Epilogue

The approaches and methods described in this book have long histories and have been employed to make major advances in research on many challenging fundamental problems. These approaches to modelling and problem reduction still form the basis for a great deal of current research on more advanced problems. As one example, the quasi-steady state assumption from Chap. 10 is used as the basis for handling large systems of complex bio-chemical reactions.

Current directions in modelling of physical systems have also put a new focus on complex systems [86] and *multiscale modelling* [33]. These are models that seek to constructively incorporate layers of understanding that come from different physical scales, often referred to as micro-, meso-, and macro-scales. Examples include (i) improved models for materials properties in continuum model settings that are derived from microscale models at the atomic scale and (ii) systems biology models of organs or entire physiological systems building from models of cells and biochemical reactions.

One of many active forums for current work on modelling is the journal *Multiscale Modeling and Simulation* published by the Society of Industrial and Applied Mathematics (SIAM). Many multiscale models make use of asymptotic approaches[1] to implement matching of descriptions at different scales in combination with numerical computations to make progress on fundamental questions in a broad array of application areas. We hope that this book has given readers a strong starting point for moving on to such advanced and challenging problems.

TW and MB

[1]Including boundary layers and other extensions of multiple scales, like averaging [73, 102] and homogenisation theory [49].

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Appendix A
Trigonometric Identities and Fourier Series

A brief summary of useful trigonometric identities:

$$\sin^2 x + \cos^2 x = 1$$

$$\sin x \cos y + \cos x \sin y = \sin(x + y) \qquad \cos x \cos y - \sin x \sin y = \cos(x + y)$$

$$\sin x \cos y = \tfrac{1}{2}\left(\sin(x - y) + \sin(x + y)\right) \qquad \sin x \sin y = \tfrac{1}{2}\left(\cos(x - y) - \cos(x + y)\right)$$

$$\cos x \cos y = \tfrac{1}{2}\left(\cos(x - y) + \cos(x + y)\right)$$

$$\sin(2t) = 2\sin t \cos t \qquad \cos(2t) = 2\cos^2 t - 1$$

$$\sin^2 t = \frac{1 - \cos 2t}{2} \qquad \cos^2 t = \frac{1 + \cos 2t}{2}$$

$$\sin^3 t = \frac{3\sin t - \sin 3t}{4} \qquad \cos^3 t = \frac{3\cos t + \cos 3t}{4}$$

$$\sin^2 t \cos t = \frac{\cos t - \cos 3t}{4} \qquad \sin t \cos^2 t = \frac{\sin t + \sin 3t}{4}$$

All trigonometric identities can be derived from successive applications of the formulas for the sum or difference of angles $(x \pm y)$ and further algebra.

The need for these identities can be eliminated by using Euler's formula, $e^{i\theta} = \cos\theta + i\sin\theta$, to replace cosine and sine by their complex representations,

$$\cos t = \mathrm{Re}(e^{it}) = \frac{e^{it} + e^{-it}}{2} \qquad \sin t = \mathrm{Im}(e^{it}) = \frac{e^{it} - e^{-it}}{2i},$$

then all results follow from algebra and re-grouping $e^{\pm int}$ to determine coefficients of $\cos(nt)$ and $\sin(nt)$ terms.

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A.1 Trigonometric Fourier Series

Analogous to the way that every nth order polynomial can be expressed as a finite Taylor polynomial, the trigonometric identities allow every nth order product of sines and cosines to be written as a sum of sines and cosines. Fourier series generalise this to represent all integrable periodic functions in terms of an infinite series of sines and cosines.

Expansions of given periodic functions, $f(x)$ on $-L < x < L$ can be written as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\tfrac{n\pi}{L}x) + \sum_{n=1}^{\infty} b_n \sin(\tfrac{n\pi}{L}x) \tag{A.1a}$$

where for $n = 0, 1, 2, \ldots$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(\tfrac{n\pi}{L}x)\,dx \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(\tfrac{n\pi}{L}x)\,dx. \tag{A.1b}$$

The Fourier series for a function f will converge to the value of the function at all points where $f(x)$ is continuous.

The Fourier expansion can also be written in *complex form* as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L} \qquad c_n = \frac{1}{2L}\int_{-L}^{L} f(x) e^{-in\pi x/L}\,dx. \tag{A.2}$$

The *Fourier cosine series* for a function $F(x)$ given on $0 \le x < L$ can be derived from the general Fourier series by defining $f(x)$ to be the even extension of $F(x)$:

$$f(x) = \begin{cases} F(x) & 0 \le x < L, \\ F(-x) & -L < x \le 0 \end{cases}$$

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\tfrac{n\pi}{L}x) \qquad a_n = \frac{2}{L}\int_0^L F(x)\cos(\tfrac{n\pi}{L}x)\,dx. \tag{A.3}$$

Similarly, the *Fourier sine series* for a function $F(x)$ given on $0 \le x < L$ can be derived from the general Fourier series by defining $f(x)$ to be the odd extension of $F(x)$:

$$f(x) = \begin{cases} F(x) & 0 \le x < L, \\ -F(-x) & -L < x \le 0 \end{cases}$$

$$F(x) = \sum_{n=1}^{\infty} b_n \sin(\tfrac{n\pi}{L}x) \qquad b_n = \frac{2}{L}\int_0^L F(x)\sin(\tfrac{n\pi}{L}x)\,dx. \tag{A.4}$$

Solutions to Selected Problems

Chapter 1

1.1 Solving the ODE for m first yields $m(t) = m_0 e^{-t}$. After substituting $m(t)$ into the equation for $z(t)$, this equation can be integrated twice and with $z'(0) = 0$ yields $z(t) = (\tau - g)(e^t - t - 1)$. From this solution we can see that lift-off occurs ($z > 0$ for $t > 0$) if $\tau > g$.

1.2 For (1.8), direct integration yields $A(t) = A_0 + kt$. For (1.9), separation of variables yields $A(t) = A_0 e^{-kt}$ and for (1.10) substitution into the equation for $B(t)$ yields $B(t) = B_0 + A_0 - A_0 e^{-kt}$. For (1.12), note that $dA/dt = dB/dt$ so $B(t) = A(t) + c$ where from the IC's $c = B_0 - A_0$, then substituting into the ODE for A, separation of variables yields $A(t) = A_0(B_0 - A_0)/(B_0 \exp(-k(A_0 - B_0)t) - A_0)$.

1.3 See (10.12a).

1.4 See Exercise 4.5 and set $\mathsf{F} = 0$ in (4.49).

1.5 (a) All solutions starting from $x_0 \neq 0$ approach either $x = \pm 1$. The *basin of attraction* of $x = -1$ is $x_0 < 0$ and all solutions having $x(t \to \infty) \to 1$ have $x_0 > 0$. (b) Second-order equilibria occur where $f = 0$ has a double root, $f'(x_*) = 1 - 3x_*^2 = 0$, namely $x_* = \pm 1/\sqrt{3}$. Substituting these x_* into $f(x_*) = 0$ yields $k_\pm = \pm 2\sqrt{3}/9$. Plotting directions on the phase lines shows these bifurcation values correspond to the changes in qualitative behaviours.

Chapter 2

2.1 Using the hint, we can express the problem as

$$\frac{df_{\text{avg}}}{dt} = \lim_{\varepsilon \to 0} \frac{d}{dt}\left(\frac{1}{\varepsilon h(t)} \int_{a(t)}^{a(t)+\varepsilon h(t)} f(x,t)\, dx\right).$$

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We are taking a derivative of a product of functions with the second factor being an integral, hence we apply Leibniz's rule (with respect to t) to it,

$$= \frac{1}{\varepsilon}\left(-\frac{h'}{h^2}\int_a^{a+\varepsilon h} f\,dx + \frac{1}{h}\left[\int_a^{a+\varepsilon h}\frac{\partial f}{\partial t}\,dx + f(a+\varepsilon h, t)\frac{d(a+\varepsilon h)}{dt} - f(a,t)\frac{da}{dt}\right]\right)$$

Now we are ready to consider the limit for $\varepsilon \to 0$. We will expand out the first two terms in the Taylor series for $\varepsilon \to 0$, using Leibniz's rule again, to take the derivative of integrals (with respect to ε),

$$\approx \frac{1}{\varepsilon}\left(-\frac{h'}{h^2}\left[\int_a^a f\,dx + \varepsilon f(a,t) - 0\right]\right.$$
$$\left. + \frac{1}{h}\left[\int_a^a \frac{\partial f}{\partial t}\,dx + \varepsilon\frac{\partial f}{\partial t}h\bigg|_{x=a} + \left(f + \varepsilon h\frac{\partial f}{\partial x}\right)\bigg|_{x=a}\left(\frac{da}{dt} + \varepsilon\frac{dh}{dt}\right) - f(a,t)\frac{da}{dt}\right]\right)$$

Eliminating null integrals, $\int_a^a g\,dx = 0$, and cancelling terms reduces this to

$$= \frac{1}{\varepsilon}\left(-\frac{\varepsilon f h'}{h} + \frac{1}{h}\left[\varepsilon\frac{\partial f}{\partial t}h + \varepsilon f\frac{dh}{dt} + \varepsilon h\frac{\partial f}{\partial x}\frac{da}{dt} + \varepsilon^2 h\frac{\partial f}{\partial x}\frac{dh}{dt}\right]\bigg|_{x=a}\right)$$
$$= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{da}{dt} + \varepsilon\frac{\partial f}{\partial x}\frac{dh}{dt}\bigg|_{x=a}$$

Hence we obtain the limit as the convective derivative at $x = a(t)$,

$$\lim_{\varepsilon\to 0}\frac{df_{\text{avg}}}{dt} = \frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x}\bigg|_{x=a}$$

In three dimensions it can be shown that

$$f_{\text{avg}}(t) = \frac{\iiint_{D(t)} f\,dV}{\iiint_{D(t)} dV} \qquad \Longrightarrow \qquad \lim_{D(t)\to 0}\frac{df_{\text{avg}}}{dt} = \frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla f\bigg|_{x=a}.$$

2.2 Expand the left-hand side of the conservation of momentum equation using the product rule

$$\begin{aligned}\frac{\partial(\rho v)}{\partial t} + \frac{\partial(\rho v^2)}{\partial x} &= \rho\frac{\partial v}{\partial t} + v\frac{\partial \rho}{\partial t} + \rho v\frac{\partial v}{\partial x} + v\frac{\partial(\rho v)}{\partial x}\\ &= \rho\frac{\partial v}{\partial t} + \rho v\frac{\partial v}{\partial x} + v\underbrace{\left[\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial x}\right]}_{=0}\\ &= \rho\left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x}\right)\end{aligned}$$

where the terms on the second line vanish by virtue of the continuity equation (the conservation of mass).

2.3 (a) Use the trigonometric identity $\cos a \cos b = 2\cos(\frac{1}{2}[a+b])\cos(\frac{1}{2}[a-b])$ to re-write ρ as

$$\rho = 2\cos\left(\tfrac{1}{2}[(k+k+\varepsilon)x - (\omega(k)+\omega(k+\varepsilon))t]x\right)\cos\left(\tfrac{1}{2}[(k+\varepsilon-k)x - (\omega(k+\varepsilon)-\omega(k))t]\right)$$

For $\varepsilon \to 0$ the Taylor series gives $\omega(k+\varepsilon) = \omega(k) + \omega'(k)\varepsilon + \cdots$; using this, for $\varepsilon \to 0$ we get

$$\omega(k+\varepsilon) - \omega(k) \approx \omega'(k)\varepsilon, \qquad \omega(k+\varepsilon) + \omega(k) \approx 2\omega(k)$$

and then the first term in the expansion of ρ is

$$\rho \approx 2\cos(kx - \omega(k)t)\cos\left(\tfrac{1}{2}\varepsilon[x - \omega'(k)t]\right) = 2\cos\left(k\left[x - \frac{\omega(k)}{k}t\right]\right)\cos\left(\tfrac{1}{2}\varepsilon[x - \omega'(k)t]\right),$$

Matching terms to the prescribed form identifies the phase and group velocities as

$$c_p(k) = \frac{\omega(k)}{k}, \qquad c_g(k) = \frac{d\omega}{dk}.$$

(b) Substituting $\rho = \cos(kx - \omega t)$ in the PDE yields

$$\rho_t + \rho_x - \rho_{xxt} = (\omega - k + \omega k^2)\sin(kx - \omega t) = 0$$

and forcing the coefficient to vanish yields the dispersion relation,

$$\omega(k) = \frac{k}{1+k^2}.$$

Similarly, substituting $\rho = \exp(kx - \tilde{\omega} t)$ yields

$$\rho_t + \rho_x - \rho_{xxt} = (-\tilde{\omega} + k + \tilde{\omega} k^2)\exp(kx - \tilde{\omega} t) = 0$$

yielding the modified dispersion relation

$$\tilde{\omega}(k) = \frac{k}{1-k^2}.$$

2.4 (a) Substituting $\rho = P(x - ct)$ in the BBM PDE yields

$$-c\frac{dP}{ds} + \frac{dP}{ds} + 6P\frac{dP}{ds} + c\frac{d^3P}{ds^3} = 0$$

(b) Note that we can integrate the ODE to yield

$$d - cP + P + 3P^2 + c\frac{d^2P}{ds^2} = 0$$

Now considering $P(s) = A\text{sech}^2(Bs)$; for $|s| \to \infty$, $P \to 0$ and similarly all derivatives $P', P'', \ldots \to 0$. Hence, evaluating the ODE for $|s| \to \infty$ we can conclude that the constant of integration is $d = 0$,

$$(1 - c)P + 3P^2 + c\frac{d^2P}{ds^2} = 0$$

Substituting in the sech^2 solution form, after some algebra the ODE gives,

$$(4A + 16cAB^2 - 4cA)\cosh(2Bs) + (24A^2 + 4A - 32cAB^2 - 4cA) = 0$$

The equation is satisfied for all s if the coefficients (in parentheses) are each zero. The first coefficient yields

$$(1 + c) + 4cB^2 = 0 \qquad \Longrightarrow \qquad B = \sqrt{\frac{c-1}{4c}},$$

then the second coefficient is obtained from

$$6A + 1 - 8cB^2 - c = 0 \qquad \Longrightarrow \qquad A = \frac{c-1}{2}.$$

Hence the soliton is

$$\begin{aligned}\rho(x,t) = P(x - ct) &= \frac{c-1}{2}\text{sech}^2\left(\sqrt{\frac{c-1}{4c}}\,(x - ct)\right)\\ &= \frac{c-1}{2}\text{sech}^2\left(\frac{1}{2}\left[\sqrt{\frac{c-1}{c}}\,x - \sqrt{c^2 - c}\,t\right]\right),\end{aligned}$$

where the final line is meant to be of the wavenumber-frequency form, $\rho = G(kx - \tilde{\omega}t)$. To check if this satisfies the modified dispersion relationship found in the previous exercise, consider if

$$k(c) = \sqrt{\frac{c-1}{c}} \qquad \tilde{\omega}(c) = \sqrt{c^2 - c},$$

are parametric equations for

$$\tilde{\omega}(k) = \frac{k}{1-k^2} = \frac{\sqrt{(c-1)/c}}{1-(c-1)/c} = c\sqrt{\frac{c-1}{c}} = \sqrt{c^2-c} = \tilde{\omega}(c),$$

so yes, this is verified. Also, the definition of the phase speed, $c = \tilde{\omega}/k = c$ holds. One of many special properties of solitons is that they satisfy dispersion relations obtained from the linearised version of the PDE while having their form determined by the full, nonlinear equation.

