

## Appendix A

# A Short Introduction to MATLAB

MATLAB<sup>1</sup> is an interactive software system for numerical computations, simulations and visualizations. It contains a large number of predefined functions and allows users to implement their programs in so-called *m-files*.

The name MATLAB originates from *MATrix LABORatory*, which indicates the matrix orientation of the software. Indeed, matrices are the major objects in MATLAB. Due to the simple and intuitive use of matrices, we consider MATLAB well suited for teaching in the field of Linear Algebra.

In this short introduction we explain the most important ways to enter and operate with matrices in MATLAB. One can learn the essential matrix operations as well as important algorithms and concepts in the context of matrices (and Linear Algebra in general) by actively using the *MATLAB-Minutes* in this book. These only use predefined functions.

A matrix in MATLAB can be entered in form of a list of entries enclosed by square brackets. The entries in the list are ordered by rows in the natural order of the indices, i.e., from “top to bottom” and “left to right”). A new row starts after every semicolon. For example, the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \text{ is entered in MATLAB by typing } A=[1 \ 2 \ 3;4 \ 5 \ 6;7 \ 8 \ 9];$$

A semicolon after the matrix A suppresses the output in MATLAB. If it is omitted then MATLAB writes out all the entered or computed quantities. For example, after entering

$$A=[1 \ 2 \ 3;4 \ 5 \ 6;7 \ 8 \ 9]$$

---

<sup>1</sup>MATLAB® is a registered trademark of The MathWorks Inc.

MATLAB gives the output

```
A =
    1  2  3
    4  5  6
    7  8  9
```

One can access parts of matrices by the corresponding indices. The list of indices from  $k$  to  $m$  is abbreviated by

$k:m$ .

A colon  $:$  means all rows for given column indices, or all columns for given row indices. If  $A$  is as above, then for example

$A(2,1)$  is the matrix  $[4]$ ,  
 $A(3,1:2)$  is the matrix  $[7\ 8]$ ,  
 $A(:,2:3)$  is the matrix  $\begin{bmatrix} 2 & 3 \\ 5 & 6 \\ 8 & 9 \end{bmatrix}$ .

There are several predefined functions that produce matrices. In particular, for given positive integers  $n$  and  $m$ ,

<code>eye(n)</code>	the identity matrix $I_n$ ,
<code>zeros(n,m)</code>	an $n \times m$ matrix with all zeros,
<code>ones(n,m)</code>	an $n \times m$ matrix with all ones,
<code>rand(n,m)</code>	an $n \times m$ “random matrix”.

Several matrices (of appropriate sizes) be combined to a new matrix. For example, the commands

```
A=eye(2); B=[4;3]; C=[2 -1]; D=[-5]; E=[A B;C D]
```

lead to

```
E =
    1  0  4
    0  1  3
    2 -1 -5
```

The help function in MATLAB is started with the command `help`. In order to get information about specific functions one adds the name of the function. For example:

Input:	Information on:
help ops	operations and operators in MATLAB (in particular addition, multiplication, transposition)
help matfun	MATLAB functions that operate with matrices
help gallery	collection of example matrices
help det	determinant
help expm	matrix exponential function

# Selected Historical Works on Linear Algebra

(We describe the content of these works using modern terms.)

