Hints, Solutions

Hints

Chapter 2

2.12 Draw the lines one by one. Every time we add a new line, we increase the number of regions by as many regions as the new line intersects. Show that this number is one greater than the previous number of lines.

2.16 Apply the inequality of arithmetic and geometric means with the numbers x, x, and $2-2x$.

2.24 Let $X = A_1 \cup \ldots \cup A_n$. Prove (using de Morgan's laws and (1.2)) that every expression $U(A_1,\ldots,A_n)$ can be reduced to the following form: $U_1 \cap U_2 \cap \ldots \cap U_N$, where for every i, $U_i = A_1^{\varepsilon_1} \cup \ldots \cup A_n^{\varepsilon_n}$. Here $\varepsilon_j = \pm 1$, $A_j^1 = A_j$, and $A_j^{-1} = X \setminus A_j$. Check that if the condition of the exercise holds for U and V of this form, then $U = V$.

Chapter 3

3.4 A finite set contains a largest element. If we add a positive number to this element, we get a contradiction.

3.10 Does the sequence of intervals $[n/x, 1/n]$ have a shared point?

3.16 Show that (a) if the number a is the smallest positive element of the set H, then $H = \{na\colon n \in \mathbb{Z}\}$; (b) if $H \neq \{0\}$ and H does not have a smallest positive element, then $H \cap (0,\delta) \neq \emptyset$ for all $\delta > 0$, and so H is everywhere dense.

3.25 Suppose that $H \neq \emptyset$ and H has a lower bound. Show that the least upper bound of the set of lower bounds of H is also the greatest lower bound of H.

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3.31 Since $b/a > 1$, there exists a rational number $n > 0$ such that $b/a = 1 + (1/n)$. Justify that $a = \big(1 + (1/n)\big)^n$ and $b = \big(1 + (1/n)\big)^{n+1}$. Let $n = p/q$, where p and q are relatively prime positive integers. Show that $\big((p+q)/p\big)^{p/q}$ can be rational only if $q = 1$.

3.33 Suppose first that b is rational. If $0 \le b \le 1$, then apply the inequality of arithmetic and geometric means. Reduce the case $b > 1$ to the case $0 < b < 1$. For irrational b, apply the definition of taking powers.

Chapter 4

4.2 Show that if the sequences (a_n) and (b_n) satisfy the recurrence, then for every $\lambda, \mu \in \mathbb{R}$, the sequence $(\lambda a_n + \mu b_n)$ does as well. Thus it suffices to show that if α is a root of the polynomial p, then the sequence (α^n) satisfies the recurrence.

4.3 (a) Let α and β be the roots of the polynomial $x^2 - x - 1$. According to the previous exercise, for every $\lambda, \mu \in \mathbb{R}$, the sequence $(\lambda\alpha^n + \mu\beta^n)$ also satisfies the recurrence. Choose λ and μ such that $\lambda\alpha^0 + \mu\beta^0 = 0$ and $\lambda\alpha^1 + \mu\beta^1 = 1$.

4.22 In order to construct a sequence oscillating at infinity, create a sequence that moves between 0 and 1 back and forth with each new step size getting closer to zero.

4.27 If the decimal expansion of $\sqrt{2}$ consisted of only a repeating digit from some point on, then $\sqrt{2}$ would be rational.

Chapter 5

5.4 Construct an infinite set $A \subset \mathbb{N}$ such that $A \cap \{kn \colon n \in \mathbb{N}\}$ is finite for all $k \in \mathbb{N}$, $k > 1$. Let $a_n = 1$ if $n \in A$ and $a_n = 0$ otherwise.

5.21 Show that the sequence $b_n = n \cdot \max\{a_i^k \colon 1 \le i, k \le n\}$ satisfies the conditions.

Chapter 6

6.3 Write the condition in the form $a_n - a_{n-1} \le a_{n+1} - a_n$. Show that the sequence is monotone from some point on.

6.4 (d) Separate the sequence into two monotone subsequences.

6.5 Show that $a_n \ge \sqrt{a}$ and $a_n \ge a_{n+1}$ for all $n \ge 2$.

6.7 Multiply the inequalities $1+1/k < e^{1/k} < 1+1/(k-1)$ for all $2 \le k \le n$.

6.9 Suppose that $(a_{n+1} - a_n)$ is monotone decreasing. Prove that $a_{2n} - a_n \ge n \cdot (a_{2n+1} - a_{2n})$ for all n.

6.17 The condition is that the finite or infinite limit $\lim_{n\to\infty} a_n = \alpha$ exists, $a_n \le \alpha$ holds for all n, and if $a_n = \alpha$ for infinitely many n, then $a_n < \alpha$ can hold for only finitely many n.

6.22 A possible construction: let $a_n = \sqrt{k}$ if $2^{2^{k-1}} \le n < 2^{2^k}$.

6.23 The statement is true. Use the same idea as the proof of the transference principle.

Chapter 7

7.2 (a) Give a closed form for the partial sums using the identity

$$\frac{1}{n^2+2n} = \frac{1}{2n} - \frac{1}{2(n+2)}.$$

A similar method can be used for the series (b), (c), and (d).

7.3 Break the rational function p/q into quotients of the form $c_i/(x-a_i)$. Show that here, $\sum_{i=1}^{k} c_i = 0$, and apply the idea used in part (c) of Exercise 7.2.

7.5 Use induction. To prove the inductive step, use the statement of Exercise 3.33.

7.8 Give the upper bound $N/10^{k-1}$ to the sum $\sum_{10^{k-1} \le a_n < 10^k} 1/a_n$, where N denotes how many numbers there are with k digits that do not contain the digit 7.

