Appendix A
Plane and Solid Angles

A.1 Plane Angles

The angle θ between two intersecting lines is shown in Fig. A.1. It is measured by drawing a circle centered on the vertex or point of intersection. The arc length s on that part of the circle contained between the lines measures the angle. In daily work, the angle is marked off in degrees.

In some cases, there are advantages to measuring the angle in *radians*. This is particularly true when trigonometric functions have to be differentiated or integrated. The angle in radians is defined by

$$\theta = \frac{s}{r}. \quad \text{(A.1)}$$

Since the circumference of a circle is $2\pi r$, the angle corresponding to a complete rotation of 360° is $2\pi r/r = 2\pi$. Other equivalences are

Degrees	Radians
360	2π
180	π
57.2958	1
1	0.01745

(A.2)

Since the angle in radians is the ratio of two distances, it is dimensionless. Nevertheless, it is sometimes useful to specify that something is measured in radians to avoid confusion.

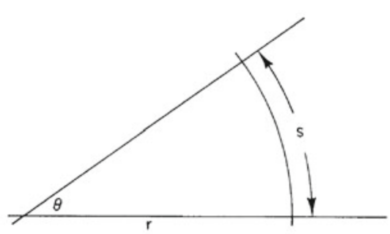

Fig. A.1 A plane angle θ is measured by the arc length s on a circle of radius r centered at the vertex of the lines defining the angle

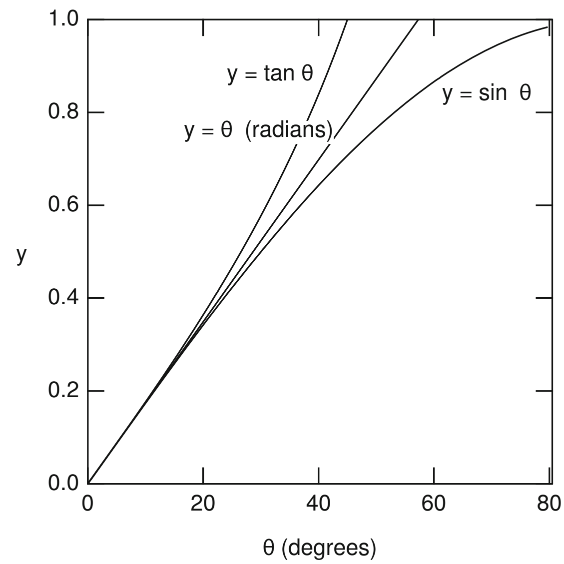

Fig. A.2 Comparison of $y = \tan\theta$, $y = \theta$ (radians), and $y = \sin\theta$

One of the advantages of radian measure can be seen in Fig. A.2. The functions $\sin\theta$, $\tan\theta$, and θ in radians are plotted vs. angle for angles less than 80°. For angles less than 15°, $y = \theta$ is a good approximation to both $y = \tan\theta$ (2.3 % error at 15°) and $y = \sin\theta$ (1.2 % error at 15°).

A.2 Solid Angles

A plane angle measures the diverging of two lines in two dimensions. *Solid angles* measure the diverging of a cone of lines in three dimensions. Figure A.3 shows a series of rays diverging from a point and forming a cone. The solid angle Ω is measured by constructing a sphere of radius r centered at the vertex and taking the ratio of the surface area S on the

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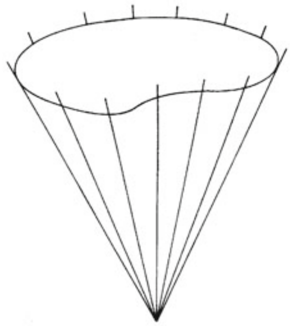

Fig. A.3 A cone of rays in three dimensions

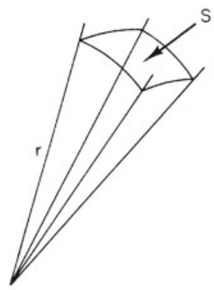

Fig. A.4 The solid angle of this cone is $\Omega = S/r^2$. S is the surface area on a sphere of radius r centered at the vertex

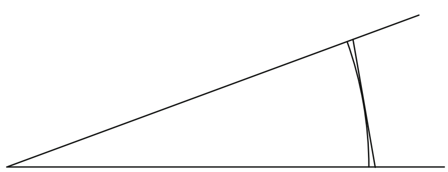

Fig. A.5 For small angles, the arc length is very nearly equal to the length of the tangent to the circle

sphere enclosed by the cone to r^2:

$$\Omega = \frac{S}{r^2}. \tag{A.3}$$

This is shown in Fig. A.4 for a cone consisting of the planes defined by adjacent pairs of the four rays shown. The unit of solid angle is the steradian (sr). A complete sphere subtends a solid angle of 4π steradians, since the surface area of a sphere is $4\pi r^2$.

When the included angle in the cone is small, the difference between the surface area of a plane tangent to the sphere and the sphere itself is small. (This is difficult to draw in three dimensions. Imagine that Fig. A.5 represents a slice through a cone; the difference in length between the circular arc and the tangent to it is small.) This approximation is often useful. A 3×5-in. card at a distance of 6 ft (72 in.) subtends a solid angle which is approximately

$$\frac{3 \times 5}{72^2} = 2.9 \times 10^{-3}\ \text{sr}.$$

It is not necessary to calculate the surface area on a sphere of 72-in. radius.

Problems

Problem 1. Convert 0.1 radians to degrees. Convert 7.5° to radians.

Problem 2. Use the fact that $\sin\theta \approx \theta \approx \tan\theta$ to estimate the sine and tangent of 3°. Look up the values in a table and see how accurate the approximation is.

Problem 3. What is the solid angle subtended by the pupil of the eye (radius = 3 mm) at a source of light 30 m away?

Problem 4. Figure A.2 suggests that $y = \theta$ is a better approximation to $\sin\theta$ than to $\tan\theta$ and that $y = \theta$ overestimates $\sin\theta$ and underestimates $\tan\theta$. Calculate (See Appendix D) or look up the Taylor expansions of $\sin\theta$ and $\tan\theta$ and use the first two nonzero terms in each expansion to verify this behavior.

Problem 5. What is the solid angle subtended by the "cap" of a sphere from the sphere center, where the "cap" is defined using spherical coordinates (Appendix L) as the surface of the sphere between $\theta = 0$ and $\theta = 30°$. Hint: In spherical coordinates an element of surface area on a sphere is $dS = r^2 \sin\theta d\theta d\phi$.

Appendix B
Vectors; Displacement, Velocity, and Acceleration

B.1 Vectors and Vector Addition

A *displacement* describes how to get from one point to another. A displacement has a magnitude (how far point 2 is from point 1 in Fig. B.1) and a direction (the direction one has to go from point 1 to get to point 2). The displacement of point 2 from point 1 is labeled **A**. Displacements can be added: displacement **B** from point 2 puts an object at point 3. The displacement from point 1 to point 3 is **C** and is the sum of displacements **A** and **B**:

$$\mathbf{C} = \mathbf{A} + \mathbf{B}. \tag{B.1}$$

A displacement is a special example of a more general quantity called a *vector*. One often finds a vector defined as a quantity having a magnitude and a direction. However, the complete definition of a vector also includes the requirement that vectors add like displacements. The rule for adding two vectors is to place the tail of the second vector at the head of the first; the sum is the vector from the tail of the first to the head of the second.

A displacement is a change of position so far in such a direction. It is independent of the starting point. To know where an object is, it is necessary to specify the starting point as well as its displacement from that point.

Displacements can be added in any order. In Fig. B.2, either of the vectors **A** represents the same displacement.

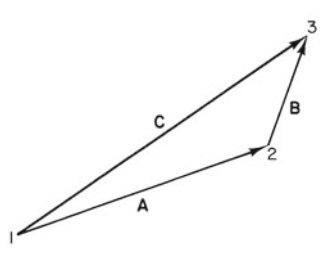

Fig. B.1 Displacement **C** is equivalent to displacement **A** followed by displacement **B**: $\mathbf{C} = \mathbf{A} + \mathbf{B}$

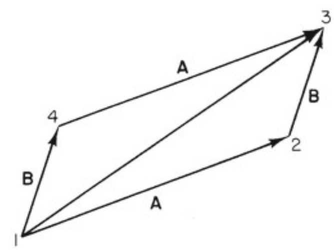

Fig. B.2 Vectors **A** and **B** can be added in either order

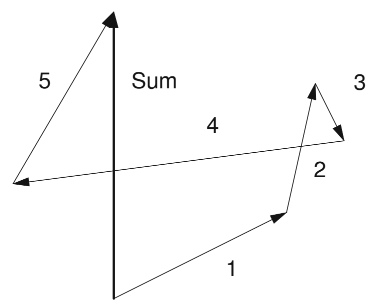

Fig. B.3 Addition of several vectors

Displacement **B** can first be made from point 1 to point 4, followed by displacement **A** from 4 to 3. The sum is still **C**:

$$\mathbf{C} = \mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}. \tag{B.2}$$

The sum of several vectors can be obtained by first adding two of them, then adding the third to that sum, and so forth. This is equivalent to placing the tail of each vector at the head of the previous one, as shown in Fig. B.3. The sum then goes from the tail of the first vector to the head of the last.

The negative of vector **A** is that vector which, added to **A**, yields zero:

$$\mathbf{A} + (-\mathbf{A}) = 0. \tag{B.3}$$

It has the same magnitude as **A** and points in the opposite direction.

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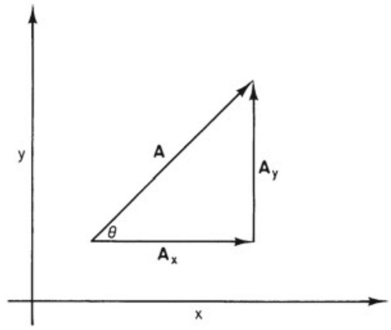

Fig. B.4 Vector **A** has components $\mathbf{A}_x$ and $\mathbf{A}_y$

Multiplying a vector **A** by a *scalar* (a number with no associated direction) multiplies the magnitude of vector **A** by that number and leaves its direction unchanged.

B.2 Components of Vectors

Consider a vector in a plane. If we set up two perpendicular axes, we can regard vector **A** as being the sum of vectors parallel to each of these axes. These vectors, $\mathbf{A}_x$ and $\mathbf{A}_y$ in Fig. B.4, are called the *components* of **A** along each axis[1]. If vector **A** makes an angle θ with the x axis and its magnitude is A, then the magnitudes of the components are

$$\begin{aligned} A_x &= A\cos\theta, \\ A_y &= A\sin\theta. \end{aligned} \tag{B.4}$$

The sum of the squares of the components is $A_x^2 + A_y^2 = A^2\cos^2\theta + A^2\sin^2\theta = A^2(\sin^2\theta + \cos^2\theta)$. Since, by Pythagoras' theorem, this must be A^2, we obtain the trigonometric identity

$$\cos^2\theta + \sin^2\theta = 1. \tag{B.5}$$

In three dimensions, $\mathbf{A} = \mathbf{A}_x + \mathbf{A}_y + \mathbf{A}_z$. The magnitudes can again be related using Pythagoras' theorem, as shown in Fig. B.5. From triangle OPQ, $A_{xy}^2 = A_x^2 + A_y^2$. From triangle OQR,

$$A^2 = A_{xy}^2 + A_z^2 = A_x^2 + A_y^2 + A_z^2. \tag{B.6}$$

In our notation, $\mathbf{A}_x$ means a vector pointing in the x direction, while A_x is the magnitude of that vector. It can become difficult to keep the distinction straight. Therefore, it is customary to write $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ to mean vectors of unit length pointing in the x, y, and z directions. (In some books, the unit vectors are denoted by $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ instead of $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$.) With this notation, instead of $\mathbf{A}_x$, one would always write $A_x\hat{\mathbf{x}}$.

[1] Some texts define the component to be a scalar, the magnitude of the component defined here.

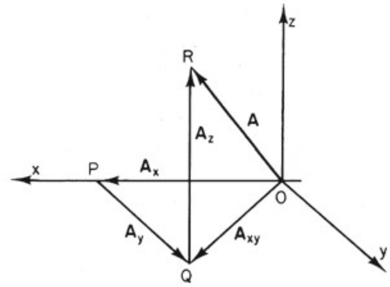

Fig. B.5 Addition of components in three dimensions

The addition of vectors is often made easier by using components. The sum $\mathbf{A} + \mathbf{B} = \mathbf{C}$ can be written as

$$\begin{aligned} A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}} + B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}} + B_z\hat{\mathbf{z}} \\ = C_x\hat{\mathbf{x}} + C_y\hat{\mathbf{y}} + C_z\hat{\mathbf{z}}. \end{aligned}$$

Like components can be grouped to give

$$\begin{aligned} (A_x + B_x)\hat{\mathbf{x}} + (A_y + B_y)\hat{\mathbf{y}} + (A_z + B_z)\hat{\mathbf{z}} \\ = C_x\hat{\mathbf{x}} + C_y\hat{\mathbf{y}} + C_z\hat{\mathbf{z}}. \end{aligned}$$

Therefore, the magnitudes of the components can be added separately:

$$\begin{aligned} C_x &= A_x + B_x, \\ C_y &= A_y + B_y, \\ C_z &= A_z + B_z. \end{aligned} \tag{B.7}$$

B.3 Position, Velocity, and Acceleration

The *position* of an object at time t is defined by specifying its displacement from an agreed-upon origin:

$$\mathbf{R}(t) = x(t)\hat{\mathbf{x}} + y(t)\hat{\mathbf{y}} + z(t)\hat{\mathbf{z}}.$$

The *average velocity* $\mathbf{v}_{\rm av}(t_1, t_2)$ between times t_1 and t_2 is defined to be

$$\mathbf{v}_{\rm av}(t_1, t_2) = \frac{\mathbf{R}(t_2) - \mathbf{R}(t_1)}{t_2 - t_1}.$$

This can be written in terms of the components as

$$\mathbf{v}_{\rm av} = \left(\frac{x(t_2) - x(t_1)}{t_2 - t_1}\right)\hat{\mathbf{x}} + \left(\frac{y(t_2) - y(t_1)}{t_2 - t_1}\right)\hat{\mathbf{y}} + \left(\frac{z(t_2) - z(t_1)}{t_2 - t_1}\right)\hat{\mathbf{z}}.$$

The *instantaneous velocity* is

$$\mathbf{v}(t) = \frac{d\mathbf{R}}{dt} = \frac{dx}{dt}\hat{\mathbf{x}} + \frac{dy}{dt}\hat{\mathbf{y}} + \frac{dz}{dt}\hat{\mathbf{z}}$$

$$= v_x(t)\hat{\mathbf{x}} + v_y(t)\hat{\mathbf{y}} + v_z(t)\hat{\mathbf{z}}. \tag{B.8}$$

The x component of the velocity tells how rapidly the x component of the position is changing.

