Appendix A
Proofs

This appendix contains proofs that were not included in the main exposition of the material in the book.

A.1 Derivation of Formula for Two-Particle, Linear Elastic Collision

Let us study a collision between two objects—drawn as carts in Fig. A.1—where cart A has mass m_A and initial velocity $v_{A,0}$, and cart be has mass m_B and initial velocity $v_{B,0}$. In this case, we assume that cart B starts at rest, $v_{B,0} = 0$. However, we can address any problem like this, since we can always place our coordinate system so that it follows the motion of cart B before the collision. We assume that there are no external forces acting on either cart during the collision—only internal forces acting between the cart.

Since there are no external forces acting, the total translational momentum, P_x, is conserved in the x-direction. The translational momentum is:

$$P = p_A + p_B = m_A v_A + m_B v_B \ . \tag{A.1}$$

The momentum is conserved throughout the collision, and it is therefore the same at the time t_0 before the collision and the time t_1 after the collision:

$$P_0 = m_A v_{A,0} + m_B v_{B,0} = m_A v_{A,1} + m_B v_{B,1} = P_1 \ , \tag{A.2}$$

where we simplify by introducing $v_{B,0} = 0\text{m/s}$, allowing us to rewrite (A.2) to:

$$m_A \left(v_{A,0} - v_{A,1} \right) = m_B v_{B,1} \ . \tag{A.3}$$

A. Malthe-Sørenssen, *Elementary Mechanics Using Python*,
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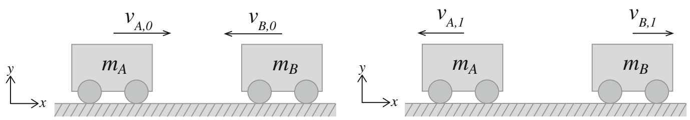

Fig. A.1 Illustration of a collision between two carts rolling along a friction-free, *horizontal* surface

This equation represents the conservation of momentum, and is true as long as the net external force is zero. We consider $v_{A,0}$ a know quantity. Then we are left with two unknowns $v_{A,1}$ and $v_{B,1}$. But we only have one equation. We do not know how the momentum is distributed between the two carts.

Now, if we assume that the collision is *elastic*, that is, that the mechanical energy is conserved throughout the collision, we can introduce an addition equation. This assumption means that we assume that the internal forces acting between the two carts are conservative, which is an additional assumption compared to what we did above. In this case, we can an additional equation for the conservation of mechanical energy:

$$\frac{1}{2}m_A v_{A,0}^2 + \frac{1}{2}m_B v_{B,0}^2 = \frac{1}{2}m_A v_{A,1}^2 + \frac{1}{2}m_B v_{B,1}^2 \ . \tag{A.4}$$

We now solve these two equations ((A.2) and (A.4)) to find $v_{A,1}$ and $v_{B,1}$. We start by rewriting (A.4) to:

$$m_A \left(v_{A,0}^2 - v_{A,1}^2\right) = m_B v_{B,1}^2 \ . \tag{A.5}$$

which can also be written as:

$$m_A \left(v_{A,0} - v_{A,1}\right)\left(v_{A,0} + v_{A,1}\right) = m_B v_{B,1}^2 \ . \tag{A.6}$$

We divide (A.6) by (A.3), getting:

$$v_{A,0} + v_{A,1} = v_{B,1} \ , \tag{A.7}$$

and we insert this for $v_{B,1}$ in (A.3):

$$m_A \left(v_{A,0} - v_{A,1}\right) = m_B \left(v_{A,0} + v_{A,1}\right) \ , \tag{A.8}$$

We solve for $v_{A,1}$:

$$m_A v_{A,0} - m_A v_{A,1} = m_B v_{A,0} + m_B v_{A,1} \ \Rightarrow \ \frac{m_A - m_B}{m_A + m_B} v_{A,0} = v_{A,1} \ , \tag{A.9}$$

We insert the result for $v_{A,1}$ into (A.3), and find:

$$m_A\left(v_{A,0}-v_{A,1}\right)=m_B v_{B,1}\ \Rightarrow\ m_A\left(1-\frac{m_A-m_B}{m_A+m_B}\right)v_{A,0}=m_B v_{B,1}\ , \tag{A.10}$$

$$v_{B,1}=\frac{2m_A}{m_A+m_B}v_{A,0}\ . \tag{A.11}$$

This proves the solutions provided in (12.67).

A.2 Derivation of Formula for Two-Particle, Linear Collisions

Let us address a general collision, elastic, inelastic, and perfectly inelastic, characterized by a coefficient of restitution r.

The coefficient of restitution is defined as the ratio of the relative velocities after the collision to the relative velocities before the collision:

$$r=-\frac{v_{A,1}-v_{B,1}}{v_{A,0}-v_{B,0}}\ . \tag{A.12}$$

The case $r=1$ corresponds to an elastic collision, $r=0$ corresponds to a perfectly inelastic collision, and the case $0<r<1$ corresponds to an inelastic collision.

From (A.12), the velocity of object B after the collision is:

$$v_{B,1}=v_{A,1}+r\left(v_{A,0}-v_{B,0}\right)\ , \tag{A.13}$$

which we combine with conservation of momentum:

$$m_A v_{A,0}+m_B v_{B,0}=m_A v_{A,1}+m_B v_{B,1}\ , \tag{A.14}$$

to get:

$$m_A v_{A,0}+m_B v_{B,0}=m_A v_{A,1}+m_B v_{A,1}+r m_B\left(v_{A,0}-v_{B,0}\right)\ . \tag{A.15}$$

We solve for $v_{A,1}$, finding:

$$v_{A,1}=\frac{\left(m_A-r m_B\right)v_{A,0}+\left(1+r\right)m_B v_{B,0}}{m_A+m_B}\ . \tag{A.16}$$

We find the velocity of object B in exactly the same way, or simply by exchanging the indexes for A and B:

$$v_{B,1} = \frac{(m_B - rm_A)\, v_{B,0} + (1 + r)\, m_A v_{A,0}}{m_A + m_B} \; . \tag{A.17}$$

These are the completely general solutions to the collision problem, including the special cases of an elastic collision ($r = 1$) and a perfectly inelastic collision ($r = 0$).

A.3 Kinetic Energy of a Multi-particle System

The total kinetic energy of a multi-particle system is:

$$K = \sum_{i=1}^{N} \frac{1}{2} m \, (\mathbf{v}_i)^2 = \sum_{i=1}^{N} \frac{1}{2} m \left(\frac{d\mathbf{r}_i}{dt} \right)^2 \; . \tag{A.18}$$

We divide the motion into the motion of the center of mass of the system and the motion relative to the center of mass:

$$\mathbf{r}_i = \mathbf{R} + \mathbf{r}_{\mathrm{cm},i} \; , \; \mathbf{v}_i = \mathbf{V} + \mathbf{v}_{\mathrm{cm},i} \; . \tag{A.19}$$

We insert this into the total kinetic energy of the system:

$$\begin{aligned} K &= \sum_{i=1}^{N} \frac{1}{2} m_i \, (\mathbf{v}_i)^2 = \frac{1}{2} \sum_{i=1}^{N} m_i \left(\mathbf{V} + \mathbf{v}_{\mathrm{cm},i} \right)^2 \\ &= \frac{1}{2} \sum_{i=1}^{N} m_i \, (\mathbf{V})^2 + \frac{1}{2} \sum_{i=1}^{N} m_i \left(2\mathbf{V} \cdot \mathbf{v}_{\mathrm{cm},i} \right) + \frac{1}{2} \sum_{i=1}^{N} m_i \left(\mathbf{v}_{\mathrm{cm},i} \right)^2 \\ &= \frac{1}{2} M \, (\mathbf{V})^2 \; + \; \mathbf{V} \cdot \underbrace{\sum_{i=1}^{N} m_i \mathbf{v}_{\mathrm{cm},i}}_{= \mathbf{P}_{\mathrm{cm}} = 0} \; + \frac{1}{2} \sum_{i=1}^{N} m_i \left(\mathbf{v}_{\mathrm{cm},i} \right)^2 \\ &= \frac{1}{2} M \, (\mathbf{V})^2 + \frac{1}{2} \sum_{i=1}^{N} m_i \left(\mathbf{v}_{\mathrm{cm},i} \right)^2 \; . \end{aligned} \tag{A.20}$$

This proves that the kinetic energy can be subdivided into the kinetic energy of the translational motion of the center of mass and the kinetic energy of the motion relative to the center of mass.

A.4 Proof of the Superposition Principle

The proof of the superposition principle follows directly from the definition of the moment of inertia: The moment of inertia of a system consisting of both system A and system B around the axis O is:

$$I_O = \sum_j m_i \rho_i^2 \ , \tag{A.21}$$

where the sum is over all particles in object A and all particles in object B. We can split this sum into two parts: One sum of all the particles in object A and one sum over all the particles in object B:

$$I_O = \sum_{j=1}^{N_A} m_i \rho_i^2 + \sum_{j=N_A+1}^{N_A+N_B} m_i \rho_i^2 = I_{O,A} + I_{O,B} \ , \tag{A.22}$$

which is a proof of the superposition principle.