7.10 It does not follow.

7.11 Yes, it follows. Prove that the given infinite series satisfies Cauchy's criterion.

Chapter 8

8.2 For all N, there are only finitely many sequences $(a_1, \ldots, a_k)$ such that $|a_1| + \cdots + |a_k| = N$.

8.4 Use the fact that every interval contains a rational point.

8.9 Every $x \in (0,1]$ has a unique form $x = 2^{-a_1} + 2^{-a_2} + \cdots$, where $a_1 < a_2 < \cdots$ are natural numbers. Apply the bijections

$$x \leftrightarrow (a_1, a_2, \ldots) \leftrightarrow (a_1, a_2 - a_1, a_3 - a_2, \ldots).$$

8.11 It suffices to prove that the set of pairs (A, B) has the cardinality of the continuum, where $A, B \subset \mathbb{N}$. Find a map that maps these pairs to subsets of $\mathbb{N}$ bijectively.

Chapter 9

9.4 Such functions exist. Construct first such a function on the four-element set $a, b, -a, -b$ for all $0 < a < b$.

9.5 Let, for example, f_c be the identically 1 function for all c. A less trivial example: let $f_c(x) = x + c$ for all $c, x \in \mathbb{R}$.

The answer to the second question is no. If, for example, $g = f_{1/2}$, then $g \circ g = f_1$, and not every function f_1 has such a g. Show that if $f(1) = -1$, $f(-1) = 1$, and $f(x) = 0$ for all $x \neq \pm 1$, then there does not exist a function $g\colon \mathbb{R} \to \mathbb{R}$ such that $g \circ g = f$. (The question of which functions can be expressed in the form $g \circ g$ has been studied extensively. See, for example, the following paper: R. Isaacs, Iterates of fractional order, *Canad. J. Math.* **2** (1950), 409–416.)

9.6 (a) Let f_1 be constant and f_2 one-to-one. (c) The answer is positive. (d) The answer is positive.

Chapter 10

10.8 Show that the infimum of the set of positive periods is positive and also a period.

10.12 First show, using the statement of Exercise 3.17, that if x is irrational, then the set of numbers $\{nx\}$ is everywhere dense in $[0, 1]$.

10.15 Suppose that $\lim_{y \to x} f(y) = \infty$ for all x. Construct a sequence of nested intervals $[a_n, b_n]$ such that $f(x) > n$ for all $x \in [a_n, b_n]$.

10.16 Suppose that $\lim_{y \to x} f(y) = 0$ for all x. Construct a sequence of nested intervals $[a_n, b_n]$ such that $|f(x)| < 1/n$ for all $x \in [a_n, b_n]$.

10.20 Construct a set $A \subset \mathbb{R}$ that is not bounded from above, but for all $a > 0$, we have $n \cdot a \notin A$ if n is sufficiently large. (We can also achieve that for every $a > 0$, at most one $n \in \mathbb{N}^+$ exists such that $n \cdot a \in A$.) Let $f(x) = 1$ if $x \in A$, and let $f(x) = 0$ otherwise.

10.57 Show that if the leading coefficient of the polynomial p of degree three is positive, then $\lim_{x \to \infty} p(x) = \infty$ and $\lim_{x \to -\infty} p(x) = -\infty$. Therefore, p takes on both positive and negative values.

10.58 Apply the Bolzano–Darboux theorem to the function $f(x) - x$.

10.60 Show that if f is continuous and $f\big(f(x)\big) = -x$ for all x, then (i) f is one-to-one, (ii) f is strictly monotone, and so (iii) $f\big(f(x)\big)$ is strictly monotone increasing.

10.71 (ii) First show that f and g are bounded in A; then apply the equality

$$f(y)g(y) - f(x)g(x) = f(y) \cdot (g(y) - g(x)) + g(x) \cdot (f(y) - f(x)).$$

10.76 Let $A = \{a_1, a_2, \ldots\}$. For an arbitrary real number x, let $f(x)$ be the sum of the numbers 2^{-n} for which $a_n < x$. Show that f satisfies the conditions.

10.80 Every continuous function satisfies the condition.

10.81 First show that if $p < q$ are rational numbers and n is a positive integer, then the set $\{a \in [-n,n] \colon \lim_{x \to a} f(x) < p < q < f(a)\}$ is countable.

10.82 Apply the ideas used in the solution of Exercise 10.17.

10.89 Not possible.

10.90 Not possible.

10.94 The function $g(x) = f(x) - f(1) \cdot x$ is additive, periodic with every rational number a period, and bounded from above on an interval. Show that $g(x) = 0$ for all x.

10.96 Apply the ideas used in the proof of Theorem 10.76.

10.102 By Exercise 10.101, it suffices to show that every point c of I has a neighborhood in which f is bounded. Let f be bounded from above on $[a,b] \subset I$. We can assume that $b < c$. Let

$$\alpha = \sup\{x \in I \colon x \geq a,\ f \text{ is bounded in } [a,x]\}.$$

Show (using the weak convexity of f) that $\alpha > c$.

Chapter 11

11.16 Use Exercise 6.7.

11.31 Use induction, using the identity

$$\cos(n+1)x + \cos(n-1)x = 2\cos nx \cdot \cos x.$$

11.35 Show that $f(0) = 1$. Prove, by induction, that if for some $a, c \in \mathbb{R}$ we have $f(a) = \cos(c \cdot a)$, then $f(na) = \cos(c \cdot na)$ for all n.

11.36 (a) Prove and use that $\sin^{-2}(k\pi/2n) + \sin^{-2}\big((n-k)\pi/2n\big) = 4\sin^{-2}(k\pi/n)$ for all $0 < k < n$. (b) Use induction. (c) Apply Theorem 11.26.

Chapter 12

12.9 There is no such point. Use the fact from number theory that for every irrational x there exist infinitely many rational p/q such that $|x-(p/q)|<1/q^2$.