- A. L. CAUCHY, *Sur l'équation à l'aide de laquelle on détermine les inégalités séculaires des mouvements des planètes*, Exercices de Mathématiques, 4 (1829).  
Proves that real symmetric matrices have real eigenvalues.
- H. GRASSMANN, *Die lineale Ausdehnungslehre, ein neuer Zweig der Mathematik*, Otto Wiegand, Leipzig, 1844.  
Contains the first development of abstract vector spaces and linear independence, including the dimension formula for subspaces.
- J. J. SYLVESTER, *Additions to the articles in the September Number of this Journal, "On a new Class of Theorems," and on Pascal's Theorem*, Philosophical Magazine, 37 (1850), pp. 363–370.  
Introduces the terms matrix and minor.
- J. J. SYLVESTER, *A demonstration of the theorem that every homogeneous quadratic polynomial is reducible by real orthogonal substitutions to the form of a sum of positive and negative squares*, Philosophical Magazine, 4 (1852), pp. 138–142.  
Proof of Sylvester's law of inertia.
- A. CAYLEY, *A memoir on the theory of matrices*, Proc. Royal Soc. of London, 148 (1858), pp. 17–37.  
First presentation of matrices as independent algebraic objects, including the basic matrix operations, the Cayley-Hamilton theorem (without a general proof) and the idea of a matrix square root.
- K. WEIERSTRASS, *Zur Theorie der bilinearen und quadratischen Formen*, Monatsber. Königl. Preußischen Akad. Wiss. Berlin, (1868), pp. 311–338.  
Proof of the Weierstrass normal form, which implies the Jordan normal form.
- C. JORDAN, *Traité des substitutions et des équations algébriques*, Paris, 1870.  
Contains the proof of the Jordan normal form independent of Weierstrass' work.
- G. FROBENIUS, *Ueber lineare Substitutionen und bilineare Formen*, J. reine angew. Math., 84 (1878), pp. 1–63.  
Contains the concept of the minimal polynomial, the (arguably) first complete proof of the Cayley-Hamilton theorem, and results on equivalence, similarity and congruence of matrices (or bilinear forms).
- G. PEANO, *Calcolo Geometrico secondo l'Ausdehnungslehre di H. Grassmann preceduto dalle operazioni della logica deduttiva*, Fratelli Bocca, Torino, 1888.  
Contains the first axiomatic definition of vector spaces, which Peano called "sistemi lineari", and studies properties of linear maps, including the (matrix) exponential function and the solution of differential equation systems.

- I. SCHUR, *Über die charakteristischen Wurzeln einer linearen Substitution mit einer Anwendung auf die Theorie der Integralgleichungen*, Math. Annalen, 66 (1909), pp. 488–510.  
Proof of the Schur form of complex matrices.
- O. TOEPLITZ, *Das algebraische Analogon zu einem Satze von Fejér*, Math. Zeitschrift, 2 (1918), pp. 187–197.  
Introduces the concept of a normal bilinear form and proves the equivalence of normality and unitary diagonalizability.
- F. D. MURNAGHAN AND A. WINTNER, *A canonical form for real matrices under orthogonal transformations*, Proc. Natl. Acad. Sci. U.S.A., 17 (1931), pp. 417–420.  
Proof of the real Schur form.
- C. ECKART AND G. YOUNG, *A principal axis transformation for non-hermitian matrices*, Bull. Amer. Math. Soc., 45 (1939), pp. 118–121.  
Proof of the modern form of the singular value decomposition of a general complex matrix.

# Bibliography

- [BryL06] K. Bryan, T. Leise, The \$25,000,000,000 eigenvector: the linear algebra behind google. *SIAM Rev.* **48**, 569–581 (2006)
- [Der03] H. Derksen, The fundamental theorem of algebra and linear algebra. *Am. Math. Monthly* **110**, 620–623 (2003)
- [Ebb91] H.-D. Ebbinghaus et al., *Numbers* (Springer, New York, 1991)
- [EstH10] E. Estrada, D.J. Higham, Network properties revealed through matrix functions. *SIAM Rev.* **52**, 696–714 (2010)
- [Hig08] N.J. Higham, *Functions of Matrices: Theory and Computation* (SIAM, Philadelphia, 2008)
- [HorJ12] R.A. Horn, C.R. Johnson, *Matrix Analysis*, 2nd edn. (Cambridge University Press, Cambridge, 2012)
- [HorJ91] R.A. Horn, C.R. Johnson, *Topics in Matrix Analysis* (Cambridge University Press, Cambridge, 1991)
- [HorO96] R.A. Horn, I. Olkin, When does  $A^*A = B^*B$  and why does one want to know? *Amer. Math. Monthly* **103**, 470–482 (1996)
- [LanT85] P. Lancaster, M. Tismenetsky, *The Theory of Matrices: With Applications*, 2nd edn. (Academic Press, San Diego, 1985)
- [Loe14] N. Loehr, *Advanced Linear Algebra* (CRC Press, Boca Raton, 2014)
- [MolV03] C. Moler, C. Van Loan, Nineteen dubious ways to compute the exponential of a matrix, twenty-five years later. *SIAM Rev.* **45**, 349 (2003)
- [Pta56] V. Pták, Eine Bemerkung zur Jordanschen Normalform von Matrizen. *Acta Sci. Math. Szeged* **17**, 190–194 (1956)
- [Sha91] H. Shapiro, A survey of canonical forms and invariants for unitary similarity. *Linear Algebra. Appl.* **147**, 101–167 (1991)