The *acceleration* is the rate of change of the velocity with time. The instantaneous acceleration is

$$\mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = \frac{dv_x}{dt}\hat{\mathbf{x}} + \frac{dv_y}{dt}\hat{\mathbf{y}} + \frac{dv_z}{dt}\hat{\mathbf{z}}.$$

Problems

Problem 1. At $t = 0$, the position of an object is given by $\mathbf{R} = 10\hat{\mathbf{x}} + 5\hat{\mathbf{y}}$, where R is in meters. At $t = 3$ s, the position is $\mathbf{R} = 16\hat{\mathbf{x}} - 10\hat{\mathbf{y}}$. What was the average velocity between $t = 0$ and 3 s?

Problem 2. The position of an object as a function of time is $\mathbf{R}(t) = (20 + 4t)\hat{\mathbf{x}} + (10 + 5t - 49t^2)\hat{\mathbf{y}}$. Determine the instantaneous velocity and acceleration as functions of time.

Problem 3. The electric field $\mathbf{E}$ is a vector (see Chap. 6). $\mathbf{E}_1$ has a magnitude of 30 V m^{-1} and is directed along the y axis. $\mathbf{E}_2$ has a magnitude of 15 V m^{-1} and is directed at an angle of $+30^\circ$ from the x axis. Calculate $\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2$. Express your answer in two ways: give the magnitude and direction of $\mathbf{E}$, and give E_x and E_y.

Appendix C
Properties of Exponents and Logarithms

In the expression a^m, a is called the *base* and m is called the *exponent*. Since $a^2 = a \times a$, $a^3 = a \times a \times a$, and

$$a^m = \underbrace{(a \times a \times a \times \cdots \times a)}_{\text{m times}},$$

it is easy to show that

$$a^m a^n = \underbrace{(a \times a \times a \times \cdots \times a)}_{\text{m times}}\underbrace{(a \times a \times a \times \cdots \times a)}_{\text{n times}},$$

$$a^m a^n = a^{m+n}. \tag{C.1}$$

If $m > n$, the same technique can be used to show that

$$\frac{a^m}{a^n} = a^{m-n}. \tag{C.2}$$

If $m = n$, this gives

$$1 = \frac{a^m}{a^m} = a^{m-m} = a^0,$$

$$a^0 = 1. \tag{C.3}$$

The rules also work for $m < n$ and for negative exponents. For example,

$$(a^{-n})(a^n) = 1$$

so

$$a^{-n} = \frac{1}{a^n}. \tag{C.4}$$

Finally,

$$\left(a^m\right)^n = \underbrace{(a^m \times a^m \times a^m \times \cdots \times a^m)}_{\text{n times}},$$

$$(a^m)^n = a^{mn}. \tag{C.5}$$

If $y = a^x$, then by definition, x is the *logarithm* of y to the base a: $x = \log_a(y)$. If the base is 10, since $100 = 10^2$, $2 = \log_{10}(100)$. Similarly, $3 = \log_{10}(1000)$, $4 = \log_{10}(10,000)$, and so forth.

The most useful property of logarithms can be derived by letting

$$y = a^m,$$

$$z = a^n,$$

$$w = a^{m+n},$$

so that

$$m = \log_a y,$$

$$n = \log_a z,$$

$$m + n = \log_a w.$$

Then, since $a^{m+n} = a^m a^n$,

$$w = yz,$$

$$\log_a(yz) = \log_a w = \log_a y + \log_a z. \tag{C.6}$$

This result can be used to show that

$$\begin{aligned} \log(y^m) &= \log(y \times y \times y \times \cdots \times y) \\ &= \log(y) + \log(y) + \log(y) + \cdots + \log(y), \\ \log(y^m) &= m \log y. \end{aligned} \tag{C.7}$$

All logarithms in this book, unless labeled with a specific base, are to base e (see Chap. 2). These are the so-called natural logarithms. We will denote the *natural logarithm* by ln, using $\log_{10}$ when we want logarithms to the base 10.

Problems

Problem 1. What is $\log_2(8)$?
Problem 2. If $\log_{10}(2) = 0.3$, what is $\log_{10}(200)$? $\log_{10}(2 \times 10^5)$?
Problem 3. What is $\log_{10}(\sqrt{10})$?

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Appendix D
Taylor's Series

Consider the function $y(x)$ shown in Fig. D.1. The value of the function at x_1, $y_1 = y(x_1)$, is known. We wish to estimate $y(x_1 + \Delta x)$.

The simplest estimate, labeled approximation 0 in Fig. D.1, is to assume that y does not change: $y(x_1 + \Delta x) \approx y(x_1)$. A better estimate can be obtained if we assume that y changes everywhere at the same rate it does at x_1. Approximation 1 is

$$y(x_1 + \Delta x) \approx y(x_1) + \left.\frac{dy}{dx}\right|_{x_1} \Delta x.$$

The derivative is evaluated at point x_1.

An even better estimate is shown in Fig. D.2. Instead of fitting the curve by the straight line that has the proper first derivative at x_1, we fit it by a parabola that matches both the first and second derivatives. The approximation is

$$y(x_1 + \Delta x) \approx y(x_1) + \left.\frac{dy}{dx}\right|_{x_1} \Delta x + \frac{1}{2}\left.\frac{d^2y}{dx^2}\right|_{x_1} (\Delta x)^2.$$

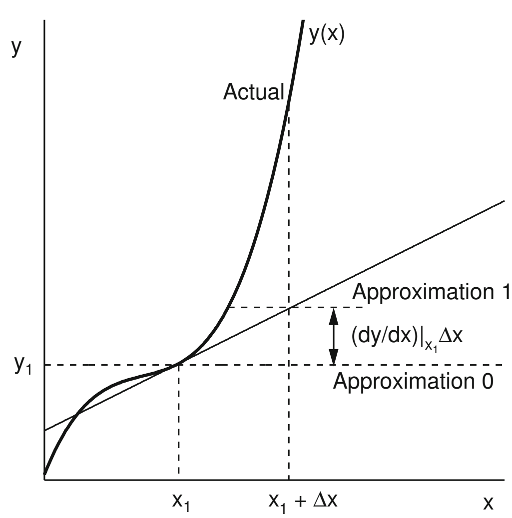

Fig. D.1 The zeroth-order and first-order approximations to $y(x)$

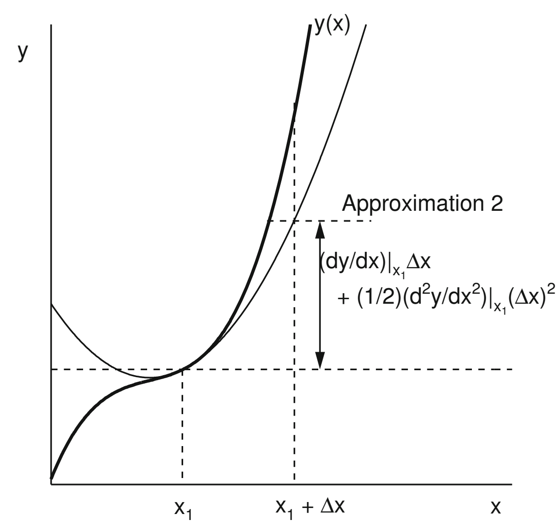

Fig. D.2 The second-order approximation fits $y(x)$ with a parabola

That this is the best approximation can be derived in the following way. Suppose the desired approximation is more general and uses terms up to $(\Delta x)^n = (x - x_1)^n$:

$$y_{\text{approx}} = A_0 + A_1(x - x_1) + A_2(x - x_1)^2 + \cdots + A_n(x - x_1)^n. \tag{D.1}$$

The constants $A_0, A_1, \ldots, A_n$ are determined by making the value of y_{approx} and its first n derivatives agree with the value of y and its first n derivatives at $x = x_1$. When $x = x_1$, all terms with $x - x_1$ in y_{approx} vanish, so that

$$y_{\text{approx}}(x_1) = A_0.$$

The first derivative of y_{approx} is

$$\frac{d(y_{\text{approx}})}{dx} = A_1 + 2A_2(x - x_1) + 3A_3(x - x_1)^2 + \cdots + nA_n(x - x_1)^{n-1}.$$

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Table D.1 $y = e^{2x}$ and its derivatives

Function or derivative	Value at $x_1 = 0$
$y = e^{2x}$	1
$\frac{dy}{dx} = 2e^{2x}$	2
$\frac{d^2y}{dx^2} = 4e^{2x}$	4
$\frac{d^3y}{dx^3} = 8e^{2x}$	8

Table D.2 Values of y and successive approximations

x	$y = e^{2x}$	$1+2x$	$1+2x+2x^2$	$1+2x +2x^2+\frac{4}{3}x^3$
−2	0.0183	−3.0	5.0	−5.67
−1.5	0.0498	−2.0	2.5	−2.0
−1	0.1353	−1.0	1.0	−0.33
−0.4	0.4493	0.2000	0.5200	0.4347
−0.2	0.6703	0.6000	0.6800	0.6693
−0.1	0.8187	0.8000	0.8200	0.8187
0	1.0000	1.0000	1.0000	1.0000
0.1	1.2214	1.2000	1.2200	1.2213
0.2	1.4918	1.4000	1.4800	1.4907
0.4	2.2255	1.8000	2.1200	2.2053
1.0	7.389	3.0000	5.0000	6.33
2.0	54.60	5.0	13.0	23.67

The second derivative is

$$2A_2 + 3 \times 2A_3(x - x_1) + \cdots + n(n-1)A_n(x - x_1)^{n-2},$$

and the nth derivative is

$$n(n-1)(n-2)\cdots 2A_n = n!A_n.$$

Evaluating these at $x = x_1$ gives

$$\left.\frac{d(y_{\text{approx}})}{dx}\right|_{x_1} = A_1,$$

$$\left.\frac{d^2(y_{\text{approx}})}{dx^2}\right|_{x_1} = 2 \times 1 \times A_2,$$

$$\left.\frac{d^3(y_{\text{approx}})}{dx^3}\right|_{x_1} = 3 \times 2 \times 1 \times A_3,$$

$$\left.\frac{d^n(y_{\text{approx}})}{dx^n}\right|_{x_1} = n!A_n.$$

Combining these expressions for A_n with Eq. D.1, we get

$$y(x_1 + \Delta x) \approx y(x_1) + \sum_{n=1}^{N} \frac{1}{n!} \left.\frac{d^n y}{dx^n}\right|_{x_1} (\Delta x)^n. \quad \text{(D.2)}$$

Tables D.1 and D.2 and Figs. D.3 and D.4 show how the Taylor's series approximation gets better over a larger and

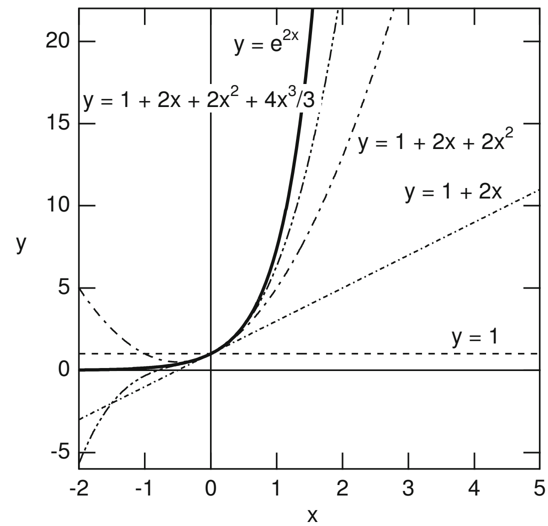

Fig. D.3 The function $y = e^{2x}$ with Taylor's series expansions about $x = 0$ of degree 0, 1, 2, and 3

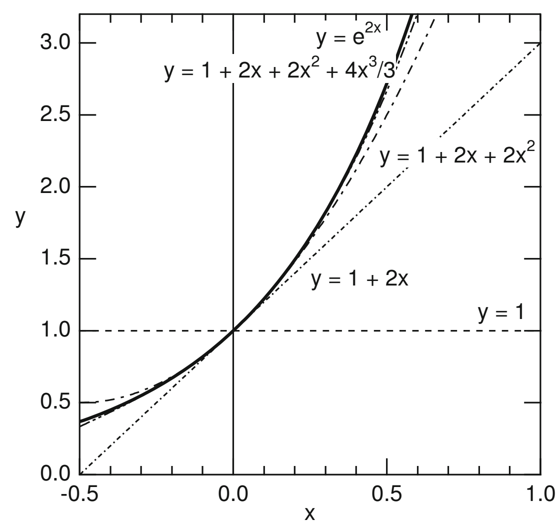

Fig. D.4 An enlargement of Fig. D.3 near $x = 0$

larger region about x_1 as more terms are added. The function being approximated is $y = e^{2x}$. The derivatives are given in Table D.1. The expansion is made about $x_1 = 0$.

Finally, the Taylor's series expansion for $y = e^x$ about $x = 0$ is often useful. Since all derivatives of e^x are e^x, the value of y and each derivative at $x = 0$ is 1. The series is

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots = \sum_{m=0}^{\infty} \frac{x^m}{m!}. \quad \text{(D.3)}$$

(Note that $0! = 1$ by definition.)

Problems

Problem 1. Make a Taylor's series expansion of $y = a + bx + cx^2$ about $x = 0$. Show that the expansion exactly reproduces the function.

Problem 2. Repeat the previous problem, making the expansion about $x = 1$.

Problem 3. (a) Make a Taylor's series expansion of the cosine function about $x = 0$. Remember that $d(\sin x)/dx = \cos x$ and $d(\cos x)/dx = -\sin x$.

(b) Make a Taylor's series expansion of the sine function.

Problem 4. The "sinc" function is defined as $\sin x/x$. Make a Taylor's series expansion of the sinc function about $x = 0$. Hint: first make a Taylor series expansion of $\sin x$ and then divide by x.

Problem 5. Derive a Taylor's series of $y = 1/(1 - x)$ about $x = 0$. Plot $y(x)$ vs x including approximation 0, approximation 1, and approximation 2 as in Figs. D.1 and D.2.

Problem 6. Derive a Taylor's series of $y(x) = \ln(1 + x)$ about $x = 0$. Plot $y(x)$ vs x, including approximation 0, approximation 1, and approximation 2, as in Figs. D.1 and D.2.

Appendix E
Some Integrals of Sines and Cosines

The average of a function of x with period T is defined to be

$$\langle f \rangle = \frac{1}{T} \int_{x'}^{x'+T} f(x)\, dx. \tag{E.1}$$

The sine function is plotted in Fig. E.1a. The integral over a period is zero, and its average value is zero. The area above the axis is equal to the area below the axis. Figure E.1b shows a plot of $\sin^2 x$. Since $\sin x$ varies between -1 and $+1$, $\sin^2 x$ varies between 0 (when $\sin x = 0$) and $+1$ (when $\sin x = \pm 1$). Its average value, from inspection of Fig. E.1b is $\frac{1}{2}$. If you do not want to trust the drawing to convince yourself of this, recall the identity $\sin^2\theta + \cos^2\theta = 1$. Since the sine function and the cosine function look the same, but are just shifted along the axis, their squares must also look similar. Therefore, $\sin^2\theta$ and $\cos^2\theta$ must have the same average. But if their sum is always 1, the sum of their averages must be 1. If the two averages are the same, then each must be $\frac{1}{2}$.