2.5 Using the given forms, the third equation, $\phi_t(x,1,t) = -f = -A\cos(kx-\omega t)$ reduces to $-\omega B(1) = -A$. Similarly, the fourth equation, $f_t = \phi_y = B'(1)\cos(kx-\omega t)$ yields $\omega A = B'(1)$. The second equation $\phi_y = B'(0)\sin(kx-\omega t)$ reduces to $B'(0) = 0$. Finally, substituting ϕ into $\phi_{xx} + \phi_{yy} = 0$ yields

$$-k^2 B + \frac{d^2B}{dy^2} = 0 \qquad \Longrightarrow \qquad B(y) = c_1 e^{ky} + c_2 e^{-ky}.$$

Applying the boundary condition $B'(0) = 0$ and $B'(1) = \omega A$ to the general solution of the ODE yields

$$c_1 k - c_2 k = 0, \qquad c_1 k e^k - c_2 k e^{-k} = \omega A \qquad\qquad c_1 = c_2 = \frac{\omega A}{2k\sinh(k)},$$

then $B(1) = A/\omega$ yields the dispersion relation,

$$\frac{\omega A}{k\sinh(k)}\cosh(k) = \frac{\omega A}{k}\tanh(k) = \frac{A}{\omega} \qquad \Longrightarrow \qquad \omega(k) = \sqrt{k\tanh(k)}.$$

2.6 (a) The initial value problems for the characteristic ODEs are

$$\frac{dX}{dt} = e^{2t} \qquad X(0) = A, \qquad \frac{dP}{dt} = P + X + t \qquad P(0) = \cos(A)$$

First solving for X yields $X(t) = A + \frac{1}{2}(e^{2t}-1)$ which can then be plugged into the equation for P to yield

$$P(t) = (A + \cos(A))e^t - A - t - \tfrac{1}{2}(1-e^{2t})$$

On each characteristic curve, we can invert $X(t,A)$ to get $A = x + \frac{1}{2}[1-e^{2t}]$, then substituting into P yields

$$\begin{aligned}\rho(x,t) = &\left(x + \tfrac{1}{2}[1-e^{2t}] + \cos(x + \tfrac{1}{2}[1-e^{2t}])\right)e^t \\ &- \left(x + \tfrac{1}{2}[1-e^{2t}]\right) - t - \tfrac{1}{2}(1-e^{2t}).\end{aligned}$$

(b) The characteristic ODEs are

$$\frac{dX}{dt} = X + 4, \qquad \frac{dP}{dt} = -2P.$$

The general solutions of these equations are

$$X(t) = c_1 e^t - 4, \qquad P(t) = c_2 e^{-2t}.$$

The side conditions give data in two parts.

For $A > 0$, we have $X(0) = A$, $P(0) = e^{-A}$ at $t = 0$. Applying this to the general solution yields

$$X(t) = (A + 4)e^t - 4, \qquad P(t) = e^{-A-2t}.$$

Inverting $x = X(t, A)$ yields $A = (x + 4)e^{-t} - 4 > 0$ and substituting this into $P(t)$ yields

$$\rho(x, t) = \exp\left(4 - (x + 4)e^{-t} - 2t\right) \qquad \text{for } x > 4(e^t - 1).$$

The second part of the data is given at $x = 0 \implies X(T) = 0$ with $P(T) = \cos(T)$ for $t > 0 \implies T > 0$. Applying these conditions to the general solution yields

$$X(t) = 4e^{t-T} - 4, \qquad P(t) = \cos(T)e^{-2t+2T}.$$

Inverting $x = X(t, T)$ yields $T = t - \ln(1 + x/4) > 0$ and substituting this into $P(t)$ yields

$$\rho(x, t) = \cos\left(t - \ln\left(1 + \frac{x}{4}\right)\right)\left(1 + \frac{x}{4}\right)^{-2} \qquad \text{for } x < 4(e^t - 1).$$

2.7 (a) Expanding out the product rule and using the given velocity yields the PDE

$$\frac{\partial \rho}{\partial t} + x^2 e^{-3t}\frac{\partial \rho}{\partial x} = -2xe^{-3t}\rho$$

and consequently the characteristic ODEs

$$\frac{dX}{dt} = X^2 e^{-3t}, \qquad X(0) = A, \qquad \text{for } 1 \le A \le 2.$$

and

$$\frac{dP}{dt} = -2XPe^{-3t}, \qquad P(0) = 1.$$

The problem for $X(t)$ yields

$$X(t, A) = \left(\frac{1}{A} + \tfrac{1}{3}e^{-3t} - \tfrac{1}{3}\right)^{-1},$$

and the problem for $P(t)$ yields

$$P(t, A) = \frac{A^2}{9}\left(e^{-3t} - 1 + \frac{3}{A}\right)^2.$$

Inverting $x = X(t, A)$ for A yields

$$A = \left(\frac{1}{x} + \tfrac{1}{3} - \tfrac{1}{3}e^{-3t}\right)^{-1},$$

and substituting this into $P(t, A)$ yields

$$\rho(x, t) = \left(\frac{x}{3}[e^{-3t} - 1] - 1\right)^{-2}.$$

2.10 (a) Substituting in the definitions of p, q in the first equation yields $\phi_{tt} - c^2 \phi_{xx} = 0$ (the wave equation) and $\phi_{xt} - \phi_{tx} = 0$, always true by the identity of mixed partial derivatives of smooth functions.
(b) The system can be written as

$$\frac{\partial}{\partial t}\begin{pmatrix} p \\ q \end{pmatrix} + \underbrace{\begin{pmatrix} 0 & -c^2 \\ -1 & 0 \end{pmatrix}}_{\mathbf{M}} \frac{\partial}{\partial x}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Following the approach given in Sect. 2.4, we obtain the wavespeeds from

$$|\mathbf{M}^T - \lambda \mathbf{I}| = \begin{vmatrix} -\lambda & -1 \\ -c^2 & -\lambda \end{vmatrix} = \lambda^2 - c^2 = 0 \qquad \Longrightarrow \qquad \lambda = \pm c.$$

The corresponding eigenvectors are then

$$\begin{aligned} \lambda_1 &= c \qquad W_1(x - ct) = p(x, t) - cq(x, t) \\ \lambda_2 &= -c \qquad W_2(x + ct) = p(x, t) + cq(x, t) \end{aligned}$$

(b) Then the solutions can be expressed in terms of W_1, W_2 are

$$p(x, t) = \frac{1}{2}\left(W_1(x - ct) + W_2(x + ct)\right) \qquad q(x, t) = \frac{1}{2c}\left(W_2(x + ct) - W_1(x - ct)\right).$$

(c) We can relate A, B to W_1, W_2 from

$$p = \frac{\partial \phi}{\partial t} = \frac{1}{2}\left(W_1(x-ct) + W_2(x+ct)\right) = -cA'(x-ct) + cB'(x+ct)$$
$$q = \frac{\partial \phi}{\partial x} = \frac{1}{2c}\left(W_2(x+ct) - W_1(x-ct)\right) = A'(x-ct) + B'(x+ct)$$

Both of which yield

$$A'(x-ct) = -\frac{1}{2c}W_1(x-ct) \qquad B'(x+ct) = \frac{1}{2c}W_2(x+ct).$$

(d) At $t = 0$ the initial condition $\phi(x,0) = f(x)$ implies $\phi_x(x,0) = q(x,0) = f'(x)$ and $\phi_t(x,0) = p(x,0) = g(x)$, and consequently

$$W_1(x) = g(x) - cf'(x) \qquad W_2(x) = g(x) + cf'(x)$$

Hence, from part (c)

$$\begin{aligned}
A'(x) &= -\frac{1}{2c}\left(g(x) - cf'(x)\right)\\
A(x) &= \int_0^x A'(\tilde{x})\,d\tilde{x} + a = \frac{f(x) - f(0)}{2} - \frac{1}{2c}\int_0^x g(\tilde{x})\,d\tilde{x} + a\\
B'(x) &= \frac{1}{2c}\left(g(x) + cf'(x)\right)\\
B(x) &= \int_0^x B'(\tilde{x})\,d\tilde{x} + b = \frac{f(x) - f(0)}{2} + \frac{1}{2c}\int_0^x g(\tilde{x})\,d\tilde{x} + b,
\end{aligned}$$

where a, b are constants of integration.

So using $\phi(x,t) = A(x-ct) + B(x+ct)$,

$$\begin{aligned}
\phi &= \tfrac{1}{2}\left(f(x-ct) - f(0)\right) - \frac{1}{2c}\int_0^{x-ct} g(\tilde{x})\,d\tilde{x} + a\\
&\quad + \tfrac{1}{2}\left(f(x+ct) - f(0)\right) + \frac{1}{2c}\int_0^{x+ct} g(\tilde{x})\,d\tilde{x} + b\\
&= \tfrac{1}{2}\left(f(x+ct) + f(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\tilde{x})\,d\tilde{x} + (a + b - f(0)).
\end{aligned}$$

Checking the initial condition, $\phi(x,0) = f(x)$,

$$\phi(x,0) = \tfrac{1}{2}(f(x) + f(x)) + \frac{1}{2c}\underbrace{\int_x^x g(\tilde{x})\,d\tilde{x}}_{=0} + (a + b - f(0)),$$

so the constants of integration should be chosen so that $a+b-f(0)=0$, yielding the D'Alembert solution.

2.12 The conservation law $p_t + q(p)_x = 0$ can be re-written as $p_t + q'(p)p_x = 0$, from which we can write the characteristic equations as

$$\frac{dP}{dt} = 0 \qquad \Longrightarrow \qquad P = f(A) = \text{constant}$$
$$\frac{dX}{dt} = q'(P) \qquad \Longrightarrow \qquad X = q'(f(A))t + A$$

(a) The two characteristic curves will intersect when $X(t_{0,1}, A_0) = X(t_{0,1}, A_1)$, namely

$$q'(f(A_0))t_{0,1} + A_0 = q'(f(A_1))t_{0,1} + A_1 \qquad \Longrightarrow \qquad t_{0,1} = -\frac{A_1 - A_0}{q'(f(A_1)) - q'(f(A_0))}$$

and subsequently using this value of $t_{0,1}$, $x_{0,1} = q'(f(A_0))t_{0,1} + A_0$ (or equivalently in terms of A_1).
(b) Minimising $t_{0,1}$ over all possible A_0, A_1 to get t_*, consider $A_1 = A_0 + \varepsilon$,

$$t_* = \min_{A_0,A_1} -\frac{A_1 - A_0}{q'(f(A_1)) - q'(f(A_0))} = \min_{A_0,\varepsilon} -\frac{\varepsilon}{q'(f(A_0+\varepsilon)) - q'(f(A_0))}$$

Taking the limit $\varepsilon \to 0$ and recalling the limit definition of the derivative,

$$t_* = \min_x \left[-\frac{1}{q''(f(x))f'(x)} \right] \geq 0.$$

2.13 (a) The characteristic ODEs are $dX/dt = P$, $dP/dt = 0$, as in (2.41). Applying the given initial conditions yields the parametric solutions

$$\begin{cases} X(t,A) = (9-A^2)t + A, \quad P(t,A) = 9 - A^2 & |A| \leq 3 \\ X(t,A) = A, \qquad P(t,A) = 0 & |A| > 3 \end{cases}$$

(b) To invert $X(t,A) = x$ for the nontrivial part of the solution, we recognise it as a quadratic equation for A,

$$A^2 t - A + (x - 9t) = 0 \qquad \Longrightarrow \qquad A = \frac{1 \pm \sqrt{1 - 4(xt - 9t^2)}}{2t}$$

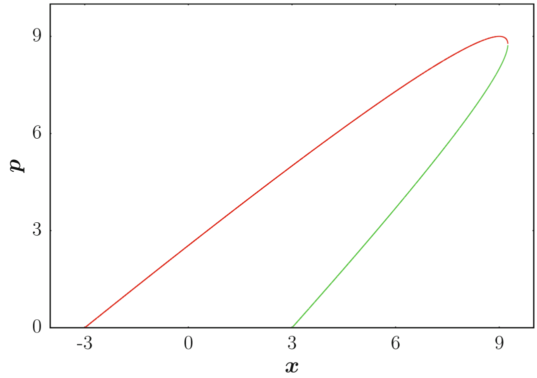
then we can substitute into P to obtain

$$p(x,t) = 9 - \left(\frac{1 \pm \sqrt{1-4(xt-9t^2)}}{2t}\right)^2$$

For more general initial conditions, $p(x, 0) = f(x)$, for the inviscid Burgers' equation we have $X = Pt + A$ with P on a characteristic curve, so we can still write $A = x - Pt$ to obtain an implicit equation for the solution, $p = f(x - pt)$, here $p = 9 - (x - pt)^2$, to yield the equivalent form

$$p(x,t) = \frac{2xt - 1 \pm \sqrt{36t^2 - 4xt + 1}}{2t^2}$$

See below for a graph of this multi-valued solution at time $t = 1$.



(c) The result of Exercise 2.12 gives that the time at which the shock forms is determined by the initial conditions,

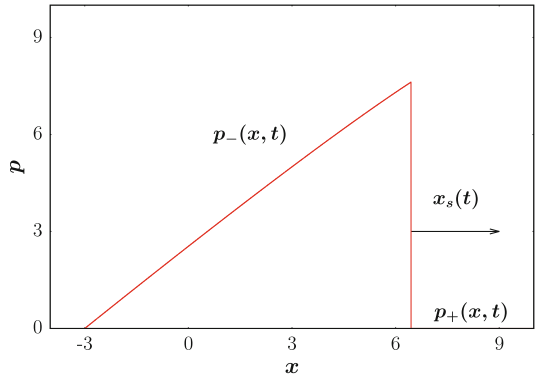
$$t_* = \min_x \left(-\frac{1}{f'(x)}\right) \quad \Longrightarrow \quad t_* = \min_{x\in[-3,3]} \left(-\frac{1}{2x}\right) = \frac{1}{6},$$

and the shock will first form at $x_* = 3$.

(d) The Rankine-Hugoniot equation for the inviscid Burgers' equation (2.49) and the result from part (c) give the ODE problem for the shock position as

$$\frac{dx_s}{dt} = -\frac{2x_s t - 1 + \sqrt{36t^2 - 4x_s t + 1}}{4t^2}, \qquad x_s(1/6) = 3,$$

where the solution ahead of the shock is $p_+(x, t) \equiv 0$ and the solution behind the shock, $p_-(x, t)$ is given by the solution above with the positive sign on the square root.



(e) The inviscid Burgers equation is a conservation law for the integral, $\int p\,dx$,

$$p_t + \left(\tfrac{1}{2}p^2\right)_x = 0 \qquad \Longrightarrow \qquad \frac{d}{dt}\left(\int p\,dx\right) = 0$$

Hence the conserved value is set by the area under the initial condition $\int(9-x^2)\,dx = 36$. Using the form of the solution with the shock, the equation that $\int p(x,t)\,dx = 36$ becomes (for $t > 1/6$, after the shock forms),

$$\int_{-3}^{x_s(t)} \frac{2xt - 1 + \sqrt{36t^2 - 4xt + 1}}{2t^2}\,dx = 36.$$

Evaluating the integral yields an implicit algebraic equation for $x_s(t)$,

$$\frac{1}{12t^3}\left(6x_s^2t^2 - (1 - 4x_st + 36t^2)^{3/2} - 6x_st + 216t^3 + 54t^2 + 1\right) = 36.$$

Chapter 3

3.1 (a, b) Both approaches give

$$-y'' + \frac{k}{x^2}y + x^2 = 0.$$

Multiplying across by x^2, this equation can be recognised as an inhomogeneous Cauchy-Euler equation, $x^2y'' - ky = -x^4$, solvable in terms of a sum of homogeneous and particular solutions.
(c) For $k = 0$ the general solution is $y(x) = x^4/12 + c_1x + c_2$. Applying the boundary conditions yields $c_1 = -1/12$, $c_2 = 1$.
(d) For $k = 2$ the general solution is $y(x) = x^4/10 + c_1x^2 + c_2/x$. Applying the first boundary conditions yields $c_1 = 9/10$, $c_2 = 0$. The condition $y(0) = 1$ cannot be satisfied because unless $c_2 = 0$ that term would diverge and the other terms in the general solution vanish for $x = 0$.