12.15 Show that $(f(y_n)-f(x_n))/(y_n-x_n)$ falls between the numbers

$$\min\left(\frac{f(x_n)-f(a)}{x_n-a},\frac{f(y_n)-f(a)}{y_n-a}\right) \quad \text{and} \quad \max\left(\frac{f(x_n)-f(a)}{x_n-a},\frac{f(y_n)-f(a)}{y_n-a}\right).$$

12.54 Suppose that $c<d$ and $f(c)>f(d)$. Let $\alpha=\sup\{x\in[c,d]: f(x)\geq f(c)\}$. Show that $\alpha<d$ and $\alpha=d$ both lead to a contradiction.

12.57 After subtracting a linear function, we can assume that $f'(c)=0$. We can also suppose that $f''(c)>0$. Thus f' is strictly locally increasing at c. Deduce that this means that f has a strict local minimum at c, and that for suitable $x_1<c<x_2$, we have $f(x_1)=f(x_2)$.

12.65 Differentiate the function $(f(x)-\sin x)^2+(g(x)-\cos x)^2$.

12.71 The statement holds for $k=n$. Prove, with the help of Rolle's theorem, that if $1\leq k\leq n$ and the statement holds for k, then it holds for $k-1$ as well.

12.76 Let $g(x)=\big(f(x+h)-f(x)\big)/h$. Then

$$\big(f(a+2h)-2f(a+h)+f(a)\big)/h^2=\big(g(a+h)-g(a)\big)/h.$$

By the mean value theorem, there exists a $c\in(a,a+h)$ such that $\big(g(a+h)-g(a)\big)/h=g'(c)=\big(f'(c+h)-f'(c)\big)/h$. Apply Theorem 12.9 to f'.

12.81 We can suppose that $a>1$. Prove that $a^x=x$ has a root if and only if the solution x_0 of $(a^x)'=a^x\cdot\log a=1$ satisfies $a^{x_0}\leq x_0$. Show that this last inequality is equivalent to the inequalities $1\leq x_0\cdot\log a$ and $e\leq a^{x_0}=1/\log a$.

12.82 (a) Let $a<1$. Show that the sequence (a_{2n+1}) is monotone increasing, the sequence (a_{2n}) is monotone decreasing, and both converge to the solution of the equation $a^x=x$. The case $a=1$ is trivial. (b) Let $a>1$. If the sequence is convergent, then its limit is the solution of the equation $a^x=x$. By the previous exercise, this has a solution if and only if $a\leq e^{1/e}$. Show that in this case, the sequence converges (monotonically increasing) to the (smaller) solution of the equation $a^x=x$.

12.83 Apply inequality (12.32).

Chapter 13

13.2 Check that the statement follows for the polynomial x^n by the binomial theorem.

13.6 Show that the value of the equation given in Exercise 13.5 does not change if we write $-x$ in place of x. Use the fact that $\binom{n}{k} = \binom{n}{n-k}$.

13.9 Use Euler's formula (11.65) for $\cos x$.

Chapter 14

14.3 Show that $s_F(f) \le S_F(g)$ for every partition F.

14.6 First show that the upper sums for partitions containing one base point form an interval. We can suppose that $f \ge 0$. Let $M = \sup\{f(t) \colon t \in [a,b]\}$. Let $g(x)$ denote the upper sum corresponding to the partition $a < x < b$, and let $g(a) = g(b) = M(b-a)$. We have to see that if $a < c < b$ and $g(c) < y < M\ (b-a)$, then g takes on the value y. One of $M_1 = \sup\{f(t) \colon t \in [a,c]\}$ and $M_2 = \sup\{f(t) \colon\ t \in [c,b]\}$ is equal to M. By symmetry, we can assume that $M_1 = M$. If $c \le x \le b$, then the first term appearing in the upper sum $g(x)$, that is, $\sup\{f(t) \colon t \in [a,x]\} \cdot (x-a) = M\ (x-a)$, is continuous. The second term, that is, $\sup\{f(t) \colon t \in [x,b]\} \cdot (b-x)$, is monotone decreasing.

Thus in the interval $[c,b]$, g is the sum of a continuous and a monotone decreasing function. Moreover, $g(b) = M(b-a) = \max g$. Show that then $g([c,d])$ is an interval.

14.9 Using Exercise 14.8, construct nested intervals whose shared point is a point of continuity.

14.16 Use the equality

$$\sin\alpha + \sin 2\alpha + \cdots + \sin n\alpha = \frac{\sin(n\alpha/2)}{\sin(\alpha/2)} \cdot \sin\big((n+1)\alpha\big)/2. \tag{1}$$

14.30 The statement is false. Find a counterexample in which f is the Dirichlet function.

14.38 Show that for every $\varepsilon > 0$, there are only finitely many points x such that $|f(x)| > \varepsilon$. Then apply the idea seen in Example 14.45.

14.42 We can assume that $c = 0$. Fix an $\varepsilon > 0$; then estimate the integrals $\int_0^\varepsilon f(tx)\,dx$ and $\int_\varepsilon^1 f(tx)\,dx$ separately.

14.44 Let $\max f = M$, and let $\varepsilon > 0$ be fixed. Show that there exists an interval $[c,d]$ on which $f(x) > M - \varepsilon$; then prove that $\sqrt[n]{\int_a^b f^n(x)\,dx} > M - 2\varepsilon$ if n is sufficiently large.

14.50 Using properties (iii), (iv), and (v), show that if $e(x) = 1$ for all $x \in (a,b)$ (the value of e can be arbitrary at the points a and b), then $\Phi\big(e;[a,b]\big) = b-a$. After this, with the help of properties (i) and (iii), show that $\Phi\big(f;[a,b]\big) = \int_a^b f(x)\,dx$ holds for every step function. Finally, use (iv) to complete the solution. (We do not need property (ii).)

14.51 Let $\alpha = \Phi(e;[0,1])$, where $e(x) = 1$ for all $x \in [0,1]$. Use (iii) and (vi) to show that if $b-a$ is rational and $e(x) = 1$ for all $x \in (a,b)$ (the value of e can be arbitrary at the points a and b), then $\Phi(e;[a,b]) = \alpha \cdot (b-a)$. Show the same for arbitrary $a < b$; then finish the solution in the same way as the previous exercise.