These same results could have been obtained analytically by using the trigonometric identity

$$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x. \tag{E.2}$$

The integrals of $\sin x$ and $\cos x$ are

$$\begin{aligned} \int \sin ax\, dx &= -\frac{1}{a}\cos ax, \\ \int \cos ax\, dx &= \frac{1}{a}\sin ax. \end{aligned} \tag{E.3}$$

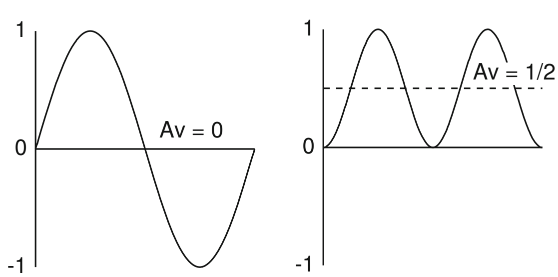

Fig. E.1 **a** Plot of $y = \sin x$. **b** Plot of $y = \sin^2 x$

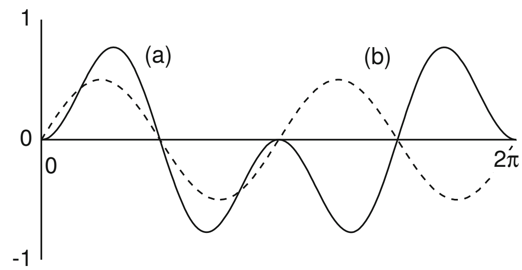

Fig. E.2 Plot of one period of **a** $y = \sin x \sin 2x$; **b** $y = \sin x \cos x$

These could be used to show that the average value of $\sin x$ or $\cos x$ is zero. Then Eq. E.2 could be used to show that the average of $\sin^2 x$ is $\frac{1}{2}$.

The integral of $\sin^2 x$ over a period is its average value times the length of the period:

$$\int_0^T \sin^2 x\, dx = \int_0^T \cos^2 x\, dx = \frac{T}{2}. \tag{E.4}$$

We will also encounter integrals like

$$\begin{aligned} &\int_0^T \sin mx\ \sin nx\, dx, \quad m \neq n, \\ &\int_0^T \cos mx\ \cos nx\, dx, \quad m \neq n, \\ &\int_0^T \cos mx\ \sin nx\, dx, \quad m = n, \quad m \neq n. \end{aligned} \tag{E.5}$$

All these integrals are zero. This can be shown using integral tables. Or, you can see why the integrals vanish by considering the specific examples plotted in Fig. E.2. Each integrand has equal positive and negative contributions to the total integral.

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Problems

Problem 1. Plot the following functions over the range 0 to 2π as in Fig. E.2, and show by inspection that the negative and positive areas cancel, giving an integral of zero: $\cos\theta \sin 2\theta$, $\sin 2\theta \sin 3\theta$, $\cos\theta \cos 2\theta$, and $\cos 2\theta \sin 2\theta$.

Problem 2. Use the trigonometric relationship

$$\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)],$$

$$\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)],$$

$$\sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)],$$

to verify that all of the integrals in Eq. E.5 are zero.

Appendix F
Linear Differential Equations with Constant Coefficients

The equation

$$\frac{dy}{dt} + by = a \tag{F.1}$$

is called a *linear differential equation* because each term involves only y or its derivatives (not $y(dy/dt)$ or $(dy/dt)^2$, etc.). A more general equation of this kind has the form

$$\frac{d^N y}{dt^N} + b_{N-1}\frac{d^{N-1}y}{dt^{N-1}} + \cdots + b_1\frac{dy}{dt} + b_0 y = f(t). \tag{F.2}$$

The highest derivative is the Nth derivative, so the equation is of order N. It has been written in standard form by dividing through by any b_N that was there originally, so that the coefficient of the highest term is one. If all the bs are constants, this is a linear differential equation with constant coefficients. The right-hand side may be a function of the independent variable t, but *not* of y. If $f(t) = 0$, it is a *homogeneous* equation; if $f(t)$ is not zero, it is an *inhomogeneous* equation.

Consider first the homogeneous equation

$$\frac{d^N y}{dt^N} + b_{N-1}\frac{d^{N-1}y}{dt^{N-1}} + \cdots + b_1\frac{dy}{dt} + b_0 y = 0. \tag{F.3}$$

The exponential e^{st} (where s is a constant) has the property that $d(e^{st})/dt = se^{st}$, $d^2(e^{st})/dt^2 = s^2e^{st}$, $d^n(e^{st})/dt^n = s^n e^{st}$. The function $y = Ae^{st}$ satisfies Eq. F.3 for any value of A and certain values of s. The equation becomes

$$A\left(s^N e^{st} + b_{N-1}s^{N-1}e^{st} + \cdots + b_1 s e^{st} + b_0 e^{st}\right) = 0,$$

$$A\left(s^N + b_{N-1}s^{N-1} + \cdots + b_1 s + b_0\right)e^{st} = 0.$$

This equation is satisfied if the polynomial in parentheses is equal to zero. The equation

$$s^N + b_{N-1}s^{N-1} + \cdots + b_1 s + b_0 = 0 \tag{F.4}$$

is called the *characteristic equation* of this differential equation. It can be written in a much more compact form using summation notation:

$$\sum_{n=0}^{N} b_n s^n = 0, \tag{F.5}$$

with $b_N = 1$.

For Eq. F.1, the characteristic equation is $s + b = 0$ or $s = -b$, and a solution to the homogeneous equation is $y = Ae^{-bt}$.

If the characteristic equation is a polynomial, it can have up to N roots. For each distinct root s_n, $y = A_n e^{s_n t}$ is a solution to the differential equation. (The question of solutions when there are not N distinct roots will be taken up below.) This is still not the solution to the inhomogeneous equation. However, one can prove[1] that the most general solution to the inhomogeneous equation is the sum of the homogeneous solution,

$$y = \sum_{n=1}^{N} A_n e^{s_n t},$$

and *any* solution to the inhomogeneous equation. The values of the arbitrary constants A_n are picked to satisfy some other conditions that are imposed on the problem. If we can guess the solution to the inhomogeneous equation, that is fine. However we get it, we need only one such solution to the inhomogeneous equation. We will not prove this assertion, but we will apply it to the first- and second-order equations and see how it works.

[1] See any calculus text.

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F.1 First-Order Equation

The homogeneous equation corresponding to Eq. F.1 has solution $y = Ae^{-bt}$. There is one solution to the inhomogeneous equation that is particularly easy to write down: when y is constant, with the value $y = a/b$, the time derivative vanishes and the inhomogeneous equation is satisfied. The most general solution is therefore of the form

$$y = Ae^{-bt} + \frac{a}{b}.$$

If the initial condition is $y(0) = 0$, then A can be determined from $0 = Ae^{-b0} + a/b$. Since $e^0 = 1$, this gives $A = -a/b$. Therefore,

$$y = \frac{a}{b}\left(1 - e^{-bt}\right). \tag{F.6}$$

A physical example of this is given in Sect. 2.8.

F.2 Second-Order Equation

The second-order equation

$$\frac{d^2y}{dt^2} + b_1\frac{dy}{dt} + b_0 y = 0 \tag{F.7}$$

has a characteristic equation $s^2 + b_1 s + b_0 = 0$ with roots

$$s = \frac{-b_1 \pm \sqrt{b_1^2 - 4b_0}}{2}. \tag{F.8}$$

This equation may have zero, one, or two solutions.

If it has two solutions s_1 and s_2, then the general solution of the homogeneous equation is $y = A_1 e^{s_1 t} + A_2 e^{s_2 t}$.

If $b_1^2 - 4b_0$ is negative, there is no solution to the equation for a real value of s. However, a solution of the form $y = Ae^{-\alpha t}\sin(\omega t + \phi)$ will satisfy the equation. This can be seen by direct substitution. Differentiating this twice shows that

$$\frac{dy}{dt} = -\alpha Ae^{-\alpha t}\sin(\omega t + \phi) + \omega Ae^{-\alpha t}\cos(\omega t + \phi),$$

$$\frac{d^2y}{dt^2} = \alpha^2 Ae^{-\alpha t}\sin(\omega t + \phi) - 2\alpha\omega Ae^{-\alpha t}\cos(\omega t + \phi) - \omega^2 Ae^{-\alpha t}\sin(\omega t + \phi).$$

If these derivatives are substituted in Eq. F.7, one gets the following results. The terms are written in two columns. One column contains the coefficients of terms with $\sin(\omega t + \phi)$, and the other column contains the coefficients of terms with $\cos(\omega t + \phi)$. The rows are labeled on the left by which term of the differential equation they came from.

Term	Coefficients	
	$\sin(\omega t + \phi)$	$\cos(\omega t + \phi)$
d^2y/dt^2	$\alpha^2 - \omega^2$	$-2\alpha\omega$
$b_1(dy/dt)$	$-b_1\alpha$	$b_1\omega$
$b_0 y$	b_0	0

The only way that the equation can be satisfied for all times is if the coefficient of the $\sin(\omega t + \phi)$ term and the coefficient of the $\cos(\omega t + \phi)$ term separately are equal to zero. This means that we have two equations that must be satisfied (call $b_0 = \omega_0^2$):

$$2\alpha\omega = b_1\omega,$$

$$\alpha^2 - \omega^2 - b_1\alpha + \omega_0^2 = 0.$$

From the first equation $2\alpha = b_1$, while from this and the second, $\alpha^2 - \omega^2 - 2\alpha^2 + \omega_0^2 = 0$, or $\omega^2 = \omega_0^2 - \alpha^2$. Thus, the solution to the equation

$$\frac{d^2y}{dt^2} + 2\alpha\frac{dy}{dt} + \omega_0^2 y = 0 \tag{F.9}$$

is

$$y = Ae^{-\alpha t}\sin(\omega t + \phi) \tag{F.10a}$$

where

$$\omega^2 = \omega_0^2 - \alpha^2, \qquad \alpha < \omega_0. \tag{F.10b}$$

Solution F.10 is a decaying exponential multiplied by a sinusoidally varying term. The initial amplitude A and the phase angle ϕ are arbitrary and are determined by other conditions in the problem. The constant α is called the *damping*. Parameter ω_0 is the undamped frequency, the frequency of oscillation when $\alpha = 0$. ω is the damped frequency.

When the damping becomes so large that $\alpha = \omega_0$, then the solution given above does not work. In that case, the solution is given by

$$y = (A + Bt)e^{-\alpha t}, \qquad \alpha = \omega_0. \tag{F.11}$$

This case is called *critical damping* and represents the case in which y returns to zero most rapidly and without multiple oscillations. The solution can be verified by substitution.

If $\alpha > \omega_0$, then the solution is the sum of the two exponentials that satisfy Eq. F.8:

$$y = Ae^{-at} + Be^{-bt}, \tag{F.12a}$$

where

$$a = \alpha + \sqrt{\alpha^2 - \omega_0^2}, \tag{F.12b}$$

$$b = \alpha - \sqrt{\alpha^2 - \omega_0^2}. \tag{F.12c}$$

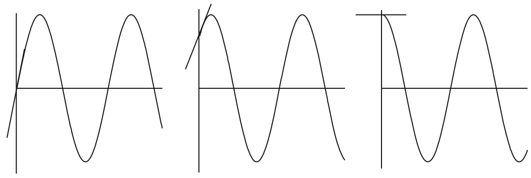

Fig. F.1 Different starting points on the sine wave give different combinations of the initial position and the initial velocity

Table F.1 Solutions of the harmonic oscillator equation

$$\frac{d^2y}{dt^2} + 2\alpha\frac{dy}{dt} + \omega_0^2 y = 0$$

Case	Criterion	Solution
Underdamped	$\alpha < \omega_0$	$y = Ae^{-\alpha t}\sin(\omega t + \phi)$ $\omega^2 = \omega_0^2 - \alpha^2$
Critically damped	$\alpha = \omega_0$	$y = (A + Bt)e^{-\alpha t}$
Overdamped	$\alpha > \omega_0$	$y = Ae^{-at} + Be^{-bt}$ $a = \alpha + (\alpha^2 - \omega_0^2)^{1/2}$ $b = \alpha - (\alpha^2 - \omega_0^2)^{1/2}$

When $\alpha = 0$, the equation is

$$\frac{d^2y}{dt^2} + \omega_0^2 y = 0. \tag{F.13}$$

The solution may be written either as

$$y = C\sin(\omega_0 t + \phi) \tag{F.14a}$$

or as

$$y = A\cos(\omega_0 t) + B\sin(\omega_0 t). \tag{F.14b}$$

The simplest physical example of this equation is a mass on a spring. There will be an equilibrium position of the mass ($y = 0$) at which there is no net force on the mass. If the mass is displaced toward either positive or negative y, a force back toward the origin results. The force is proportional to the displacement and is given by $F = -ky$. The proportionality constant k is called the *spring constant*. Newton's second law, $F = ma$, is $m(d^2y/dt^2) = -ky$ or, defining $\omega_0^2 = k/m$,

$$\frac{d^2y}{dt^2} + \omega_0^2 y = 0.$$

This (as well as the equation with $\alpha \neq 0$) is a second-order differential equation. Integrating it twice introduces two constants of integration: C and ϕ, or A and B. They are usually found from two initial conditions. For the mass on the spring, they are often the initial velocity and initial position of the mass.

The equivalence of the two solutions can be demonstrated by using Eqs. F.14a and a trigonometric identity to write $C\sin(\omega_0 t + \phi) = C[\sin\omega_0 t\cos\phi + \cos\omega_0 t\sin\phi]$. Comparison with Eq. F.14b shows that $B = C\cos\phi$, $A = C\sin\phi$. Squaring and adding these gives $C^2 = A^2 + B^2$, while dividing one by the other shows that $\tan\phi = A/B$.

Changing the initial phase angle changes the relative values of the initial position and velocity. This can be seen from the three plots of Fig. F.1, which show phase angles 0, $\pi/4$, and $\pi/2$. When $\phi = 0$, the initial position is zero, while the initial velocity has its maximum value. When $\phi = \pi/4$, the initial position has a positive value, and so does the initial velocity. When $\phi = \pi/2$, the initial position has its maximum value and the initial velocity is zero. The values of A and B are determined from the initial position and velocity. At $t = 0$, Eq. F.14b and its derivative give $y(0) = A$, $dy/dt(0) = \omega_0 B$.

The term in the differential equation equal to $2\alpha(dy/dt)$ corresponds to a drag force acting on the mass and damping the motion. Increasing the damping coefficient α increases the rate at which the oscillatory behavior decays. Figure F.2 shows plots of y and dy/dt for different values of α.

The second-order equation we have just studied is called the *harmonic oscillator* equation. Its solution is summarized in Table F.1.