3.2

$$\delta^2 J = \tfrac{1}{2}\int_0^1 \frac{(h')^2}{(1+(y')^2)^{3/2}}\,dx \qquad \Longrightarrow \qquad \delta^2 J_* = \tfrac{1}{2}\int_0^1 \frac{(h')^2}{(1+b^2)^{3/2}}\,dx \geq 0,$$

where $\delta^2 J_*$ will be strictly positive unless $h'(x) \equiv 0$, but from the boundary conditions on y, the boundary conditions on h are $h(0) = h(1) = 0$. Hence $h' \equiv 0$ only if $h \equiv 0$.

3.4 (a) The $x(t)$ Euler-Lagrange equation can be simplified to

$$(y''x'y' - x''(y')^2)(1 + 4k^2(x^2+y^2)) + 4k^2((x')^2 + (y')^2)(x'y - xy')y' = 0.$$

The y Euler-Lagrange equation takes the same form after interchanging $x \leftrightarrow y$.
(b) The Euler-Lagrange equation for $y(x)$ can be reduced to

$$y'' = \frac{4k^2(1+(y')^2)(xy'-y)}{1+4k^2(x^2+y^2)}.$$

(c) For (ii), since z is constant on the semicircle, the distance is just the arclength of the semi-circle, $J_{ii} = \pi$ for all k.

For (i), we have $y = 0$ and $z = kx^2$, hence the arclength can be calculated from

$$J_i(k) = \int_{-1}^1 \sqrt{1+4k^2x^2}\,dx = \sqrt{1+4k^2} + \frac{1}{2k}\operatorname{arcsinh}(2k).$$

For $k = 0$, $J_i(0) = 2$ giving the length of the straight line in the xy plane, while for $k > 0$ J_i is an unbounded monotone increasing function of k, $J_i(k \to \infty) \to \infty$. Let k_* be the value of k for which $J_i(k_*) = \pi$. Then for $0 \leq k < k_*$, the straight-line path will be shorter than the semi-circle detours. But for $k > k_*$, the semi-circle is shorter.

3.5 (a) After the substitution, the action in terms of $\theta(t)$ is

$$I = \int \tfrac{1}{2}m\ell^2(\theta')^2 + mg\ell\cos\theta\,dt$$

and from (3.21) applied to $\theta(t)$ the resulting Euler-Lagrange is the equation of the pendulum,

$$\theta'' + \frac{g}{\ell}\sin\theta = 0.$$

(b) See Exercise 3.27 for the Euler-Lagrange equation for $x(t)$.

3.6 Substituting the parametric equation for x, y into the general Lagrangian, $L = \frac{1}{2}m((x')^2 + (y')^2) - mgy$ yields

$$L = \tfrac{1}{2}m\ell^2(\theta')^2 + mg\ell\cos\theta - \sigma m\left(\ell\omega\theta'\cos(\omega t)\sin\theta - g\sin(\omega t)\right) + \tfrac{1}{2}m\sigma^2\omega^2\cos^2(\omega t)$$

Applying (3.21) then yields the (vertically-oscillated) parametrically-driven pendulum equation,

$$\theta'' + \frac{g}{\ell}\sin\theta = -\frac{\omega^2\sigma}{\ell}\sin(\omega t)\sin\theta.$$

3.7 (a) Starting from $H = -L + y'\partial_{y'}L$,

$$\begin{aligned}
\frac{dH}{dt} &= \frac{d}{dt}\left(-L + y'\frac{\partial L}{\partial y'}\right) \\
&= -\left(\frac{\partial L}{\partial t} + \frac{\partial L}{\partial y}\frac{dy}{dt} + \frac{\partial L}{\partial y'}\frac{d^2y}{dt^2}\right) + \frac{d^2y}{dt^2}\frac{\partial L}{\partial y'} + y'\frac{d}{dt}\left(\frac{\partial L}{\partial y'}\right) \\
&= -\frac{\partial L}{\partial t} - y'\left[\frac{\partial L}{\partial y} - \frac{d}{dt}\left(\frac{\partial L}{\partial y'}\right)\right] = 0,
\end{aligned}$$

where the final step makes use of the assumption that $\partial_t L = 0$ to leave $-y'$ times the Euler-Lagrange equation. Hence $H' = 0$ and H is a constant, justifying (3.59).
(b) Writing the Lagrangian as $L = T(t, y, y') - V(t, y)$ the condition that the Hamiltonian is the total energy is

$$H = -T + V + y'\frac{\partial T}{\partial y'} = T + V \qquad \Longrightarrow \qquad y'\frac{\partial T}{\partial y'} = 2T$$

This is a first order separable ODE for T as a function of y' and yields $T(t, y, y') = A(t, y)(y')^2$ where $A(t, y)$ is any function of t and y.

3.8 (a) At $t = 0$, the mass starts at $x = 0$, $y = 1$ from rest, $v = 0$, hence the initial energy is $E_0 = mg$. Equating the energy at later times with E_0 yields

$$\tfrac{1}{2}mv^2 + mgy = mg \qquad \Longrightarrow \qquad v(y) = \sqrt{2g(1-y)}$$

Consequently the functional for the time of travel is

$$T = \int_0^1 \sqrt{\frac{1 + (y')^2}{2g(1-y)}}\,dx \qquad L(y, y') = \sqrt{\frac{1 + (y')^2}{2g(1-y)}}$$

(b) The Euler-Lagrange equation for $y(x)$ can be simplified down to

$$\frac{1}{2}\left(\frac{1+(y')^2}{(1-y)^3}\right)^{1/2} - \frac{d}{dx}\left(\frac{y'}{\sqrt{(1+(y')^2)(1-y)}}\right) = 0,$$

but this is a difficult looking equation, so we turn to a different approach.
(c) The Beltrami identity (3.59) is applicable since L does not explicitly depend on x, so

$$\begin{aligned} H &= L - y'\frac{\partial L}{\partial y'} \\ &= \sqrt{\frac{1+(y')^2}{2g(1-y)}} - \frac{(y')^2}{\sqrt{2g(1+(y')^2)(1-y)}} \\ &= \frac{1}{\sqrt{2g(1+(y')^2)(1-y)}} = C \end{aligned}$$

At $x = 0$, $y = 1$ and to make C finite, the slope would have to diverge, $y' \to -\infty$, so this form is indeterminate. At $x = 1$, we have $y = 0$, so $C = 1/\sqrt{2g(1 + y'(1)^2)}$, but we don't know $y'(1)$. Still, we can write a singular ODE problem for $y(x)$,

$$\frac{dy}{dx} = -\sqrt{\frac{1}{2gC^2(1-y)} - 1}, \qquad y(0) = 1, \qquad y(1) = 0.$$

(d) Substituting the parametric equations of the cycloid (3.60) into the ODE gives a necessary relation between C and k, $1 = 4gC^2k$. The initial conditions at $\theta = 0$, $x(0) = 0$ and $y(0) = 1$ are satisfied automatically for any k. The final state, $x = 1$ and $y = 0$ at some $\theta = \theta_*$ yield two coupled equations for finding (k, θ_*):

$$k(\theta_* - \sin\theta_*) = 0 \qquad 1 - k(1 - \cos\theta_*) = 0.$$

3.10 Following the approach given in Sect. 3.4.2 with $f(x) = 1+(x-1)^2$, $\tilde{y} = y_* + \varepsilon h$ and $\tilde{b} = b_* + \varepsilon c$, we obtain the first variation as

$$\delta J_* = y'_* h\Big|_0^{b_*} - \int_0^{b_*} y''_* h\, dx + \tfrac{1}{2} y'_*(b)^2 c.$$

The boundary condition at the origin determines that $h(0) = 0$. Expanding out the boundary condition $\tilde{y}(\tilde{b}) = f(\tilde{b})$ yields

$$y_*(b_*) = f(b_*), \qquad y'_*(b_*)c + h(b_*) = f'(b_*)c$$

consequently, $c = h(b_*)/(f'(b_*) - y'_*(b_*))$. Using this, the first variation can be re-written as

$$\delta J_* = y'_*(b_*)\left(1 + \frac{y'_*(b_*)}{2(f'(b_*) - y'_*(b))}\right)h(b_*) - \int_0^{b_*} y''_* h\,dx.$$

There are two possible choices for natural boundary conditions that would make the boundary term vanish for all possible choices of $h(b)$: either $y'_*(b_*) = 0$ or (after some algebra) $y'_*(b_*) = 2f'(b_*)$ (we will explore both possibilities). Subject to natural boundary conditions, applying the fundamental lemma to the critical point condition $\delta J_* = 0$, we obtain the Euler-Lagrange equation $y''_* = 0$. After applying the boundary condition $y(0) = 0$, viable solutions are of the form $y_*(x) = Ax$. The boundary condition $y'_*(b_*) = 0$ would yield $A = 0$, but this option must be rejected since the solution would not reach the curve $y = f(x)$. Hence, using $y'_*(b_*) = 2f'(b_*)$, we get $A = 4(b_* - 1)$ and then $y_*(b_*) = f(b_*)$ yields the quadratic equation, $3b_*^2 - 2b_* - 2 = 0$ and the solution

$$y_*(x) = \frac{4}{3}(\sqrt{7} - 2)x \qquad 0 \le x \le \frac{1 + \sqrt{7}}{3}.$$

3.12 Beginning by substituting $\tilde{y} = y_* + \varepsilon h$ and expanding for $\varepsilon \to 0$ yields

$$\tilde{J} = J_* + \varepsilon \int_0^1 h' y'''_* + y'_* h''' - 240xh\,dx + \cdots$$

Using integration by parts to get the derivatives off of the h's, we arrive at a form where the fundamental lemma can be applied to critical point condition, yielding the ODE,

$$-2y'''_* - 240x = 0.$$

Note that all of the boundary terms vanish thanks to

$$y'(0) \text{ given} \implies h'(0) = 0 \qquad y''(1) \text{ given} \implies h''(1) = 0$$

$$y'''(0) \text{ given} \implies h'''(0) = 0 \qquad y(1) \text{ given} \implies h(1) = 0$$

The ODE can be integrated directly to give the solution as a polynomial, after applying the four boundary conditions, the final solution is

$$y_*(x) = -x^5 + 10x^2 - 4.$$

3.13 Beginning by substituting $\tilde{y} = y_* + \varepsilon h$ and expanding for $\varepsilon \to 0$ yields

$$\tilde{J} = J_* + \varepsilon \int_0 \left[2y_*' h' + (1-2x) \int_0^x 2y_*(t)h(t)dt \right] dx + \cdots$$

Applying integration by parts yields the first variation as

$$\left(2y_*' h + (x - x^2) \int_0^x 2y_* h\, dt \right) \Big|_0^1 - \int_0^1 \left(2y_*'' + 2(x-x^2)y_* \right) hdx$$

The second boundary term vanishes automatically at $x = 0$ and $x = 1$ due to the $x(1-x)$ factor. The first boundary term vanishes under the natural boundary conditions

$$2y_*'(0)h(0) = 0 \qquad \Longrightarrow \qquad \{y_*'(0) = 0 \quad \text{or} \quad h(0) = 0 \implies y_*(0) = A\}$$

and

$$2y_*'(1)h(1) = 1 \qquad \Longrightarrow \qquad \{y_*'(1) = 0 \quad \text{or} \quad h(1) = 0 \implies y_*(1) = B\}$$

3.14 Beginning by substituting $\tilde{y} = y_* + \varepsilon h$ and expanding for $\varepsilon \to 0$ yields

$$\tilde{J} = J_* + \varepsilon \left(\int_0^1 2y_* y_*'' h'' + (y_*'')^2 h + 2y_* h\, dx + y_*'(0)h'(1) + y_*'(1)h'(0) \right) + \cdots$$

Integrating by parts yields the first variation as

$$\left(2y_*(1)y_*''(1) + y_*'(0) \right) h'(1) + \left(-2y_*(0)y_*''(0) + y_*'(1) \right) h'(0) \\ + \int_0^1 \left(2(y_* y_*'')'' + (y_*'')^2 + 2y \right) h\, dx$$

Hence the Euler-Lagrange equation is

$$\left(\frac{d^2 y_*}{dx^2} \right)^2 + 2\frac{d^2}{dx^2} \left(y_* \frac{d^2 y_*}{dx^2} \right) + 2y_* = 0$$

and the further natural boundary conditions needed are

$$\{2y_*(1)y_*''(1) + y_*'(0) = 0 \quad \text{or} \quad h'(1) = 0 \implies y_*'(1) = A\}$$

and

$$\{-2y_*(0)y_*''(0) + y_*'(1) = 0 \quad \text{or} \quad h'(0) = 0 \implies y_*'(0) = B\}$$

3.15 (a) Using $v = c/n(x)$, functional (3.6) becomes

$$T(y) = \frac{1}{c}\int_0^1 n(x)\sqrt{1+(y')^2}\,dx \qquad \Longrightarrow \qquad L(x,y') = \frac{n(x)}{c}\sqrt{1+(y')^2}$$

Applying (3.21) to this Lagrangian yields

$$0 - \frac{d}{dx}\left(\frac{n(x)y'}{\sqrt{1+(y')^2}}\right) = 0,$$

which can be integrated once, and applying the initial condition yields

$$\frac{n(x)y'}{\sqrt{1+(y')^2}} = C = \frac{n(0)}{\sqrt{2}} \qquad \Longrightarrow \qquad \frac{dy}{dx} = \frac{n(0)}{\sqrt{2n(x)^2 - n(0)^2}}$$

(b) Yes, $n_1 \sin\theta_1 = n_2 \sin\theta_2$.
(c) $L(x,y,y') = n(x,y)\sqrt{1+(y')^2}/c$ yields the Euler-Lagrange equation

$$\sqrt{1+(y')^2}\,\frac{\partial n}{\partial y} - \frac{d}{dx}\left(\frac{n(x,y)y'}{\sqrt{1+(y')^2}}\right) = 0.$$

3.16 (a) The total kinetic and potential energies are given by

$$T = \int_0^\ell \tfrac{1}{2}\rho u_t^2\,dx \qquad V = \int_0^\ell \tfrac{1}{2}EIu_x^2\,dx$$

(b) $L = T - V$ then the action is

$$J = \int_{t_0}^{t_1} L\,dt = \int_{t_0}^{t_1}\int_0^\ell \left(\tfrac{1}{2}\rho u_t^2 - \tfrac{1}{2}EIu_x^2\right)\,dx\,dt$$

(c) Substituting $\tilde{u}(x,t) = u_*(x,t) + \varepsilon h(x,t)$ and expanding for $\varepsilon \to 0$ yields

$$\tilde{J} = J_* + \varepsilon\int_{t_0}^{t_1}\int_0^\ell \left(\rho u_{*t}h_t - EIu_{*xx}h_{xx}\right)\,dx\,dt + \cdots$$

Splitting the $O(\varepsilon)$ term into separate integrals, we interchange the order of integration on the first,

$$\int_0^\ell\int_{t_0}^{t_1} \rho u_{*t}h_t\,dt\,dx - \int_{t_0}^{t_1}\int_0^\ell EIu_{*xx}h_{xx}\,dx\,dt$$

Applying integration by parts with respect to the inner integrals respectively yields

$$\rho\left(\int_0^{\ell}\left[u_{*t}h\Big|_{t_0}^{t_1}\right]dx - \int_{t_0}^{t_1}\int_0^{\ell} u_{*tt}h\,dx\,dt\right)$$
$$+EI\left(\int_{t_0}^{t_1}\left[u_{*xx}h_x - u_{*xxx}h\Big|_0^{\ell}\right]dt - \int_{t_0}^{t_1}\int_0^{\ell} u_{*xxxx}h\,dx\,dt\right)$$

The perturbation in the solution at initial and final times $t = t_0, t_1$ can be assumed to be zero $h(x, t_0) \equiv h(x, t_1) \equiv 0$ to eliminate the first boundary terms. Eliminating the second boundary term defines natural boundary conditions to remove the $u_{*xx}h_x$ and $u_{*xxx}h$ terms at each time,

$$\{u_{*xx}(0,t) = 0 \quad \text{or} \quad h_x(0) = 0 \quad \Longrightarrow \quad u_{*x}(0,t) = A(t)\}$$

and

$$\{u_{*xxx}(0,t) = 0 \quad \text{or} \quad h(0) = 0 \quad \Longrightarrow \quad u_*(0,t) = B(t)\}$$

and similarly for the boundary conditions at $x = 1$.