Chapter 15

15.3 Is it true that the function $\sqrt{x}$ is Lipschitz on $[0,1]$?

15.4 Apply the idea behind the proof of Theorem 15.1.

15.9 Not possible. See Exercise 14.9.

15.17 Compute the derivative of the right-hand side.

15.25 Use the substitution $y = \pi - x$.

15.32 Let $q = c \cdot r_1^{n_1} \cdots r_k^{n_k}$, where $r_1,\dots,r_k$ are distinct polynomials of degree one or two. First of all, show that in the partial fraction decomposition of p/q, the numerator A of the elementary rational function $A/r_k^{n_k}$ is uniquely determined. Then show that the degree of the denominator of $(p/q) - (A/r_k^{n_k})$ is smaller than the degree of q, and apply induction.

15.37 By Exercise 15.36, it is enough to show that $\int_2^x dt/\log^{n+1} t = o(x/\log^n x)$. Use L'Hôpital's rule for this.

Chapter 16

16.13 Let the two cylinders be $\{(x,y,z)\colon y^2+z^2 \le R^2\}$ and $\{(x,y,z)\colon x^2+z^2 \le R^2\}$. Show that if their intersection is A, then the sections $A^z = \{(x,y)\colon (x,y,z) \in A\}$ are squares, and then apply Theorem 16.11.

16.27 Let N be the product of the first n primes. Choose a partition whose base points are the points i/N ($i = 0,\dots,N$), as well as an irrational point between each of $(i-1)/N$ and i/N. Show that if $c \le 2$, then the length of the corresponding inscribed polygonal path tends to infinity as $n \to \infty$. (We can use the fact that the sum of the reciprocals of the first n prime numbers tends to infinity as $n \to \infty$; see Corollary 18.16.) Thus if $c \le 2$, then the graph is not rectifiable.

If $c > 2$, then the graph is rectifiable. To prove this, show that it suffices to consider the partitions F for which there exists an N such that the rational base points of F are exactly the points i/N ($i = 0,\dots,N$). When finding bounds for the inscribed polygonal paths, we can use the fact that if $b > 0$, then the sums $\sum_{k=1}^N 1/k^{b+1}$ remain smaller than a value independent of N.

16.29 Let $F\colon a = t_0 < t_1 < \cdots < t_n = b$ be a fine partition, and let r_i be the smallest nonnegative number such that a disk D_i centered at $g(t_{i-1})$ with radius r_i covers the set $g([t_{i-1}, t_i])$. Show that the sum of the areas of the disks D_i is small.

16.31 (d) and (e) Show that there exist a point P and a line e such that the points of the curve are equidistant from P and e. Thus the curve is a parabola.

16.35 The graph of the function f agrees with the image of the curve if we know that $f(a\cdot\varphi\cdot\cos\varphi) = a\cdot\varphi\cdot\sin\varphi$ for all $0 \le \varphi \le \pi/4$. Check that the function $g(x) = a\cdot x\cdot\cos x$ is strictly monotone increasing on $[0, \pi/4]$. Thus $f(x) = a\cdot g^{-1}(x)\cdot\sin\left(g^{-1}(x)\right)$ for all $x \in [0, a\cdot\pi\cdot\sqrt{2}/8]$. To compute the integrals $\int f\,dx$ and $\int f^2\,dx$, use the substitution $x = g(t)$.

16.36 The graph of the function f agrees with the image of the curve if we know that $f(ax - a\sin x) = a - a\cos x$ for all $x \in [0, 2\pi]$. Check that the function $g(x) = ax - a\sin x$ is strictly monotone increasing on $[0, 2a\pi]$; then use the ideas from the previous question.

16.40 Apply the formula for the surface area of a segment of the sphere.

Chapter 17

17.2 Let $a = x_0 < x_1 < \cdots < x_n = b$ be a uniform partition of $[a,b]$ into n equal parts. Show that if $j \le k$, $c \in [x_{j-1}, x_j]$ and $d \in [x_{k-1}, x_k]$, then

$$\sum_{i=j}^{k} \omega(f;[x_{i-1},x_i])\cdot(x_i - x_{i-1}) \ge |f(d) - f(c)|\cdot(b-a)/n. \tag{2}$$

17.3 Show that if $k \le (\sqrt{n})/2$, then the numbers $2/\left((2i+1)\pi\right)$ $(i = 1,\ldots,k)$ fall into different subintervals of the partition F_n. Deduce from this, using (2), that

$$\Omega_{F_n}(f) \ge \frac{1}{2}\cdot\sum_{i=1}^{[(\sqrt{n})/2]} \frac{2}{(2i+1)\pi}\cdot\frac{1}{n} \ge c\cdot\frac{\log n}{n}.$$

17.5 Suppose that the value $V_{F_0}(f)$ is the largest, where $F_0\colon a = c_0 < c_1 < \cdots < c_k = b$. Show that f is monotone on each interval $[c_{i-1}, c_i]$ $(i = 1,\ldots,k)$.

17.10 See Exercise 16.27.

17.14 Show that $f' \equiv 0$.

17.15 If $\alpha \ge \beta + 1$, then check that f' is bounded, and so f is Lipschitz. In the case $\alpha < \beta + 1$, show that if $0 \le x < y \le 1$, then

$$|f(x) - f(y)| \le |x^\alpha - y^\alpha| + y^\alpha\cdot\min\left(2, \left|x^{-\beta} - y^{-\beta}\right|\right).$$

Then prove that $y^\alpha - x^\alpha \le C \cdot (y-x)^\gamma$, and $y^\alpha \cdot \min\left(2, (x^{-\beta} - y^{-\beta})\right) \le C \cdot (y-x)^\gamma$ with a suitable constant C. In the proof of the second inequality, distinguish two cases based on whether $y-x$ is small or large compared to y.