A.5 Proof of the Parallel-Axis Theorem:

The moment of inertia of an object around the axis A is:

$$I_A = \sum_i m_i \rho_{A,i}^2 \ , \tag{A.23}$$

where $\rho_{A,i}$ is the distance from a point i to the axis. Since the vector $\mathbf{s}$ points from axis A to axis C through the center of mass, we can write the vector from axis A to point i as (see Fig. A.2):

$$\boldsymbol{\rho}_{A,i} = \mathbf{s} + \boldsymbol{\rho}_{C,i} \ , \tag{A.24}$$

where $\boldsymbol{\rho}_{C,i}$ is a vector from axis C to point i. We insert this into the sum in (A.23):

$$\begin{aligned} I_A &= \sum_i m_i \left(\boldsymbol{\rho}_{A,i}\right)^2 = \sum_i m_i \left(\mathbf{s} + \boldsymbol{\rho}_{C,i}\right)^2 = \sum_i m_i \left(s^2 + 2\boldsymbol{\rho}_{C,i} \cdot \mathbf{s} + \rho_{C,i}^2\right) \\ &= \sum_i m_i s^2 + \sum_i m_i 2\boldsymbol{\rho}_{C,i} \cdot \mathbf{s} + \sum_i m_i \rho_{C,i}^2 = Ms^2 + 2\mathbf{s} \cdot \underbrace{\sum_i m_i \boldsymbol{\rho}_{C,i}}_{=0} + \underbrace{\sum_i m_i \rho_{C,i}^2}_{=I_C} \\ &= I_C + Ms^2 \ . \end{aligned} \tag{A.25}$$

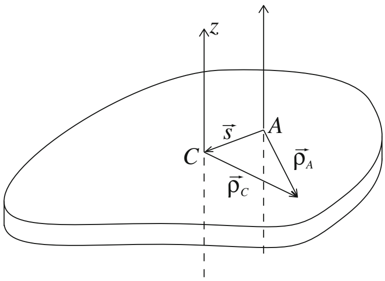

Fig. A.2 Illustration of the parallel-axis theorem. The moment of inertia around an axis C through the center of mass can be used to find the moment of inertia around an axis A—if the axis A is parallel to the axis C through the center of mass. The vector $\mathbf{s}$ is perpendicular to both axes, and points from the origin of axis C to the origin of axis A

Here we have used that the position of the center of mass in the center of mass system is zero, hence $\sum_i m_i \boldsymbol{\rho}_{C,i} = 0$.

A.6 Rotational Momentum and Newton's Second Law for a Point Particle

The angular momentum of the point particle with mass m, velocity $\mathbf{v}$, and position, $\mathbf{r}$ is defined as:

$$\mathbf{l} = \mathbf{r} \times \mathbf{p} = \mathbf{r} \times m\mathbf{v} \ . \tag{A.26}$$

We find Newton's second law by taking the time derivative of this equation:

$$\frac{d\mathbf{l}}{dt} = \frac{d}{dt}(\mathbf{r} \times \mathbf{p}) = \underbrace{\frac{d\mathbf{r}}{dt} \times \mathbf{p}}_{=0} + \mathbf{r} \times \underbrace{\frac{d\mathbf{p}}{dt}}_{=\sum \mathbf{F}} \ , \tag{A.27}$$

where we have used that $d\mathbf{r}/dt = \mathbf{v}$ is parallel to $\mathbf{p}$ and that the cross-product $\mathbf{v} \times \mathbf{p}$ is zero. This gives Newton's second law for a point particle on an alternative form:

$$\frac{d\mathbf{l}}{dt} = \sum_j \mathbf{r} \times \mathbf{F}_j \ , \tag{A.28}$$

where all the forces are acting in the point $\mathbf{r}$, since the point particles does not have any physical extent.

A.7 Rotational Momentum of a Multiparticle System

From the momentum $\mathbf{p}_i$ of a particle i, we introduced the total momentum,

$$\mathbf{P} = \sum_i \mathbf{p}_i \ , \tag{A.29}$$

of a multiparticle system. This allowed us to formulate a generalized version of Newton's second law for translational motion. Similarly, we can introduce the rotational momentum of a system of particles, simple as the sum of the rotational momentum of each particle. Particle i is in position $\mathbf{r}_i$ relative to the origin, hence the rotational momentum of particle i is:

$$\mathbf{l}_i = \mathbf{r}_i \times m_i \mathbf{v}_i \ , \tag{A.30}$$

and the total rotational momentum of a system of particles around a point O, the origin, is defined as:

$$\mathbf{L}_O = \sum_i \mathbf{l}_i = \sum_i \mathbf{r}_i \times m_i \mathbf{v}_i \ . \tag{A.31}$$

A.8 Newton's Second Law for Rotation Around a Fixed Axis

Since we have alread found that for a single point particle, Newton's second law can be written as

$$\frac{d\mathbf{l}_i}{dt} = \boldsymbol{\tau}_i \ , \tag{A.32}$$

we can try to find a similar relation for a multiparticle system. We start from the definition of the total angular momentum $\mathbf{L}_O$:

$$\mathbf{L}_O = \sum_{i=1}^{N} \mathbf{r}_i \times m_i \mathbf{v}_i \ , \tag{A.33}$$

and take the time derivative on both sides:

$$\frac{d\mathbf{L}_O}{dt} = \frac{d}{dt} \sum_{i=1}^{N} \mathbf{r}_i \times m_i \mathbf{v}_i = \sum_{i=1}^{N} \underbrace{\frac{d\mathbf{r}_i}{dt} \times m_i \mathbf{v}_i}_{=0} + \sum_{i=1}^{N} \mathbf{r}_i \times \underbrace{\frac{d m_i \mathbf{v}_i}{dt}}_{=\mathbf{F}_i^{\text{net}}} \ . \tag{A.34}$$

Here, the net force on part i is the sum of the external forces acting on part i and the internal forces. An internal force must originate in one of the other parts of the system. We can therefore write all the internal forces as: $\mathbf{F}_{j,i}$, meaning the force from part j on part i. The net force on part i is therefore:

$$\mathbf{F}_i^{\text{net}} = \mathbf{F}_i^{\text{ext}} + \sum_{j \neq i} \mathbf{F}_{j,i} \ , \tag{A.35}$$

which we insert into the equations, getting:

$$\frac{d\mathbf{L}_O}{dt} = \sum_{i=1}^{N} \mathbf{r}_i \times \left(\mathbf{F}_i^{\text{ext}} + \sum_{j \neq i} \mathbf{F}_{j,i} \right) = \sum_{i=1}^{N} \mathbf{r}_i \times \mathbf{F}_i^{\text{ext}} + \sum_{i=1}^{N} \sum_{j \neq i} \mathbf{r}_i \times \mathbf{F}_{j,i} \ . \tag{A.36}$$

Let us look at the last term, which is the sum of the torques on part i from all parts j in the system. From Newton's third law, we know that $\mathbf{F}_{j,i} = -\mathbf{F}_{i,j}$. Therefore, we rewrite the equation so that we explicitely include action/reaction terms. We do this through a "trick" you will often meet in physics: We realize that the sum over all the internal torques:

$$\sum_{i=1}^{N} \sum_{j \neq i} \mathbf{r}_i \times \mathbf{F}_{j,i} \ , \tag{A.37}$$

is a sum over all pairs, i, j, so that $i \neq j$. We could also write this as:

$$\sum_{i=1}^{N} \sum_{j \neq i} \mathbf{r}_i \times \mathbf{F}_{j,i} = \sum_{i,j:i \neq j} \mathbf{r}_i \times \mathbf{F}_{j,i} \ , \tag{A.38}$$

where the last sum is over all possible values of i and j as long as they are not equal. However, in this sum, there are pairs of torques that are related by action/reaction forces. There is a torque on particle i due to the force $\mathbf{F}_{j,i}$ from particle j on particle i, but there is also a torque on particle j due to the force $\mathbf{F}_{i,j}$ from particle i on particle j. If we want to include both of these terms explicitely in the sum, the sum must only be over half of the i and j values so that we do not include any term twice. We ensure this by summing over all pairs i and j so that $i < j$:

$$\sum_{i,j:i \neq j} \mathbf{r}_i \times \mathbf{F}_{j,i} = \sum_{i,j:i<j} \left(\mathbf{r}_i \times \mathbf{F}_{j,i} + \mathbf{r}_j \times \mathbf{F}_{i,j} \right) \ . \tag{A.39}$$

We now use that $\mathbf{F}_{j,i} = -\mathbf{F}_{i,j}$:

$$\sum_{i,j:i<j} \left(\mathbf{r}_i \times \mathbf{F}_{j,i} + \mathbf{r}_j \times \mathbf{F}_{i,j}\right) = \sum_{i,j:i<j} \left(\mathbf{r}_i \times \mathbf{F}_{j,i} + \mathbf{r}_j \times \left(-\mathbf{F}_{j,i}\right)\right)$$
$$= \sum_{i,j:i<j} \left(\mathbf{r}_i - \mathbf{r}_j\right) \times \mathbf{F}_{j,i} \ . \tag{A.40}$$

Hmmm. What can we do about this sum? We realize that if the force on particle i from particle j acts along the line between particle i and j:

$$\mathbf{F}_{j,i} = C_{j,i}\left(\mathbf{r}_i - \mathbf{r}_j\right) \ , \tag{A.41}$$

then the last term in (A.40) is zero. We call such forces central forces. In a rigid body, we assume that all forces are central. However, in many other types of systems the forces are also central. For example, gravitational forces, typical two-particle interatomic forces (given by a two-particle potential energy), and electro-static forces are central forces: This means that many forces from the atomic to the galactic scale are indeed central forces. It is therefore a reasonable assumption to assume that the forces are central forces, and that the sum of internal torques is zero.

The rate of change of the total angular momentum of the system is therefore equal to the net external torque on the system:

$$\frac{d\mathbf{L}_O}{dt} = \sum_{i=1}^{N} \mathbf{r}_i \times \mathbf{F}_i^{\text{ext}} = \boldsymbol{\tau}_{\text{ext}} \ , \tag{A.42}$$

which is what we call the net external torque around the point O.

