$$

This can be solved by multiplying both sides by an *integrating factor* of the form

$$P(x) = e^{\int_a^x p(s)ds}.$$

This transforms the left side of the equation into a total derivative, and it becomes

$$\frac{d}{dx}\left(P(x)y\right) = P(x)q(x).$$

Now, both sides can be integrated from a to x to find y.

For example, to find an expression for the solution to the initial value problem

$$y' + 2xy = \sqrt{x}, \quad y(0) = 3,$$

We note that the integrating factor is

$$P(x) = \exp\left(\int_0^x 2sds\right) = e^{x^2}.$$

Multiplying both sides of the equation by the integrating factor gives

$$\left(ye^{x^2}\right)' = \sqrt{x}e^{x^2}.$$

Now, integrating from 0 to x (while changing the dummy variable of integration to s) gives

$$y(x)e^{x^2} - y(0) = \int_0^x \sqrt{s}e^{s^2}\, ds.$$

Solving for y gives

$$y(x) = e^{-x^2}\left(3 + \int_0^x \sqrt{s}e^{s^2}\, ds\right) = 3e^{-x^2} + \int_0^x \sqrt{s}e^{s^2-x^2}\, ds.$$

Bernoulli Equations. Bernoulli equations are nonlinear equations having the form

$$y' + p(x)y = q(x)y^n.$$

The transformation of dependent variables $w = y^{1-n}$ transforms a Bernoulli equation into a first-order linear equation for w.

Second-Order Equations

Constant-Coefficient Equations. The linear equation

$$ay'' + by' + cy = 0,$$

where a, b, and c are constants, occurs frequently in applications. We recall that the general solution of a linear, second-order, homogeneous equation is a linear combination of two independent solutions. That is, if $y_1(x)$ and $y_2(x)$ are independent solutions, then the general solution is

$$y = c_1y_1(x) + c_2y_2(x),$$

where c_1 and c_2 are arbitrary constants. If we try a solution of the form $y = e^{mx}$, where m is to be determined, then substitution into the equation gives the so-called *characteristic equation*

$$am^2 + bm + c = 0$$

for m. This is a quadratic polynomial that will have two roots, m_1 and m_2, called *eigenvalues*. Three possibilities can occur: unequal real roots, equal real roots, and complex roots (which must be complex conjugates).
Case (I). m_1, m_2 real and unequal. In this case two independent solutions are

$$y_1 = e^{m_1x} \quad y_2 = e^{m_2x}.$$

Case (II). m_1, m_2 real and equal, i.e., $m_1 = m_2 \equiv m$. In this case two independent solutions are

$$y_1 = e^{mx} \quad y_2 = xe^{mx}.$$

Case (III). $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$ are complex conjugate roots. In this case two real, independent solutions are

$$y_1 = e^{\alpha x} \sin \beta x \quad y_2 = e^{\alpha x} \cos \beta x.$$

Of particular importance are the two equations

$$y'' + a^2 y = 0, \quad y'' - a^2 y = 0$$

which have general solutions

$$y(x) = c_1 \cos ax + c_2 \sin ax, \quad y = c_1 e^{-ax} + c_2 e^{ax},$$

respectively. Equivalently, this second solution can be written

$$y = c_1 \cosh ax + c_2 \sinh ax.$$

These equations occur so frequently that it is best to memorize their solutions.

Cauchy–Euler Equations. It is difficult to solve second-order linear equations with variable coefficients. The reader may recall power series methods are generally applied. However, there is a special equation that can be solved with a simple formula, namely, a Cauchy–Euler equation of the form

$$ax^2 y'' + bxy' + cy = 0.$$

This equation admits power functions as solutions. Hence, if we try a solution of the form $y = x^m$, where m is to be determined, then we obtain upon substitution the characteristic equation

$$am(m-1) + bm + c = 0.$$

This quadratic equation has two roots, m_1 and m_2. Thus, there are three cases: Case (I). m_1, m_2 real and unequal. In this case two independent solutions are

$$y_1 = x^{m_1} \text{ and } y_2 = x^{m_2}.$$

Case (II). m_1, m_2 real and equal, i.e., $m_1 = m_2 \equiv m$. In this case two independent solutions are

$$y_1 = x^m \text{ and } y_2 = x^m \ln x.$$

Case (III). $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$ are complex conjugate roots. In this case two real, independent solutions are

$$y_1 = x^\alpha \sin(\beta \ln x) \text{ and } y_2 = x^\alpha \cos(\beta \ln x).$$

Particular Solutions

The general solution of the nonhomogeneous equation

$$y'' + p(x)y' + q(x)y = f(x)$$

is

$$y = c_1 y_1(x) + c_2 y_2(x) + y_{\mathrm{P}}(x),$$

where y_1 and y_2 are independent solutions of the homogeneous equation (when $f(x) \equiv 0$), and $y_{\mathrm{P}}(x)$ is any particular solution to the inhomogeneous equation. For constant-coefficient equations a particular solution can sometimes be found by judiciously 'guessing' the form from the form of $f(x)$; this method is called the method of *undetermined coefficients*. In all cases, however, there is a general formula, called the *variation of parameters* formula, which gives the particular solution in terms of the two linearly independent solutions y_1 and y_2 of the homogeneous equation. The formula, which is derived in elementary texts, is given by

$$y_P(x) = y_2(x)\int_a^x \frac{y_1(s)f(s)}{W(s)}ds - y_1(x)\int_a^x \frac{y_2(s)f(s)}{W(s)}ds,$$

where

$$W(x) = y_1(x)y_2'(x) - y_2(x)y_1'(x)$$

is the *Wronskian*.