17.16 If a function is Hölder α with some constant $\alpha \ge 1$, then f is Hölder 1, so Lipschitz, and so it has bounded variation. If $\alpha < 1$, then there exists a function that is Hölder α that does not have bounded variation. Look for the example in the form $x^n \cdot \sin x^{-n}$ where n is big.

17.19 To prove the "only if" statement, show that if f does not have bounded variation on $[a,b]$, then either there exists a point $a \le c < b$ such that f does not have bounded variation in any right-hand neighborhood of c, or there exists a point $a < c \le b$ such that f does not have bounded variation on any left-hand neighborhood of c. If, for example, f does not have bounded variation in any right-hand neighborhood of c, then look for a strictly decreasing sequence tending to c with the desired property.

Chapter 18

18.4 Let c be a shared point of discontinuity. Then there exists an $\varepsilon > 0$ such that no good δ exists for the continuity of f or g at c. Show that for every $\delta > 0$, there exists a partition with mesh smaller than δ such that the approximating sums formed with different inner points differ by at least ε^2. By symmetry, we can suppose that $c < b$ and that f is discontinuous from the right at c. Distinguish two cases based on whether g is discontinuous or continuous from the right at c.

18.5 Show that for all $\delta > 0$, there exists a partition $0 = x_0 < x_1 < \cdots < x_n = 1$ with mesh smaller than δ, and there exist inner points c_i and d_i such that

$$\sum_{i=1}^{n} f(c_i)\big(f(x_i) - f(x_{i-1})\big) > 1 \quad \text{and} \quad \sum_{i=1}^{n} f(d_i)\big(f(x_i) - f(x_{i-1})\big) < -1.$$

18.6 We give hints for two different proofs. (i) Choose a sequence of partitions F_n satisfying $\lim_{n\to\infty} \delta(F_n) = 0$, and for each n, fix the inner points. Show that the sequence of approximating sums $\sigma_{F_n}(f,g)$ is convergent. Let $\lim_{n\to\infty} \sigma_{F_n}(f,g) = I$. Show that $\int_a^b f\,dg$ exists and its value is I. (ii) Let A_n denote the set of approximating sums corresponding to partitions with mesh smaller than $1/n$ with arbitrary inner points, and let J_n be the smallest closed interval containing the set A_n. Show that $J_1 \supset J_2 \supset \ldots$, and as $n \to \infty$, the length of J_n tends to zero. Thus the intervals J_n have exactly one shared point. Let this shared point be I. Show that $\int_a^b f\,dg$ exists, and its value is I.

18.8 Use the statement of Exercise 18.4.

18.9 Use the statement of Exercise 17.19.

18.11 Suppose first that g is Lipschitz and monotone increasing, and apply the idea used in the proof of Theorem 18.10. Prove that every Lipschitz function can be expressed as the difference of two monotone increasing Lipschitz functions.

Chapter 19

19.7 By Cauchy's criterion, $\lim_{x\to\infty}\int_x^{2x} f\,dt = 0$.

19.23 First, with the help of Theorem 19.22 or by the method seen in Example 19.20.3, show that the integral $\int_1^\infty \sin x/\sqrt{x}\,dx$ is convergent; then apply the substitution $x^2 = t$.

19.24 The integrals are convergent for all $c > 0$. We can prove this with the help of Theorem 19.22 or by the method seen in Example 19.20.3. If $c \le 0$, then the integrals are divergent. Show that in this case, Cauchy's criterion is not satisfied.

19.27 Use Cauchy's criterion.

19.30 Apply the construction in the solution of Exercise 19.28, with the change of choosing the values $f_n(a_n)$ to be large.

19.33 In both exercises, choose the function g to be piecewise constant; that is, constant on the intervals (a_{n-1}, a_n), where (a_n) is a suitable sequence that tends to infinity.

19.35 Let $g(x) = f(x)^{1-1/\log x}$. Show that if at a point x, we have $f(x) < 1/x^2$, then $g(x) \le c/x^2$; if $f(x) \ge 1/x^2$, then $g(x) \le c \cdot f(x)$. Then use the majorization principle.

19.37 Only three cases are possible.

19.38 Use integration by parts.

19.39 Use induction with the help of the previous question.

19.42 Prove the statement by induction on n. (Let the nth statement be that (19.8) holds for every $c > 0$.) Prove the induction step with the help of integration by parts.

19.43 Use the fact that $(1 - t/n)^n \le e^{-t}$ for every $0 < t \le n$, and show from this that $\Gamma(c) > n^c \cdot n!/\big(c(c+1)\cdots(c+n)\big)$.

Show that $e^t \cdot (1 - t/n)^n$ is monotone decreasing on $[0, n]$, and deduce that

$$e^{-t} \le (1+\varepsilon)\cdot\left(1 - \frac{t}{n}\right)^n$$

for all $t \in [0, n]$ if n is sufficiently large. Show from this that for sufficiently large n, $\Gamma(c) < \varepsilon + (1+\varepsilon)\cdot n^c \cdot n!/\big(c(c+1)\cdots(c+n)\big)$.

19.46 Apply (19.9) and Wallis formula.

Solutions

Chapter 2

2.3 Let $\mathscr{H}_n$ denote the system of sets $H \subset \{1,\ldots,n\}$ satisfying the condition, and let a_n be the number of elements in $\mathscr{H}_n$. It is easy to check that $a_1 = 2$ and $a_2 = 3$ (the empty set works, too). If $n > 2$, then we can show that for every $H \subset \{1,\ldots,n\}$,

$$\begin{aligned} H \in \mathscr{H}_n \iff & [(n \notin H) \wedge (H \cap \{1,\ldots,n-1\} \in \mathscr{H}_{n-1})] \vee \\ & \vee [(n \in H) \wedge (H \cap \{1,\ldots,n-2\} \in \mathscr{H}_{n-2}) \wedge (n-1 \notin H)]. \end{aligned}$$

It is then clear that for $n > 2$, we have $a_n = a_{n-1} + a_{n-2}$. Thus the sequence of numbers a_n is $2, 3, 5, 8, \ldots$. This is called the Fibonacci sequence, which has an explicit formula (see Exercise 4.3).