A.9 Rotational Momentum of a Ridig Body

We have now found a general formulation for Newton's second law for rotational motion of a multiparticle system. However, we are often interested in rigid bodies. Can we simplify the relation by first finding the rotational momentum $\mathbf{L}_O$, of a rigid body rotating around a fixed axis, and then use this to find a simplified expression for Newton's second law for rotation of rigid bodies?

First, we find the rotational momentum of a rigid body. Figure A.3 shows a rigid body rotating around the z-axis with an angular velocity $\boldsymbol{\omega} = \omega\,\mathbf{k}$. The rigid body consists of a set of mass point, m_i, located at positions $\mathbf{r}_i$. The rotational momentum of this system is then:

$$\mathbf{L}_O = \sum_i \mathbf{l}_i = \sum_i \mathbf{r}_i \times m_i \mathbf{v}_i \ . \tag{A.43}$$

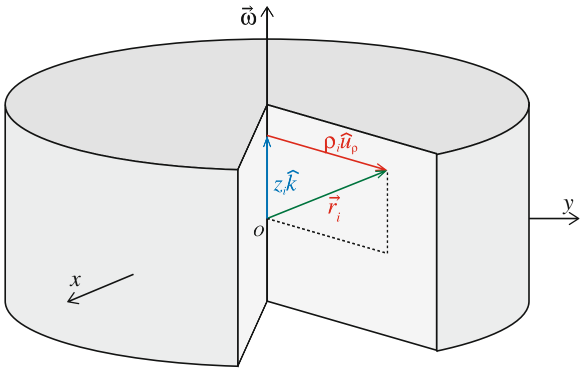

Fig. A.3 Illustration of a *rigid* body rotating around the z-axis. The cylindrical coordinate system is illustrated

Since the rigid body is rotating with angular velocity $\boldsymbol{\omega}$, the velocity of point i at $\mathbf{r}_i$ is

$$\mathbf{v}_i = \boldsymbol{\omega} \times \mathbf{r}_i \ , \tag{A.44}$$

where we can decompose the position $\mathbf{r}_i$ using a cylidrical coordinate system with a radius vector, $\boldsymbol{\rho} = \rho \hat{u}_\rho$, from the z-axis and out to the point $\mathbf{r}_i$ and a coordinate z_i along the z-axis:

$$\mathbf{r}_i = \boldsymbol{\rho}_i + z_i \, \mathbf{k} \ . \tag{A.45}$$

We insert this into the expression for $\mathbf{l}_i$, getting:

$$\mathbf{l}_i = \mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{r}_i) = \mathbf{r}_i \times (\omega \mathbf{k} \times (\boldsymbol{\rho}_i + z_i \, \mathbf{k})) \ , \tag{A.46}$$

where we notice that the $z_i \mathbf{k}$ term is parallel to $\omega \mathbf{k}$, and therefore this part of the cross product is zero, giving:

$$\mathbf{l}_i = \mathbf{r}_i \times \left(\boldsymbol{\omega} \times \boldsymbol{\rho}_i + \underbrace{\boldsymbol{\omega} \times z_i \, \mathbf{k}}_{=0} \right) = \mathbf{r}_i \times (\boldsymbol{\omega} \times \boldsymbol{\rho}_i) \ . \tag{A.47}$$

We use Lagrange's formula for the cross product:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \, (\mathbf{a} \cdot \mathbf{c}) - \mathbf{c} \, (\mathbf{a} \cdot \mathbf{b}) \ , \tag{A.48}$$

getting:

$$\begin{aligned} \mathbf{l}_i &= m_i \left(\boldsymbol{\omega} \, (\mathbf{r} \cdot \boldsymbol{\rho}) - \boldsymbol{\rho} \, (\mathbf{r} \cdot \boldsymbol{\omega}) \right) = m_i \left(\boldsymbol{\omega} \, ((\boldsymbol{\rho} + z \, \mathbf{k}) \cdot \boldsymbol{\rho}) - \boldsymbol{\rho} \, ((\boldsymbol{\rho} + z \, \mathbf{k}) \cdot \boldsymbol{\omega}) \right) \\ &= m_i \left(\rho_i^2 \boldsymbol{\omega} - \boldsymbol{\omega} z_i \boldsymbol{\rho}_i \right) \ . \end{aligned} \tag{A.49}$$

The total rotational momentum around the point O is therefore:

$$\begin{aligned}\mathbf{L}_O &= \sum_i \mathbf{l}_i = \sum_i m_i \left(\rho_i^2 \boldsymbol{\omega} - \boldsymbol{\omega} z \boldsymbol{\rho}_i\right) \\ &= \underbrace{\sum_i m_i \rho_i^2}_{=I_{O,z}} \boldsymbol{\omega} - \sum_i \boldsymbol{\omega} m_i z_i \boldsymbol{\rho}_i = I_{O,z} \boldsymbol{\omega} - \boldsymbol{\omega} \sum_i m_i z_i \boldsymbol{\rho}_i \ .\end{aligned} \quad \text{(A.50)}$$

Notice the second part of this equation. This term will be zero if the object is rotationally symmetric around the rotation axis. Otherwise, we will need to include this term.

However, if we are only interested in the z-component of the total rotational momentum of a rigid body, then we get a simplified result:

$$L_{O,z} = \mathbf{L}_O \cdot \mathbf{k} = \left(I_{O,z} \boldsymbol{\omega} - \boldsymbol{\omega} \sum_i m_i z_i \boldsymbol{\rho}_i \right) \cdot \mathbf{k} = I_{O,z} \underbrace{\boldsymbol{\omega} \cdot \mathbf{k}}_{=\omega} - \boldsymbol{\omega} \sum_i m_i z_i \underbrace{\boldsymbol{\rho}_i \cdot \mathbf{k}}_{=0} = I_{O,z} \boldsymbol{\omega} \ . \quad \text{(A.51)}$$

Although we must be careful with this expression, because it is tempting to generalize it to a vector equations, which we found above is only correct if the object is rotationally symmetric around the rotation axis.

A.10 Subdivision of Rotational Momentum

The total rotational momentum $\mathbf{L}_O$ around a fixed point O can be decomposed into the rotational momentum of the center of mass moving as a point particle relative to O and the rotational momentum relative to the center of mass:

$$\mathbf{L}_O = \mathbf{R} \times \mathbf{R} + \mathbf{L}_{cm} \ , \quad \text{(A.52)}$$

where $\mathbf{R}$ is the position and $\mathbf{P} = M\mathbf{V}$ is the momentum of the center of mass. We can show this by starting from the definition of the rotational momentum around a fixed point O:

$$\mathbf{L}_O = \sum_{i=1}^{N} \mathbf{l}_i = \sum_{i=1}^{N} \mathbf{r}_i \times m_i \mathbf{v}_i \ . \quad \text{(A.53)}$$

We decompose the position of mass m_i into:

$$\mathbf{r}_i = \mathbf{R} + \mathbf{r}_{cm,i} \quad \text{(A.54)}$$

where $\mathbf{R}$ is the position of the center of mass, and $\mathbf{r}_{cm,i}$ is the position of mass i relative to the center of mass. Inserted into (A.53), we get:

$$\begin{aligned}
\mathbf{L}_O &= \sum_{i=1}^{N} m_i \left(\mathbf{r}_i \times \frac{d\mathbf{r}_i}{dt} \right) \\
&= \sum_{i=1}^{N} m_i \left[\left(\mathbf{R} + \mathbf{r}_{cm,i} \right) \times \frac{d}{dt} \left(\mathbf{R} + \mathbf{r}_{cm,i} \right) \right] \\
&= \sum_{i=1}^{N} m_i \left[(\mathbf{R} \times \mathbf{V}) + \left(\mathbf{R} \times \mathbf{v}_{cm,i} \right) + \left(\mathbf{r}_{cm,i} \times \mathbf{V} \right) + \left(\mathbf{r}_{cm,i} \times \mathbf{v}_{cm,i} \right) \right] \\
&= M\mathbf{R} \times \mathbf{V} + \mathbf{R} \times \underbrace{\frac{d}{dt} \sum_{i=1}^{N} m_i \mathbf{r}_{cm,i}}_{=0} + \underbrace{\left(\sum_{i=1}^{N} m_i \mathbf{r}_{cm,i} \right)}_{=0} \times \mathbf{V} + \sum_{i=1}^{N} \left(m_i \mathbf{r}_{cm,i} \times \mathbf{v}_{cm,i} \right) \\
&= \mathbf{R} \times \mathbf{P} + \sum_{i=1}^{N} \mathbf{r}_{cm,i} \times \mathbf{p}_{cm,i}
\end{aligned} \tag{A.55}$$

A.11 Newton's Second Law for Rotation Around the Center of Mass

We can always describe the motion of a particle i in a multiparticle system as:

$$\mathbf{r}_i = \mathbf{R} + \mathbf{r}_{\mathrm{cm},i} \ , \tag{A.56}$$

where $\mathbf{R}$ is the position of the center of mass, and $\mathbf{r}_{\mathrm{cm},i}$ is the position of the particle relative to the center of mass, as illustrated in Fig. A.4.

