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Hans Peters

# Game Theory

A Multi-Leveled Approach

Second Edition

 Springer

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ISSN 2192-4333                      ISSN 2192-4341 (electronic)  
Springer Texts in Business and Economics  
ISBN 978-3-662-46949-1            ISBN 978-3-662-46950-7 (eBook)  
DOI 10.1007/978-3-662-46950-7

Library of Congress Control Number: 2015941154

Springer Heidelberg New York Dordrecht London  
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*Voor Lenie, Nina en Remco*



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## Preface

This book is a compilation of much of the material I used for various game theory courses over the past decades. The first part, *Thinking Strategically*, is intended for undergraduate students in economics or business, but can also serve as an introduction for the subsequent parts of the book. The second and third parts go deeper into the various topics treated in the first part. These parts are intended for more mathematically oriented undergraduate students, or for graduate students in (for instance) economics. Part II is on noncooperative games and Part III on cooperative games. Part IV consists of a mathematical tools chapter, a chapter with review problems for Part I, and a chapter with hints and solutions to the problems of all chapters. Every chapter has a section with problems.

The material draws heavily on game theory texts developed by many others, often in collaboration. I mention in particular Jean Derks, Thijs Jansen, Andrés Perea, Ton Storcken, Frank Thuijsman, Stef Tijs, Dries Vermeulen, and Koos Vrieze. I am also seriously indebted to a large number of introductory, intermediate, and advanced texts and textbooks on game theory, and hope I have succeeded in giving sufficient credits to the authors of these works in all due places.

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### About the Second Edition

In this second edition, I have corrected mistakes, omissions, and typos from the first edition, and tried to improve the exposition throughout the book. I have added extra problems to some chapters, and also a chapter with review problems for Part I. In Chap. 6, I have added a few sections on auctions with incomplete information. With only few exceptions, the references to the literature are now collected in Notes sections, which conclude every chapter in the book.

This second edition has benefitted tremendously from extensive comments of Piotr Frackiewicz and Peter Wakker. The list of people from who I received comments also includes Krzysztof Apt, Maikel Bosschaert, Yukihiro Funaki, Ali Ihsan Ozkes, Mahmut Parlar, Thijs Ruijgrok, Steffen Sagave, Judith Timmer, Mark Voorneveld, and others.

## How to Use This Book

Part I of the book is intended, firstly, for undergraduate students in economics and business and, secondly, as preparation and background for Parts II–IV. Part I is preceded by Chap. 1, which is a general introduction to game theory by means of examples. The first chapter of Part I, Chap. 2 of the book, is on zero-sum games. This chapter is included, not only for historical reasons—the minimax theorem of von Neumann (1928) was one of the first formal results in game theory—but also since zero-sum games (all parlor games) require basic, strictly competitive, game-theoretic thinking. The heart of Part I consists of Chaps. 3–6 on noncooperative games and applications, and Chap. 9 as a basic introduction to cooperative games. These chapters can serve as a basics course in game theory. Chapters 7 and 8 on repeated games and evolutionary games can serve as extra material, as well as Chap. 10 on cooperative game models and Chap. 11, which is an introduction to the related area of social choice theory.

Although this book can be used for self-study, it is not intended to replace the teacher. Part I is meant for students who are knowledgeable in basic calculus, and does not try to avoid the use of mathematics on that basic level. Moreover, (almost) all basic game theory models are described in a formally precise manner, although I am aware that some students may have a blind spot for mathematical notation that goes beyond simple formulas for functions and equations. This formal presentation is included especially since many students have always been asking questions about it: leaving it out may lead to confusion and ambiguities. On the other hand, a teacher may decide to drop these more formal parts and go directly to the examples of concretely specified games. For example, in Chap. 5, the game theory teacher may decide to skip the formal Sect. 5.1 and go directly to the worked out examples of games with incomplete information—and perhaps later return to Sect. 5.1.

Parts II–IV require more mathematical sophistication and are intended for graduate students in economics, or for an elective game theory course for students in (applied) mathematics. In my experience, it works well to couple the material in these parts to related chapters in Part I. In particular, one can combine Chaps. 2 and 12 on zero-sum games, Chaps. 3 and 13 on finite games, Chaps. 4, 5, and 14 on games with incomplete information and games in extensive form, and Chaps. 8 and 15 on evolutionary games. For cooperative game theory, one can combine Chap. 9 with Part III.

Each chapter contains a problems section. Moreover, Chap. 23 contains review problems for Part I. Hints, answers and solutions are provided at the end of the book

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in Chap. 24. For a complete set of solutions for teachers, as well as any comments, please contact me by email.<sup>1</sup>

Maastricht, The Netherlands  
January 2015

Hans Peters

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## Reference

von Neumann, J. (1928). Zur Theorie der Gesellschaftsspiele. *Mathematische Annalen*, 100, 295–320.

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