2.11 The inductive step does not work when $n = 3$; we cannot state that $P = Q$ here.

2.14 Let $a \geq -1$. The inequality $(1+a)^n \geq 1+na$ is clearly true if $1+na < 0$, so we can assume that $1+na \geq 0$. Apply the inequality of the arithmetic and geometric means consisting of the n numbers $1+na, 1, \ldots, 1$. We get that

$$\sqrt[n]{1+na} \leq \big((1+na)+n-1\big)/n = 1+a.$$

If we raise both sides to the nth power, then we get the inequality we want to prove.

Chapter 3

3.27 If the set $\mathbb{N}$ were bounded from above, then it would have a least upper bound. Let this be a. Then $n \leq a$ for all $n \in \mathbb{N}$. Since if $n \in \mathbb{N}$, then $n+1 \in \mathbb{N}$, we must have $n+1 \leq a$, that is, $n \leq a-1$ for all $n \in \mathbb{N}$. This, however, is impossible, since then $a-1$ would also be an upper bound. This shows that $\mathbb{N}$ is not bounded from above, so the axiom of Archimedes is satisfied.

Let $[a_n, b_n]$ be nested closed intervals. The set $A = \{a_n \colon n \in \mathbb{N}\}$ is bounded from above, because each b_n is an upper bound. If $\sup A = c$, then $a_n \leq c$ for all n. Since b_n is also an upper bound of A, $c \leq b_n$ for all n. Thus $c \in [a_n, b_n]$ for every n, which is Cantor's axiom.

3.32 If $b/a = c$ then $c \geq 1$. By Theorems 3.23 and 3.24, we have that $b^r/a^r = c^r \geq c^0 = 1$.

3.33 First let b be rational and $0 \le b \le 1$. Then $b = p/q$, where $q > 0$ and $0 \le p \le q$ are integers. By the inequality of arithmetic and geometric means,

$$(1+x)^b = \sqrt[q]{(1+x)^p} = \sqrt[q]{(1+x)^p \cdot 1^{q-p}} \le \frac{p(1+x)+q-p}{q} = 1+bx. \tag{3}$$

Now let $b > 1$ be rational. If $1+bx \le 0$, then $(1+x)^b \ge 1+bx$ holds. Thus we can suppose that $bx > -1$. Since $0 < 1/b < 1$, we can apply (3) to bx instead of x, and $1/b$ instead of b, to get that

$$(1+bx)^{1/b} \le 1+(1/b)\cdot bx = 1+x.$$

Then applying Exercise 3.32, we get that $1+bx \le (1+x)^b$. Thus we have proved the statement for a rational exponent b.

In the proof of the general case, we can assume that $x \ne 0$ and $b \ne 0,1$, since the statement is clear when $x = 0$ or $b = 0,1$. Let $x > 0$ and $0 < b < 1$. If $b < r < 1$ is rational, then by Theorem 3.27, $(1+x)^b \le (1+x)^r$, and by (3), $(1+x)^r \le 1+rx$. Thus $(1+x)^b \le 1+rx$ for all rational $b < r < 1$. It already follows from this that $(1+x)^b \le 1+bx$. Indeed, if $(1+x)^b > 1+bx$, then we can chose a rational number $b < r < 1$ such that $(1+x)^b > 1+rx$, which is impossible.

Now let $-1 < x < 0$ and $0 < b < 1$. If $0 < r < b$ is rational, then by Theorem 3.27, $(1+x)^b < (1+x)^r$, and by (3), $(1+x)^r \le 1+rx$. Thus $(1+x)^b \le 1+rx$ for all rational numbers $0 < r < b$. It then follows that $(1+x)^b \le 1+bx$, since if $(1+x)^b > 1+bx$, then we can chose a rational number $0 < r < b$ such that $(1+x)^b > 1+rx$, which is impossible.

We can argue similarly when $b > 1$.

Chapter 4

4.1 $a_n = 1+(-1)^n \cdot 2^{2-n}$.

4.3 (a) The roots of the polynomial x^2-x-1 are $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$. In the sense of the previous question, for every $\lambda, \mu \in \mathbb{R}$, the sequence $(\lambda\alpha^n + \mu\beta^n)$ also satisfies the recurrence. Choose λ and μ such that $\lambda+\mu = \lambda\alpha^0 + \mu\beta^0 = 0 = u_0$ and $\lambda\alpha^1 + \mu\beta^1 = 1 = u_1$ hold. It is easy to see that the choice $\lambda = 1/\sqrt{5}, \beta = -1/\sqrt{5}$ works. Thus the sequence

$$v_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right) \tag{4}$$

satisfies the recurrence, and $v_0 = u_0,\ v_1 = u_1$. Then by induction, it follows that $v_n = u_n$ for all n, that is, u_n is equal to the right-hand side of (4) for all n.

4.13 For a given $\varepsilon > 0$, there exists an N such that if $n \geq N$, then $|a_n - a| < \varepsilon$. Let $|a_1 - a| + \cdots + |a_N - a| = K$. If $n \geq N$, then

$$|s_n - a| = \left| \frac{(a_1 - a) + \cdots + (a_n - a)}{n} \right| \leq \frac{|a_1 - a| + \cdots + |a_n - a|}{n} \leq \frac{K + n\varepsilon}{n} < 2\varepsilon,$$

given that $n > K/\varepsilon$. Thus $s_n \to a$. It is clear that the sequence $a_n = (-1)^n$ satisfies $s_n \to 0$.