Similarly, we may split the total rotational momentum of a system into the rotational momentum of the center of mass, and the rotational momentum relative to the center of mass:

$$\mathbf{L}_O = \sum_{i=1}^{N} \mathbf{l}_i = \sum_{i=1}^{N} \mathbf{r}_i \times m_i \mathbf{v}_i \ , \tag{A.57}$$

where we now introduce $\mathbf{r}_i = \mathbf{R} + \mathbf{r}_{\mathrm{cm},i}$ and $\mathbf{v}_i = \mathbf{V} + \mathbf{v}_{\mathrm{cm},i}$:

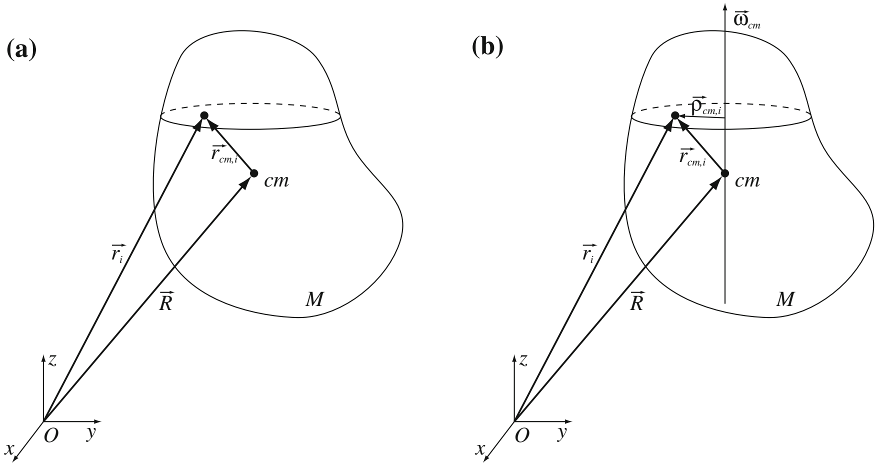

Fig. A.4 Illustration of an object rotating around the center of mass. **a** Position of a point relative to the center of mass. **b** Position of a point relative to the axis of rotation

$$
\begin{aligned}
\mathbf{L}_O &= \sum_{i=1}^{N} m_i \left(\mathbf{r}_i \times \frac{d\mathbf{r}_i}{dt} \right) \\
&= \sum_{i=1}^{N} m_i \left[\left(\mathbf{R} + \mathbf{r}_{cm,i}\right) \times \frac{d}{dt}\left(\mathbf{R} + \mathbf{r}_{cm,i}\right) \right] \\
&= \sum_{i=1}^{N} m_i \left[(\mathbf{R} \times \mathbf{V}) + \left(\mathbf{R} \times \mathbf{v}_{cm,i}\right) + \left(\mathbf{r}_{cm,i} \times \mathbf{V}\right) + \left(\mathbf{r}_{cm,i} \times \mathbf{v}_{cm,i}\right) \right] \\
&= M\mathbf{R} \times \mathbf{V} + \mathbf{R} \times \frac{d}{dt} \underbrace{\sum_{i=1}^{N} m_i \mathbf{r}_{cm,i}}_{=0} + \underbrace{\left(\sum_{i=1}^{N} m_i \mathbf{r}_{cm,i} \right)}_{=0} \times \mathbf{V} + \sum_{i=1}^{N} \left(m_i \mathbf{r}_{cm,i} \times \mathbf{v}_{cm,i} \right) \\
&= \mathbf{R} \times \mathbf{P} + \sum_{i=1}^{N} \mathbf{r}_{cm,i} \times \mathbf{p}_{cm,i}
\end{aligned}
\tag{A.58}
$$

The first term is the rotational momentum of the center of mass motion around the axis O, the second term is the rotational momentum of the object relative to the center of mass:

$$
\mathbf{L}_{\text{cm}} = \sum_{i=1}^{N} \mathbf{r}_{cm,i} \times \mathbf{p}_{cm,i} \; . \tag{A.59}
$$

Let us now use Newton's second law for rotations of multiparticle systems to find out what determines the change in rotational momentum around the center of mass.

From Newton's second law for rotations, we know that the rotational momentum and torque around the *fixed point* O are related by:

$$\frac{d\mathbf{L}_O}{dt} = \boldsymbol{\tau}_O = \sum_{i=1}^{N} \mathbf{r}_i \times \mathbf{F}_i^{\text{ext}} , \tag{A.60}$$

First, we insert the result for the rotational momentum to determine what is on the left hand side:

$$\begin{aligned} \frac{d}{dt}\mathbf{L}_O &= \frac{d}{dt}(\mathbf{R} \times \mathbf{P} + \mathbf{L}_{\text{cm}}) = \frac{d\mathbf{R}}{dt} \times \mathbf{P} + \mathbf{R} \times \frac{d\mathbf{P}}{dt} + \frac{\mathbf{L}_{\text{cm}}}{dt} \\ &= \underbrace{\mathbf{V} \times M\mathbf{V}}_{=0} + \mathbf{R} \times \mathbf{F}^{\text{ext}} + \frac{\mathbf{L}_{\text{cm}}}{dt} = \mathbf{R} \times \mathbf{F}^{\text{ext}} + \frac{\mathbf{L}_{\text{cm}}}{dt} . \end{aligned} \tag{A.61}$$

Second, we rewrite the right-hand side of (A.60) by introducing the center of mass coordinates:

$$\mathbf{r}_i = \mathbf{R} + \mathbf{r}_{\text{cm},i} , \tag{A.62}$$

We can therefore rewrite

$$\begin{aligned} \boldsymbol{\tau}_O &= \sum_{i=1}^{N} \mathbf{r}_i \times \mathbf{F}_i^{\text{ext}} = \sum_{i=1}^{N} \left(\mathbf{R} + \mathbf{r}_{\text{cm},i}\right) \times \mathbf{F}_i^{\text{ext}} = \sum_{i=1}^{N} \mathbf{R} \times \mathbf{F}_i^{\text{ext}} + \sum_{i=1}^{N} \mathbf{r}_{\text{cm},i} \times \mathbf{F}_i^{\text{ext}} \\ &= \mathbf{R} \times \sum_{i=1}^{N} \mathbf{F}_i^{\text{ext}} + \sum_{i=1}^{N} \mathbf{r}_{\text{cm},i} \times \mathbf{F}_i^{\text{ext}} = \mathbf{R} \times \mathbf{F}^{\text{ext}} + \sum_{i=1}^{N} \mathbf{r}_{\text{cm},i} \times \mathbf{F}_i^{\text{ext}} , \end{aligned} \tag{A.63}$$

where we have introduce the net external force as:

$$\mathbf{F}^{\text{ext}} = \sum_{i=1}^{N} \mathbf{F}_i^{\text{ext}} . \tag{A.64}$$

We insert this result and the result from (A.61) into (A.60), getting:

$$\begin{aligned} \frac{d\mathbf{L}_O}{dt} &= \boldsymbol{\tau}_O \mathbf{R} \times \mathbf{F}^{\text{ext}} + \frac{\mathbf{L}_{\text{cm}}}{dt} = \mathbf{R} \times \mathbf{F}^{\text{ext}} + \sum_{i=1}^{N} \mathbf{r}_{\text{cm},i} \times \mathbf{F}_i^{\text{ext}} \frac{\mathbf{L}_{\text{cm}}}{dt} \\ &= \sum_{i=1}^{N} \mathbf{r}_{\text{cm},i} \times \mathbf{F}_i^{\text{ext}} \frac{\mathbf{L}_{\text{cm}}}{dt} = \boldsymbol{\tau}_{\text{cm}} , \end{aligned} \tag{A.65}$$

where we have introduced:

$$\boldsymbol{\tau}_{\text{cm}} = \sum_{i=1}^{N} \mathbf{r}_{\text{cm},i} \times \mathbf{F}_i^{\text{ext}} \ , \tag{A.66}$$

as the torque around the center of mass.

We have therefore proven Newton's second law for rotational motion around the center of mass:

$$\frac{d\mathbf{L}_{\text{cm}}}{dt} = \boldsymbol{\tau}_{\text{cm}} \ . \tag{A.67}$$

We notice that in this derivation we have not assumed anything about the motion of the center of mass—it can be accelerated or executing any type of motion—the law is still valid!

Appendix B
Solutions

Chapter 2

B.1 Seconds.

(**a**) `s = 3600*h, print s` (**b**) 5400 s, 43200 s, 86499 s

B.5 Plotting the normal distribution.

(**a**)

```
def normal(x,mu,sigma):
    P = 1.0/sqrt(2*pi*sigma**2)*exp(-(x-mu)**2/2*sigma**2)
    return P
```

(**b**)

```
x = linspace(-5,5,1000)
P = norpdf(x,0,1)
plot(x,P), show()
```

(**c**)

```
plot(x,P), hold('on')
P = normpdf(x,0,2); plot(x,P,'-r')
P = normpdf(x,0,0.5); plot(x,P,'-g')
hold('off'), show()
```

(**d**)

```
x = linspace(-5,5,1000)
subplot(3,1,1)
P = normpdf(x,0,1)
plot(x,P,'-b')
subplot(3,1,2)
P = normpdf(x,1,1);
plot(x,P,'-g')
subplot(3,1,3)
P = normpdf(x,2,);
plot(x,P,'-g'), show()
```


A. Malthe-Sørenssen, *Elementary Mechanics Using Python*,
Undergraduate Lecture Notes in Physics, DOI 10.1007/978-3-319-19596-4

B.6 Plotting $1/x^n$.

(a)

```
def fvalue(x,n):
    f = 1.0/x**n
    return f
```

(b)

```
x = linspace(-1,1,1000)
f1 = fvalue(x,1)
plot(x,f1), hold('on')
f2 = fvalue(x,2)
plot(x,f2)
f3 = fvalue(x,3)
plot(x,f3), hold('off'), show()
```

B.7 Plotting $\sin(x)/x^n$.

(a)

```
def gvalue(x,n):
    g = 1.0/x**n
    return g
```

(b)

```
x = linspace(-5,5,1000)
g1 = gvalue(x,1)
plot(x,g1), hold('on')
g2 = gvalue(x,2)
plot(x,g2)
g3 = gvalue(x,3)
plot(x,g3), hold('off'), show()
```

B.8 Logistic map.