# Index

## A

Abuse of notation, 142  
Adjacency matrix, 259  
Adjoint, 188  
    Euclidean vector space, 190  
    unitary vector space, 192  
Adjunct matrix, 91  
Adjugate matrix, 91  
Algebraic multiplicity, 202  
Alternating, 89  
Angle between vectors, 173  
Annihilator, 164, 230  
Assertion, 10

## B

Backward substitution, 48  
Basis, 119  
    dual, 156  
Basis extension theorem, 120  
Bessel's identity, 181  
Bijective, 15  
Bilinear form, 159  
    non-degenerate, 159  
    positive definite, 290  
    symmetric, 159  
Binomial formula, 51  
Bivariate polynomial, 132  
Block matrix, 39  
Block multiplication, 48

## C

Canonical basis of  $K^{n,m}$ , 120  
Cartesian product, 18  
Cauchy-Schwarz inequality, 171  
Cayley-Hamilton theorem, 105

Centralizer, 33

Characteristic polynomial  
    of a matrix, 102  
    of an endomorphism, 202

Chemical reaction, 262

Cholesky factorization, 289

Circuit simulation, 6, 267

Codomain, 14

Column vector, 116

Commutative, 24

Commutative diagram, 146, 148

Companion matrix, 103

Complex numbers, 30  
    absolute value, 31  
    modulus, 31

Composition, 16

Congruent matrices, 161

Conjunction, 10

Contraposition, 11

Coordinate map, 146

Coordinates, 124

Coordinate transformation matrix, 128, 144

Cosine theorem, 173

Cramer's rule, 96

Cross product, 182

Cycle, 97

Cyclic decomposition, 237

Cyclic subspace, 228

## D

De Morgan law, 21

Derivative of a polynomial, 152

Determinant, 82

    alternating, 89

    computation via  $LU$ -decomposition, 91

    computational formulas, 88

continuous, 83  
 linear, 90  
 multiplication theorem, 90  
 normalized, 86  
 Diagonal matrix, 46  
 Diagonalizable, 203  
 Dimension formula  
   for linear maps, 140  
   for subspaces, 129  
 Dimension of a vector space, 123  
 Direct sum, 130, 226  
 Disjoint, 13  
 Disjunction, 10  
 Division with remainder, 214  
 Domain, 14  
 Dual basis, 156  
 Dual map, 157  
 Dual pair, 159  
 Dual space, 155  
 Duhamel integral, 266

**E**

Echelon form, 57  
 Eigenspace, 200  
 Eigenvalue  
   algebraic multiplicity, 202  
   geometric multiplicity, 200  
   of a matrix, 106  
   of an endomorphism, 199  
 Eigenvector  
   of a matrix, 106  
   of an endomorphism, 199  
 Elementary matrices, 55  
 Elementary row operations, 57  
 Empty list, 118  
 Empty map, 14  
 Empty product, 27  
 Empty set, 12  
 Empty sum, 27, 118  
 Endomorphism, 135  
   diagonalizable, 203  
   direct sum, 226  
   nilpotent, 229  
   normal, 225, 271  
   orthogonal, 277  
   positive (semi-)definite, 288  
   selfadjoint, 195  
   simultaneous triangulation, 223  
   triangulation, 207  
   unitarily diagonalizable, 226, 272  
   unitary, 277  
   unitary triangulation, 210