Chapter 5

5.9 Let $\max_{1 \leq i \leq k} a_i = a$. It is clear that

$$a = \sqrt[n]{a^n} \leq \sqrt[n]{a_1^n + \cdots + a_k^n} \leq \sqrt[n]{k \cdot a^n} = \sqrt[n]{k} \cdot a \to a,$$

and so the statement follows by the squeeze theorem.

Chapter 6

6.15 If the set $\mathbb{N}$ were bounded from above, then the sequence $a_n = n$ would also be bounded, so by the Bolzano–Weierstrass theorem, it would have a convergent subsequence. This, however, is impossible, since the distance between any two terms of this sequence is at least 1. This shows that $\mathbb{N}$ is not bounded from above, so the axiom of Archimedes holds.

Let $[a_n, b_n]$ be nested closed intervals. The sequence (a_n) is bounded, since a_1 is a lower bound and every b_i is an upper bound for it. By the Bolzano–Weierstrass theorem, we can choose a convergent subsequence a_{n_k}. If $a_{n_k} \to c$, then $c \leq b_i$ for all i, since $a_{n_k} \leq b_i$ for all i and k. On the other hand, if $n_k \geq i$, then $a_{n_k} \geq a_i$, so $c \geq a_i$. Thus $c \in [a_i, b_i]$ for all i, so Cantor's axiom holds.

6.21 Consider a sequence $a_k \to \infty$ such that $a_{k+1} - a_k \to 0$. Repeating the terms of this sequence enough times gives us a suitable sequence. For example, starting from the sequence $a_k = \sqrt{k}$: (a) Let $a_n = \sqrt{k}$ if $2^{k-1} \leq n < 2^k$. (d) Let (t_k) be a strictly monotone increasing sequence of positive integers such that

$$t_{k+1} > t_k + \max_{n < t_k} s_n$$

for all k, and let $a_n = \sqrt{k}$ if $t_{k-1} \leq n < t_k$. Then (a_n) is monotone increasing and tends to infinity. If $t_{k-1} \leq n < t_k$, then $n + s_n < t_{k+1}$, so $a_{s_n} - a_n \leq \sqrt{k+1} - \sqrt{k}$, which implies $a_{s_n} - a_n \to 0$.

Chapter 7

7.2

(a) Since $1/(n^2+2n) = (1/2)\cdot(1/n - 1/(n+2))$, we have that

$$\sum_{n=1}^{N} \frac{1}{n^2+2n} = \frac{1}{2}\cdot\sum_{n=1}^{N}\left(\frac{1}{n} - \frac{1}{n+2}\right) = \frac{1}{2}\cdot\left(1+\frac{1}{2} - \frac{1}{N+1} - \frac{1}{N+2}\right).$$

Thus the partial sums of the series tend to $3/4$, so the series is convergent with sum $3/4$.

(b) If we leave out the first term in the series in (a), then we get the series in (b). Thus the partial sums of this new series tend to $(3/4)-(1/3) = 5/12$, so it is convergent with sum $5/12$.

(c) Since $1/(n^3-n) = (1/2)\cdot(1/(n-1) - 2/n + 1/(n+1))$, we have that

$$\sum_{n=2}^{N} \frac{1}{n^3-n} = \frac{1}{2}\cdot\sum_{n=2}^{N}\left(\frac{1}{n-1} - \frac{2}{n} + \frac{1}{n+1}\right) = \frac{1}{2}\cdot\left(1-\frac{1}{2} - \frac{1}{N} + \frac{1}{N+1}\right).$$

Thus the partial sums of the series tend to $1/4$, so the series is convergent with sum $1/4$.

7.5 We prove this by induction. The statement holds for $n = 1$. To prove the inductive step, we need to show that if $n \geq 1$, then

$$\frac{1}{(n+1)^{b+1}} \leq \left(1+\frac{1}{b} - \frac{1}{b\cdot(n+1)^b}\right) - \left(1+\frac{1}{b} - \frac{1}{b\cdot n^b}\right).$$

After multiplying this through by $b\cdot(n+1)^b$ and rearranging, this takes the form

$$1+\frac{b}{n+1} \leq \left(1+\frac{1}{n}\right)^b. \tag{5}$$

If $b \geq 1$, then (5) is clear from the Bernoulli inequality for real powers (the first statement of Exercise 3.33).

If $0 < b < 1$, then take the reciprocal of both sides of (5), then raise them to the power $1/b$. After some rearrangement, (5) becomes the inequality

$$\left(1-\frac{b}{n+1+b}\right)^{1/b} \geq 1 - \frac{1}{n+1}. \tag{6}$$

Since $1/b > 1$, (6) again follows from the first statement of Exercise 3.33.

7.14 Let $\lim_{n\to\infty} s_n = A$, where s_n is the nth partial sum of the series. Since

$$\begin{aligned} a_1 + 2a_2 + \cdots + na_n &= (a_1 + \cdots + a_n) + (a_2 + \cdots + a_n) + \cdots + (a_n) = \\ &= s_n + (s_n - s_1) + \cdots + (s_n - s_{n-1}) = \\ &= (n+1)s_n - (s_1 + \cdots + s_n), \end{aligned}$$

we have that

$$\frac{a_1 + 2a_2 + \cdots + na_n}{n} = \frac{n+1}{n} s_n - \frac{s_1 + \cdots + s_n}{n}. \tag{7}$$

Since $s_n \to A$, we have $(s_1 + \cdots + s_n)/n \to A$ (see Exercise 4.13), and so the right-hand side of (7) tends to zero.

Chapter 9

9.6 (b), (c), (d): see the following paper: W. Sierpiński, Sur les suites infinies de fonctions définies dans les ensembles quelconques, *Fund. Math.* **24** (1935), 09–212. (See also: W. Sierpiński: *Oeuvres Choisies* (Warsaw 1976) volume III, 255–258.)