(a)

```
def logistic(x,r):
    g = r*x*(1-x)
    return g
```

(b)

```
r = 1.0
n = 100
x = zeros(n,float)
for i in range(n-1):
    x[i+1] = logistic(x,r)
i = r_[0:n-1], plot(i,x), show()
```

B.9 Euler's method.

(a)

```
def acceleration(v,x,k,C):
    a = -k*x - C*v
    return a
```

(b)

```
k = 10, C = 5, n = 100, deltat = 0.01, n = 100
x = zeros(n,float), v = zeros(n,float)
a = zeros(n,float), t = zeros(n,float)
x[0] = x0, v[0] = v0
for i in range(n-1):
```

```
    a[i] = acceleration(v[i],x[i],k,C)
    v[i+1] = v[i] + a[i]*deltat
    x[i+1] = x[i] + v[i]*deltat
    t[i+1] = t[i] + deltat
subplot(3,1,1)
plot(t,a), xlabel('t'), ylabel('a')
subplot(3,1,2)
plot(t,v), xlabel('t'), ylabel('v')
subplot(3,1,3)
plot(t,x), xlabel('t'), ylabel('x')
show()
```

(**c**) You only need to change the function `acceleration`.

```
def acceleration(v,x,k,C):
    a = k*sin(x)-C*v
    return a
```

B.10 Throwing two dice.

(**a**) In vectorized notation:

```
def dice(n):
    x1 = randint(1,6,n)
    x2 = randint(1,6,n)
    z = x1+x2
    return z
```

Using loops:

```
def dice(n):
z = zeros(n,float)
for i in range(n):
   x1 = randint(6)
   x2 = randint(6)
   z[i] = x1 + x2
return
```

B.11 Reading data.

(**a**)

```
t,x,y=loadtxt('trajectory.d',usecols=[0,1,2],unpack=True)
```

(**b**)

```
subplot(2,1,1)
plot(t,x), xlabel('t (s)'), ylabel('x (m)')
subplot(2,1,2)
plot(t,y), xlabel('t (s)'), ylabel('y (m)')
```

(**c**)

```
plot(x,y), xlabel('x (m)'), ylabel('y (m)')
```

B.12 Numerical integration of a data-set.

(**a**)

```
t,v=loadtxt('velocityy.d',usecols=[0,1],unpack=True)
```

(**b**)

```
plot(t,v), xlabel('t (s)'), ylabel('v (m/s)')
```

(**c**)

```
n = len(t)
y = zeros(n,float)
y0 = 0.0
y[0] = y0
for i in range(n-1):
    y[i+1] = y[i] + v[i]*(t[i+1]-t[i])
```

(**d**)

```
subplot(2,1,1)
plot(t,y), xlabel('t (s)'), ylabel('y (m)')
subplot(2,1,2)
plot(t,v), xlabel('t (s)'), ylabel('v (m/s)')
```

Chapter 3

B.1 Kilometers per hour.

40 m/s

B.2 Miles per hour.

(**a**) 43 mph (**b**) 89 km/h

B.3 Acceleration of gravity.

(**a**) $g = 32.2\,\text{ft/s}^2$ (**b**) $g = 1.3\,10^5\,\text{km/h}^2$

B.4 Bacterial volume.

(**a**) $4\pi\ (\mu m)^3$ (**b**) $4\pi\ 10^{-}18\,\text{m}^3$ (**c**) 4π femtoliter

B.5 Ruler length.

2.2 m

B.6 Sphere mass and volume.

(**a**) $7.2\,\text{mm}^3$ (**b**) $5.6\,10^{-5}\,\text{kg} = 56\,\text{milligram}$

B.7 Laserlength.

11.2 m

B.8 Salmon speed.

(**a**) 3.05 m/s (**b**) 3.05 m/s (**c**) Works if you are limited by your accuracy in time.

Chapter 4

B.12 Capturing the motion of a falling ball.

(**d**) $v_{\text{max}} \simeq -5.75\,\text{m/s}$

B.17 The fastest indian.

(**a**) 898 m (**b**) 11.1 s

B.18 Meeting trains.

(**a**) 12 minutes (**b**) 10 km

B.19 Catching up.
(**a**) 900 m (**b**) 0 m (**d**) 2160 s (**e**) 36 m (**l**) 2.32 km

B.20 Electron in electric field.
(**a**) $\sqrt{14000\,\text{m}^2/\,\text{s}^2} = 118\,\text{m/s}$

B.21 Archery.
(**a**) $1.8\,\text{km/s}^2$

B.22 Collision.
(**a**) $a = -50\,\text{m/s}^2$

B.23 Braking distance.
(**a**) $x = v_0^2/(10\,\text{m/s}^2)$ (**b**) $x_{\text{old tires}} = (3/2)\ x_{\text{new tires}}$ (**c**) $x_{\text{stop, new tires}} = 26.2\,\text{m}$, $x_{\text{stop, old tires}} = 35.9\,\text{m}$

B.28 A swimming bacterium.
(**a**) $v = v_0 + a_0 T/(2\pi)\,[1 - \cos(2\pi t/T)]$ (**b**) $x = v_0 t + a_0(T/2\pi)\,[t - (T/2\pi)\sin(2\pi t/T)]$ (**c**) $v_{av} = v_0 + a_0(T/2\pi)$

Chapter 5

B.18 Pulling a train.
(**a**) $2\,\text{m/s}^2$ (**b**) $1.66\,\text{m/s}^2$

B.19 Firing a bullet.
$v = 141\,\text{m/s}$

B.20 Jumping into snow.
$6mg$

B.22 Vertical throw.
(**b**) $a = -g$ (**c**) $t = \left(v_0 + \sqrt{2gh_0 + v_0^2}\right)/g$ (**d**) $v = -\sqrt{v_0^2 + 2gh_0}$ (**e**) $v = -\sqrt{v_0^2 + 2gh_0}$

B.23 Reaction time.
(**b**) $x(t) = -(1/2)gt^2$ (**c**) $t = \sqrt{2h/g}$ (**d**) $x_{\text{car}} = v_{\text{car}}\sqrt{2h/g}$

B.24 Terminal velocity of heavy and large objects.
(**b**) $a = -g + Dv^2/m$ (**c**) Largest mass has largest acceleration (**d**) $a = -g + (6C_0 v^2)/(\pi\rho d)$ where ρ is the mass density (**e**) The object with the largest diameter has the largest magnitude of the acceleration.

B.25 Space shuttle with air resistance.
(**b**) $a = F/m - g$ (**c**) 153.8 m/s, 1538 m

B.26 Experiments in Pisa.
(**a**) Gravity and air resistance. (**b**) Air resistance is the same for both spheres (**c**) $a = g - f(v)/m$ (**d**) The solid sphere reaches the ground first.

B.27 Stretching an aluminum wire.
$k = 98\,\text{kN/m}$

B.28 Two masses and a spring.
$k = 98100\,\text{N/m}$

Chapter 6

B.12 Alpha particle.
(**a**) 2235 m/s (**b**) $\mathbf{r} = \mathbf{v}t = 1000\,\text{m/s}\,t\,\mathbf{i} + 2000\,\text{m/s}\,t\,\mathbf{j}$ (**c**) 2235 m

B.13 Airplane collision.
(**a**) $x(t) = 0$, $y(t) = 472.2\,\text{m/s}\,t$ (**b**) $x(t) = -1.0e4\,\text{m} + 29.2\,\text{m/s}\,t$, $y(t) = 8.0e4\,\text{m} + 251.4\,\text{m/s}\,t$, (**d**) No (**e**) Yes

B.14 Motion of spaceship.
(**b**)

$$\mathbf{v}(t) = \begin{cases} 1000\,\text{m/s}\,\mathbf{i} + 10\,\text{m/s}^2\,t\,\mathbf{j}\ , \text{when } t < 10\,\text{s} \\ 1000\,\text{m/s}\,\mathbf{i} + 100\,\text{m/s}\,\mathbf{j}\ \ , \text{when } t \geq 10\,\text{s} \end{cases} \tag{B.1}$$

(**c**)

$$\mathbf{r}(t) = \begin{cases} 1000\,\text{m/s}\,t\,\mathbf{i} + 5\,\text{m/s}^2\,t^2\,\mathbf{j} & \text{when} t < 10\,\text{s} \\ 1000\,\text{m/s}\,t\,\mathbf{i} + 500\,\text{m}\,\mathbf{j} + 100\,\text{m/s}\,t\,\mathbf{j} & \text{when} t \geq 10\,\text{s} \end{cases} \tag{B.2}$$

B.15 Controlling the electron beam.
(**a**) $v_x(t) = 100\,\text{m/s}$, $v_y(t) = -20\,\text{m/s}^2\,t - 5\,\text{m/s}^3\,t^2$. (**b**) $x(t) = 100\,\text{m/s}\,t$, $y(t) = -10\,\text{m/s}^2\,t^2 - (5/3)\,\text{m/s}^3\,\text{t}^3$. (**c**) $t = 1/50\,\text{s}$ (**d**) $y = -4.01 \times 10^{-3}\,\text{m}$ (**e**) $\alpha = -0.23°$.