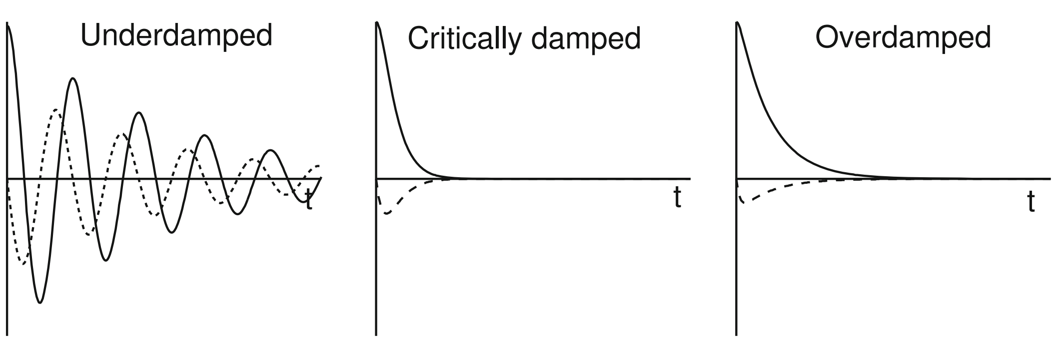

Fig. F.2 Plot of $y(t)$ (solid line) and dy/dt (dashed line) for different values of α

Problems

Problem 1. From Eq. F.14 with $\omega_0 = 10$, find A, B, C, and ϕ for the following cases:

(a) $y(0) = 5$, $(dy/dt)(0) = 0$.
(b) $y(0) = 5$, $(dy/dt)(0) = 5$.
(c) $y(0) = 0$, $(dy/dt)(0) = 50$.
(d) What values of A, B, and C would be needed to have the same ϕ as in case (b) and the same amplitude as in case (a)?

Problem 2. Verify Eq. F.11 in the critically damped case.

Problem 3. Find the general solution of the equation

$$\frac{d^2y}{dt^2} + 2\alpha\frac{dy}{dt} + \omega_0^2 y = \begin{cases} 0, & t \le 0 \\ \omega_0^2 y_0, & t \ge 0 \end{cases}$$

subject to the initial conditions $y(0) = 0, \quad (dy/dt)(0) = 0$

(a) for critical damping, $\alpha = \omega_0$,
(b) for no damping, and
(c) for overdamping, $\alpha = 2\omega_0$.

Problem 4. Show using numerical examples or physical arguments that the overdamped and critically damped solutions can cross the $y = 0$ axis at most once. Draw a plot of one such case.

Problem 5. Start with Eq. F.9. Add the function $f(t) = \sin\omega_1 t$ to the right-hand side so you have an inhomogeneous equation. Search for a solution to the inhomogeneous equation (sometimes called a *particular solution*) by guessing that $y(t) = A\sin\omega_1 t + B\cos\omega_1 t$. Put this back in the differential equation and find values of A and B that satisfy the equation. For what values of ω_1 will A and B be largest? This is an example of *resonance*: when the system is driven at its natural frequency, the response is largest.

Appendix G
The Mean and Standard Deviation

In many measurements in physics or biology there may be several possible outcomes to the measurement. Different values are obtained when the measurement is repeated. For example, the measurement might be the number of red cells in a certain small volume of blood, whether a person is right handed or left handed, the number of radioactive disintegrations of a certain sample during a 5-min interval, or the scores on a test.

Table G.1 gives the scores on an examination administered to 30 people. These results are also plotted as a histogram in Fig. G.1.

Table G.1 Quiz scores

Student No.	Score	Student No.	Score
1	80	16	71
2	68	17	83
3	90	18	88
4	72	19	75
5	65	20	69
6	81	21	50
7	85	22	81
8	93	23	94
9	76	24	73
10	86	25	79
11	80	26	82
12	88	27	78
13	81	28	84
14	72	29	74
15	67	30	70

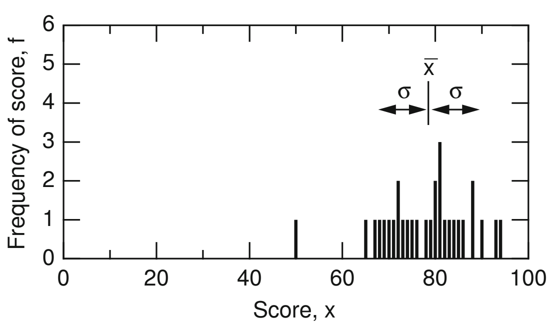

Fig. G.1 Histogram of the quiz scores in Table G.1

The table and the histogram give all the information that there is to know about the experiment unless the result depends on some variable that was not recorded, such as the age of the student or where the student was sitting during the test.

In many cases the frequency distribution gives more information than we need. It is convenient to invent some quantities that will answer the questions: Around what values do the results cluster? How wide is the distribution of results? Many different quantities have been invented for answering these questions. Some are easier to calculate or have more useful properties than others.

The *mean* or *average* shows where the distribution is centered. It is familiar to everyone: add up all the scores and divide by the number of students. For the data given above, the mean is $\overline{x} = 77.8$.

It is often convenient to group the data by the value obtained, along with the frequency of that value. The data of Table G.1 are grouped this way in Table G.2. The mean is calculated as

$$\overline{x} = \frac{1}{N}\sum_i f_i x_i = \frac{\sum_i f_i x_i}{\sum_i f_i},$$

where the sum is over the different values of the test scores that occur. For the example in Table G.2, the sums are $\sum_i f_i = 30$, $\sum_i f_i x_i = 2335$, so $\overline{x} = 2335/30 = 77.8$. If a large number of trials are made, f_i/N can be called the *probability* p_i of getting result x_i. Then

$$\overline{x} = \sum_i x_i p_i. \tag{G.1}$$

Note that $\sum p_i = 1$.

The average of some function of x is

$$\overline{g(x)} = \sum_i g(x_i) p_i. \tag{G.2}$$

For example,

$$\overline{x^2} = \sum_i (x_i)^2 p_i.$$

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Table G.2 Quiz scores grouped by score

Score number i	Score x_i	Frequency of score, f_i	$f_i x_i$
1	50	1	50
2	65	1	65
3	67	1	67
4	68	1	68
5	69	1	69
6	70	1	70
7	71	1	71
8	72	2	144
9	73	1	73
10	74	1	74
11	75	1	75
12	76	1	76
13	78	1	78
14	79	1	79
15	80	2	160
16	81	3	243
17	82	1	82
18	83	1	83
19	84	1	84
20	85	1	85
21	86	1	86
22	88	2	176
23	90	1	90
24	93	1	93
25	94	1	94

The width of the distribution is often characterized by the *dispersion* or *variance*:

$$\overline{(\Delta x)^2} = \overline{(x - \overline{x})^2} = \sum_i p_i (x_i - \overline{x})^2. \tag{G.3}$$

This is also sometimes called the mean square variation: the mean of the square of the variation of x from the mean. A measure of the width is the square root of this, which is called the *standard deviation* σ. The need for taking the square root is easy to see since x may have units associated with it. If x is in meters, then the variance has the units of square meters. The width of the distribution in x must be in meters.

A very useful result is

$$\overline{(x - \overline{x})^2} = \overline{x^2} - \overline{x}^2.$$

To prove this, note that $(x_i - \overline{x})^2 = x_i^2 - 2x_i\overline{x} + \overline{x}^2$. The variance is then

$$\overline{(\Delta x)^2} = \sum_i p_i x_i^2 - 2\sum_i x_i\,\overline{x}\,p_i + \sum_i p_i \overline{x}^2.$$

The first sum is the definition of $\overline{x^2}$. The second sum has a number $\overline{x}$ in every term. It can be factored in front of the sum, to make the second term $-2\overline{x}\sum x_i p_i$, which is just $-2(\overline{x})^2$. The last term is $(\overline{x})^2 \sum p_i = (\overline{x})^2$. Combining all three sums gives Eq. G.4. In summary,

$$\sigma = \sqrt{\overline{(\Delta x)^2}},$$
$$\sigma^2 = \overline{(\Delta x)^2} = \overline{(x - \overline{x})^2} = \overline{x^2} - \overline{x}^2. \tag{G.4}$$

This equation is true as long as the p_is are accurately known. If the p_is have only been estimated from N experimental observations, the best estimate of σ^2 is $N/(N-1)$ times the value calculated from Eq. G.4.

For the data of Fig. G.1, $\sigma = 9.4$. This width is shown along with the mean at the top of the figure.

Problems

Problem 1. Calculate the variance and standard deviation for the data in Table G.2.

Problem 2. Use the data in Table G.1 to calculate the mean of the squares and then verify that $\sigma^2 = \overline{x^2} - \overline{x}^2$.

Problem 3. Another way to characterize the distribution of values is the *mode*: the most common value recorded, or the one with the highest probability. Find the mode of the data in Table G.1.

Problem 4. Still another way to describe a distribution is the median: line up all the values in order from the smallest to largest, and find the middle value. (If you have an even number of values, average the middle two.) Find the median of the data in Table G.1.

Problem 5. Find the mean and standard deviation of the following data: 14, 8, 12, 13, 7, 7, 11 and 9. You do not have enough data to know the probabilities accurately, so use the factor $N/(N-1)$ to calculate the variance.

Problem 6. Imagine that the data in Table G.1 represent 30 measurements of some quantity. The measurements contain errors, which explain why the values are not all the same. One property of the mean and standard deviation is that approximately two-third of the measurements should fall within the range $\overline{x} \pm \sigma$. (This is true for a Gaussian distribution of data and is approximately true for many others.) Check whether this is approximately true for the data in Table G.1.

Problem 7. Suppose that you make a set of measurements that have mean $\overline{x}$ and standard deviation σ. You now repeat this set of measurements N times, so that you have N mean values. These mean values will have a distribution that is narrower than the distribution of the values in a single set of measurements. The *standard deviation of the mean* is denoted by $\sigma_{mean} = \sigma/\sqrt{N}$. Calculate the standard deviation of the mean for the data in Table G.1.

Appendix H
The Binomial Probability Distribution

Consider an experiment with two mutually exclusive outcomes, which is repeated N times, with each repetition being independent of every other. One of the outcomes is labeled "success"; the other is called "failure." The experiment could be throwing a die with success being a three, flipping a coin with success being a head, or placing a particle in a box with success being that the particle is located in a subvolume v.

In a single try, call the probability of success p and the probability of failure q. Since one outcome must occur and both cannot occur at the same time,

$$p + q = 1. \tag{H.1}$$

Suppose that the experiment is repeated N times. The probability of n successes out of N tries is given by the *binomial probability distribution*, which is stated here without proof.[1] We can call the probability $P(n; N)$, since it is a function of n and depends on the parameter N. Strictly speaking, it depends on two parameters, N and p: $P(n; N, p)$. It is[2]

$$P(n; N) = P(n; N, p) = \left(\frac{N!}{n!(N-n)!}\right) p^n (1-p)^{N-n}. \tag{H.2}$$

The factor $N!/[n!(N-n)!]$ counts the number of different ways that one can get n successful outcomes; the probability of each of these ways is $p^n(1-p)^{N-n}$. In the example of three particles in Sect. 3.1, there are three ways to have one particle in the left-hand side. The particle can be either particle a or particle b or particle c. The factor gives directly

$$\left(\frac{N!}{n!(N-n)!}\right) = \frac{3!}{1!2!} = \frac{3 \times 2 \times 1}{(1)(2 \times 1)} = \frac{6}{2} = 3.$$

[1] A detailed proof can be found in many places. See, for example, F. Reif (1964). *Statistical Physics*. Berkeley Physics Course, Vol. 5. New York, McGraw-Hill, p. 67.

[2] $N!$ is N factorial and is $N(N-1)(N-2)\cdots 1$. By definition, $0! = 1$.

The remaining factor, $p^n(1-p)^{N-n}$, is the probability of taking n tries in a row and having success and taking $N-n$ tries in a row and having failure.

The binomial distribution applies if each "try" is independent of every other try. Such processes are called *Bernoulli processes* (and the binomial distribution is often called the Bernoulli distribution). In contrast, if the probability of an outcome depends on the results of the previous try, the random process is called a *Markov process*. Although such processes are important, they are more difficult to deal with and are not discussed here.

Some examples of the use of the binomial distribution are given in Chap. 3. As another example, consider the problem of performing several laboratory tests on a patient. In the 1970s it became common to use automated machines for blood-chemistry evaluations of patients; such machines automatically performed (and reported) 6, 12, 20, or more tests on one small sample of a patient's blood serum, for less cost than doing just one or two of the tests. But this meant that the physician got a large number of results—many more than would have been asked for if the tests were done one at a time. When such test batteries were first done, physicians were surprised to find that patients had many more abnormal tests than they expected. This was in part because some tests were not known to be abnormal in certain diseases, because no one had ever looked at them in that disease. But there still was a problem that some tests were abnormal in patients who appeared to be perfectly healthy.

We can understand why by considering the following idealized situation. Suppose that we do N independent tests, and suppose that in healthy people, the probability that each test is abnormal is p. (In our vocabulary, having an abnormal test is "success"!). The probability of not having the test abnormal is $q = 1 - p$. In a perfect test, p would be 0 for healthy people and would be 1 in sick people; however, very few tests are that discriminating. The definition of normal vs abnormal involves a compromise between false positives (abnormal test results in healthy people) and false negatives (normal test results in sick people). Good reviews of this

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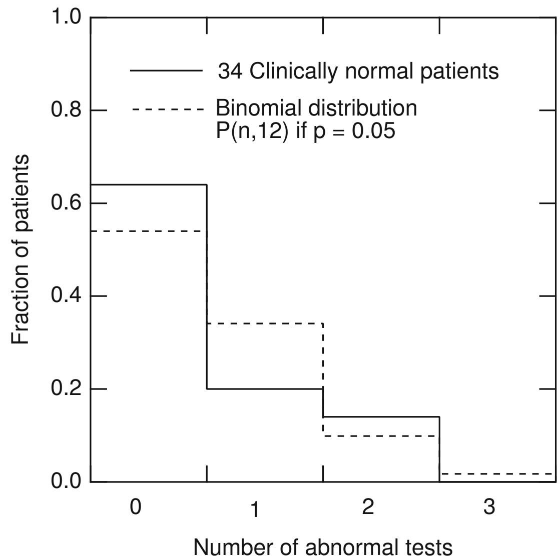

Fig. H.1 Measurement of the probability that a clinically normal patient having a battery of 12 tests done has n abnormal tests (*solid line*) and a calculation based on the binomial distribution (*dashed line*). The calculation assumes that $p = 0.05$ and that all 12 tests are independent. Several of the tests in this battery are not independent, but the general features are reproduced

problem have been written by Murphy and Abbey[3] and by Feinstein.[4] In many cases, p is about 0.05. Now suppose that p is the same for all the tests and that the tests are independent. Neither of these assumptions is very good, but they will show what the basic problem is. Then, the probability for all of the N tests to be normal in a healthy patient is given by the binomial probability distribution:

$$P(0; N, p) = \frac{N!}{0!N!} p^0 q^N = q^N.$$

If $p = 0.05$, then $q = 0.95$, and $P(0; N, p) = 0.95^N$. Typical values are $P(0; 12) = 0.54$ and $P(0; 20) = 0.36$. If the assumptions about p and independence are right, then only 36 % of healthy patients will have all their tests normal if 20 tests are done.