Equivalence, 10  
 Equivalence class, 19  
 Equivalence normal form, 69  
 Equivalence relation, 18  
   congruent matrices, 161  
   equivalent matrices, 69  
   left equivalent matrices, 71  
   normal form, 19  
   similar matrices, 108  
 Equivalent matrices, 69, 148  
 Euclidean theorem, 217  
 Evaluation homomorphism, 152  
 Exchange lemma, 121  
 Exchange theorem, 122  
 Extended coefficient matrix, 75

**F**

Field, 28  
 Finite dimensional, 123  
 Fourier expansion, 180  
 Fundamental Theorem of Algebra, 218

**G**

Gaussian elimination algorithm, 57  
 Generalized eigenvector, 244  
 Geometric multiplicity, 200  
 Givens rotation, 280  
 $GL_n(R)$ , 46  
 Grade of a vector, 227  
 Gram-Schmidt method, 175  
 Graph, 258  
 Group, 23  
   additive, 25  
   homomorphism, 25  
   multiplicative, 25  
 Group of units, 33

**H**

Hermitian, 162, 163  
 Hilbert matrix, 64, 71, 98  
 Homogeneous, 73, 263  
 Homomorphism, 135  
 Hooke's law, 268  
 Householder matrix, 185, 280

**I**

Identity, 15  
 Identity matrix, 38  
 Image, 15, 137  
 Implication, 10

- Index set, 13
- Inertia, 286
- Initial value problem, 261
- Injective, 15
- Inner product, 167
- Insurance premiums, 3, 43
- Integral domain, 34
- Invariant subspace, 200
- Inverse, 17
- Inverse map, 17
- Invertible, 17, 28, 45
- Isomorphism, 135
  
- J**
- Jordan block, 234
- Jordan canonical form, 238
  - algorithm for computing, 245
- Jordan chain, 244
  
- K**
- Kernel, 137
- Kronecker delta-function, 38
- Kronecker product, 303
- Krylov subspace, 228
  
- L**
- Lagrange basis, 153
- Laplace expansion, 95
- Least squares approximation, 6, 179, 301
- Left adjoint, 188
- Left ideal, 51
- Linear, 2, 135
- Linear factor, 202
- Linear form, 155
- Linear functional, 155
- Linearly independent, 118
- Linear map, 135
  - change of bases, 148
  - dual, 157
  - matrix representation, 144
  - rank, 149
  - transpose, 158
- Linear matrix equation, 308
- Linear optimization problem, 5
- Linear regression, 6, 178, 301
- Linear span, 117
- Linear system, 73
  - homogeneous, 73
  - non-homogeneous, 73
  - solution algorithm, 76
  - solution set, 73, 139
  
- Logical values, 11
- Low rank approximation, 299
- $LU$ -decomposition, 61
- Lyapunov equation, 309
  
- M**
- Map, 14
- MATLAB-Minute, 42, 49, 61, 64, 91, 108, 210, 223, 243, 258, 281, 298
- Matrix, 37
  - (non-)singular, 45
  - block, 39
  - column-stochastic, 109
  - complex symmetric, 196
  - diagonal, 46
  - diagonal entries, 38
  - diagonalizable, 203
  - diagonally dominant, 94
  - empty, 38
  - Hermitian, 163
  - Hermitian part, 291
  - Hermitian transpose, 163
  - invertibility criteria, 94, 108
  - invertible, 45, 64, 71, 93
  - negative (semi-)definite, 288
  - nilpotent, 112
  - normal, 271
  - orthogonal, 177
  - positive, 110
  - positive (semi-)definite, 288
  - row-stochastic, 4
  - skew-Hermitian part, 291
  - skew-symmetric, 42
  - square, 38
  - symmetric, 42
  - transpose, 42
  - triangular, 46
  - triangulation, 208
  - unitarily diagonalizable, 272
  - unitary, 177
  - unitary triangulation, 210
  - zero divisor, 45, 69
- Matrix exponential function, 257
- Matrix function, 253
- Matrix operations, 39
- Matrix representation
  - adjoint map, 195
  - bilinear form, 160
  - dual map, 157
  - linear map, 144
  - sesquilinear form, 163
- Minimal polynomial, 241

Minor, 91  
 Möbius transformation, 291  
 Monic, 103  
 Moore-Penrose inverse, 300  
 Multiplication theorem for determinants, 90