B.17 Running inside a bus.
(**a**) 40 km/h (**b**) 60 km/h

B.18 Jumping onto a running train.
(**a**) $-10\,\text{m/s}$ (**b**) $5\,\text{m/s}^2$ (**c**) $v = -10\,\text{m/s} + 5\,\text{m/s}^2\,t$, when $t < 2\,\text{s}$, $v = 0\,\text{m/s}$, when $t > 2\,\text{s}$ (**d**) $v = 5\,\text{m/s}^2\,t$, when $t < 2\,\text{s}$, $v = 10\,\text{m/s}$ when $t > 2\,\text{s}$

B.19 A plane in crosswinds.
(**a**) 78.5 degrees over west (**b**) $v = 293.9\,\text{km/h}$

Chapter 7

B.11 Chandelier.
(**b**) $T = 490.5 * \sqrt{(}h^2 + 8)/h$ (**c**) $h = 0.6424\,\text{m}$

B.12 Three-pointer.
(**b**) $x(t) = 4.7\,textm/s\,t$, $y(t) = y_0 + 8.1\,\text{m/s}\,t - 4.9\,\text{m/s}^2\,t^2$ (**c**) 2.2 m (**d**) $-6.5\,\text{m/s}$

B.13 Hitting an apple.
(**b**) $x(t) = 50\,\text{m/s} \cdot t$ (**c**) 1.23 m (**d**) 3.675 m (**e**) 4.5 m

B.14 Hitting the target.
$v = 3.50\,\text{m/s}$

B.15 Long jump world record.
9.20 m

B.18 Weather balloon.
(**b**) $a = (B/m) - g$ (**c**) $v(t) = v(0) + (B/m - g)\,t$, $z(t) = z(0)(1/2)\,(B/m - g)\,t^2$. (**f**) $v_z^2 = (B/m - g)/(D/m)$ (**g**) $\mathbf{F}_D = -D\,|\mathbf{v} - \mathbf{w}|\,(\mathbf{v} - \mathbf{w})$. (**i**) $\mathbf{a} = (B/m)\mathbf{k} - g\mathbf{k} - (D/m)\,|\mathbf{v} - w\,\mathbf{i}|\,(\mathbf{v} - w\,\mathbf{i})$. (**m**) $v_z = ((B/m) - g)/(D/m)$ (**o**) It is the same.

Chapter 8

B.5 Skier pulled up a slope.
(**a**) $v(t) = at$ (**b**) $s(t) = \frac{1}{2}at^2$ (**c**) $\mathbf{r}(t) = s(t)\,(\cos\alpha\,\mathbf{i} + \sin\alpha\,\mathbf{j})$ (**d**) $|\mathbf{v}(t)| = |at|$

B.6 Skiing down a slope.
(**a**) $v(t) = a\,t$ (**b**) $s(t) = (1/2)\,a\,t^2$ (**c**) $\mathbf{r}(t) = h\,\mathbf{j} + (g/2)\sin\alpha t^2\,(\cos\alpha\,\mathbf{i} - \sin\alpha\,\mathbf{j})$ (**d**) $t = \sqrt{h/g}\,(1/\sin\alpha)$

B.7 Bead on a line.
(**a**) $v(t) = a\,t$ (**b**) $s(t) = (1/2)\,a\,t^2$ (**c**) $h(t) = -s(t)\ \cos\alpha$

B.8 Acceleration of 200 m sprinter.
(**a**) $R = 100\,\text{m}/\pi$ (**b**) $a = v^2/R$ toward the center of the circle

B.9 Velocity of point on helicopter rotor blade.
(**a**) $v \simeq 105\,\text{m/s}$ (**b**) $a = 2.2\,\text{km/s}$ towards the center of the blade

B.10 Turning a high-speed train.
(**a**) $a = v^2/R$ where R is the radius of the circle (**b**) $R = v^2/a \simeq 3.15\,\text{km}$ (**c**) $t = \pi R/(2v) \simeq 89\,\text{s}$

B.11 Acceleration on the equator.
(**a**) $v \simeq 464\,\text{m/s}$ (**b**) $a = v^2/R \simeq 0.03\,\text{m/s}^2 = 0.0034\,g$

B.12 Artificial gravity in space travel.
(**a**) $n \simeq 4.2$ (**b**) $\Delta a = (2\pi/T)^2 \cdot 2\,\text{m} = 0.4\,\text{m/s}^2$

B.13 Probe in tornado.
(**a**) $\bar{\mathbf{a}} = -3.3\,\text{m/s}^2\,\mathbf{i} - 18.1\,\text{m/s}^2\,\mathbf{j}$ (**b**) $R \simeq 40$, $\mathbf{r}_{circle} \simeq 5\,\text{m}\,\mathbf{i} + 10\,\text{m}\,\mathbf{j}$

B.14 Bead on ring.
(**a**) $v = R\cos\theta\ (2\pi n)/(60\text{s})$ (**b**) $a = R\cos\theta(2\pi n/(60\text{s}))^2$

B.16 Car in a wire.
(**a**) $v = a_t t$ (**b**) $a_r = v^2/R = a_t^2 t^2/R$ (**c**) $v = \sqrt{100 a_t R}$

Chapter 9

B.6 Rope with finite mass.
(**a**) $S = mg/(2\sin\alpha)$ (**b**) $S = mg/(2\sin\alpha)$ (**c**) No

B.7 Fireman on pole.
(**b**) $F_\mu = mg$ (**c**) $N = mg/\mu_d$

B.8 Pulling a box.
(**b**) $N = mg - T\sin(\alpha)$ (**c**) $a = (T/m)(\cos(\alpha) + \mu\sin(\alpha)) - \mu g$, d) $\alpha = \pi/4$
(**d**) $\alpha = \pi/4$

B.9 Hanging rope.
(**b**) $T = x\,(M/L)\,g$ (**c**) $N = (L - x)\,/L\,Mg$ (**d**) $x = \mu/\,(\mu + 1)\;L$

B.10 Pulling out a book.
(**a**) $F > \mu_2(m_1 + m_2)g$ (**b**) $F > (\mu_1(m_1 + m_2) + \mu_2 m2)\,g$

B.11 Forces on a 200 m runner.
(**a**) $f = mv^2/R$ (**b**) $\mu = v^2/(gR)$

B.12 Rope through a hole.
$v = \sqrt{MgR/m}$

B.13 Bead on a wire.
$\alpha = \sin^{-1}\left(T^2 g\right)/\left(R(2\pi)^2\right)$

B.14 Man in a wheel.
$v = \sqrt{gR/\mu_s}$

B.15 Motorcycle in a loop.
$v \geq \sqrt{gR}$

B.16 Stick-slip friction.
(**b**) $x_b(t) = b + ut$ (**e**) $N = mg$ (**f**) $a = 0$ (**g**) $\Delta L = (\mu_d mg)/k$ (**h**) $x(t) = x_b(t) - b - (\mu_d mg)/k$ (**j**) $\Delta L = (\mu_s mg)/k$ (**k**) $f = k\,u\,t$

B.17 Feather in tornado.
(**b**) $a = -g + (D/m)v^2$ (**d**) $D/mg = (t/h)^2 = (4.8\,\text{s}/2.4\,\text{m}) = 4.0\,\text{s}^2\,\text{m}^{-2}$
(**e**) $a = d^2z/dt^2 = -g - D|v_z|v_z$, $y(0) = h$, and $v(0) = 0$ (**f**)

```
# Program for a falling feather
from pylab import *
h = 2.4
Dmg = 4.0
g = 9.8
time = 10.0
dt = 0.001
n = int(round(time/dt))
t = zeros(n,float)
x = zeros(n,float)
v = zeros(n,float)
a = zeros(n,float)
```

```
x[0] = h
v[0] = 0.0
i = 1
while (i<n) and (x[i]>=0.0):
    a[i] = -g - g*Dmg*v[i]*abs(v[i])
    v[i+1] = v[i] + a[i]*dt
    x[i+1] = x[i] + v[i+1]*dt
    t[i+1] = t[i] + dt
    i = i + 1
i = i - 1, x(i), t(i)
subplot(2,1,1)
plot(t[0:i],x[0:i])
xlabel('t [s]'), ylabel('x [m]')
subplot(2,1,2)
plot(t[0:i],v[0:i])
xlabel('t [s]'), ylabel('v [m/s]')
```

(**h**) $\mathbf{a} = -g\hat{z} g(D/mg)\,|\mathbf{v}-\mathbf{w}|\,(\mathbf{v}-\mathbf{w})$ (**i**) $w_T = v_T = v_0$, $\mathbf{a} \simeq v_0^2/r_0$ (**j**) No. (**k**)

```
from pylab import *
vT = 0.18 # Terminal velocity
Dmg = 4.0
R = 20.0 # Size in meters
U = 100.0 # Velocity in m/s
g = 9.8
time = 15.0
dt = 0.001
n = int(round(time/dt))
t = zeros((n,1),float)
r = zeros((n,3),float)
v = zeros((n,3),float)
a = zeros((n,3),float)
t[0] = 0.0;
r[0,:] = array([-1.0*R,0.0,2.4])
v[0,:] = array([0.0,0.0,0.0])
i = 1
while ((r(i,3)>=0.0)and(i<n-1)):
    rr = norm(r[i,:])
    u = U*array([-r[i,1)],r[i,0],0.0])*exp(-rr/R)/R
    vrel = v[i,:] - u
    aa = -g*array([0,0,1]) - g*Dmg*norm(vrel)*vrel
    a[i = aa
    v[i+1] = v[i] + aa*dt
    r[i+1] = r[i] + v[i+1]*dt
    t[i+1] = t[i] + dt
    i = i + 1
imax = i
figure(3)
i = range(imax)
ii = range(0,imag,100)
plot(r[i,0],r[i,1],'-',r[i,0],r[ii,1],'o')
axis('equal'), xlabel('x [m]'), ylabel('y [m]')
```

B.18 Modelling Atomic Interactions.