Figure H.1 shows a plot of the number of patients in a series who were clinically normal but who had abnormal tests. The data have the general features predicted by this simple model.

We can derive simple expressions to give the mean and standard deviation if the probability distribution is binomial. The mean value of n is defined to be

$$\overline{n} = \sum_{n=0}^{N} n P(n; N) = \sum_{n=0}^{N} \frac{N!n}{n!(N-n)!} p^n (1-p)^{N-n}.$$

The first term of each sum is for $n = 0$. Since each term is multiplied by n, the first term vanishes, and the limits of the sum can be rewritten as

$$\sum_{n=1}^{N} \frac{N!n}{n!(N-n)!} p^n (1-p)^{N-n}.$$

To evaluate this sum, we use a trick. Let $m = n-1$ and $M = N-1$. Then we can rewrite various parts of this expression as follows:

$$\frac{n}{n!} = \frac{1}{(n-1)!} = \frac{1}{m!},$$

$$p^n = pp^m,$$

$$N! = (N)(N-1)!,$$

$$(N-n)! = [N-1-(n-1)]! = (M-m)!.$$

The limits of summation are $n = 1$ or $m = 0$, and $n = N$ or $m = M$. With these substitutions

$$\overline{n} = Np \sum_{m=0}^{M} \frac{M!}{m!(M-m)!} p^m (1-p)^{M-m}.$$

This sum is exactly the sum of a binomial distribution over all possible values of m and is equal to one. We have the result that, for a binomial distribution,

$$\overline{n} = Np. \tag{H.3}$$

This says that the average number of successes is the total number of tries times the probability of a success on each try. If 100 particles are placed in a box and we look at half the box so that $p = \frac{1}{2}$, the average number of particles in that half is $100 \times \frac{1}{2} = 50$. If we put 500 particles in the box and look at $\frac{1}{10}$ of the box, the average number of particles in the volume is also 50. If we have 100,000 particles and $v/V = p = 1/2000$, the average number is still 50.

For the binomial distribution, the variance σ^2 can be expressed in terms of N and p using Eq. G.4. The average of n^2 is

$$\overline{n^2} = \sum_n P(n; N) n^2 = \sum_{n=0}^{N} \frac{N!}{n!(N-n)!} n^2 p^n (1-p)^{N-n}.$$

The trick to evaluate this is to write $n^2 = n(n-1) + n$. With this substitution we get two sums:

$$\overline{n^2} = \sum_{n=0}^{N} \frac{N!}{n!(N-n)!} n(n-1) p^n q^{N-n} + \sum_{n=0}^{N} \frac{N!n}{n!(N-n)!} p^n q^{N-n}.$$

[3] E. A. Murphy and H. Abbey (1967). The normal range—a common misuse. *J Chronic Dis* **20**: 79.

[4] A. R. Feinstein (1975). Clinical biostatistics XXVII. The derangements of the normal range. *Clin Pharmacol Therap* **15**: 528.

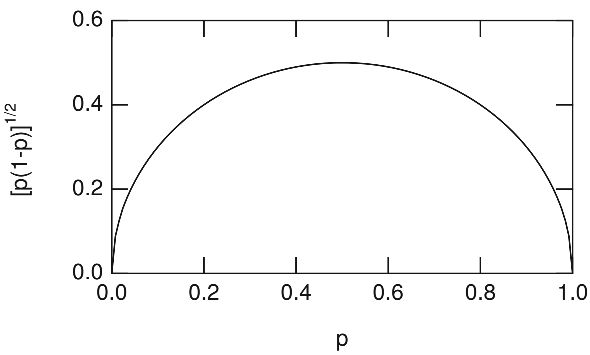

Fig. H.2 Plot of $\left[p(1-p)\right]^{1/2}$

The second sum is $\overline{n} = Np$. The first sum is rewritten by noticing that the terms for $n = 0$ and $n = 1$ both vanish. Let $m = n - 2$ and $M = N - 2$:

$$\begin{aligned}\overline{n^2} &= Np + N(N-1)\sum_{m=0}^{M}\frac{M!}{m!(M-m)!}p^2p^mq^{M-m}\\ &= Np + N(N-1)p^2 = Np + N^2p^2 - Np^2.\end{aligned}$$

Therefore,

$$\overline{(\Delta n)^2} = \overline{n^2} - \overline{n}^2 = Np - Np^2 = Np(1-p) = Npq.$$

For the binomial distribution, then,

$$\sigma = \sqrt{Npq} = \sqrt{\overline{n}q}.$$

The standard deviation for the binomial distribution for fixed p goes as $N^{1/2}$. For fixed N, it is proportional to $\sqrt{p(1-p)}$, which is plotted in Fig. H.2. The maximum value of σ occurs when $p = q = \frac{1}{2}$. If p is very small, the event happens rarely; if p is close to 1, the event nearly always happens. In either case, the variation is reduced. On the other hand, if N becomes large while p becomes small in such a way as to keep $\overline{n}$ fixed, then σ increases to a maximum value of $\sqrt{\overline{n}}$. This variation of σ with N and p is demonstrated in Fig. H.3. Figure H.3a–c shows how σ changes as N is held fixed and p is varied. For $N = 100$, p is 0.05, 0.5, and 0.95. Both the mean and σ change. Comparing Fig. H.3b with H.3d shows two different cases where $\overline{n} = 50$. When p is very small because N is very large in Fig. H.3d, σ is larger than in Fig. H.3b.

Problems

Problem 1. Calculate the probability of throwing 0, 1, ..., 9 heads out of a total of nine throws of a coin.

Problem 2. Assume that males and females are born with equal probability. What is the probability that a couple will have four children, all of whom are girls? The couple has had three girls. What is the probability that they will have a fourth girl? Why are these probabilities so different?

Problem 3. The Mayo Clinic reported that a single stool specimen in a patient known to have an intestinal parasite yields positive results only about 90 % of the time (R. B. Thomson, R. A. Haas, and J. H. Thompson, Jr. (1984). Intestinal parasites: The necessity of examining multiple stool specimens. *Mayo Clin Proc* **59**: 641–642). What is the probability of a false negative if two specimens are examined? Three?

Problem 4. The *Minneapolis Tribune* on October 31, 1974, listed the following incidence rates for cancer in the Twin Cities greater metropolitan area, which at that time had a total population of 1.4 million. These rates are compared to those in nine other areas of the country whose total population is 15 million. Assume that each study was for 1 year. Are the differences statistically significant? Show calculations to support your answer. How would your answer differ if the study were for several years?

Type of cancer	Incidence per 100,000 per year	
	Twin Cities	Other
Colon	35.6	30.9
Lung (women)	34.2	40.0
Lung (men)	63.6	72.0
Breast (women)	81.3	73.8
Prostate (men)	69.9	60.8
Overall	313.8	300.0

Problem 5. The probability that a patient with cystic fibrosis gets a bad lung illness is 0.5 % per day. With treatment, which is time consuming and not pleasant, the daily probability is ten times less.[5] Show that the probability of not having an illness in a year is 16 % without treatment and 83 % with treatment.

[5] These numbers are from W. Warwick, MD, private communication. See also A. Gawande, The bell curve. *The New Yorker,* December 6, 2004, pp. 82–91.

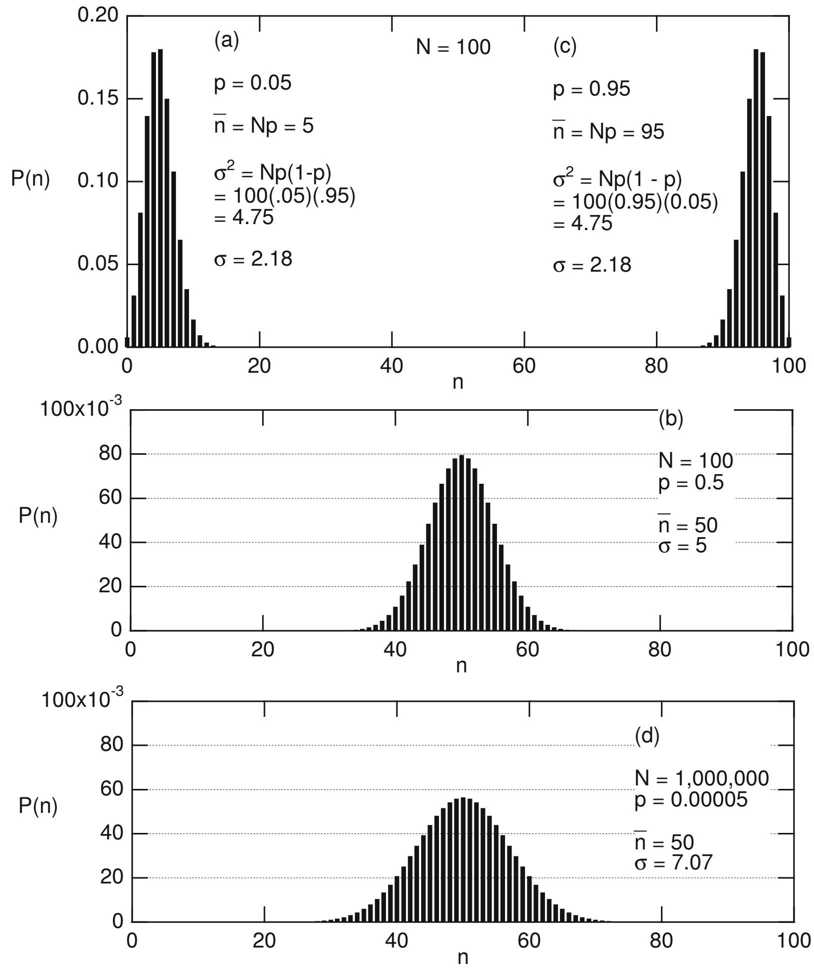

Fig. H.3 Examples of the variation of σ with N and p. (a), (b), and (c) show variations of σ with p when N is held fixed. The maximum value of σ occurs when $p = 0.5$. Note that (a) and (c) are both in the top panel. Comparison of (b) and (d) shows the variation of σ as p and N change together in such a way that $\bar{n}$ remains equal to 50

Appendix I
The Gaussian Probability Distribution

Appendix H considered a process that had two mutually exclusive outcomes and was repeated N times, with the probability of "success" on one try being p. If each try is independent, then the probability of n occurrences of success in N tries is

$$P(n; N, p) = \frac{N!}{n!(N-n)!} p^n (1-p)^{N-n}. \qquad \text{(I.1)}$$

This probability distribution depends on two parameters N and p. We have seen two other parameters, the mean, which roughly locates the center of the distribution, and the standard deviation, which measures its width. These parameters, $\overline{n}$ and σ, are related to N and p by the equations

$$\overline{n} = Np,$$

$$\sigma^2 = Np(1-p).$$

It is possible to write the binomial distribution formula in terms of the new parameters instead of N and p. At best, however, it is cumbersome, because of the need to evaluate so many factorial functions. We will now develop an approximation that is valid when N is large and which allows the probability to be calculated more easily.

The procedure is to take the log of the probability, $y = \ln(P)$ and expand it in a Taylor's series (Appendix D) about some point. Since there is a value of n for which P has a maximum and since the logarithmic function is monotonic, y has a maximum for the same value of n. We will expand about that point; call it n_0. Then the form of y is

$$y = y(n_0) + \left.\frac{dy}{dn}\right|_{n_0} (n - n_0) + \frac{1}{2}\left.\frac{d^2y}{dn^2}\right|_{n_0} (n - n_0)^2 + \cdots .$$

Since y is a maximum at n_0, the first derivative vanishes and it is necessary to keep the quadratic term in the expansion.

To take the logarithm of Eq. I.1, we need a way to handle the factorials. There is a very useful approximation to the factorial, called *Stirling's approximation*:

$$\ln(n!) \approx n \ln n - n. \qquad \text{(I.2)}$$

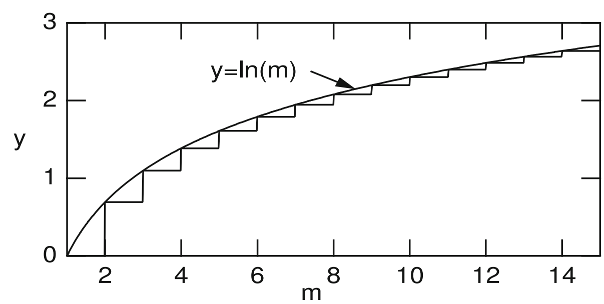

Fig. I.1 Plot of $y = \ln m$ used to derive Stirling's approximation

Table I.1 Accuracy of Stirling's approximation

n	$n!$	$\ln(n!)$	$n \ln n - n$	Error	% Error
5	120	4.7875	3.047	1.74	36
10	3.6×10^6	15.104	13.026	2.08	14
20	2.4×10^{18}	42.336	39.915	2.42	6
100	9.3×10^{157}	363.74	360.51	3.23	0.8

To derive it, write $\ln(n!)$ as

$$\ln(n!) = \ln 1 + \ln 2 + \cdots + \ln n = \sum_{m=1}^{n} \ln m.$$

The sum is the same as the total area of the rectangles in Fig. I.1, where the height of each rectangle is $\ln m$ and the width of the base is one. The area of all the rectangles is approximately the area under the smooth curve, which is a plot of $\ln m$. The area is approximately

$$\int_1^n \ln m \, dm = [m \ln m - m]_1^n = n \ln n - n + 1.$$

This completes the proof of Eq. I.2. Table I.1 shows values of $n!$ and Stirling's approximation for various values of n. The approximation is not too bad for $n > 100$.