Applying the fundamental lemma to $\iint(\rho u_{*tt} + EIu_{*xxxx})h\,dx\,dt = 0$ yields the beam equation PDE as the Euler-Lagrange equation,

$$\rho\frac{\partial^2 u}{\partial t^2} + EI\frac{\partial^4 u}{\partial x^4} = 0.$$

3.17 Substituting $\tilde{u}(x, y) = u_*(x, y) + \varepsilon h(x, y)$ and expanding for $\varepsilon \to 0$ yields

$$\tilde{J} = J_* + \varepsilon\iint_D k\left(u_{*x}h_x + u_{*y}h_y\right)\,dA + \cdots$$

The first variation can be written in vector form as $\iint k\nabla u_* \cdot \nabla h\,dA$ then matching to the product rule with $\mathbf{g} = k\nabla u_*$ and $f = h$, it can be expressed as

$$\delta J_* = \iint_D \nabla\cdot(hk\nabla u_*)\,dA - \iint_D \nabla\cdot(k\nabla u_*)h\,dA$$

Applying the divergence theorem to the first integral changes it to an integral on the boundary

$$\delta J_* = \oint_{\partial D} h(k\mathbf{n}\cdot\nabla u_*)\,ds - \iint_D \nabla\cdot(k\nabla u_*)h\,dA$$

Eliminating the boundary integral leads to two choices for natural boundary conditions: (i) homogeneous Neumann conditions, $\mathbf{n} \cdot \nabla u = \frac{\partial u}{\partial n} = 0$, or (ii) specified Dirichlet conditions, $h(\partial D) = 0 \implies u(\partial D) = f(\partial D)$ given. Subsequently, the fundamental lemma can be applied to the remaining double integral to yield the elliptic PDE

$$\nabla \cdot (k \nabla u) = 0.$$

3.19 (a) The augmented Lagrangian for this problem is

$$\mathscr{L} = 1 + (y')^2 - \lambda \left(y^2 - \tfrac{80}{\pi} \right),$$

then the Euler-Lagrange problem is

$$\frac{d^2 y}{dx^2} + \lambda y = 0, \qquad y(0) = 0, \qquad y(\pi) = 1, \qquad \int_0^\pi y^2 \, dx = 80.$$

The ODE is a linear-constant coefficient equation; it breaks down into two cases, depending on the value of $\lambda \gtrless 0$.

If $\lambda = -\alpha^2 < 0$ then the solution of the boundary value problem for the ODE is

$$y_-(x) = \frac{\sinh(\alpha x)}{\sinh(\alpha \pi)}.$$

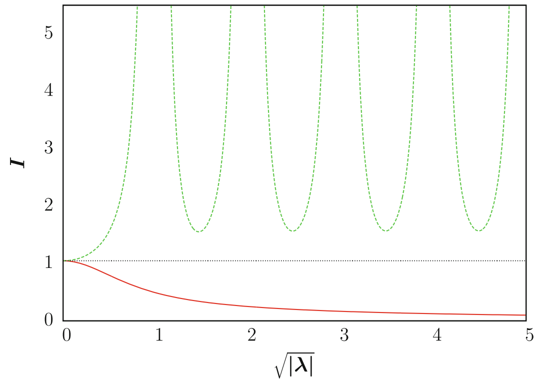
If $\lambda = \alpha^2 > 0$ then the solution is

$$y_+(x) = \frac{\sin(\alpha x)}{\sin(\alpha \pi)}.$$

In the two cases, the integral becomes

$$I_\pm(\alpha) = \int_0^\pi y_\pm^2 \, dx = \left\{ \frac{2\alpha\pi - \sin(2\alpha\pi)}{4\alpha \sin^2(\alpha\pi)}, \ \frac{\sinh(2\alpha\pi) - 2\alpha\pi}{4\alpha \sinh^2(\alpha\pi)} \right\}$$

$I_+(\alpha) \geq \pi/3$ and has multiple solutions for any value of the constraint greater than $\pi/3 \approx 1.047$. Meanwhile $0 \leq I_-(\alpha) \leq \pi/3$ and is monotone decreasing, so there is a single solution for each value of the constraint.



3.20 (a) Recall Green's theorem,

$$\oint_{\partial D} P(x,y)\,dx + Q(x,y)\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

We have

$$\frac{1}{2}\int_0^1 \left[x(t)y'(t) - y(t)x'(t)\right] dt = \oint \tfrac{1}{2}x\,dy - \tfrac{1}{2}y\,dx.$$

We can identify $P = -\frac{1}{2}y$, $Q = \frac{1}{2}x$, therefore,

$$\frac{1}{2}\int_0^1 \left[x(t)y'(t) - y(t)x'(t)\right] dt = \iint \left(\tfrac{1}{2} + \tfrac{1}{2}\right) dA = \text{Area}.$$

(b) The Euler-Lagrange equations are

$$\frac{d}{dt}\left(y + \frac{\lambda x'}{\sqrt{(x')^2 + (y')^2}}\right) = 0 \qquad \frac{d}{dt}\left(-x + \frac{\lambda y'}{\sqrt{(x')^2 + (y')^2}}\right) = 0$$

(c) Integrating once yields

$$y + \frac{\lambda x'}{\sqrt{(x')^2 + (y')^2}} = c_1 \qquad -x + \frac{\lambda y'}{\sqrt{(x')^2 + (y')^2}} = c_2$$

and can be re-arranged to yield

$$(c_1 - y)^2 + (c_2 + x)^2 = \lambda^2 \left(\frac{(x')^2 + (y')^2}{(x')^2 + (y')^2}\right)^2 = \lambda^2$$

Hence we have the equation of a circle with radius $\lambda = P/(2\pi)$ and centre $(-c_2, c_1)$. Having the origin on the circle means that $c_1^2 + c_2^2 = \lambda^2$, hence we can take $c_1 = \lambda \sin\theta$, $c_2 = \lambda\cos\theta$ for any θ.

3.23 (a) The augmented functional is

$$I = \int_1^2 \left[6y^2 + x^2 \left(\frac{dy}{dx}\right)^2 + x^7 - \lambda(24xy - 5) \right] dx.$$

(b) Substituting $\tilde{y}(x) = y_*(x) + \varepsilon h(x)$ and expanding for $\varepsilon \to 0$ yields the first variation as

$$\delta I = \int_1^2 12y_* h + 2x^2 y_*' h' - 24xh\, dx$$

Examining the boundary condition $\tilde{y}(2) = \tilde{y}(1) + 3$ gives that $h(2) = h(1)$, and applied to the boundary terms produced by integration by parts, $2x^2 y_*' h|_1^2 = 2(4y_*'(2) - y_*'(1))h(1)$, hence we determine the natural boundary condition

$$4y_*'(2) - y_*'(1) = 0$$

and then the Euler-Lagrange equation is

$$12y - (2x^2y')' - 24\lambda x = 0 \qquad \Longrightarrow \qquad x^2y'' + 2xy' - 6y = -12\lambda x.$$

This is an inhomogeneous Cauchy-Euler equation with solution

$$y = c_1x^2 + c_2x^{-3} + 3\lambda x$$

(c) Substituting into the boundary condition, the natural boundary condition and the integral constraint yields three equations for c_1, c_2, λ,

$$3c_1 - \frac{7}{8}c_2 + 3\lambda = 3, \qquad 14c_1 + \frac{9}{4}c_2 + 9\lambda = 0, \qquad 90c_1 + 12c_2 + 168\lambda = 5.$$

3.24 The augmented functional for the constrained problem is

$$I = \int_1^2 \frac{3y^2}{x^5} - \frac{(y')^2}{x^3} - \lambda(y+3)\, dx$$

The resulting Euler-Lagrange equation for $y = y_*(x)$ is

$$x^2y'' - 3xy' + 3y = \tfrac{1}{2}\lambda x^5.$$

The solution of this inhomogeneous Cauchy-Euler equation is

$$y = c_1 x + c_2 x^3 + \frac{\lambda}{16} x^5.$$

Applying the boundary conditions and the integral constraint determine $c_1 = 11$, $c_2 = -8$, $\lambda = 16$.

3.26 (a) The Hamiltonian is

$$\mathscr{H} = L + \lambda f = 4x^2 + 3xu + u^2 + \lambda(3x + u).$$

The PMP yields

$$\frac{dx}{dt} = 3x + u, \qquad \frac{d\lambda}{dt} = -8x - 3u - 3\lambda, \qquad 3x + 2u + \lambda = 0.$$

Eliminating λ reduces the problem to a system of two linear ODEs in x, u,

$$\frac{dx}{dt} = 3x + u \qquad \frac{du}{dt} = -5x - 3u$$

having the general solution

$$x(t) = c_1 e^{2t} + c_2 e^{-2t} \qquad u(t) = -c_1 e^{2t} - 5c_2 e^{-2t}$$

Substituting these into the Hamiltonian yields $\mathscr{H} = 16c_1c_2$. The Hamiltonian is indeed a constant, and to enforce $\mathscr{H} = 0$ we need either $c_1 = 0$ or $c_2 = 0$. The possibility of $c_1 = c_2 = 0$ can be excluded because it would give the trivial solution and could not satisfy the initial condition $x(0) = 2$. Consider $c_2 = 0$, then $c_1 = 2$; this yields a monotone increasing function for $x(t) \geq 2$ and could never satisfy the target condition $x(T) = 1$. Hence $c_1 = 0$ yielding $x(t) = 2e^{-2t}$ and the target condition determines $T_* = \frac{1}{2} \ln 2$.
(b) Everything remains the same, but since $T = \frac{1}{4}$ is imposed, we lose the natural boundary condition, that $\mathscr{H} = 0$, so imposing $x(0) = 2$ and $x(\frac{1}{4}) = 1$ determine the constants, and

$$x(t) = \left(\frac{e^{1/2} - 2}{e - 1}\right) e^{2t} + \left(\frac{2e - e^{1/2}}{e - 1}\right) e^{-2t} \qquad u(t) = -10e^{-2t}.$$

For this solution, the value of the Hamiltonian is

$$\mathscr{H} = -\frac{16(e-2)(2e^{1/2} - 1)}{(e-1)^2} \approx -7.21 < 0.$$

This illustrates the *maximum* in PMP, at the optimal solution (having the optimal stopping time), the value of the Hamiltonian will be maximised; in general $\mathscr{H} \leq 0$.

Chapter 4

4.1 For cases (i, ii, iii) the scaled problem is

$$\frac{d^2y}{dt^2} = -\frac{\Pi_1}{(1+\Pi_2 y)^2} \qquad y(0) = \Pi_3 \qquad y'(0) = -\Pi_4$$

$$\Pi_1 = \frac{4\pi G\rho_\mathsf{E}\mathsf{R}_\mathsf{E}\mathrm{T}^2}{3\mathrm{L}} \qquad \Pi_2 = \frac{\mathrm{L}}{\mathsf{R}_\mathsf{E}} \qquad \Pi_3 = \frac{2}{\mathrm{L}} \qquad \Pi_4 = \frac{\mathsf{V}_0\mathrm{T}}{\mathrm{L}}$$

(i) Starting with $\Pi_4 = 1$ we get $\mathrm{L} = \mathsf{V}_0\mathrm{T}$ then $\Pi_3 = 1$ sets $\mathrm{L} = 2, \mathrm{T} = 2/\mathsf{V}_0$, $\Pi_1 = 8\pi G\rho_\mathsf{E}\mathsf{R}_\mathsf{E}/(3\mathsf{V}_0^2) = \varepsilon \to 0$

$$\frac{d^2y}{dt^2} = -\frac{\varepsilon}{(1+\Pi_2 y)^2} \qquad y(0) = 1 \qquad y'(0) = -1$$

(ii) Starting with $\Pi_1 = 1$ we get $\mathrm{L} = 4\pi G\rho_\mathsf{E}\mathsf{R}_\mathsf{E}\mathrm{T}^2/3$ then $\Pi_3 = 1$ sets $\mathrm{T} = \sqrt{3/(2\pi G\rho_\mathsf{E}\mathsf{R}_\mathsf{E})}$ and $\mathrm{L} = 2$ and $\Pi_4 = \mathsf{V}_0\sqrt{3/(8\pi G\rho_\mathsf{E}\mathsf{R}_\mathsf{E})} = \varepsilon \to 0$

$$\frac{d^2y}{dt^2} = -\frac{1}{(1+\Pi_2 y)^2} \qquad y(0) = 1 \qquad y'(0) = -\varepsilon$$

(iii) Starting with $\Pi_3 = 1$, we get $\mathrm{L} = 2$, then $\Pi_4 = 1$ sets $\mathrm{T} = 2/\mathsf{V}_0$ and $\Pi_1 = 8\pi G\rho_\mathsf{E}\mathsf{R}_\mathsf{E}/(3\mathsf{V}_0^2) = \varepsilon \to 0$

$$\frac{d^2y}{dt^2} = -\frac{\varepsilon}{(1+\Pi_2 y)^2} \qquad y(0) = 1 \qquad y'(0) = -1$$

For (iv) the scaled problem is

$$\frac{d^2y}{dt^2} = -\frac{\Pi_1}{(y+\Pi_2)^2} \qquad y(0) = \Pi_3 \qquad y'(0) = -\Pi_4$$

$$\Pi_1 = \frac{G\mathsf{M}_\mathsf{E}\mathrm{T}^2}{\mathrm{L}^3} \qquad \Pi_2 = \frac{\mathsf{R}_\mathsf{E}}{\mathrm{L}} \qquad \Pi_3 = \frac{2}{\mathrm{L}} \qquad \Pi_4 = \frac{\mathsf{V}_0\mathrm{T}}{\mathrm{L}}$$

Setting $\Pi_3 = \Pi_4 = 1$ yields $\mathrm{L} = 2$ and $\mathrm{T} = 2/\mathsf{V}_0$ and $\Pi_2 = \mathsf{R}_\mathsf{E}/2 = \varepsilon \to 0$

$$\frac{d^2y}{dt^2} = -\frac{\Pi_1}{(y+\varepsilon)^2} \qquad y(0) = 1 \qquad y'(0) = -1$$