(**a**) $x = 0$ is an unstable equilibrium point, while $x = \pm d$ are stable equilibrium points. (**c**) $v_0 = \pm\sqrt{18U_0/m}$ (**d**) $|v| \geq \sqrt{2U_0/m}$ (**f**) $a = F/m$; $v(t+\Delta t) = v(t) + a\,\Delta t$; $x(t+\Delta t) = x(t) + v(t+\Delta t)\,\Delta t$ (**g**)

```
from pylab import *
m = 1 #pkg
d = 0.1 #nm
U_0 = 1 #nJ
dt = 0.01 #ns
T = 10 #ns
```

```
n = int(round(T/dt))
t = zeros(n,float);
v = zeros(n,float);
x = zeros(n,float);
for i in range(n-1):
        F = - ((4*U_0)/(d**4))*(x[i]^3-x[i]*d**2)
        a = F/m
        t[i+1] = t[i] + dt
        v[i+1] = v[i] + a*dt
        x[i+1] = x[i] + v[i+1]*dt
```

(**j**) $\mathbf{a} = -(4U_0) / \left(md^4\right) \left(r^3 - rd^2\right) (\mathbf{r}/r)$ (**k**)

```
for i in range(n-1):
    rr = norm(r[i])
    F = -(4*U_0)/(d**4)*(rr**3-rr*d**2)*(r[i]/rr)
    a = F/m
    t[i+1] = t[i] + dt
    v[i+1] = v[i] + a*dt
    r[i+1] = r[i] + v[i+1]*dt
```

(**m**) For the atom to move in a circular orbit with a constant radius, the initial conditions have to be $r > d$ and $v = \sqrt{((4U_0)/(md^4)\,(r^3 - rd^2)r}$, with the velocity directed orthogonal to the position.

Chapter 10

B.7 Dragging a cart.

(**a**) $W_F = F\,s = 90\,\text{N} \times 10\,\text{m} = 900\,\text{J}$ (**b**) $W_f = f\,s = -30\,\text{N} \times 10\,\text{m} = -300\,\text{J}$ (**c**) $v = \sqrt{2(W_F + W_f)/m} = 11\,\text{m/s}$

B.8 Toboggan slide.

(**a**) $W_f = (1/2)mv^2 - mgh = -275\,\text{J}$.

B.9 Crate on conveyor belt.

(**b**) $s = \left(v_0^2\right) / (2\mu_d g)$ (**d**) $W = \int_0^x -f\,dx = -fx = -\mu_d mgx$ (**e**) $x = -\left(v_c^2\right) / (2\mu_d g)$

B.10 Volleyball smash.

(**a**) $x \simeq 0.043\,\text{m}$

B.11 A bouncing ball.

(**a**) $v_1 = -\sqrt{2gh}$ (**c**) $W_G = -mgy$, $W_k = -k\,(2/5)\,(-y)^{5/2}$ (**d**) $mg\,(-y) - (2k/5)\,(-y)^{5/2} = -mgh$ (**e**) $v_3 = -v_1 = \sqrt{2gh}$

B.12 Power of the heart.

(**a**) $W = 70.53\,\text{kJ}$ (**b**) $P = 0.816\,\text{W}$

B.13 Power station.

(**a**) $P = 9800\,\text{W}$

B.14 Accelerating car.

(**a**) $t = (1/2)\,(mv^2)/P = 2.48\,\text{s}$

B.15 An accelerating motorbike.
(**a**) $v = \sqrt{2\,P\,t/m}$ (**b**) $a(t) = dv/dt = \sqrt{P/2mt}$ (**c**) $x(t) = (2/3)\sqrt{2Pt^3/m}$

B.16 Driving efficiently.
(**c**) $a = \left((P_0/v) - Dv^2\right)/m$ (**g**) $W_E = mv^2/2$ (**h**) $W_D = Dv^2 L$

Chapter 11

B.8 The loop.
(**a**) $v_B = \sqrt{2gh}$ (**b**) $v_C = \sqrt{2g(h-2R)}$ (**c**) $v_C \geq \sqrt{gR}$ (**d**) $h \geq (5/2)R$ (**e**) $s = h/\mu$

B.9 Sliding on a cylinder.
(**a**) $v = \sqrt{2gR\,(1-\cos(\theta))}$ (**b**) $\cos\theta = 2/3$

B.10 Vertical pendulum.
(**a**) $v = \sqrt{v_0^2 - 4gL}$ (**b**) $v_0 \geq \sqrt{5\,g\,L}$

B.11 Two-point pendulum.
(**a**) $v_A = \sqrt{2gL}$ (**b**) $v_B = \sqrt{2g(2h-L)}$ (**c**) $h > (2/3)\,L$

B.12 Lennard-Jones Potential.
(**a**) $F = U_0\left(12\left(a^{12}/r^{13}\right) - 6\left(b^6/r^7\right)\right)$ (**c**) $r_{1,2} = \pm 2^{(1/6)}\,(a^2/b)$

B.13 A bouncing ball—part 1.
(**a**) R (**b**) $v = \sqrt{2g(h-R)}$ (**c**) $\delta y = \sqrt{(2mg/k)\;(h-R)}$

B.14 A bouncing ball—part 2.
(**a**) $\mathbf{v} = (v_0, -\sqrt{2g(h-R)})$ (**b**) $\mathbf{v} = (v_0, 0)$ (**c**) $\delta y = \sqrt{(2mg/k)\;(h-R)}$
(**d**) $\mathbf{v} = (v_0, +\sqrt{2g(h-R)})$

B.15 Shooting Ions.
(**b**) $x_1 = C/\left((1/2)mv_0^2 + C/b\right)$ (**c**) $v_\infty^2 = v_0^2 + 2\,C/(m\,b)$ (**e**) $\mathbf{a} = (C/m)\mathbf{r}/r^3$, $\mathbf{r}(0) = b\,\mathbf{i} + d\,\mathbf{j}$, $\mathbf{v}(0) = v_0\,\mathbf{i}$. (**g**)

```
m = 1.0    # mass in dimensionless units
b = 1.0    # length in dimensionless units
d = 0.2    # length in dimensionless units
C = 1.0
v0 = 2.5
time = 4.0/v0 # time in dimensionless units
dt = 0.001 # dt
n = int(round((time/dt))
r = zeros((n,2),float)
v = zeros((n,2),float)
t = zeros(n,float)
# Initial conditions
r[0] = array([b,d])
v[0] = array([-v0,0.0])
# Solve eqns. of motion
for i in range(n-1):
     rnorm3 = norm(r[i])**3
     F = C/rnorm3*r[i]
     a = F/m
     v[i+1] = v[i] + a*dt
     r[i+1] = r[i] + v[i+1]*dt
     t[i+1] = t[i] + dt
plot(r(:,1),r(:,2),'-')
xlabel('x/b'), ylabel('y/b'), axis('equal')
```

B.4 A bike and a car.
(**a**) $v = 600\,\text{km/h}$

Chapter 12

B.5 Kicking a ball.
(**a**) $\Delta p = 8.6\,\text{kg m/s}$ (**b**) $J = 8.6\,\text{kg m/s}$ (**c**) $F_{\text{avg}} = 86\,\text{N}$ (**d**) $F_{\text{avg}} = 172\,\text{N}$

B.6 Stopping a car.
(**a**) $F = 100\,\text{kN}$ (**b**) $F = 3.3\,\text{kN}$

B.7 Ball reflected from wall.
(**a**) $\Delta p = 2mv_0 \sin\theta$ (**b**) $J = 2mv_0 \sin\theta$ (**c**) $F = 2mv_0 \sin\theta/\Delta t$ (**d**) $\theta = 90°$

B.8 Snowball on ice.
(**a**) $\mathbf{p} = 34.6\,\text{kg m/s}\,\mathbf{i} + 20\,\text{kg m/s}\,\mathbf{j}$ (**b**) $\mathbf{v}_{\text{you}} = -0.43\,\text{m/s}\,\mathbf{i}$, $\mathbf{v}_{\text{son}} = 0\,\text{m/s}\,\mathbf{i}$ (**c**) $\mathbf{v}_{\text{you}} = -0.43\,\text{m/s}\,\mathbf{i}$, $\mathbf{v}_{\text{son}} = 1.73\,\text{m/s}\,\mathbf{i}$

B.9 Toppling a book.
(**a**) You should choose the ball that bounces back

B.10 Bullet and a block.
(**a**) $v_0 = 20.8\,\text{m/s}$ (**b**) $\Delta E_k = -20.6\,\text{J}$

B.11 Stopping a ball.
(**a**) Yes

B.12 Pendulum and block.
(**a**) $v = -\sqrt{2gL}/3$, $V = 2\sqrt{2gL}/3$ (**b**) $h = L/9$

B.14 Newton's cradle.
(**a**) $v_0 = \sqrt{2gh_0}$ (**b**) $v_1^A = 0$ and $v_1^B = v_0$. (**c**) $h_1 = h_0$. (**d**) $h_1 = h_0/4$. (**e**) $v_0 = v_1^A + (1+r)\,v_0/2$, and $v_1^A = (1-r)\,v_0/2$ (**f**) The result of the first collision is to give ball B velocity v_0 and ball A velocity 0. The result of the second collision is to give ball C velocity v_0 and ball B velocity 0. (**g**) There are two equations with three unknowns.