## N

Negative (semi-)definite, 288  
 Network, 260
 

- centrality, 260
- communicability, 260

 Neutral element, 23  
 Nilpotency index, 229  
 Nilpotent, 33, 112, 229  
 Non-homogeneous, 73, 263  
 Norm, 168
 

- $L^2$ -, 169
- $\infty$ -, 170
- $p$ -, 169
- Euclidean, 169
- Frobenius, 169
- induced by a scalar product, 172
- maximum column sum, 170
- maximum row sum, 170
- unitarily invariant, 301

 Normal, 225, 271  
 Normal form, 19  
 Normed space, 169  
 $n$ -tuple, 18  
 Nullity, 140  
 Null ring, 38  
 Null space, 137  
 Null vector, 116

## O

One-form, 155  
 Ordered pair, 18  
 Ordinary differential equation, 261  
 Orthogonal basis, 173  
 Orthogonal complement, 182  
 Orthogonal endomorphism, 277  
 Orthogonal matrix, 177  
 Orthogonal vectors, 173  
 Orthonormal basis, 173

## P

PageRank algorithm, 1, 109  
 Parallelogram identity, 184  
 Parseval's identity, 181  
 People
 

- Abel, Niels Henrik (1802–1829), 24

- Bessel, Friedrich Wilhelm (1784–1846), 181
- Bézout, Étienne (1730–1783), 216
- Cantor, Georg (1845–1918), 9, 13, 17
- Cauchy, Augustin Louis (1789–1857), 98, 171, 257
- Cayley, Arthur (1821–1895), 37, 105
- Cholesky, André-Louis (1875–1918), 289
- Collatz, Lothar (1910–1990), 1
- Cramer, Gabriel (1704–1752), 95
- Descartes, René (1596–1650), 17
- Dodgson, Charles Lutwidge (1832–1898), 37
- Duhamel, Jean-Marie Constant (1797–1872), 266
- Eckart, Carl Henry (1902–1973), 296
- Euclid of Alexandria (approx. 300 BC), 167
- Fourier, Jean Baptiste Joseph (1768–1830), 180
- Frobenius, Ferdinand Georg (1849–1917), 66, 105, 169
- Gauß, Carl Friedrich (1777–1855), 57, 218
- Givens, Wallace (1910–1993), 280
- Gram, Jørgen Pedersen (1850–1916), 175
- Graßmann, Hermann Günther (1809–1877), 23, 122
- Hamilton, Sir William Rowan (1805–1865), 41, 105, 116
- Hermite, Charles (1822–1901), 57, 162
- Hilbert, David (1862–1943), vii, 23, 64
- Hooke, Sir Robert (1635–1703), 268
- Householder, Alston Scott (1904–1993), 185
- Jordan, Marie Ennemond Camille (1838–1922), 238, 296
- Kirchhoff, Gustav Robert (1824–1887), viii, 7
- Kronecker, Leopold (1832–1891), 38, 303
- Krylov, Aleksey Nikolaevich (1863–1945), 228
- Lagrange, Joseph-Louis (1736–1813), 153
- Laplace, Pierre-Simon (1749–1827), 95
- Leibniz, Gottfried Wilhelm (1646–1716), 82
- Lyapunov, Alexandr Mikhailovich (1857–1918), 309