4.2 Defining the nondimensionalized solution as $\mathsf{X} = \mathrm{L}x(t)$ with $\mathsf{T} = \mathrm{T}t$, we can write the scaled problem as

$$x'' + x = \Pi_1 \sin(\Pi_2 t), \qquad x(0) = 1, \qquad x'(0) = \Pi_3$$

with $\mathrm{L} = \mathsf{X}_0$ being an imposed scale set by the IC and derived timescale $\mathrm{T} = \sqrt{\mathsf{M}/\mathsf{K}}$ being the inverse of the natural frequency $\omega_0 = \sqrt{\mathsf{K}/\mathsf{M}}$ and

$$\Pi_1 = \frac{\mathsf{F}}{\mathsf{K}\mathsf{X}_0}, \qquad \Pi_2 = \frac{\Omega}{\omega_0}, \qquad \Pi_3 = \frac{\omega_0 \mathsf{V}_0}{\mathsf{X}_0}.$$

4.3 The choice of scalings $\mathrm{L} = \mathsf{B}/\sqrt{\mathsf{KM}}$, $\mathrm{T} = \mathsf{M}/\mathsf{B}$ yields the nondimensional problem

$$x'' + x' + x^3 = \Pi_1 \sin(\Pi_2 t), \qquad x(0) = \Pi_3, \qquad x'(0) = \Pi_4,$$

with parameters

$$\Pi_1 = \frac{\mathsf{F}\mathsf{K}^{1/2}\mathsf{M}^{3/2}}{\mathsf{B}^3}, \qquad \Pi_2 = \omega\frac{\mathsf{M}}{\mathsf{B}}, \qquad \Pi_3 = \frac{\mathsf{A}\mathsf{K}^{1/2}\mathsf{M}^{1/2}}{\mathsf{B}^1}, \qquad \Pi_4 = \frac{\mathsf{C}\mathsf{K}^{1/2}\mathsf{M}^{3/2}}{\mathsf{B}^2}.$$

4.5 (a) After some algebra, we get the scalings

$$\mathrm{X} = \frac{\mathsf{DE}^2}{\mathsf{CFH}}, \qquad \mathrm{Y} = \frac{\mathsf{E}}{\mathsf{F}}, \qquad \mathrm{Z} = \frac{\mathsf{E}^2}{\mathsf{FH}}, \qquad \mathrm{T} = \frac{\mathsf{DE}}{\mathsf{BCH}},$$

then the dimensionless parameters follow as

$$\alpha = \frac{\mathsf{AF}}{\mathsf{BE}}, \qquad \beta = \frac{\mathsf{BCH}^2}{\mathsf{D}^2\mathsf{E}^2}, \qquad \gamma = \frac{\mathsf{BC}}{\mathsf{DE}}, \qquad \delta = \frac{3\mathsf{GE}}{\mathsf{F}^2}$$

and $\mu = \mathsf{X}_0/\mathrm{X}$, $\sigma = \mathsf{Y}_0/\mathrm{Y}$, $\omega = \mathsf{Z}_0/\mathrm{Z}$
(b) $x' = \alpha - y$, $\beta y' = x - z$ and $0 = y - y^2 + \frac{1}{3}\delta y^3 - z$. Solving the last equation for $z = f(y)$ yields a phase plane system for (x, y)

$$x' = \alpha - y \qquad \beta y' = x - y + y^2 - \tfrac{1}{3}\delta y^3$$

A mismatch will occur unless the initial conditions satisfy $\omega = f(\sigma)$.
(c) $x' = \alpha - y$, $0 = x - z$, $\gamma z' = y - y^2 + \frac{1}{3}\delta y^3 - z$. Using the second equation, the other two reduce to

$$x' = \alpha - y \qquad \beta x' = y - y^2 + \tfrac{1}{3}\delta y^3 - x.$$

Equating the two expressions for x' yields $\gamma(\alpha - y) = y - y^2 + \frac{1}{3}\delta y^3 - x$. Finally, implicitly differentiating and using the equation for x' yields

$$y' = \frac{\alpha - y}{1 + \gamma - 2y + \delta y^2}$$

A mismatch will occur unless the initial conditions satisfy $\omega = \mu$.

4.7 The final nondimensionalized system is

$$h_t + hu_x + uh_x = 0 \qquad u_t + uu_x + \frac{1}{\mathrm{Fr}^2}h_x = 0$$

where the Froude number is defined as $\mathrm{Fr} = \mathsf{U}/\sqrt{g\mathsf{H}}$.

4.9 (a) Let $\delta = \mathsf{W}/\mathsf{L}$ then $\Pi_1 = 4(1+\delta)(1+1/\delta)$.
(b) $\Pi_1 = \pi((3(1+\delta) - \sqrt{(3+\delta)(1+3\delta)})(3(1+1/\delta) - \sqrt{(3/\delta+1)(1/\delta+3)}$. $\Pi_{1,\text{ellipse}} > \Pi_{1,\text{rect}}$ if the aspect ratio is sufficiently large or small.
(c) Let $x = \Pi_2, y = \Pi_3$ and $z = 1/\Pi_1^2$ for a more convenient calculations: $z = g(x, y) = \frac{1}{2}(\frac{1}{2} - x)(\frac{1}{2} - y)(x + y - \frac{1}{2})$. Using multivariable calculus, determine the critical points of g: $g_x(x, y) = g_y(x, y) = 0$ yielding $(x, y) = (0, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, 0)$ or $(\frac{1}{3}, \frac{1}{3})$. Of these possibilities, $(\frac{1}{3}, \frac{1}{3})$ maximises $g = 1/432$ and hence minimises $\Pi_{1,\text{tri}} \geq 12\sqrt{3}$. Note that $(12\sqrt{3} \approx 20.78) > (12.56 \approx 4\pi)$ as is expected from the result that the circle minimises the perimeter-to-area ratio, $\Pi_{1,\text{circ}} = 4\pi$.

4.11 $\Pi_1 = \mathsf{AT}/\mathsf{B}$, $\Pi_2 = \mathsf{AC}/\mathsf{B}^2$, $\Pi_3 = \mathsf{A}^2\mathsf{D}/\mathsf{B}^3$, $\mathsf{T} = (\mathsf{B}/\mathsf{A})f(\mathsf{AC}/\mathsf{B}^2, \mathsf{A}^2\mathsf{D}/\mathsf{B}^3)$

4.12 (a) The equations for the dimensional exponents for cancelling out units in the Π's are:

$$\begin{aligned} A - 3B - C + E + F &= 0 \qquad [\mathrm{m}] \\ B + C + D &= 0 \qquad [\mathrm{kg}] \\ -2A - C - 2D - F &= 0 \qquad [\mathrm{s}] \end{aligned}$$

These three equations are linearly independent (as can be seen by reducing them to echelon form), so $\tilde{r} = 3$. There are 6 given quantities and 3 base units ($r = 3$), so we get $n - \tilde{r} = 6 - 3 = 3$ free parameters.

Chapter 5

5.1 Using the rescaled solution (5.1), we can rewrite Burgers' equation as

$$u_t + \left(\frac{\mathsf{UT}}{\mathsf{L}}\right)uu_x = \left(\frac{\mathsf{T}}{\mathsf{L}}\right)\kappa u_{xx}.$$

To make this scale invariant, the coefficient factors need to be set to one. The normalising the first yields $\mathsf{U} = \mathsf{L}/\mathsf{T}$. Normalising the second yields $\mathsf{L} = \mathsf{T}^{1/2}$ and subsequently $\mathsf{U} = \mathsf{T}^{-1/2}$. A scale-invariant similarity variable can be obtained from $\Pi_1 = \mathsf{L}\mathsf{T}^c = \mathsf{T}^{1/2}\mathsf{T}^{1/2} = \mathsf{T}^0$, hence $c = 1/2$ and hence determines $\eta = xt^{-1/2}$. Likewise a scale-invariant similarity function is determined by $\Pi_2 = \mathsf{U}\mathsf{T}^d = \mathsf{T}^{-1/2}\mathsf{T}^d = \mathsf{T}^0$, hence $d = 1/2$ and $f(\eta) = t^{1/2}u$ yielding the similarity solution form $u(x, t) = t^{-1/2}f(\eta)$. Substituting this into Burgers' equation yields

$$\tfrac{1}{2}\left(f + \eta\frac{df}{d\eta}\right) + f\frac{df}{d\eta} = \kappa\frac{d^2f}{d\eta^2}.$$

5.2 Applying (5.1)–(5.3) and the integral yields

$$u_t + \left(\frac{\mathsf{UT}}{\mathsf{L}}\right)uu_x = 0, \qquad (\mathsf{U}^2\mathsf{L})\int_0^\infty u^2\,dx = 1.$$

Making the PDE scale invariant selects the scaling relation $\mathsf{U} = \mathsf{L}/\mathsf{T}$. Then making the integral scale-invariant determines $\mathsf{L} = \mathsf{T}^{2/3}$ and consequently $\mathsf{U} = \mathsf{T}^{-1/3}$. Further, this determines the similarity variable $\eta = x/t^{2/3}$ and the similarity solution as $u(x, t) = t^{-1/3}f(\eta)$. Substituting this into the inviscid Burgers equation yields the ODE for $f(\eta)$,

$$-\tfrac{1}{3}\left(f + \eta\frac{df}{d\eta}\right) + f\frac{df}{d\eta} = 0 \qquad \int_0^\infty f^2\,d\eta = 1.$$

The similarity solution likewise reduces the generalised Burgers equation, $u(u_t + uu_x) = \kappa u_{xx}$, to the ODE

$$-\tfrac{1}{3}f\left(f + 2\eta\frac{df}{d\eta}\right) + f^2\frac{df}{d\eta} = \kappa\frac{d^2f}{d\eta^2}.$$

5.3 (a, b) Applying (5.1) to the PDE yields

$$\frac{\partial u}{\partial t} + \left(\frac{\mathsf{UT}}{\mathsf{L}}\right)u\frac{\partial u}{\partial x} = (\mathsf{T}^{\sigma+1}\mathsf{U}^3)t^\sigma u^4.$$

To make this equation scale invariant, we need $\mathsf{U} = \mathsf{L}/\mathsf{T}$ (from the first coefficient) and $\mathsf{T}^{\sigma+1}\mathsf{U}^3 = 1$ in the second one. The boundary condition must be scale invariant as well,

$$u(0, t) = \left(\frac{\mathsf{T}^3}{\mathsf{U}}\right)t^3 \qquad \Longrightarrow \qquad \mathsf{U} = \mathsf{T}^3,$$

consequently $\mathsf{L} = \mathsf{T}^4$ (from $\mathsf{L} = \mathsf{UT}$). Satisfying the second condition for the scale invariance of the PDE, $\mathsf{T}^{\sigma+1}\mathsf{U}^3 = \mathsf{T}^{\sigma+1}\mathsf{T}^9 = \mathsf{T}^0$ determines $\sigma = -10$.

(c) The scale-invariant similarity variable is $\eta = x/t^4$ and the form of the similarity solution is $u(x,t) = t^3 f(\eta)$. Substituting this form into the PDE reduces it to the ODE,

$$3f - 4\eta \frac{df}{d\eta} + f\frac{df}{d\eta} = f^4.$$

5.4 Satisfying scale-invariance in the heat equation determines $\mathsf{L} = \mathsf{T}^{1/2}$ and the similarity variable $\eta = x/t^{1/2}$.

(a) The boundary condition will be scale invariant for $\mathsf{U} = \mathsf{T}^2$. This yields the form of the similarity solution as $u(x,t) = t^2 f(\eta)$. Substituting in the PDE yields the ODE problem

$$2f - \tfrac{1}{2}\eta f' = f'', \qquad f(0) = 1, \qquad f(\eta \to \infty) \to 0.$$

(b) Here the boundary condition will be scale-invariant if $\mathsf{U} = \mathsf{L} = \mathsf{T}^{1/2}$ yielding the similarity solution $u(x,t) = t^{1/2} f(\eta)$, satisfying the ODE problem,

$$\tfrac{1}{2}(f - \eta f') = f'', \qquad f'(0) = 2, \qquad f(\eta \to \infty) \to 0.$$

(c) Here the boundary condition will be scale-invariant if $\mathsf{U} = 1/\mathsf{L} = \mathsf{T}^{-1/2}$ yielding the similarity solution $u(x,t) = t^{-1/2} f(\eta)$, satisfying the ODE problem,

$$-\tfrac{1}{2}(f + \eta f') = f'', \qquad f'(0) = f(0)^2, \qquad f(\eta \to \infty) \to 0.$$

(d) Here the integral condition will be scale-invariant if $\mathsf{L}^3\mathsf{U} = 1$, or $\mathsf{U} = \mathsf{T}^{-3/2}$ yielding the similarity solution $u(x,t) = t^{-3/2} f(\eta)$, satisfying the ODE problem,

$$-\tfrac{1}{2}(3f + \eta f') = f'', \qquad \int_0^\infty \eta^2 f \, d\eta = 1, \qquad f(\eta \to \infty) \to 0.$$

(e) Here $\mathsf{L} = \mathsf{T} = 1$ but U is a free parameter. The solution can be written as $u(x,t) = Ce^t e^{-x}$, which is actually a travelling wave, $u = Ce^{-(x-t)}$.

5.5 Applying (5.1) to the PDE yields

$$\frac{\partial u}{\partial t} = \left(\frac{\mathsf{T}}{\mathsf{L}^2}\right) \frac{\partial^2 u}{\partial x^2} + \left(\mathsf{U}^3\mathsf{T}\right) u^4.$$

The two coefficients must be normalised in order to make the PDE scale-invariant. The first coefficient determines $\mathsf{L} = \mathsf{T}^{1/2}$ and the second sets $\mathsf{U} = \mathsf{T}^{-1/3}$. The scale-invariant similarity variable is $\eta = x/t^{1/2}$ and the form of the similarity solution is $u(x,t) = t^{-1/3} f(\eta)$, satisfying the ODE

$$-\tfrac{1}{3}f - \tfrac{1}{2}\eta f' = f'' + f^4.$$

5.6 (a) Applying (5.1) to the PDE yields

$$\frac{\partial u}{\partial t} = -\left(\frac{\mathsf{U}^3\mathsf{T}}{\mathsf{L}^2}\right)\frac{\partial}{\partial x}\left(u^3\frac{\partial u}{\partial x}\right) - \left(\frac{\mathsf{UT}}{\mathsf{L}^4}\right)\frac{\partial}{\partial x}\left(u\frac{\partial^3 u}{\partial x^3}\right)$$

The two coefficients must be normalised in order to make the PDE scale-invariant. The first coefficient determines $\mathsf{L}^2 = \mathsf{U}^3\mathsf{T}$; substituting this into $\mathsf{L}^4 = \mathsf{UT}$ determines $\mathsf{U} = \mathsf{T}^{-1/5}$ and subsequently $\mathsf{L} = \mathsf{T}^{1/5}$. The scale-invariant similarity variable is then $\eta = x/t^{1/5}$ and the form of the similarity solution is $u(x, t) = t^{-1/5}f(\eta)$, satisfying the ODE

$$-\tfrac{1}{5}(f + \eta f') = -(f^3 f')' - (ff''')'.$$

Note that the form and scaling of the similarity solution has already been determined, the additional integral condition happens to be consistent, $\int f\, d\eta = 1$.
(b) Applying

$$u(x, t) = \mathsf{U}\tilde{u}(\tilde{x}, \tilde{t}), \qquad x = \mathsf{L}\tilde{x}, \qquad t = t_c + \mathsf{T}\tilde{t}$$

to the PDE yields the same scalings, $\mathsf{U} = \mathsf{T}^{-1/5}$ and $\mathsf{L} = \mathsf{T}^{1/5}$. However the similarity variable and solution now take a modified form,

$$\eta = \frac{x}{(t_c - t)^1/5} \qquad u(x, t) = (t_c - t)^{-1/5}f(\eta),$$

and the similarity function satisfies the modified ODE

$$\tfrac{1}{5}(f + \eta f') = -(f^3 f')' - (ff''')'.$$

In part (a), the similarity solution evolves to a limiting behaviour as $\mathsf{T} \to \infty$ ($t \to \infty$), while for this finite-time blow-up case, divergence occurs as $\mathsf{T} \to 0$ ($t \to t_c$).