B.15 Catching an atom.
(**b**) $F(x) = -k\,(x - b)$ when $b - d < x < b + d$, $F(x) = 0$ when $x > b + d$ and the atom cannot move to $x < b - d$. (**c**) $v_{A,1} = -\sqrt{v_{A,0}^2 + (2U_0/m)}$ (**d**) $v_2 = \frac{1}{2}v_{A,1}$. (**e**) $v_0 \geq \sqrt{U_0/m}$. (**g**)

```
k = 100.0
m = 1.0
b = 1.0
d = 0.5
r0 = array([1.0,0.0])
v0 = array([0.0,2.8])
time = 5.0
dt = 0.001
n = round(time/dt)
```

```
t = zeros(n,float)
r = zeros((n,2),float)
v = zeros((n,2),float)
a = zeros((n,2),float)
v[0] = v0
r[0] = r0
for i in range(n-1):
    rr = norm(r[i])
    if (rr>b+d):
        F = array([0.0,0.0]);
    elseif (rr>b-d):
        F = -k*(rr-b)*r[i]/rr
    else # Collision - reverse velocity in radial direction
        ur = r[i]/rr
        vprojur = dot(v[i],ur)
        v[i] = v[i] - vprojur*ur + abs(vprojur)*ur
    a[i] = F/m
    v[i+1] = v[i] + a[i]*dt
    r[i+1] = r[i] + v[i]*dt
    t[i+1] = t[i] + dt
plot(r[:,0],r[:,1])
xlabel('x/b'), ylabel('y/b')
```

(**l**) Not possible.

Chapter 13

B.5 Two-particle system.

(**a**) $x = 14/3\,\text{m}$

B.6 Center of mass of Earth-Moon system.

(**a**) 0.763 Earth-radii from the centre of the Earth

B.7 Carbon-monoxide.

(**a**) 48.37 pm from the Oxygen molecule

B.8 Three-particle system.

(**a**) $\mathbf{r} = 2\,\text{m}\,\mathbf{i} + 3\,\text{m}\,\mathbf{j}$ (**b**) By placing the particle at the center of mass of the system

B.9 Tetrahedron.

(**a**) $\mathbf{R} = (0, 0, 0)$ (**b**) $\mathbf{R} = (0, 0.4, 0.4)$

B.10 Cubic hole.

(**a**) $\mathbf{R} = -(L - d/2)\,(d/L)^3/\left(1 - (d/L)^3\right)\,\mathbf{i}$, where the origin is at the centre of the large cube and the small cube is cut out on the positive side of the x-axis

B.11 Triangle.

(**a**) $R_{CM} = (0, (2/3)a)$, where the origin is at the bottom centre.

B.12 Triangle.

(**a**) $R_{CM} = (0, (b/\sqrt{3}))$, where the origin is at the bottom centre

B.13 A piece of pie.

(**a**) $X = (2/3)\ (R\sin\theta)\,/\theta$, $Y = (2/3)\ (R(1 - \cos\theta))\,/\theta$

B.14 Person in a boat.
(**a**) 2.4 m in the opposite direction of John

B.15 Car on a train.
(**a**) 5 m in the opposite direction

Chapter 14

B.4 Flywheel position.
(**a**) $\omega = (c_1/t_1) + 2c_2\left(t/t_2^2\right)$ (**b**) $\alpha = \left(2c_2/t_2^2\right)$

B.5 Unbalanced wheel.
(**a**) $\omega = 2.5\cos(t/(2\,\mathrm{s}))$ rad/s (**b**) $\alpha = -1.25\sin(t/(2\,\mathrm{s}))$ rad/s^2

B.6 Earth and Sun.
(**a**) 1.99×10^{-7} rad/s (**b**) 7.27×10^{-5} rad/s

B.7 Engine.
(**a**) 6.98 rad/s^2 (**b**) 375

B.8 Spinning down.
(**a**) $\omega(t) = 10\,\mathrm{rad/s^2}\ t$ (**b**) $\theta(t) = 5\,\mathrm{rad/s^2} t$ (**c**) $\omega(t) = 30\,\mathrm{rad/s} - 0.1\,\mathrm{rad/s^2}\ t$ (**d**) $\theta(t) = 45\,\mathrm{rad} - 0.05\,\mathrm{rad/s^2}\ t^2$ (**e**) 300 s (**f**) 600 s

B.9 A slippery wheel.
(**a**) $\omega = \omega_0 \exp(-k_\omega t)$ (**b**) 23.0 s

B.10 Running the curve.
(**a**) $\omega = 0.20$ rad/s (**b**) $\alpha = 0$ (**c**) $a = 2$ m/s^2

B.11 Rotating Earth.
(**a**) $\omega_0 = 7.27\,10^{-5}$ rad/s (**b**) ω_0 (**c**) $v = \omega_0 R = 463.8$ m/s (**d**) ω_0 (**e**) $v = \omega_0(R\cos\alpha)$ (**f**) $\alpha = 0$ (**g**) $a = \frac{v^2}{R} = \omega_0^2 R = 0.034$ m/s^2 (**h**) $a = \omega_0^2\rho = \omega_0^2(R\cos\alpha) = 0.017$ m/s^2 directed in towards the rotational axis

B.12 Rolling wheel.
(**b**) 0 m/s (**c**) $2v$ (**d**) 0 m/s^2 along the surface and v^2/R normal to the surface toward the center of the wheel (**e**) 0 m/s^2 along the surface and v^2/R normal to the surface, toward the center of the wheel

Chapter 15

B.4 Three-particle system.
(**a**) $\mathbf{R} = (0, -a/3)$ (**b**) $I_{cm} = 6ma^2$ (**c**) $I_{0,z} = (6 + 1/9)ma^2$ (**d**) $I_{0,x} = 3ma^2$ (**e**) $I_{0,Y} = 2ma^2$

B.5 Compound system.
(**a**) $(1/12)mL^2 + (4/5)MR^2 + 2M(L/2)^2$ (**b**) $(4/5)MR^2$ (**c**) $(4/5)MR^2 + (1/12)mL^2$
$+ m(L/2)^2 + ML^2$

B.6 Water molecule.
(**a**) $I_{cm} = 1.92\,\mathrm{u}\,a^2$ (**b**) $I_O = 2\,\mathrm{u}\,a^2$

B.7 Compound system.
(**a**) $(4/5)MR^2 + 4MR^2$ (**b**) $\boldsymbol{\omega} = \sqrt{(5/6)\,(g/R)\,sin(\theta)}$

B.8 Atwood's fall machine.
(**a**) $v = \sqrt{(gh(m_1 - m_2))\,/\,(M + m_1 + m_2)}$
(**b**) $\boldsymbol{\omega} = (1/R)\sqrt{(g\,h(m_1 - m_2))\,/\,(M + m_1 + m_2)}$

B.9 Triangular pendulum.
(**a**) $I_O = 2mL^2$ (**c**) $\boldsymbol{\omega} = \left(\left(\sqrt{3}/2\right)(g/L)\right)^{1/2}$ (**d**) It continues with the same angular velocity around a center of mass that follows a parabolic path.

B.10 Spinning toy car.
(**c**) $\boldsymbol{\omega} = \omega_0 - \mu\,(g/Rc)\,t$ (**d**) $t = (\boldsymbol{\omega}_0 R)/(\mu g)\;1/\;(1/(2+c) + (1/c))$

B.11 Micro-electromechanical system.
(**a**) $X = L/2$, $Y = L/2$. (**c**) $I_y = ML^2/3$
(**h**) $\boldsymbol{\omega} = \sqrt{(15\,g\,sin\theta)/(11\,L) - (3\,\kappa\,\theta^2)/(22\,M\,L^2)}$ (**j**) $\theta = 10\,(M\,L\,g/\kappa)$

Chapter 16

B.5 Motion of rod during a collision-like process.
(**a**) $v_0 = -\sqrt{2gh}$, $\boldsymbol{\omega}_0 = 0$ (**c**) $\boldsymbol{\omega}_1 = -(3/2)\,(v_0/L)$ (**d**) $p_1 = (3/4)p_0$ (**e**) $\alpha = (3/2)(g/L)\cos(\theta) - (3\kappa)/(ML^2)\,\theta$ (**f**) $I_{O,z}\boldsymbol{\omega}_1^2 = \kappa\theta^2 - MgL\sin(\theta)$ (**g**) $\boldsymbol{\omega}_2 = -\boldsymbol{\omega}_1$ (**h**) $v_2 = (3/4)v_0$ (**k**) $y_4 = (9/16)\,h$

B.6 Collision between a rod and a block.
(**a**) $I_O = (1/3)\,M\,L^2$ (**b**) $E_{k,1} = (MgL)/2\,(\cos(\theta) - \cos(\theta_0))$
(**c**) $\boldsymbol{\omega}_0 = \sqrt{(3g/L)(1 - \cos(\theta_0))}$ (**g**) The rod stops completely, and the block gains the "velocity" of the rod. (**h**) $v_1 = (\boldsymbol{\omega}_0\,L)\,/\,(1 + 3\,(m/M))$

B.7 A model of two rods colliding.
(**b**) $v_1 = v_0/2$ (**c**) 0 (**d**) $v_1 = v_0/2$ (**e**) $\boldsymbol{\omega}_1 = -dv_0/\left(d^2 + (L^2/3)\right)\,\mathbf{k}$ (**f**) $K_0 - K_1 = (Mv_0^2/4)\,\left(1 - d^2/\left(d^2 + (L^2/3)\right)\right)$

B.9 Tarzan's swing.
(**a**) $v_{x1} = v_0$, $v_{y1} = \sqrt{2gh}$ (**b**) $I_{O,z} = ML^2/3$ (**d**) $y_3 = \left((1/2)I_{O,z}\boldsymbol{\omega}_2^2\right)/\,((m + (M/2))\,g)$ (**e**) The same height.

B.10 Rolling up a slope.
(**c**) $a_x = g\,(\mu\cos\theta - \sin\theta)$ (**d**) $v(t) = g\,(\mu\cos\theta - \sin\theta)\,t$ (**e**) $\alpha = fR/I$ (**f**) $\boldsymbol{\omega}(t) = \boldsymbol{\omega}_0 + (f\;R/I)\,t$ (**g**) $t = R\boldsymbol{\omega}_0/\,[(f/I) + g\,(\mu\cos\theta - \sin\theta)]$

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