We can now return to the task of deriving the binomial distribution. Taking logarithms of Eq. I.1, we get

$$y = \ln P = \ln(N!) - \ln(n!) - \ln(N-n)!$$
$$+ n \ln p + (N-n)\ln(1-p).$$

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With Stirling's approximation, this becomes

$$y = N \ln N - n \ln n - N \ln(N-n) \\ + n \ln(N-n) + n \ln p + (N-n)\ln(1-p). \qquad \text{(I.3)}$$

The derivative with respect to n is

$$\frac{dy}{dn} = -\ln n + \ln(N-n) + \ln p - \ln(1-p).$$

The second derivative is

$$\frac{d^2y}{dn^2} = -\frac{1}{n} - \frac{1}{N-n}.$$

The point of expansion n_0 is found by making the first derivative vanish:

$$0 = \ln \frac{(N-n)p}{n(1-p)}.$$

Since $\ln 1 = 0$, this is equivalent to $(N-n_0)p = n_0(1-p)$ or $n_0 = Np$. The maximum of y occurs when n is equal to the mean. At $n = n_0$, the value of the second derivative is

$$\frac{d^2y}{dn^2} = -\frac{1}{Np} - \frac{1}{N(1-p)} = -\frac{1}{Np(1-p)}.$$

It is still necessary to evaluate $y_0 = y(n_0)$. If we try to do this by substitution of $n = n_0$ in Eq. I.3, we get zero. The reason is that the Stirling approximation we used is too crude for this purpose. (There are additional terms in Stirling's approximation that make it more accurate.) The easiest way to find $y(n_0)$ is to call it y_0 for now and determine it from the requirement that the probability be normalized. Therefore, we have

$$y = y_0 - \frac{1}{2Np(1-p)}(n-Np)^2$$

so that, in this approximation,

$$P(n) = e^y = e^{y_0} e^{-(n-Np)^2/[2Np(1-p)]}.$$

With $Np = \overline{n}$, $e^{y_0} = C_0$, and $Np(1-p) = \sigma^2$, this is

$$P(n) = C_0 e^{-(n-\overline{n})^2/2\sigma^2}.$$

To evaluate C_0, note that the sum of $P(n)$ for all n is the area of all the rectangles in Fig. I.2. This area is approximately the area under the smooth curve, so that

$$1 = C_0 \int_{-\infty}^{\infty} e^{-(n-\overline{n})^2/2\sigma^2}\, dn.$$

It is shown in Appendix K that half of this integral is

$$\int_0^{\infty} dx\, e^{-bx^2} = \frac{1}{2}\sqrt{\frac{\pi}{b}}.$$

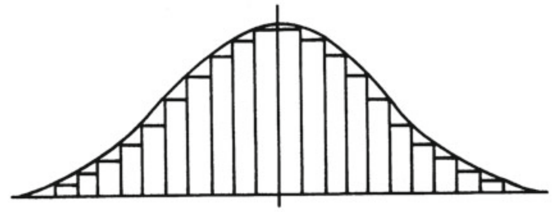

Fig. I.2 Evaluating the normalization constant

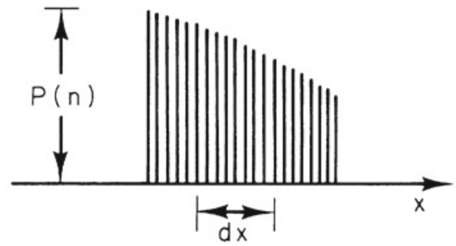

Fig. I.3 The allowed values of x are closely spaced in this case

Therefore the normalization integral is (letting $x = n - \overline{n}$)

$$\int_{-\infty}^{\infty} e^{-x^2/2\sigma^2} dx = \sqrt{2\pi\sigma^2}.$$

The normalization constant is $C_0 = 1/\sqrt{2\pi\sigma^2}$, so that the Gaussian or normal probability distribution is

$$P(n) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(n-\overline{n})^2/2\sigma^2}. \qquad \text{(I.4)}$$

It is possible, as in the case of the random-walk problem, that the measured quantity x is proportional to n with a very small proportionality constant, $x = kn$, so that the values of x appear to form a continuum. As shown in Fig. I.3, the number of different values of n (each with about the same value of $P(n)$) in the interval dx is proportional to dx. The easiest way to write down the Gaussian distribution in the continuous case is to recognize that the mean is $\overline{x} = k\overline{n}$, and the standard deviation is $\sigma_x^2 = \overline{(x-\overline{x})^2} = \overline{x^2} - \overline{x}^2 = k^2\overline{n^2} - k^2\overline{n}^2 = k^2\sigma^2$. The term $P(x)dx$ is given by $P(n)$ times the number of different values of n in dx. This number is dx/k. Therefore,

$$P(x)dx = P(n)\frac{dx}{k} = dx \frac{1}{k\sqrt{2\pi\sigma^2}} e^{-(x/k-\overline{x}/k)^2/2\sigma^2} \\ = dx \frac{1}{\sqrt{2\pi}\sigma_x} e^{-(x-\overline{x})^2/2\sigma_x^2}. \qquad \text{(I.5)}$$

To recapitulate: the binomial distribution in the case of large N can be approximated by Eq. I.4, the Gaussian or normal distribution, or Eq. I.5 for continuous variables. The original parameters N and p are replaced in these approximations by $\overline{n}$ (or $\overline{x}$) and σ.

Problems

Problem 1. An improved approximation to Stirling's formula[1] is

$$\ln n! \approx n \ln n - n + \frac{\ln(2\pi n)}{2}$$

Expand Table I.1 to include entries using this approximation.

Problem 2. Let $y = (x - \overline{x})/\sigma$. Express the Gaussian probability distribution as a function of y. Calculate the mean and standard deviation of this distribution.

[1] For more about Stirling's formula, see N. D. Mermin (1994) Stirling's formula! *Am J Phys* **52**: 362–365.

Appendix J
The Poisson Distribution

Appendix H discussed the binomial probability distribution. If an experiment is repeated N times, and has two possible outcomes, with "success" occurring with probability p in each try, the probability of getting that outcome x times in N tries is

$$P(x; N, p) = \frac{N!}{x!(N-x)!} p^x (1-p)^{N-x}.$$

The distribution of possible values of x is characterized by a mean value $\overline{x} = Np$ and a variance $\sigma^2 = Np(1-p)$. It is possible to specify $\overline{x}$ and σ^2 instead of N and p to define the distribution.

Appendix I showed that it is easier to work with the Gaussian or normal distribution when N is large. It is specified in terms of the parameters $\overline{x}$ and σ^2 instead of N and p:

$$P(x; \overline{x}, \sigma^2) = \frac{1}{(2\pi\sigma^2)^{1/2}} e^{-(x-\overline{x})^2/2\sigma^2}.$$

The *Poisson distribution* is an approximation to the binomial distribution that is valid for large N and for small p (when N gets large and p gets small in such a way that their product remains finite). To derive it, rewrite the binomial probability in terms of $p = \overline{x}/N$:

$$\begin{aligned} P(x) &= \frac{N!}{x!(N-x)!} (\overline{x}/N)^x (1-\overline{x}/N)^{N-x} \\ &= \frac{N!}{x!(N-x)!} \frac{1}{N^x} \overline{x}^x \left(1 - \frac{\overline{x}}{N}\right)^N \left(1 - \frac{\overline{x}}{N}\right)^{-x}. \end{aligned} \tag{J.1}$$

It is necessary next to consider the behavior of some of these factors as N becomes very large. The factor $(1-\overline{x}/N)^N$ approaches $e^{-\overline{x}}$ as $N \to \infty$, by definition (see p. 34). The factor $N!/(N-x)!$ can be written out as

$$\frac{N(N-1)(N-2)\cdots 1}{(N-x)(N-x-1)\cdots 1} = N(N-1)(N-2)\cdots(N-x+1).$$

If these factors are multiplied out, the first term is N^x, followed by terms containing N^{x-1}, N^{x-2}, ..., down to N^1. But there is also a factor N^x in the *denominator* of the expression for P, which, combined with this gives

$$1 + (\text{something})N^{-1} + (\text{something})N^{-2} + \cdots.$$

As long as N is very large, all terms but the first can be neglected. With these substitutions, Eq. J.1 takes the form

$$P(x) = \frac{1}{x!} \overline{x}^x e^{-\overline{x}} \left(1 - \frac{\overline{x}}{N}\right)^{-x}. \tag{J.2}$$

The values of x for which $P(x)$ is not zero are near $\overline{x}$, which is much less than N. Therefore, the last term, which is really $\left[1/(1-p)\right]^x$, can be approximated by one, while such a term raised to the Nth power had to be approximated by e^{-x}. If this is difficult to understand, consider the following numerical example. Let N=10,000 and $p = 0.001$, so $\overline{x} = 10$. The two terms we are considering are $(1 - 10/10,000)^{10,000} = 4.517 \times 10^{-5}$, which is approximated by $e^{-10} = 4.54 \times 10^{-5}$, and terms like $(1 - 10/10,000)^{-10} = 1.001$, which are approximated by 1.

With these approximations, the probability is $P(x) = [(\overline{x})^x/x!]e^{-\overline{x}}$ or, calling $\overline{x} = m$,

$$P(x) = \frac{m^x}{x!} e^{-m}. \tag{J.3}$$

This is the Poisson distribution and is an approximation to the binomial distribution for large N and small p, such that the mean $\overline{x} = m = Np$ is defined (that is, it does not go to infinity or zero as N gets large and p gets small).

This probability, when summed over all values of x, should be unity. This is easily verified. Write

$$\sum_{x=0}^{\infty} P(x) = e^{-m} \sum_{x=0}^{\infty} \frac{m^x}{x!}.$$

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Table J.1 Comparison of the binomial, Gaussian, and Poisson distributions

Binomial	$P(x;N,p) = \frac{N!}{x!(N-x)!} p^x (1-p)^{N-x}$ $\bar{x} = m = Np$ $\sigma^2 = Np(1-p) = m(1-p)$
Gaussian	$P(x;m,\sigma) = \frac{1}{(2\pi\sigma^2)^{1/2}} e^{-(x-m)^2/2\sigma^2}$
Poisson	$P(x;m) = \frac{m^x}{x!} e^{-m}$ $m = Np$ $\sigma^2 = m$

But the sum on the right is the series for e^m and $e^{-m}e^m = 1$. The same trick can be used to verify that the mean is m:

$$\sum_{x=0}^{\infty} xP(x) = \sum_{x=0}^{\infty} x \frac{m^x}{x!} e^{-m} = \sum_{x=1}^{\infty} x \frac{m^x}{x!} e^{-m}.$$

The index of summation can be changed from x to $y = x - 1$:

$$\sum_{x=0}^{\infty} xP(x) = \sum_{y=0}^{\infty} \frac{(y+1)}{(y+1)!} m^y m e^{-m} = m \sum_{y=0}^{\infty} \frac{m^y}{y!} e^{-m} = m.$$

One can show that the variance for the Poisson distribution is $\sigma^2 = \overline{(x-m)^2} = m$.

Table J.1 compares the binomial, Gaussian, and Poisson distributions. The principal difference between the binomial and Gaussian distributions is that the latter is valid for large N and is expressed in terms of the mean and standard deviation instead of N and p. Since the Poisson distribution is valid for very small p, there is only one parameter left, and $\sigma^2 = m$ rather than $m(1-p)$.

The Poisson distribution can be used to answer questions like the following:

1. How many red cells are there in a small square in a hemocytometer? The number of cells N is large; the probability p of each cell falling in a particular square is small. The variable x is the number of cells per square.
2. How many gas molecules are found in a small volume of gas in a large container? The number of tries is the total number of molecules. The probability that an individual molecule is in the smaller volume is $p = V/V_0$, where V is the small volume and V_0 is the volume of the entire box.
3. How many radioactive nuclei (or excited atoms) decay (or emit light) during a time dt? The probability of decay during time dt is proportional to how long dt is: $p = \lambda dt$. The number of tries is the N nuclei that might decay during that time.

The last example is worth considering in greater detail. The probability p that each nucleus decays in time dt is proportional to the length of the time interval: $p = \lambda dt$. The average number of decays if many time intervals are examined is

$$m = Np = N\lambda\, dt.$$

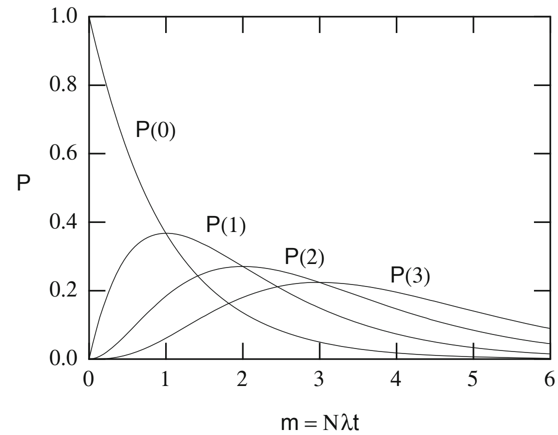

Fig. J.1 Plot of $P(0)$ through $P(3)$ vs. $N\lambda t$

The probability of x decays in time dt is

$$P(x) = \frac{(N\lambda dt)^x}{x!} e^{-N\lambda dt}.$$

As $dt \to 0$, the exponential approaches one, and

$$P(x) \to \frac{(N\lambda dt)^x}{x!}.$$

The overwhelming probability for $dt \to 0$ is for there to be no decays: $P(0) \approx (N\lambda dt)^0/0! = 1$. The probability of a single decay is $P(1) = Ndt$; the probability of two decays during dt is $(N\lambda dt)^2/2$, and so forth.

If time interval t is finite, it is still possible for the Poisson criterion to be satisfied, as long as $p = \lambda t$ is small. Then the probability of no decays is

$$P(0) = e^{-m} = e^{-N\lambda t}.$$

The probability of one decay is

$$P(1) = (N\lambda t)e^{-N\lambda t}.$$

This probability increases linearly with t at first and then decreases as the exponential term begins to decay. The reason for the lowered probability of one decay is that it is now more probable for two or more decays to take place in this longer time interval. As t increases, it is more probable that there are two decays than one or none; for still longer times, even more decays become more probable. The probability that n decays occur in time t is $P(n)$. Figure J.1 shows plots of $P(0)$, $P(1)$, $P(2)$, and $P(3)$, vs $m = N\lambda t$.

Problems

Problem 1. In the USA 400,000 people were killed or injured one year in automobile accidents. The total population was 200,000,000. If the probability of being killed or injured is independent of time, what is the probability that you will escape unharmed from 70 years of driving?

Problem 2. Large proteins consist of a number of smaller subunits that are stuck together. Suppose that an error is made in inserting an amino acid once in every 10^5 tries; $p = 10^{-5}$. If a chain has length 1000, what is the probability of making a chain with no mistakes? If the chain length is 10^5?

Problem 3. The muscle end plate has an electrical response whenever the nerve connected to it is stimulated. I. A. Boyd and A. R. Martin (The end plate potential in mammalian muscle. *J Physiol* **132**: 74–91 (1956)) found that the electrical response could be interpreted as resulting from the release of packets of acetylcholine by the nerve. In terms of this model, they obtained the following data:

Number of packets reaching the end plate	Number of times observed
0	18
1	44
2	55
3	36
4	25
5	12
6	5
7	2
8	1
9	0

Analyze these data in terms of a Poisson distribution.