5.8 The similarity solutions are

$$h(x, t) = t^{-1/2}f(xt^{-3/8}), \qquad u(x, t) = t^{-5/8}g(xt^{-3/8}).$$

Chapter 6

6.2 At leading order it is straightforward to identify $\delta_0 = 1$ and $x_0 = 3$ as a triple root of $(x_0 - 3)^3 = 0$. The next iteration, $x \sim 3 + \delta_1 x_1$ yields $\delta_1^3 x_1^3 = 216\varepsilon$ hence $\delta_1 = \varepsilon^{1/3}$ and x_1 is given by the one of the three cube roots of 216, namely $x_1 = 6e^{i2\pi k/3}$ for $k = 0, 1, 2$. Going on, we will get $\delta_2 = \varepsilon^{2/3}$ and

$$x \sim 3 + 6\varepsilon^{1/3} + 8\varepsilon^{2/3}, \qquad x \sim 3 - (3 \pm i3\sqrt{3})\varepsilon^{1/3} - (4 \mp i4\sqrt{3})\varepsilon^{2/3}$$

6.3 Observe that setting $\varepsilon = 0$ in the equation yields a contradiction (‘60 = 0’), hence the solutions must be singular in order to yields valid balances. Let $x = \delta X$ to yield

$$\underbrace{\varepsilon^6\delta^3X^3}_{(1)} - \underbrace{5\varepsilon^3\delta^2X^2}_{(2)} - \underbrace{20\varepsilon\delta X}_{(3)} + \underbrace{60}_{(4)} = 0.$$

Consider the different possibilities for dominant balances to determine the distinguished limits yielding solutions. This is a third order polynomial, so there must be exactly three roots that must represented within the set of distinguished limits. These distinguished limits are given by

- Balancing terms (3, 4): $\delta = \varepsilon^{-1}, x \sim 3/\varepsilon$
- Balancing terms (2, 3): $\delta = \varepsilon^{-2}, x \sim -4/\varepsilon^2$
- Balancing terms (1, 2): $\delta = \varepsilon^{-3}, x \sim 5/\varepsilon^3$

6.5 (a) The solution is given by

$$v(t) \sim -\tfrac{1}{2}t^2 - \tfrac{1}{20}\varepsilon t^5 - \tfrac{1}{160}\varepsilon^2 t^8.$$

(b) Note that after factoring out t^2, the magnitude of the terms follows

$$O(1) \gg O(\varepsilon t^3) \gg O(\varepsilon^2 t^6)$$

with the ordering being preserved if $\varepsilon t^3 \ll 1$, namely $0 \leq t \ll O(\varepsilon^{-1/3})$.

6.6 Observe that setting $\varepsilon = 0$ in the equations yields $-y = 1$ and $y = 4$, hence we get a contradiction (y is overdetermined). Rescale each variable independently, $x = \delta(\varepsilon)X$ and $y = \sigma(\varepsilon)Y$ and carry out dominant balance of the system to obtain $\delta = \varepsilon^{-1}$ (from a dominant balance in the first equation) and $\sigma = \varepsilon^0$ (from a dominant balance in the second equation). Subsequently,

$$x \sim \frac{1}{\varepsilon}(5 - 5\varepsilon) \qquad y \sim 4 - 5\varepsilon.$$

6.7 Taking the logarithm of the equation, we get

$$\ln(2) + 2\ln(x) - 5x = 3\ln(2) + \ln(\varepsilon)$$

It is easy to identify $\ln(\varepsilon) \to -\infty$ as the dominant term on the righthand side. Now, substitute $x = \delta_0 x_0$ with the assumption that $\delta_0 \to \infty$,

$$\ln(2) + 2\ln(\delta_0) + 2\ln(x_0) - 5\delta_0 x_0 = 3\ln(2) + \ln(\varepsilon)$$

Since $x_0 = O(1)$, so is its log. Since $z \gg \ln(z)$ as $z \to \infty$ the $5\delta_0 x_0$ term is the largest term on the left-hand side of the equation. Hence we get that $x_0 = 1/5$ and $\delta_0 = -\ln(\varepsilon)$. Note that we have put the negative sign in δ_0 rather than in the coefficient so that the gauge function is positive ($\ln(z) \to -\infty$ for $z \to 0$). In general for dealing with logarithms, it may be better to write them as $\delta_0 = \ln(\frac{1}{\varepsilon})$. The next iteration, $x \sim \delta_0 x_0 + \delta_1 x_1$ (with $\delta_0 \gg \delta_1$) yields a dominant balance between terms coming from the original $2\ln(x)$ and $5x$ terms to determine $\delta_1 = \ln(\ln(\frac{1}{\varepsilon}))$ and the solution as

$$x \sim \tfrac{1}{5}\ln(\tfrac{1}{\varepsilon}) + \tfrac{2}{5}\ln(\ln(\tfrac{1}{\varepsilon})).$$

6.10 (a) Substituting $h = 1 + \varepsilon\eta$ and $u = 1 + \varepsilon\nu$ into the PDEs and then collecting $O(\varepsilon)$ terms yields the linearised system

$$\eta_t + \nu_x + \eta_x = 0 \qquad \nu_t + \nu_x + \mathrm{Fr}^{-2}\eta_x = 0$$

(b) We can then write the system in the form

$$\frac{\partial}{\partial t}\begin{pmatrix}\eta\\ \nu\end{pmatrix} + \underbrace{\begin{pmatrix}1 & 1\\ \mathrm{Fr}^{-2} & 1\end{pmatrix}}_{\mathbf{M}}\frac{\partial}{\partial x}\begin{pmatrix}\eta\\ \nu\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$$

Consequently the determinant condition for the wavespeeds is

$$|\mathbf{M}^T - \lambda\mathbf{I}| = \begin{vmatrix}1-\lambda & \mathrm{Fr}^{-2}\\ 1 & 1-\lambda\end{vmatrix} = 0 \qquad \Longrightarrow \qquad (1-\lambda)^2 = \mathrm{Fr}^{-2}$$

and hence the waves generated will have speeds

$$\lambda = 1 \pm \mathrm{Fr}^{-1}$$

and if $\mathrm{Fr} > 1$ then $\lambda_\pm > 0$ and all waves will move to the right (downstream, subcritical), while if $\mathrm{Fr} < 1$ then $\lambda_\pm \gtrless 0$ and the two classes of waves move in opposite directions.

Chapter 7

7.3 Noting that the inhomogeneous boundary conditions are $O(1)$, the dominant is finite and the coefficient $3 \le (4 - x^2) \le 4$ is bounded on the domain, we expect the solution to be finite and bounded, hence we will assume $\beta = 0$ and seek solutions of the form $y = Y(X)$. Letting $X = (x - x_*)/\varepsilon^\alpha$ for $\alpha \ge 0$, equivalently we have $x = x_* + \varepsilon^\alpha X$. The inhomogeneous term then becomes

$$\cos(\tfrac{\pi}{2}x) = \cos(\tfrac{\pi}{2}x_* + \varepsilon^\alpha\tfrac{\pi}{2}X)$$

There are two cases that need to be considered for the scaling of this term with respect to the position of the boundary layer:

$$\begin{aligned} x_* = 0 \qquad &\Longrightarrow \qquad \cos(\tfrac{\pi}{2}x) = \cos(\varepsilon^\alpha\tfrac{\pi}{2}X) \sim 1\\ x_* = 1 \qquad &\Longrightarrow \qquad \cos(\tfrac{\pi}{2}x) = -\sin(\varepsilon^\alpha\tfrac{\pi}{2}X) \sim -\varepsilon^\alpha\tfrac{\pi}{2}X \end{aligned}$$

To determine the dominant balances, examine

$$\underbrace{\varepsilon^{1-2\alpha}Y''}_{(1)}+\underbrace{\varepsilon^{0}(4-x_*^2)Y}_{(2)}=\underbrace{\cos(\tfrac{\pi}{2}x)}_{(3)},$$

where term (3) is either $O(\varepsilon^0)$ or $O(\varepsilon^\alpha)$ as described above.

For $\alpha=0$, we recover the outer distinguished limit which yields the leading order outer solution

$$y_0(x)=-\frac{\cos(\frac{\pi}{2}x)}{4-x^2}.$$

Note that this solution does not satisfy either boundary condition (with $y_0(0)=-1/4$ and $y_0(1)=0$, so we will need boundary layers at both $x_*=0$ and $x_*=1$. First consider the case that the boundary layer is on the left edge of the domain, $x_*=0$, making term (3) be $O(1)$. Then the dominant balance for the singular distinguished limit is between terms (1, 2) yielding $\alpha=\frac{1}{2}$. Term (3) is of the same order, $O(1)$, and hence also takes part in the dominant balance. The resulting leading order inner problem for $y=Y(X)$ with $X=x/\varepsilon^{1/2}$ is

$$Y_0''-4Y_0=1,\qquad Y_0(0)=-1\qquad\Longrightarrow\qquad Y_0^L=-\frac{3}{4}e^{-2X}-\frac{1}{4},$$

where we note that we have eliminated an un-matchable exponentially growing term for $X\to\infty$. Applying the matching condition between this inner solution and the outer solution,

$$\lim_{X\to\infty}Y_0^L=\lim_{x\to 0}y_0=-\frac{1}{4},$$

which fortunately yields consistency and gives the overlap as $-1/4$.

For the boundary layer at $x_*=1$, term (3) is $O(\varepsilon^\alpha)$. The singular distinguished limit is still $\alpha=\frac{1}{2}$ but now only terms (1, 2) are involved in the dominant balance, yielding the leading order inner problem for $y=Y(X)$ with $X=(x-1)/\varepsilon^{1/2}$,

$$Y_0''-3Y_0=0,\qquad Y_0(0)=2\qquad\Longrightarrow\qquad Y_0^R=2e^{\sqrt{3}X},$$

and again, an unmatchable exponentially growing term (for $X\to-\infty$) has be excluded. Here, applying the matching condition between this inner solution and the outer solution,

$$\lim_{X\to-\infty}Y_0^R=\lim_{x\to 1}y_0=0,$$

again yielding consistency of the construction process, here with zero overlap.

Forming the left and right boundary layer corrections by subtracting the respective matching overlaps from their inner solutions, we can write the composite solution (7.22) as

$$y \sim -\frac{\cos(\frac{\pi}{2}x)}{4-x^2} - \frac{3}{4}e^{-2x/\sqrt{\varepsilon}} + 2e^{\sqrt{3/\varepsilon}(x-1)}.$$

7.4 (a) Given that the boundary occurs at $x_* = 0$, the singular solution will have the form $y = \varepsilon^{\beta} Y(X)$ with $X = x/\varepsilon^{\alpha}$ and $\alpha > 0$. The homogeneous initial condition provides no information. The second initial condition gives

$$y'(0) = \frac{4}{\varepsilon^2} \qquad \Longrightarrow \qquad \varepsilon^{\beta-\alpha} Y'(0) = 4\varepsilon^{-2},$$

hence $Y'(0) = 0$ and $\beta - \alpha = -2$. Using this in the ODE, we get

$$\underbrace{\varepsilon^{-1-\alpha} Y''}_{(1)} + \underbrace{2\varepsilon^{-2} Y'}_{(2)} - \underbrace{6\varepsilon^{-2+\alpha} Y}_{(3)} = \underbrace{5\varepsilon^{\alpha} X}_{(4)}.$$

It can be shown that the only consistent singular dominant balance is between terms $(1, 2)$, yielding $\alpha = 1$, hence $\beta = -1$. Consequently, the inner problem is

$$Y'' + 2Y' - 6\varepsilon Y = 5\varepsilon^3 X, \qquad Y(0) = 0, \qquad Y'(0) = 4,$$

which reduces to the leading order problem

$$Y_0'' + 2Y_0' = 0 \qquad \Longrightarrow \qquad Y_0(X) = 2(1 - e^{-2X}),$$

satisfying both initial conditions.
(b) Applying the matching condition,

$$\lim_{X\to\infty} \varepsilon^{\beta_{\text{inner}}} Y_{\text{inner}}(X) = \lim_{x\to 0} \varepsilon^{\beta_{\text{outer}}} y_{\text{outer}}(x),$$

we get from the inner solution that the limit is $2/\varepsilon$ hence the outer solution must be scaled by $\beta_{\text{outer}} = -1$ and satisfy initial condition $y_{\text{outer}}(0) = 2$.
(c) Using $\beta = -1, \alpha = 0$ for the scaling of the outer solution yields the equation for $y = y_{\text{outer}}$

$$\varepsilon^0 y'' + 2\varepsilon^{-1} y' - 6\varepsilon^{-1} y = 5x, \qquad y(0) = 2$$

which at leading order reduces to

$$2y_0' - 6y_0 = 0 \qquad y_0(x) = 2e^{3x}.$$

Consequently, we can construct the composite solution from the outer solution plus the inner solution minus the overlap as

$$y \sim \frac{2}{\varepsilon}e^{3x} + \frac{2}{\varepsilon}\left(1 - e^{-2x/\varepsilon}\right) - \frac{2}{\varepsilon} = \frac{2}{\varepsilon}\left(e^{3x} - e^{-2x/\varepsilon}\right).$$

7.5 (a) The leading order outer solution is given by

$$2y' = e^{y_0} = 0 \qquad \Longrightarrow \qquad y_0(x) = -\ln\left(\tfrac{1}{2}(x-c)\right),$$

where the choice of the constant of integration depends on which boundary condition applies to the outer solution.
(b) Assuming the solution to be finite, $y = O(1)$, hence $\beta = 0$, dominant balances are determined by

$$\underbrace{\varepsilon^{1-2\alpha}Y''}_{(1)} + \underbrace{2\varepsilon^{-\alpha}Y'}_{(2)} - \underbrace{\varepsilon^{0}e^{Y}}_{(3)} = 0.$$

The outer distinguished limit is obtained by balancing terms (2,3) at $\alpha = 0$. The singular distinguished limited follows from balancing terms (1,2) at $\alpha = 1$. The leading order inner problem and its general solution is

$$Y_0'' + 2Y_0' = 0 \qquad \Longrightarrow \qquad Y_0(X) = C_1 + C_2 e^{-2X}.$$

As yet we have not determined the position of the boundary layer (x_*), but from the form of $Y_0(X)$ we can see that only the choice $x_* = 0$ can yield a nontrivial inner solution. Consequently, applying the boundary condition

$$Y_0(0) = 0 \qquad \Longrightarrow \qquad Y_0(X) = C_1(1 - e^{-2X}).$$

(c) Meanwhile we know that the other boundary condition, $y(1) = 0$ must apply to the outer solution (since there is no boundary possible at $x_* = 1$), hence the final form of the outer solution becomes

$$y_0(x) = -\ln\left(\tfrac{1}{2}(x+1)\right).$$

Asymptotic matching to the inner solution then determines that $C_1 = \ln(2)$. Consequently the leading order composite solution is

$$y \sim -\ln\left(\tfrac{1}{2}(x+1)\right) - \ln(2)e^{-2x/\varepsilon}.$$

(d) Expanding the outer solution as $y \sim y_0 + \varepsilon y_1$, the exponential term can be written as

$$e^{y} \sim e^{y_0+\varepsilon y_1} = e^{y_0}e^{\varepsilon y_1} \sim e^{y_0}\left(1 + \varepsilon y_0 + \tfrac{1}{2}\varepsilon^2 y_1^2 + \cdots\right).$$