There are several introductory texts on differential equations. A classic text is Boyce and DiPrima (1995), as well as later editions. Birkhoff and Rota (1978) and Kelley and Peterson (2004) are two more advanced texts.

References

Birkhoff, G., & Rota, J.-C. (1989). *Ordinary differential equations* (4th ed.). NY: Wiley.

Boyce, W. E., & DiPrima, R. C. (1995). *Elementary differential equations* (5th ed.). NY: Wiley.

Britton, N. F. (2003). *Essential mathematical biology*. NY: Springer.

Chorin, A. J., & Marsden, J. E. (1993). *A mathematical introduction to fluid mechanics* (3rd ed.). NY: Springer.

Churchill, R. V. (1960). *Complex variables and applications* (2nd ed.). NY: McGraw-Hill.

Churchill, R. V. (1969). *Fourier series and boundary value problems* (2nd ed.). NY: McGraw-Hill.

Churchill, R. V. (1972). *Operational mathematics* (3rd ed.). NY: McGraw-Hill.

Debnath, L. (1997). *Nonlinear partial differential equations*. Boston: Birkhauser.

Edelstein-Keshet, L. (1989). *Mathematical models in biology*. NY: McGraw-Hill.

Farlow, S. J. (1993). *Partial differential equations for scientists and engineers*. NY: Dover Publications.

Fetter, C. W. (1993). *Contaminant hydrogeology*. NY: Macmillan.

Griffiths, D. J. (2005). *Introduction to quantum mechanics*. Englewood Cliffs: Pearson Prentice-Hall.

Haberman, R. (2013). *Applied partial differential equations* (5th ed.). Upper Saddle River: Pearson.

Holmes, M. H. (2007). *Introduction to numerical methods in differential equations*. NY: Springer.

J. D. Logan, *Applied Partial Differential Equations,* 3rd edition.
Undergraduate Texts in Mathematics, DOI 10.1007/978-3-319-12493-3

Holmes, M. H. (2009). *Introduction to the foundations of applied mathematics.* NY: Springer.

Hughes-Hallet, D., et al. (2013). *Calculus* (6th ed.). Hoboken: Wiley.

Kelley, W., & Peterson, A. (2004). *The theory of differential equations.* Upper Saddle River: Pearson Education.

Kot, M. (2001). *Elements of mathematical ecology.* Cambridge: Cambridge University Press.

Lin, C. C., & Segel, L. A. (1989). *Mathematics applied to deterministic models in the natural sciences*, reprinted by Soc. Appl. and Industr. Math. (SIAM), Philadelphia.

Logan, J. D. (1987). *Applied mathematics: A contemporary approach.* NY: Wiley-Interscience. (See Chapter 8 on finite difference methods.).

Logan, J. D. (2001). *Transport modeling in hydrogeochemical systems.* NY: Springer.

Logan, J. D. (2009). *Introduction to nonlinear partial differential equations* (2nd ed.). NY: Wiley-Interscience.

Logan, J. D. (2015). *A first course in differential equations* (3rd ed.). NY: Springer.

Logan, J. D. (2013). *Applied mathematics: Fourth edition.* Hoboken: Wiley.

Logan, J. D., & Wolesensky, W. R. (2008). *Mathematical methods in biology.* Hoboken: Wiley.

McOwen, R. C. (2003). *Partial differential equations* (2nd ed.). Upper Saddle River: Prentice-Hall.

Murray, J. D. (2003). *Mathematical biology* (Vol. II). NY: Springer.

Parker, D. F. (2003). *Fields, flows, and waves.* NY: Springer.

Segel, L. A., & Handelman, G. H. (1977). *Mathematics applied to continuum mechanics.* NY: Macmillan. (Reprinted by Dover Publications, NY, 1987).

Smith, G. D. (1978). *Numerical solution of partial differential equations: Finite difference methods* (2nd ed.). Oxford: Oxford University Press.

Spivak, M. (2008). *Calculus* (4th ed.). Houston: Publish or Perish.

Strauss, W. (1992). *Introduction to partial differential equations.* NY: Wiley.

Vretblad, A. (2006). *Fourier analysis and its applications.* NY: Springer.

Index

J. D. Logan, *Applied Partial Differential Equations,* 3rd edition.
Undergraduate Texts in Mathematics, DOI 10.1007/978-3-319-12493-3