Appendix K
Integrals Involving e^{-ax^2}

Integrals involving e^{-ax^2} appear in the Gaussian distribution. The integral

$$I = \int_{-\infty}^{\infty} e^{-ax^2} dx$$

can also be written with y as the dummy variable:

$$I = \int_{-\infty}^{\infty} e^{-ay^2} dy.$$

These can be multiplied together to get

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx dy\, e^{-ax^2} e^{-ay^2}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx dy\, e^{-a(x^2+y^2)}.$$

A point in the xy plane can also be specified by the polar coordinates r and θ (Fig. K.1). The element of area $dxdy$ is replaced by the element $rdrd\theta$:

$$I^2 = \int_0^{2\pi} d\theta \int_0^{\infty} r\, dr\, e^{-ar^2} = 2\pi \int_0^{\infty} r\, dr\, e^{-ar^2}.$$

To continue, make the substitution $u = ar^2$, so that $du = 2ardr$. Then

$$I^2 = 2\pi \int_0^{\infty} \frac{1}{2a} e^{-u}\, du = \frac{\pi}{a}\left[-e^{-u}\right]_0^{\infty} = \frac{\pi}{a}.$$

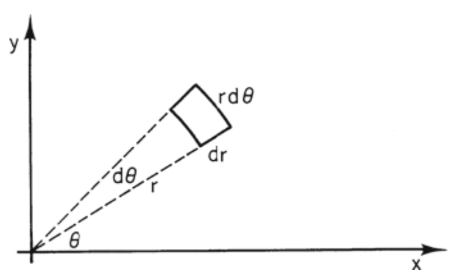

Fig. K.1 An element of area in polar coordinates

The desired integral is, therefore,

$$I = \int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}. \qquad (K.1)$$

This integral is one of a sequence of integrals of the general form

$$I_n = \int_0^{\infty} x^n e^{-ax^2} dx.$$

From Eq. K.1, we see that

$$I_0 = \frac{I}{2} = \frac{1}{2}\sqrt{\frac{\pi}{a}}. \qquad (K.2)$$

The next integral in the sequence can be integrated directly with the substitution $u = ax^2$:

$$I_1 = \int_0^{\infty} x e^{-ax^2} dx = \frac{1}{2a}\int_0^{\infty} e^{-u} du = \frac{1}{2a}. \qquad (K.3)$$

A value for I_2 can be obtained by integrating by parts:

$$I_2 = \int_0^{\infty} x^2 e^{-ax^2} dx.$$

Let $u = x$ and $dv = xe^{-ax^2}dx = -(1/2a)d(e^{-ax^2})$. Since $\int u dv = uv - \int v du$,

$$\int_0^{\infty} x^3 e^{-ax^2} dx = -\frac{xe^{-ax^2}}{2a} + \frac{1}{2a}\int e^{-ax^2} dx.$$

This expression is evaluated at the limits 0 and ∞. The term xe^{-ax^2} vanishes at both limits. The second term is $I_0/2a$. Therefore,

$$I_2 = \frac{1}{2 \times 2a}\sqrt{\frac{\pi}{a}}.$$

This process can be repeated to get other integrals in the sequence. The even members build on I_0; the odd members

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build on I_1. General expressions can be written. Note that $2n$ and $2n+1$ are used below to assure even and odd exponents:

$$\int_0^\infty x^{2n} e^{-ax^2} dx = \frac{1 \times 3 \times 5 \times (2n-1)}{2^{n+1} a^n} \sqrt{\frac{\pi}{a}}, \quad \text{(K.4)}$$

$$\int_0^\infty x^{2n+1} e^{-ax^2} dx = \frac{n!}{2a^{n+1}}, \qquad (a > 0). \quad \text{(K.5)}$$

The integrals in Appendix I are of the form

$$\int_{-\infty}^{\infty} e^{-x^2/2\sigma^2} dx.$$

This integral is $2I_0$ with $a = 1/(2\sigma^2)$. Therefore, the integral is $\sqrt{2\pi\sigma^2}$.

Integrals of the form

$$J = \int_0^\infty x^n e^{-ax} dx,$$

can be transformed to the forms above with the substitution $y = x^{1/2}$, $x = y^2$, $dx = 2y\,dy$. Then

$$J = \int_0^\infty y^{2n} e^{-ay^2} 2y\,dy = 2\int_0^\infty y^{2n+1} e^{-ay^2} dy.$$

Therefore,

$$\int_0^\infty x^n e^{-ax} dx = \frac{n!}{a^{n+1}} = \frac{\Gamma(n+1)}{a^{n+1}}. \quad \text{(K.6)}$$

The gamma function $\Gamma(n) = (n-1)!$ if n is an integer. Unlike $n!$, it is also defined for noninteger values. Although we have not shown it, Eq. K.6 is correct for noninteger values of n as well, as long as $a > 0$ and $n > -1$.

Problems

Problem 1. Use integration by parts to evaluate

$$I_3 = \int_0^\infty x^3 e^{-ax^2} dx.$$

Compare this result with Eq. K.5.

Problem 2. Show that $\int_{-\infty}^{\infty} x e^{-ax^2} dx = 0$. Note the lower limit is $-\infty$, not 0. There is a hard way and an easy way to show this. Try to find the easy way.

Appendix L
Spherical and Cylindrical Coordinates

It is possible to use coordinate systems other than the rectangular (or Cartesian) (x, y, z): In spherical coordinates (Fig. L.1), the coordinates are radius r and angles θ and ϕ:

$$\begin{aligned} x &= r \sin\theta \cos\phi, \\ y &= r \sin\theta \sin\phi, \\ z &= r \cos\theta. \end{aligned} \tag{L.1}$$

In Cartesian coordinates a volume element is defined by surfaces on which x is constant (at x and $x + dx$), y is constant, and z is constant. The volume element is a cube with edges dx, dy, and dz. In spherical coordinates, the cube has faces defined by surfaces of constant r, constant θ, and constant ϕ (Fig. L.2). A volume element is then

$$dV = (dr)(r\,d\theta)(r \sin\theta\,d\phi) = r^2 \sin\theta\,d\theta\,d\phi\,dr. \tag{L.2}$$

To calculate the divergence of vector $\mathbf{J}$, resolve it into components $\mathbf{J}_r$, $\mathbf{J}_\theta$, and $\mathbf{J}_\phi$, as shown in Fig. L.2. These components are parallel to the vectors defined by small displacements in the r, θ, and ϕ directions. A detailed calculation[1] shows that the divergence is

$$\begin{aligned} \operatorname{div}\mathbf{J} = \nabla\cdot\mathbf{J} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 J_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\,J_\theta) \\ + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(J_\phi). \end{aligned} \tag{L.3}$$

The gradient, which appears in the three-dimensional diffusion equation (Fick's first law), can also be written in spherical coordinates. The components are

$$\begin{aligned} (\nabla C)_r &= \frac{\partial C}{\partial r}, \\ (\nabla C)_\theta &= \frac{1}{r}\frac{\partial C}{\partial\theta}, \\ (\nabla C)_\phi &= \frac{1}{r\sin\theta}\frac{\partial C}{\partial\phi}. \end{aligned} \tag{L.4}$$

Figure L.2 also shows that the element of area on the surface of the sphere is $(r\,d\theta)(r\sin\theta\,d\phi) = r^2\sin\theta\,d\theta\,d\phi$. The element of solid angle is therefore

$$d\Omega = \sin\theta\,d\theta\,d\phi.$$

This is easily integrated to show that the surface area of a sphere is $4\pi r^2$ or that the solid angle is 4π sr.

$$\begin{aligned} S &= r^2\int_0^\pi \sin\theta\,d\theta \int_0^{2\pi} d\phi = 2\pi r^2 \int_0^\pi \sin\theta\,d\theta \\ &= 2\pi r^2\,[-\cos\theta]_0^\pi = 4\pi r^2. \end{aligned}$$

Similar results can be written down in cylindrical coordinates (r, ϕ, z), shown in Fig. L.3.

Table L.1 shows the divergence, gradient, and curl in rectangular, cylindrical, and spherical coordinates, along with the Laplacian operator ∇^2.

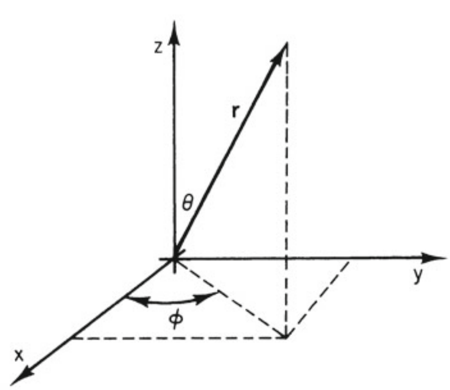

Fig. L.1 Spherical coordinates.

[1] H. M. Schey (2005). *Div, Grad, Curl, and All That*. 4th. ed. New York, Norton.

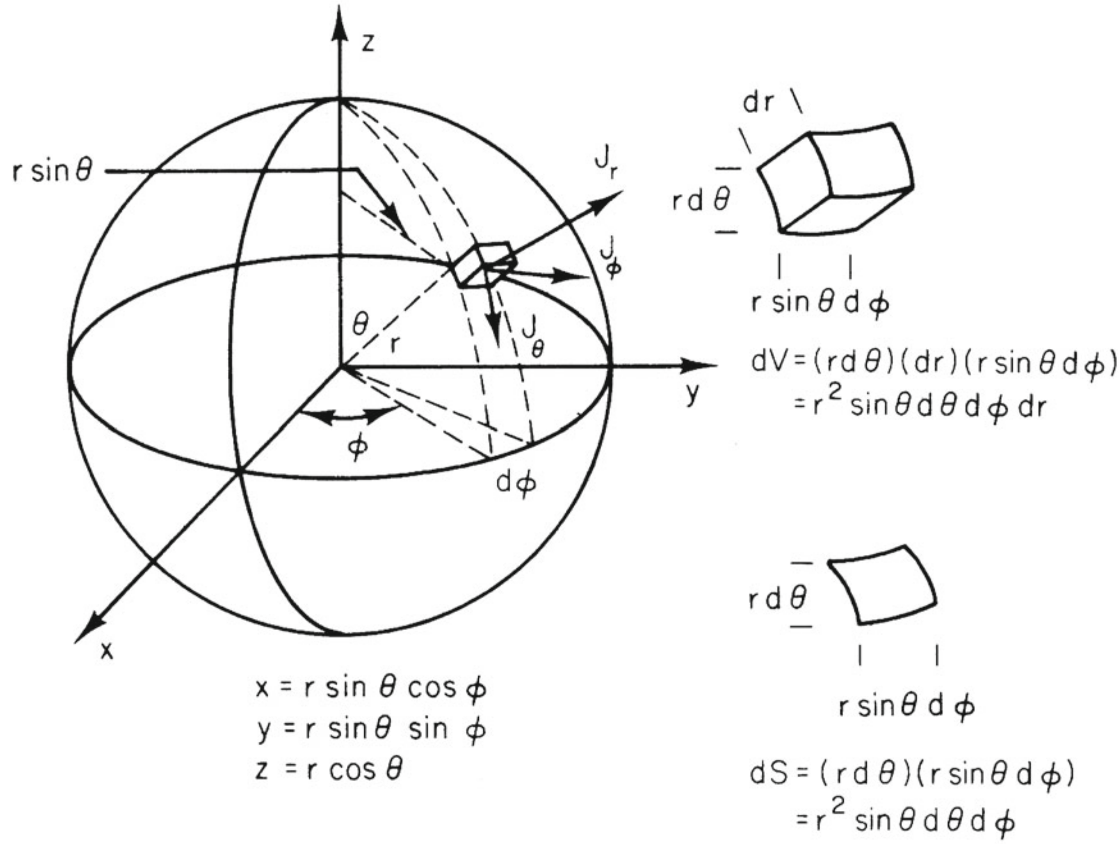

Fig. L.2 The volume element and element of surface area in spherical coordinates

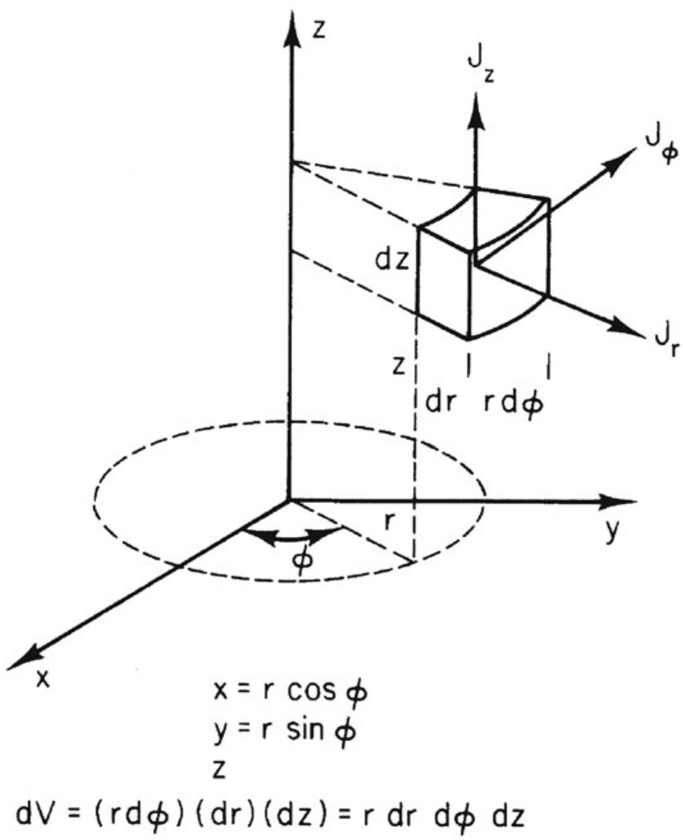

Fig. L.3 A cylindrical coordinate system

Table L.1 The vector operators in rectangular, cylindrical, and spherical coordinates

Rectangular x, y, z	Cylindrical r, ϕ, z	Spherical r, θ, ϕ
Gradient		
$(\nabla C)_x = \frac{\partial C}{\partial x}$	$(\nabla C)_r = \frac{\partial C}{\partial r}$	$(\nabla C)_r = \frac{\partial C}{\partial r}$
$(\nabla C)_y = \frac{\partial C}{\partial y}$	$(\nabla C)_\phi = \frac{1}{r}\frac{\partial C}{\partial \phi}$	$(\nabla C)_\theta = \frac{1}{r}\frac{\partial C}{\partial \theta}$
$(\nabla C)_z = \frac{\partial C}{\partial z}$	$(\nabla C)_z = \frac{\partial C}{\partial z}$	$(\nabla C)_\phi = \frac{1}{r\sin\theta}\frac{\partial C}{\partial \phi}$
Laplacian		
$\nabla^2 C = \frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2} + \frac{\partial^2 C}{\partial z^2}$	$\nabla^2 C = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial C}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 C}{\partial \phi^2} + \frac{\partial^2 C}{\partial z^2}$	$\nabla^2 C = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial C}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial C}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 C}{\partial \phi^2}$
Divergence		
$\nabla\cdot\mathbf{j} = \frac{\partial j_x}{\partial x} + \frac{\partial j_y}{\partial y} + \frac{\partial j_z}{\partial z}$	$\nabla\cdot\mathbf{j} = \frac{1}{r}\frac{\partial (r j_r)}{\partial r} + \frac{1}{r}\frac{\partial j_\phi}{\partial \phi} + \frac{\partial j_z}{\partial z}$	$\nabla\cdot\mathbf{j} = \frac{1}{r^2}\frac{\partial (r^2 j_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial (\sin\theta\, j_\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial j_\phi}{\partial \phi}$
Curl		
$(\nabla\times\mathbf{j})_x = \frac{\partial j_z}{\partial y} - \frac{\partial j_y}{\partial z}$	$(\nabla\times\mathbf{j})_r = \frac{1}{r}\frac{\partial j_z}{\partial \phi} - \frac{\partial j_\phi}{\partial z}$	$(\nabla\times\mathbf{j})_r = \frac{1}{r\sin\theta}\times\left[\frac{\partial(\sin\theta\, j_\phi)}{\partial\theta} - \frac{\partial(j_\theta)}{\partial\phi}\right]$
$(\nabla\times\mathbf{j})_y = \frac{\partial j_x}{\partial z} - \frac{\partial j_z}{\partial x}$	$(\nabla\times\mathbf{j})_\phi = \frac{\partial j_r}{\partial z} - \frac{\partial j_z}{\partial r}$	$(\nabla\times\mathbf{j})_\theta = \frac{1}{r\sin\theta}\times\left[\frac{\partial j_r}{\partial\phi} - \frac{\sin\theta\,\partial(r j_\phi)}{\partial r}\right]$
$(\nabla\times\mathbf{j})_z = \frac{\partial j_y}{\partial x} - \frac{\partial j_x}{\partial y}$	$(\nabla\times\mathbf{j})_z = \frac{1}{r}\frac{\partial (r j_\phi)}{\partial r} - \frac{1}{r}\frac{\partial j_r}{\partial \phi}$	$(\nabla\times\mathbf{j})_{\partial\phi} = \frac{1}{r}\left[\frac{\partial(r j_\theta)}{\partial r} - \frac{\partial j_r}{\partial\theta}\right]$

Appendix M
Joint Probability Distributions

In both physics and medicine, the question often arises of what is the probability that x has a certain value x_i while y has the value y_j. This is called a *joint probability*. Joint probability can be extended to several variables. This appendix derives some properties of joint probabilities for discrete and continuous variables.