Collecting $O(\varepsilon)$ terms in the ODE, the equation for y_1 is then

$$y_0'' + 2y_1' + y_1 e^{y_0} = 0 \qquad \Longrightarrow \qquad y_1(x) = -\frac{\ln\left(\frac{1}{2}(x+1)\right)}{2(x+1)}.$$

7.7 (a) The outer solution is given by

$$y \sim \pm\sqrt{4-x^2} + \frac{\varepsilon}{2(4-x^2)}.$$

(b) At $x_* = 2$, the limiting behaviour of the outer solution for $x \to 2$ determines that $\beta = \frac{1}{2}\alpha$ and the dominant balance of terms in the ODE determines the singular distinguished limit as $\alpha = \frac{2}{3}$, $\beta = \frac{1}{3}$ with the inner problem being

$$4X + \varepsilon^{2/3}X^2 + Y^2 = -Y' \qquad X = \frac{x-2}{\varepsilon^{2/3}}.$$

At $x_* = 0$, there is one inner solution that must be matchable to the outer solution, $y(x \to 0) \sim \pm 2 = O(1)$, hence it must have $\beta = 0$. A dominant balance in the ODE yields $\alpha = 1$ with the inner problem

$$\varepsilon^2 X^2 + Y^2 = 4 - Y', \qquad X = \frac{x}{\varepsilon}.$$

At $x_* = 0$ there is a different inner solution that satisfies the initial condition $y(0) = 3/\varepsilon$, hence having $\beta = -1$. Determination of the dominant balance with this value of β yields $\alpha = 2$ and the ODE

$$\varepsilon^6 X^2 + Y^2 = 4\varepsilon^2 - Y' \qquad X = \frac{x}{\varepsilon^2},$$

since this solution exists on a narrower domain ($\varepsilon^2 \ll \varepsilon$), this is sometimes called an *inner-inner* solution.

Chapter 8

8.1 (a) Using the method of separation of variables, we get (8.8),

$$\mathsf{U}(\mathsf{X},\mathsf{Y}) = \sum_{n=1}^{\infty} \mathsf{c}_n \sinh\left(\tfrac{n\pi}{\mathsf{L}}\mathsf{Y}\right)\sin\left(\tfrac{n\pi}{\mathsf{L}}\mathsf{X}\right)$$

Evaluating the boundary condition along $\mathsf{Y} = \mathsf{H}$ yields

$$\left.\frac{\partial \mathsf{U}}{\partial \mathsf{Y}}\right|_{\mathsf{Y}=\mathsf{H}} = \sum_{n=1}^{\infty} \left(\frac{n\pi \mathsf{c}_n}{\mathsf{L}} \cosh(\tfrac{n\pi}{\mathsf{L}}\mathsf{H})\right) \sin(\tfrac{n\pi}{\mathsf{L}}\mathsf{X}) = e^{-3\mathsf{X}/\mathsf{L}}$$

and the Fourier sine series determines the final form of the solution as

$$\mathsf{U}(\mathsf{X}, \mathsf{Y}) = \sum_{n=1}^{\infty} \left(\frac{2}{n\pi \cosh\left(\frac{n\pi \mathsf{H}}{\mathsf{L}}\right)} \int_0^L e^{-3\tilde{\mathsf{X}}/\mathsf{L}} \sin(\tfrac{n\pi}{\mathsf{L}}\tilde{\mathsf{X}})\, d\tilde{\mathsf{X}} \right) \sinh(\tfrac{n\pi}{\mathsf{L}}\mathsf{Y}) \sin(\tfrac{n\pi}{\mathsf{L}}\mathsf{X})$$

(b) The leading order outer problem, scaled with $\bar{\mathsf{U}} = 1\,\mathrm{m}$ is

$$u_{0yy} = 0, \qquad u_0(x, 0) = 0, \qquad u_{0y} = \mathsf{H}e^{-3x}$$

yielding $u_0(x, y) = \mathsf{H}e^{-3x}y$.
(c) Substituting $\mathsf{X} = \mathsf{L}x$, $\mathsf{Y} = \mathsf{H}y$ and taking the $\varepsilon \to 0$ limit of the separation of variables solution yields $\mathsf{U} =$ (Fourier sine series of e^{-3x})$\mathsf{H}y$.

8.2 (a) The leading order outer problem is

$$u_{0yy} = 0, \qquad u_0(x, 0) = \sin(5\pi x), \qquad u_0(x, 1) = \cos(3\pi x)$$

yielding $u_0(x, y) = (\cos(3\pi x) - \sin(5\pi x))y + \sin(5\pi x)$.
(b) Since $\lim_{x\to 0} u_0(x, y) = y$, the boundary layer correction at $x_*^L = 0$ can be expressed as $V^L(X, y) = U(x, y) - y$ with $V^L(0, y) = \sin(\frac{1}{2}\pi y) - y$ and homogeneous Dirichlet boundary conditions on rest of the boundary of the semi-infinite strip, $0 \le y \le 1$, $x \ge 0$. Then

$$V^L(X^L, y) = -2\sum_{n=1}^{\infty} \frac{(-1)^n}{(4n^2 - 1)n\pi} e^{-n\pi X^L} \sin(n\pi y) \qquad X^L = \frac{x}{\varepsilon} \ge 0.$$

Similarly, the right boundary layer correction can be determined to be

$$V^R(X^R, y) = \sum_{k=0}^{\infty} \frac{8}{(2k+1)^3\pi^3} e^{(2k+1)\pi X^R} \sin((2k+1)\pi y) \qquad X^R = \frac{x-1}{\varepsilon} \le 0.$$

(d) The nondimensional form of the flux is

$$J^R = \frac{1}{\mathsf{L}} \int_0^1 u_x(1, y)\, dy \sim \frac{1}{\mathsf{L}} \int_0^1 5\pi(y - 1) + \frac{1}{\varepsilon} \sum_{k=0}^{\infty} \frac{8\sin((2k+1)\pi y)}{(2k+1)^2\pi^2}\, dy.$$

The value of the flux is dominated by the $O(1/\varepsilon)$ contribution from the boundary layer correction

$$J^R \sim \frac{1}{\varepsilon \mathsf{L}} \sum_{k=0}^{\infty} \frac{16}{(2k+1)^3 \pi^3} \approx \frac{16}{\mathsf{H}\pi^3}(1.0518) \approx \frac{0.543}{\mathsf{H}}.$$

8.3 Let $f(x) = F(\mathsf{X})$. (a) The scaled form of the no-flux condition on the top boundary is

$$u_y + \varepsilon^2 15\pi \sin(3\pi x) u_x = 0 \qquad \text{at } y = f(x) = 15 + 5\cos(3\pi x).$$

(b) The $O(1)$ outer problem is

$$u_{0yy} = 0 \qquad u_{0y}(x, 0) = 0 \qquad u_{0y}(x, f(x)) = 0$$

The $O(\varepsilon)$ outer problem is

$$u_{0xx} + u_{1yy} = 0 \qquad u_{1y}(x, 0) = 0 \qquad u_{1y}(x, f) + 15\pi \sin(3\pi x) u_{0x}(x, f) = 0$$

yielding the solutions

$$u_0(x, y) = C_2(x), \qquad u_1(x, y) = -\tfrac{1}{2} C_2''(x) y^2 + C_4(x)$$

where the top boundary condition gives the compatibility condition

$$-\frac{d}{dx}\left((15 + 5\cos(3\pi x)) \frac{dC_2}{dx} \right) = 0$$

(d) Applying matching of the outer solution to the boundary layers implies zeroing the $n = 0$ coefficients in the cosine series expansions of the boundary layer corrections yields the boundary conditions on $C_2(x)$,

$$c_0 = \frac{1}{20} \int_0^{20} \left[-\frac{y^3}{100} - C_2(0) \right] dy = 0 \qquad \Longrightarrow \qquad C_2(0) = -20,$$

$$d_0 = \frac{1}{10} \int_0^{10} \left[3y^2 - C_2(1) \right] dy = 0 \qquad \Longrightarrow \qquad C_2(1) = 100.$$

8.5 (a) The $O(1)$ problem is

$$\phi_{0yy} = 0, \qquad \phi_{0y}(x, 0) = 0 \qquad \Longrightarrow \qquad \phi_0(x, y) = C_0(x, t).$$

The $O(\varepsilon^2)$ problem is

$$\phi_{1yy} + \phi_{0xx} = 0, \qquad \phi_{1y}(x, 0) = 0 \qquad \Longrightarrow \qquad \phi_1(x, y) = -\tfrac{1}{2} C_{0xx}(x, t) + C_1(x, t).$$

The $O(\varepsilon^4)$ problem is

$$\phi_{2yy} + \phi_{1xx} = 0, \qquad \phi_{2y}(x,0) = 0 \quad \Longrightarrow \quad \phi_1(x,y) = \tfrac{1}{24}C_{0xxxx}(x,t) - \tfrac{1}{2}C_{1xx}(x,t)y^2 + C_2(x,t).$$

(b, c) See Exercise 9.12.

Chapter 9

9.1 Using the initial conditions, $x_0(t) = \cos(t) + 2\sin(t)$. Using the $O(\varepsilon)$ equation with $x_0(t)$ specified and the initial conditions $x_1(0) = -1, x_1'(0) = 1$, $x_1(t) = [115\sin(t) - 12\cos(t)] + [60t\sin(t) - 120t\cos(t)] + [11\cos(3t) + 2\sin(3t)]$ with the homogeneous, secular and non-resonant respond terms grouped respectively.

9.2 (a) $\tilde{x}_0 = a\cos\theta$ with $\theta \sim (2 + \frac{3}{16}\varepsilon a^2)t$ works for any $a > 0$. (b) $\tilde{x}_0 = a\cos\theta$ with $\omega_0 = 3, \omega_1 = 0$ satisfies the problem only for $a = 2$.

9.3 (a) The amplitude equations are

$$\frac{dA}{d\tau} = \tfrac{1}{2}A - \frac{A}{8}(A^2 + B^2), \qquad \frac{dB}{d\tau} = \tfrac{1}{2}B - \frac{B}{8}(B^2 + A^2).$$

(b) Note that $\frac{d(R^2)}{d\tau} = 2A\frac{dA}{d\tau} + 2B\frac{dB}{d\tau} = 2R\frac{dR}{d\tau}$. Expanding out $2R\frac{dR}{d\tau}$ yields

$$\frac{dR}{d\tau} = \tfrac{1}{2}R(1 - \tfrac{1}{4}R^2)$$

which has $R_* = 2$ as an equilibrium. Note that this matches the periodic solution determined in Exercise 9.2(b), but now, using MMTS, we have a prediction for rate of convergence to the limit cycle.

9.5 (b) Suppressing the $e^{\pm it}$ resonant forcing terms in the $O(\varepsilon)$ equation for X_1 yields

$$\frac{dC}{d\tau} = -\tfrac{1}{2}\beta C + \frac{3}{2}i\alpha|C|^2C + \tfrac{1}{2}e^{i\gamma\tau},$$

with the second solvability condition being exactly the complex conjugate of the above equation (and hence yielding no independent information).
(c) Substituting into the amplitude ODE for C reduces to the algebraic equation

$$\gamma = \tfrac{1}{2}i\beta + \frac{3}{2}\alpha M^2 - \frac{1}{4M}e^{-i\theta}.$$

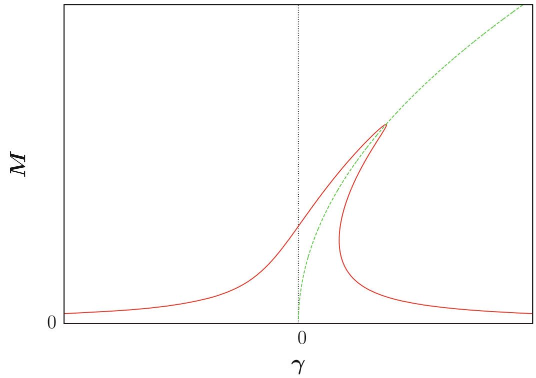
Separating real and imaginary parts of this equation yields

$$\gamma = \tfrac{3}{2}\alpha M^2 - \frac{\cos\theta}{4M}, \qquad 0 = \tfrac{1}{2}\beta + \frac{\sin\theta}{4M}.$$

For fixed α, β, expressing $\sin\theta = \pm\sqrt{1-\cos^2\theta}$, we can solve the second equation to get $\cos\theta = \pm\sqrt{1-4\beta^2M^2}$. Substituting this into the first equation gives the final (real-valued) form of the detuning relation:

$$\gamma(M) = \tfrac{3}{2}\alpha M^2 \pm \frac{\sqrt{1-4\beta^2M^2}}{4M}.$$

Entrained solutions exist up to a critical amplitude set by the damping, $M \le M_c = 1/(2\beta)$. Noting that the terms $f = x + \varepsilon\alpha x^3$ can be interpreted as a restoring force due to a spring, the case of $\alpha > 0$ is called a *hardening spring* since the restoring force is stronger than for the corresponding linear Hooke's law spring [54]. Conversely, the case $\alpha < 0$ is called a *softening spring*. In contrast to the case of a linear oscillator near resonance (see Fig. 4.1), for $\alpha \neq 0$, the amplitude of the forced solution is shifted away from the oscillator's natural frequency by the amplitude-dependent detuning, $\gamma(M)$. For hardening (softening) springs, the resonant peak gets shifted above (below) the (zero-amplitude) linear natural frequency. See below for a plot of the detuning relation plotted for a hardening spring case, $\alpha > 0$ along with the (undamped) resonant "backbone" curve $\gamma = \frac{3}{2}\alpha M^2$.



9.6 The amplitude equations are $\frac{d\Phi}{d\tau} = 0$ and $\frac{dR}{d\tau} = -\frac{4R^2}{3\pi}$ yielding the solution

$$x(t) \sim \frac{\sin(t)}{1 + \frac{4\varepsilon t}{3\pi}}.$$

9.7 The amplitude equations are $\frac{dA}{d\tau} + \frac{1}{8}A = 0$, $\frac{dB}{d\tau} - \frac{1}{8}B = 0$.

9.8 (c, d) The amplitude equation is determined by forcing the coefficient of the resonant forcing term (e^{-i4t}) in the equation for X_1 to vanish, $\frac{dA}{d\tau} - \frac{1}{2}A^2 = 0$, yielding the solution $x \sim \frac{2}{1-i-\varepsilon t}e^{-i4t}$.