- Möbius, August Ferdinand (1790–1868), 291
- Moore, Eliakim Hastings (1862–1932), 300
- Parseval, Marc-Antoine (1755–1836), 181
- Peano, Giuseppe (1858–1932), 13
- Penrose, Sir Roger (1931–), 300
- Perron, Oskar (1880–1975), 111
- Pták, Vlastimil (1925–1999), 227
- Pythagoras of Samos (approx. 570–495 BC), 173
- Ruffini, Paolo (1765–1822), 215
- Sarrus, Pierre Frédéric (1798–1861), 83
- Schmidt, Erhard (1876–1959), 175, 296
- Schur, Issai (1875–1941), 209, 292
- Schwarz, Hermann Amandus (1843–1921), 171
- Steinitz, Ernst (1871–1928), 122
- Sylvester, James Joseph (1814–1897), 37, 91, 105, 140, 287, 296, 308
- Toeplitz, Otto (1881–1940), 240, 271
- Vandermonde, Alexandre-Théophile (1735–1796), 98
- Weierstraß, Karl Theodor Wilhelm (1815–1897), 29, 238
- Wilkinson, James Hardy (1919–1986), 98
- Zehfuss, Johann Georg (1832–1901), 303
- Permutation, 81
  - associated permutation matrix, 87
  - inversion, 82
- Permutation matrix, 49
- Perron eigenvector, 111
- Pivot positions, 64
- Polar decomposition, 297
- Polarization identity, 165
- Polynomial, 31
  - common root, 224
  - constant, 213
  - coprime, 214
  - degree, 31, 103, 213
  - divisor, 214
  - irreducible, 214
  - monic, 103
  - multiplicity of a root, 216
- Positive definite, 167, 288, 290
- Positive semidefinite, 288
- Power set, 14
- Predicate, 10
- Pre-image, 15, 137
- Principal axes transformation, 282
- Principal vector, 244
- Projection, 197, 249
- Proof by contraposition, 11
- Pseudoinverse, 300
- Pythagorean theorem, 173
- Q**
- $QR$ -decomposition, 176
- Quadratic form, 165, 283
- Quadric, 285
- Quantifiers, 11
- Quotient field, 34
- Quotient set, 19
- R**
- Rank, 66, 149
- Rank-nullity theorem, 140
- Rational functions, 34
- Reflection matrix, 280
- Reflexive, 18
- Relation, 18
- Residue class, 20
- Restriction, 15
- Right adjoint, 188
- Right-hand rule, 183
- Ring, 26
  - multiplicative inverse, 28
  - of matrices, 45
  - of polynomials, 31
- Ring homomorphism, 51
- Root of a polynomial, 107
  - simple, 250
- Rotation matrix, 280
- Row vector, 116
- S**
- Sarrus rule, 83
- Scalar product, 167
- Schur complement, 52, 292
- Schur form
  - of a matrix, 210
  - of an endomorphism, 210
  - real, 274
- Schur's theorem, 209
- Selfadjoint, 195
- Sesquilinear form, 162
- Set, 9
  - cardinality, 14
  - difference, 13
  - intersection, 13
  - union, 13

Sherman-Morrison-Woodbury formula, 52  
 Sign, 82  
 Signature, 82  
 Signature formula of Leibniz, 82  
 Similar matrices, 108, 148  
 Singular value decomposition (SVD), 296  
 Skew-symmetric, 42  
 Standard basis of  $K^{n,m}$ , 120  
 Standard scalar product of  $\mathbb{C}^{n,1}$ , 168  
 Standard scalar product of  $\mathbb{R}^{n,1}$ , 168  
 Stephanos' theorem, 307  
 Subfield, 29  
 Subgroup, 25  
 Subring, 34  
 Subset, 11  
 Subspace, 117  
     complement, 133  
     invariant, 200  
 Surjective, 15  
 Sylvester equation, 308  
 Sylvester's law of inertia, 287  
 Symmetric, 18, 42, 159  
 Symmetric group, 82  
 System of linear differential equations, 263

**T**

Tensor product, 303  
 Toeplitz matrix, 240  
 Trace, 103, 113, 168  
 Transitive, 18  
 Transposition, 42, 84  
 Triangle inequality, 169  
 Triangular matrix, 46

Triangulation, 207

**U**

Unit circle, 170  
 Unit vectors, 120  
 Unitarily diagonalizable, 226, 272  
 Unitary endomorphism, 277  
 Unitary matrix, 177  
 Unitary triangulation, 210

**V**

Vandermonde matrix, 98  
 Vec map, 305  
 Vector product, 182  
 Vector space, 115  
     Euclidean, 167  
     of bilinear forms, 164  
     of continuous functions, 116  
     of homomorphisms, 136  
     of matrices, 116  
     of polynomials, 116  
     unitary, 167

**W**

Wilkinson matrix, 91, 98

**Z**

Zero divisor, 29, 45, 69  
 Zero matrix, 38  
 Zero vector space, 118