M.1 Discrete Variables

Consider two variables. For simplicity assume that each can have only two values. The first might be the patient's health with values *healthy* and *sick*; the other might be the results of some laboratory test, with results *normal* and *abnormal*. Table M.1 shows the values of the two variables for a sample of 100 patients. The joint probability that a patient is healthy *and* has a normal test result is $P(x = 0, y = 0) = 0.6$; the probability that a patient is sick and has an abnormal test is $P(1, 1) = 0.15$. The probability of a false positive test is $P(0, 1) = 0.20$; the probability of a false negative is $P(1, 0) = 0.05$.

The probability that a patient is healthy regardless of the test result is obtained by a summing over all possible test outcomes: $P(x = 0) = P(0, 0)+P(0, 1) = 0.6+0.2 = 0.8$.

In a more general case, we can call the joint probability $P(x, y)$, the probability that x has a certain value independent of y, $P_x(x)$, and so forth. Then

$$P_x(x) = \sum_y P(x, y)$$
$$P_y(y) = \sum_x P(x, y). \tag{M.1}$$

Since any measurement must give some value for x and y, we can write

$$1 = \sum_x P_x(x) = \sum_x \sum_y P(x, y),$$
$$1 = \sum_y P_y(y) = \sum_y \sum_x P(x, y). \tag{M.2}$$

Table M.1 The results of measurements on 100 patients showing whether they are healthy or sick and whether a laboratory test was normal or abnormal

	Healthy ($x = 0$)	Sick ($x = 1$)
Normal test ($y = 0$)	60	5
Abnormal test ($y = 1$)	20	15

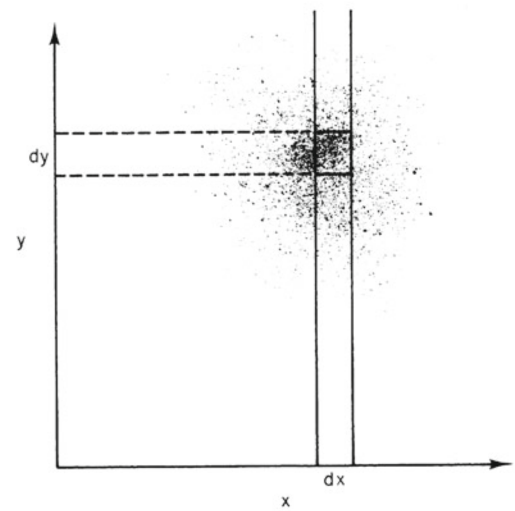

Fig. M.1 The results of measuring two continuous variables simultaneously. Each experimental result is shown as a point

M.2 Continuous Variables

When a variable can take on a continuous range of values, it is quite unlikely that the variable will have *precisely* the

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value x. Instead, there is a probability that it is in the interval (x, dx), meaning that it is between x and $x + dx$. For small values of dx, the probability that the value is in the interval is proportional to the width of the interval. We will call it $p_x(x)dx$. The extension to joint probability in two dimensions is $p(x, y)dxdy$. This is the probability that x is in the interval (x, dx) and y is in the interval (y, dy). Figure M.1 shows each outcome of a joint measurement as a dot in the xy plane. The probability that x is in (x, dx) regardless of the value of y is

$$p_x(x)dx = \left(\int p(x, y)dy \right) dx. \qquad \text{(M.3)}$$

It is proportional to the total number of dots in the vertical strip in Fig. M.1. Normalization requires that

$$1 = \int p_x(x)dx = \int dx \int dy\, p(x, y). \qquad \text{(M.4)}$$

The first strip could be taken horizontally:

$$1 = \int p_y(y)dy = \int dy \int dx\, p(x, y).$$

Figure M.2 shows a perspective drawing of $p(x, y)$. The volume of the shaded column is $p(x, y)dxdy$. The volume of the slice is $p_x(x)dx$. The entire volume under the surface is equal to 1.

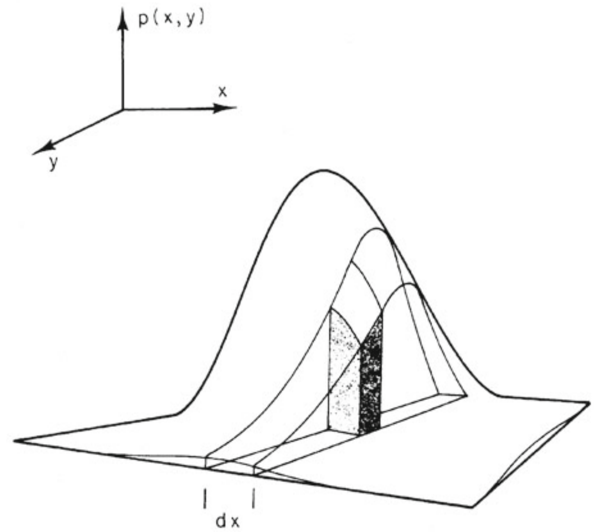

Fig. M.2 Perspective drawing of $p(x, y)$

Appendix N
Partial Derivatives

When a function depends on several variables, we may want to know how the value of the function changes when one or more of the variables is changed. For example, the volume of a cylinder is

$$V = \pi r^2 h.$$

How does V change when r is changed while the height of the cylinder is kept fixed?

$$V(r + \Delta r) = \pi(r + \Delta r)^2 h = \pi(r^2 + 2r\Delta r + \Delta r^2)h.$$

Subtracting the original volume, we have

$$\Delta V = \pi(2r\Delta r + \Delta r^2)h.$$

In the limit of small Δr, this is

$$dV = 2\pi hr dr.$$

This is the same answer we would have gotten if h had been regarded as a constant. The *partial derivative* of V with respect to r is defined to be

$$\left(\frac{\partial V}{\partial r}\right)_h = \lim_{\Delta r \to 0}\left(\frac{V(r + \Delta r, h) - V(r, h)}{\Delta r}\right) = 2\pi rh.$$

The subscript h in the partial derivative symbol means that h is held fixed during the differentiation. Sometimes it is omitted; when it is not there, it is understood that all variables except the one following the ∂ are held fixed.

If the cylinder radius is held fixed while the height is varied, we can write

$$\Delta V = V(r, h + \Delta h) - V(r, h) = \pi r^2 \Delta h.$$

The partial derivative is

$$\left(\frac{\partial V}{\partial h}\right)_r = \lim_{\Delta h \to 0}\left(\frac{V(r, h + \Delta h) - V(r, h)}{\Delta h}\right) = \pi r^2.$$

Suppose now that we allow small changes in both r and h. The difference in volume is

$$\Delta V = V(r + \Delta r, h + \Delta h) - V(r, h).$$

We can add and subtract the term $V(r, h + \Delta h)$:

$$\begin{aligned}\Delta V &= V(r + \Delta r, h + \Delta h) - V(r, h + \Delta h) \\ &\quad + V(r, h + \Delta h) - V(r, h) \\ &= \frac{V(r + \Delta r, h + \Delta h) - V(r, h + \Delta h)}{\Delta r}\Delta r \\ &\quad + \frac{V(r, h + \Delta h) - V(r, h)}{\Delta h}\Delta h.\end{aligned}$$

In the limit as Δr and $\Delta h \to 0$, the first term is

$$\left(\frac{\partial V}{\partial r}\right)_h \Delta r,$$

evaluated at $h + \Delta h$. If the derivatives are continuous at (r, h), the derivative evaluated at $(r, h + \Delta h)$ is negligibly different from the derivative evaluated at (r, h). Therefore, we can write

$$dV = \left(\frac{\partial V}{\partial r}\right)_h dr + \left(\frac{\partial V}{\partial h}\right)_r dh.$$

This result is true for several variables. For a function $w(x, y, z)$,

$$dw = \left(\frac{\partial w}{\partial x}\right)_{y,z} dx + \left(\frac{\partial w}{\partial y}\right)_{x,z} dy + \left(\frac{\partial w}{\partial z}\right)_{x,y} dz. \quad \text{(N.1)}$$

The derivatives are evaluated as though the variables being held fixed were ordinary constants. If $w = 3x^2yz^4$,

$$\left(\frac{\partial w}{\partial x}\right)_{y,z} = 6xyz^4,$$

$$\left(\frac{\partial w}{\partial y}\right)_{x,z} = 3x^2z^4,$$

$$\left(\frac{\partial w}{\partial z}\right)_{x,y} = 12x^2yz^3.$$

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It is also possible to take higher derivatives, such as $\partial^2 w/\partial x^2$ or $\partial^2 w/\partial x \partial y$. One important result is that the order of differentiation is unimportant, if the function, its first derivatives, and the derivatives in question are continuous at the point where they are evaluated. Without filling in all the details of a rigorous proof, we will simply note that

$$f = \frac{\partial w}{\partial x} = \lim_{\Delta x \to 0} \left(\frac{w(x + \Delta x, y) - w(x, y)}{\Delta x} \right)$$

$$g = \frac{\partial w}{\partial y} = \lim_{\Delta y \to 0} \left(\frac{w(x, y + \Delta y) - w(x, y)}{\Delta y} \right).$$

The mixed partials are

$$\frac{\partial^2 w}{\partial y \partial x} = \frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \left(\frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} \right)$$
$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \left(\frac{w(x + \Delta x, y + \Delta y) - w(x, y + \Delta y) - w(x + \Delta x, y) + w(x, y)}{\Delta x \, \Delta y} \right)$$

$$\frac{\partial^2 w}{\partial x \partial y} = \frac{\partial g}{\partial x}$$
$$= \lim_{\substack{\Delta y \to 0 \\ \Delta x \to 0}} \left(\frac{w(x + \Delta x, y + \Delta y) - w(x + \Delta x, y) - w(x, y + \Delta y) + w(x, y)}{\Delta x \, \Delta y} \right).$$

The right side of each of these equations is the same, except for the order of the terms. Thus,

$$\frac{\partial}{\partial x} \frac{\partial w}{\partial y} = \frac{\partial}{\partial y} \frac{\partial w}{\partial x}.$$

Problems

Problem 1. If $w = 12x^3 y + z$, find the three partial derivatives $\partial w/\partial x$, $\partial w/\partial y$, and $\partial w/\partial z$.

Problem 2. If $V = xyz$ and $x = 5$, $y = 6$, $z = 2$, find dV when $dx = 0.01$, $dy = 0.02$, and $dz = 0.03$. Make a geometrical interpretation of each term.

Appendix O
Some Fundamental Constants and Conversion Factors

The values of the fundamental constants are from the 2010 least-squares adjustment, available at http://www.nist.gov/pml/data/index.cfm

Symbol	Constant	Value	SI units
c	Velocity of light in vacuum	2.997925×10^{8}	$m\ s^{-1}$
e	Elementary charge	1.602177×10^{-19}	C
F	Faraday constant	9.64853×10^{4}	$C\ mol^{-1}$
g	Standard acceleration of free fall	9.80665	$m\ s^{-2}$
h	Planck's constant	6.626070×10^{-34}	J s
$\hbar$	Planck's constant (reduced)	1.054572×10^{-34}	J s
		6.582119×10^{-16}	eV s
k_B	Boltzmann's constant	1.380649×10^{-23}	$J\ K^{-1}$
		8.617343×10^{-5}	$eV\ K^{-1}$
m_e	Electron rest mass	9.109383×10^{-31}	kg
$m_e c^2$	Electron rest energy	8.187105×10^{-14}	J
		5.10999×10^{5}	eV
m_p	Proton rest mass	1.672622×10^{-27}	kg
N_A	Avogadro's number	6.022141×10^{23}	mol^{-1}
r_e	Classical electron radius	2.817940×10^{-15}	m
R	Gas constant	8.31446	$J\ mol^{-1}\ K^{-1}$
u	Mass unit (^{12}C standard)	1.660539×10^{-27}	kg
uc^2	Mass unit (energy units)	9.31494×10^{8}	eV
ϵ_0	Electrical permittivity of free space	8.85419×10^{-12}	$C^2\ N^{-1}\ m^{-2}$
$1/4\pi\epsilon_0$		8.98755×10^{9}	$N\ m^2\ C^{-2}$
σ_{SB}	Stefan Boltzmann constant	5.67037×10^{-8}	$W\ m^{-2}\ K^{-4}$
λ_C	Compton wavelength of electron	2.42631×10^{-12}	m
μ_B	Bohr magneton	9.274010×10^{-24}	$J\ T^{-1}$
μ_0	Magnetic permeability	$4\pi \times 10^{-7}$	$T\ m\ A^{-1}$
	of space	$\approx 12.566 \times 10^{-7}$	
μ_N	Nuclear magneton	5.050784×10^{-27}	$J\ T^{-1}$

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Some of the more useful conversion factors for converting from older units to SI units are listed. (Taken from *Standard for Metric Practice,* ASTM E 380-76, Copyright 1976 by the American Society for Testing and Materials, Philadelphia)

To convert from	To	Multiply by
Angstrom	Meter	1.000000×10^{-10}
Atmosphere (standard)	Pascal	1.013250×10^{5}
Bar	Pascal	1.000000×10^{5}
Barn	Meter^2	1.000000×10^{-28}
Calorie (thermochemical)	Joule	4.184000
Centimeter of mercury (0 °C)	Pascal	1.33322×10^{3}
Centimeter of water (4 °C)	Pascal	9.80638×10^{1}
Centipoise	Pascal second	1.000000×10^{-3}
Curie	Becquerel	3.700000×10^{10}
Dyne	Newton	1.000000×10^{-5}
Electron volt	Joule	1.60218×10^{-19}
Erg	Joule	1.000000×10^{-7}
Fermi (femtometer)	Meter	1.000000×10^{-15}
Gauss	Tesla	1.000000×10^{-4}
Liter	Meter^3	1.000000×10^{-3}
Mho	Siemens	1.000000
Millimeter of mercury	Pascal	1.33322×10^{2}
Poise	Pascal second	1.000000×10^{-1}
Roentgen	Coulomb per kilogram	2.58×10^{-4}
Torr	Pascal	1.33322×10^{2}

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