9.10 Substituting the expansion into the ODE, at leading order we get

$$X_{0tt} + X_0 = \varepsilon^{1-\beta}X_0^2 + \varepsilon^{1+\beta}\cos t.$$

The inhomogeneous forcing terms will be higher order if $-1 < \beta < 1$ to yield $X_0(t,\tau) = A(\tau)e^{it} + B(\tau)e^{-it}$. In the equation for X_1, deferring the resonant terms to higher order requires $\alpha - \gamma > 0$ and $1+\beta-\gamma > 0$ with $1-\beta-\gamma = 0$ needed to make the remaining forcing term be $O(1)$. Finally, in the equation for X_2 balancing resonant terms leads to $\alpha = 4/3$, $\beta = 1/3$, $\gamma = 2/3$ with amplitude equations

$$-2i\frac{dA}{d\tau} + \frac{1}{2} + \frac{10}{3}A^2B = 0 \qquad 2i\frac{dB}{d\tau} + \frac{1}{2} + \frac{10}{3}AB^2 = 0,$$

yielding a stable limit cycle with $X_0(t) = -(6/5)^{1/3}\cos t$.

9.11 (a) The trivial solution $x \equiv 0$ is an equilibrium state. Linear stability analysis yields $\lambda^2 + 1 + \varepsilon e^{-2\lambda} = 0$ with $O(1)$ solutions $\lambda \sim \pm i \pm \varepsilon\frac{1}{2}ie^{\mp 2i}$, both of these have $\mathrm{Re}(\lambda) > 0$ so the trivial solution is unstable. (b) The amplitude equations are

$$\frac{dA}{d\tau} - \tfrac{1}{2}(A\sin 2 + B\cos 2) = 0, \qquad \frac{dB}{d\tau} + \tfrac{1}{2}(A\cos 2 - B\sin 2) = 0.$$

9.12 (a) Taking ∂_t of the first equation in (9.36) yields $F_{0t} = -C_{0tt}$ and hence $C_{0tt} = C_{0xx}$ from the second equation. Taking ∂_{xx} of the first equation yields $F_{0xx} = -C_{0txx}$ then using this in ∂_t of the second equation yields $F_{0tt} = F_{0xx}$.
(b) The $O(1)$ equations determine that $f_0 = c_{0z}$ at leading order. There is one $\varepsilon^2\partial_\tau$ term from the $O(1)$ equations that gets added to the $O(\varepsilon^2)$ (9.37) equations. The sum of ∂_z of the first equation plus the second cancels the c_1, f_1 terms to yield the KdV equation (9.38) depending only on $f_0(z,\tau)$.

9.13 $\lambda^{(1)} \sim 2 - \frac{1}{2}\varepsilon$, $\lambda^{(2)} \sim -4 + \frac{3}{2}\varepsilon$

Chapter 10

10.1 (a) $f(x) = -x^3 + 9x + 3x^2$,
(b) $(x_*, y_*) = (0,0)$ is the only equilibrium point; the eigenvalues from its linear stability are $\lambda = (9 \pm \sqrt{81-16\varepsilon})/(2\varepsilon) > 0$, so it is an unstable node.
(c) The slow manifold is $y_0(x_0) = (x_0^3 - 3x_0^2 - 9x_0)/4$.
(e) The slow manifold has local extrema at $y(-1) = 5/4$ and $y(3) = -27/4$. In the fast evolution stages, y is constant and x evolves from the extrema value to the other value on the slow manifold, satisfying $y = S(x)$. Solving the cubic $S(x) = 5/4$ yields $x_{max} = 5$ and solving $S(x) = -27/4$ yields $x_{min} = -3$.

(f) Implicitly differentiating the equation for the slow manifold, $\frac{d}{dt}(y = S(x))$, and applying $\frac{dy}{dt} = -x$ yields the equation for evolution on the slow manifold,

$$\frac{dx}{dt} = -\frac{4x}{3x^2 - 6x - 9} \qquad \to \qquad \int \frac{dx}{g(x)} = -\int \frac{3x^2 - 6x - 9}{4x}\,dx$$

The limit cycle is composed of two fast and two slow stages, $P = T_{\text{fast},1} + T_{\text{slow},2} + T_{\text{fast},3} + T_{\text{slow},4} \sim T_{\text{slow},2} + T_{\text{slow},4}$. Stage 2 starts at $x = 5$ and ends at $x = 3$,

$$T_{\text{slow},2} = \frac{1}{4}\int_5^3 \frac{9}{x} + 6 - 3x\,dx = 3 + \frac{9}{4}\ln(3/5).$$

Similarly, stage 4 starts at $x = -3$ and ends at $x = -1$,

$$T_{\text{slow},4} = \frac{1}{4}\int_{-3}^{-1} \frac{9}{x} + 6 - 3x\,dx = 6 - \frac{9}{4}\ln(3).$$

Therefore the period is $P \sim 9 - (9/4)\ln(5) \approx 5.38$.

10.2 (a) Setting $\varepsilon = 0$ in the z' equation yields the slow manifold as $z = S(y) = y - y^2 + \frac{1}{3}y^3$ and then the leading order slow system is

$$\frac{dx_0}{dt} = 2 - y_0 \qquad \frac{dy_0}{dt} = x_0 - y_0 + y_0^2 - \tfrac{1}{3}y_0^3$$

The only equilibrium point of the slow system is $(x_*, y_*) = (\frac{2}{3}, 2)$, which is a stable spiral point with eigenvalues $\lambda = \frac{1}{2}(-1 \pm 3i)$. Consequently the $t \to \infty$ solution will be $(x, y, z) \sim (\frac{2}{3}, 2, \frac{2}{3} = S(2))$.

Note that the initial condition is not on the slow manifold since $0 \neq S(3)$, therefore there will be an initial layer governed by the fast system (for the distinguished limit with $\alpha = 1$ set by the z' equation),

$$\frac{dX}{dT} = \varepsilon(2 - Y), \qquad \frac{dY}{dT} = \varepsilon(X - Z), \qquad \frac{dZ}{dT} = Y - Y^2 + \tfrac{1}{3}Y^3 - Z$$

At leading order, the initial layer will have $X_0(T) \equiv 1$, $Y_0(T) \equiv 3$, constants set by the initial conditions. Then the last relation is $Z_0' = 3 - Z_0$ yielding $Z_0(T) = 3(1 - e^{-T})$ which connections the initial condition to the point $(x, y, z) = (1, 3, 3)$ on the slow manifold.

(b) Setting $\varepsilon = 0$ in the y' equation yields the relation $x = z$. Using that result in the x' gives a second expression for z' that could be matched to the z' equation to yield $2 - y = y - y^2 + \frac{1}{3}y^3 - z$. The expression for $z = z(y)$ is monotone increasing and hence could be inverted to yield $y = y(z)$ to describe the slow manifold as a parametric curve in 3D. Implicitly differentiation gives motion on this curve as

$$\frac{dy}{dt} = \frac{2-y}{2-2y+y^2},$$

which has a stable equilibrium point at $y_* = 2$. Consequently the solution for $t \to \infty$ will approach $(x, y, z) \sim (\frac{2}{3} = x(\frac{2}{3}), 2, \frac{2}{3} = z(2))$.

Since the initial condition does not lie on the slow manifold, there will be an initial layer governed by the fast system,

$$\frac{dX}{dT} = \varepsilon(2-Y), \qquad \frac{dY}{dT} = X - Z, \qquad \frac{dZ}{dT} = \varepsilon\left(Y - Y^2 + \tfrac{1}{3}Y^3 - Z\right).$$

At leading order, the initial layer will have $X_0(T) \equiv 0$, $Z_0(T) \equiv 1$, constants set by the initial conditions. Then the last relation is $Y_0' = -1$, describing linear decrease of Y_0 from the initial value $Y_0(0) = 3$ until it reaches the slow manifold, at $z(y) = 1$ (at $y \approx 2.1517$).

10.3 (a) The dimensional rate equations are

$$\frac{d\mathsf{A}}{d\mathsf{T}} = -2\mathsf{k}_1\mathsf{A}^2 + 2\mathsf{k}_2\mathsf{B} - \mathsf{k}_3\mathsf{AB}, \qquad \frac{d\mathsf{B}}{d\mathsf{T}} = \mathsf{k}_1\mathsf{A}^2 - \mathsf{k}_2\mathsf{B} - \mathsf{k}_3\mathsf{AB}, \qquad \frac{d\mathsf{C}}{d\mathsf{T}} = \mathsf{k}_3\mathsf{AB}.$$

(c) The scaled problem is

$$\frac{da}{dt} = -2a^2 + 2\Pi_1 b - \Pi_2 ab, \qquad \varepsilon\frac{db}{dt} = a^2 - \Pi_1 b - \Pi_2 ab, \qquad \frac{dc}{dt} = \Pi_2 ab$$

where

$$\varepsilon = \frac{\mathsf{B}_0}{\mathsf{A}_0}, \qquad \Pi_1 = \frac{\mathsf{k}_2\mathsf{B}_0}{\mathsf{k}_1\mathsf{A}_0^2}, \qquad \Pi_2 = \frac{\mathsf{k}_3\mathsf{B}_0}{\mathsf{k}_1\mathsf{A}_0}.$$

(d) By setting $\varepsilon = 0$ in the b' equation, we get the slow manifold, $b = S(a) = a^2/(\Pi_1 + \Pi_2 a)$. Substituting the slow manifold for b in the rate equations for a', c' yields

$$\frac{da}{dt} = -\frac{3\Pi_2 a^3}{\Pi_1 + \Pi_2 a} \qquad \frac{dc}{dt} = \frac{\Pi_2 a^3}{\Pi_1 + \Pi_2 a},$$

which is consistent with the overall expectation of the sum $c + \frac{1}{3}a$ being conserved. Undoing the dimensional scalings, we get

$$\frac{d\mathsf{A}}{d\mathsf{T}} = -\frac{3\mathsf{k}_1\mathsf{k}_3\mathsf{A}^3}{\mathsf{k}_2 + \mathsf{k}_3\mathsf{A}} = -\mathsf{G}(\mathsf{A}) \qquad \text{and } \mathsf{F}(\mathsf{A}) = \tfrac{1}{3}\mathsf{G}(\mathsf{A}).$$

10.5 The reactions take the form

$$A + B \to 2P \qquad B \rightleftharpoons 2C \qquad C + A \rightleftharpoons D + P \qquad D + B \to P + C$$

The rate equations are

$$\frac{dA}{dT} = -k_3 AC + k_4 DP \qquad \frac{dB}{dT} = -k_1 B + k_2 C^2 - k_5 BD$$
$$\frac{dC}{dT} = 2k_1 B - 2k_2 C^2 - k_3 AC + k_4 DP + k_5 BD$$
$$\frac{dD}{dT} = k_3 AC - k_4 DP - k_5 BD \qquad \frac{dP}{dT} = k_3 AC - k_4 DP + k_5 BD$$

Using the QSSA applied to the intermediates sets $C' = D' = 0$ yielding the slow manifold-type relations

$$C = \sqrt{\frac{k_1 B}{k_2}} \qquad D = \frac{k_3 A}{k_4 P + k_5 B}\sqrt{\frac{k_1 B}{k_2}}$$

Substituting these into the rate equation for P' yields

$$\frac{dP}{dT} = \frac{\alpha A B^{3/2}}{B + \beta A} \qquad \alpha = 2k_3\sqrt{\frac{k_1}{k_2}} \qquad \beta = \frac{k_4}{k_5}.$$

Chapter 11

11.1 Integrals of the Gaussian yield

$$\int_{-\infty}^{\infty} \rho(x,t)\,dx = C_1, \qquad \int_{-\infty}^{\infty} x\rho(x,t)\,dx = C_1 C_3, \qquad \int_{-\infty}^{\infty} x^2\rho(x,t)\,dx = C_1\left(C_3^2 + 2(t + C_2)\right).$$

Matching these results with (11.3), (11.4) and (11.7) determines

$$C_1 = M_0 = \int f\,dx, \qquad C_3 = \frac{M_1}{M_0} = \frac{1}{M_0}\int xf\,dx, \qquad C_2 = \frac{1}{2M_0}\int x^2 f\,dx - \frac{M_1^2}{2M_0^2}.$$

11.2 (a) Integrating the PDE against x^n and applying the boundary conditions yields (noting that $\rho(|x| \to \infty) \to 0$ implies that $\partial_x \rho(|x| \to \infty) \to 0$)

$$\frac{dM_0}{dt} = 4M_0, \qquad \frac{dM_1}{dt} - 2M_0 = 4M_1, \qquad \frac{dM_2}{dt} - 4M_1 = 6M_0 + 4M_2.$$

Solving this system of linear ODEs and applying the initial conditions, $M_0(0) = \sqrt{\pi}$, $M_1(0) = 0$, $M_2(0) = \frac{1}{2}\sqrt{\pi}$ yields

$$M_0(t) = \sqrt{\pi}e^{4t}, \qquad M_1(t) = 2\sqrt{\pi}te^{4t}, \qquad M_2(t) = \sqrt{\pi}\left(4t^2 + 6t + \tfrac{1}{2}\right)e^{4t}.$$

(b) Integrating the PDE against x^0, x^1 and applying the boundary conditions yields

$$\frac{dM_0}{dt} - 2 = -3\frac{\partial \rho}{\partial x}\bigg|_{x=0} + 4M_0, \qquad \frac{dM_1}{dt} - 2M_0 = 3 + 4M_1.$$

11.3 (a) Integrating the PDE against x^0, x^1 and applying the boundary conditions yields

$$\frac{dM_0}{dt} = -M_0, \qquad \frac{dM_1}{dt} = -M_1 - (\rho(\pi, t) - \rho(0, t)),$$

(b) Substituting the separation of variables form into the PDE yields that $\lambda_k = 1 + k^2$ for $k = 0, 1, 2, \ldots$. Namely, all modes decay exponentially since $\lambda_k > 0$, but $\lambda_0 = 1$ decays the least rapidly.

Examining the initial condition at $t = 0$, $\rho(x, 0) = f(x)$ and obtaining the Fourier cosine series coefficients (see Appendix A) yields

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f\,dx, \qquad a_k = \frac{2}{\pi}\int_0^{\pi} f\cos(kx)\,dx.$$

Retaining the slowest-decaying mode from the series for $\rho(x, t)$ then yields

$$\rho(x, t) \sim \frac{e^{-t}}{\pi}\int_0^{\pi} f\,dx \qquad \text{as } t \to \infty.$$

(b) Integrating the PDE against x^0, x^1 and applying the boundary conditions ($\rho \to 0$ as $|x| \to \infty$) yields

$$\frac{dM_0}{dt} = -M_0, \qquad \frac{dM_1}{dt} = -M_1, \qquad \frac{dM_2}{dt} = 2M_0 - M_2.$$

11.4 (a) Integrating the PDE directly yields $M_0'(t) + \rho(\infty, t) - \rho(0, t) = -2M_0(t)$, then applying the boundary conditions yields

$$\frac{dM_0}{dt} = (\beta - 2)M_0.$$

(b) Integrating the PDE directly yields the same initial form as in part (a), but the new birth condition yields the ODE

$$\frac{dM_0}{dt} = -2M_0 + M_1.$$

Integrating the PDE against e^{-3a} similarly yields

$$\frac{dM_1}{dt} = -4M_1.$$

11.5 (a) $M_0 = \int f\,dx$, $M_1 = \int xf\,dx$.
(b) Note that the porous medium equation can be re-written as $\rho_t = \frac{1}{4}(\rho^4)_{xx}$ to make integration by parts more convenient,

$$\frac{dM_2}{dt} = \frac{1}{4}\int x^2(\rho^4)_{xx}\,dx = \frac{1}{4}\left(x^2(\rho^4)_x - 2x\rho^4\Big| + 2\int \rho^4\,dx\right) \geq 0,$$

where the boundary terms vanish due to the boundary conditions and the integral is positive-definite.

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