9.18 Suppose that the graph of f intersects the line $y = ax + b$ at more than two points. Then there exist numbers $x_1 < x_2 < x_3$ such that $f(x_i) = ax_i + b$ $(i = 1, 2, 3)$. By the strict convexity of f, $f(x_2) < h_{x_1,x_3}(x_2)$. It is easy to see that both sides are equal to $ax_2 + b$ there, which is a contradiction.

Chapter 10

10.17 Let $\{r_n\}$ be an enumeration of the rational numbers. Let $r \in \mathbb{Q}$ and $\varepsilon = 1/2$. Since f is continuous in r, there exists a closed and bounded interval I_1 such that $\sup\{f(x) : x \in I_1\} < \inf\{f(x) : x \in I_1\} + 1$. We can assume that $r_1 \notin I_1$, since otherwise, we can take a suitable subinterval of I_1. Suppose that $n > 1$ and we have already chosen the interval I_{n-1}. Choose an arbitrary rational number r from the interior of I_{n-1}. Since f is continuous in r, there exists a closed and bounded interval $I_n \subset I_{n-1}$ such that $\sup\{f(x) : x \in I_n\} < \inf\{f(x) : x \in I_n\} + 1/n$. We can assume that $r_n \notin I_n$, since otherwise, we can choose a suitable subinterval of I_n. We can also assume that I_n is in the interior of I_{n-1} (that is, they do not share an endpoint).

Thus we have defined the nested closed intervals I_n for each n. Let $x_0 \in \bigcap_{n=1}^{\infty} I_n$. Then x_0 is irrational, since $x_0 \neq r_n$ for all n. Let $u_n = \inf\{f(x) : x \in I_n\}$ and $v_n = \sup\{f(x) : x \in I_n\}$. Clearly, $u_n \le f(x_0) \le v_n$ and $v_n - u_n < 1/n$ for all n. Let $\varepsilon > 0$ be fixed. If $n > 1/\varepsilon$, then $f(x_0) - \varepsilon < u_n \le v_n < f(x_0) + \varepsilon$, from which it is clear that $|f(x) - f(x_0)| < \varepsilon$ for all $x \in I_n$. Since $x_0 \in I_{n+1}$ and I_{n+1} is in the interior of I_n, there exists a $\delta > 0$ such that $(x_0 - \delta, x + \delta) \subset I_n$. Thus $|f(x) - f(x_0)| < \varepsilon$ whenever $|x - x_0| < \delta$, which proves that f is continuous at x_0.

10.21 See the following paper: H.T. Croft, A question of limits, *Eureca* **20** (1957), 11–13. For the history and a generalization of the problem, see L. Fehér, M. Laczkovich, and G. Tardos, Croftian sequences, *Acta Math. Hung.* **56** (1990), 353–359.

10.40 Let $f(0)=1$ and $f(x)=0$ for all $x \neq 0$. Moreover, $g(x)=0$ for all x. Then $\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = 0$. On the other hand, $f\big(g(x)\big)=1$ for all x.

10.61 If I is degenerate, then so is $f(I)$. If I is not degenerate, then let $\alpha = \inf f(I)$ and $\beta = \sup f(I)$. We show that $(\alpha,\beta) \subset f(I)$. If $\alpha < c < \beta$, then for suitable $a,b \in I$, we have $\alpha < f(a) < c < f(b) < \beta$. Since f is continuous on $[a,b]$, by Theorem 10.57 f attains the value c over $[a,b]$, that is, $c \in f(I)$. This shows that $(\alpha,\beta) \subset f(I)$.

If $\alpha = -\infty$ and $\beta = \infty$, then by the above, $f(I) = \mathbb{R}$, so the statement holds. If $\alpha \in \mathbb{R}$ and $\beta = \infty$, then $(\alpha,\infty) \subset f(I) \subset [\alpha,\infty)$, so $f(I)$ is one of the intervals (α,∞), $[\alpha,\infty)$, so the statement holds again. We can argue similarly if $\alpha = -\infty$ and $\beta \in \mathbb{R}$. Finally, if $\alpha,\beta \in \mathbb{R}$, then by $(\alpha,\beta) \subset f(I) \subset [\alpha,\beta]$, we know that $f(I)$ can only be one of the following sets: (α,β), $[\alpha,\beta)$, $(\alpha,\beta]$, $[\alpha,\beta]$. Thus $f(I)$ is an interval.

10.88 Such is the function $-x$, for example.

10.91 It is easy to see by induction on k that

$$f\left(\frac{x_1+\cdots+x_{2^k}}{2^k}\right) \le \frac{f(x_1)+\cdots+f(x_{2^k})}{2^k} \tag{8}$$

for all (not necessarily distinct) numbers $x_1,\ldots,x_{2^k} \in I$. If $x_1,\ldots,x_n \in I$ are fixed numbers, then let $s = (x_1+\cdots+x_n)/n$ and $x_i = s$ for all $n < i \le 2^n$. By (8), we have

$$f(s) \le \frac{f(x_1)+\cdots+f(x_n)+(2^n-n)\cdot f(s)}{2^n},$$

and so

$$f(s) \le \frac{f(x_1)+\cdots+f(x_n)}{n}.$$

10.99 Suppose that $f(x) \le K$ for all $|x-x_0| < \delta$. If $|h| < \delta$, then

$$f(x_0) \le \frac{f(x_0-h)+f(x_0+h)}{2} \le \frac{K+f(x_0+h)}{2},$$

so $f(x_0+h) \ge 2f(x_0)-K$. This means that $2f(x_0)-K$ is a lower bound of the function f over $(x_0-\delta,x_0+\delta)$. Thus f is also bounded from below, and so it is bounded in $(x_0-\delta,x_0+\delta)$.

10.100 If we apply inequality (10.24) with the choices $a = x$ and $b = x + 2^k h$, then we get that

$$f\left(x+2^{k-1}h\right) \leq \frac{1}{2}\left[f(x)+f\left(x+2^k h\right)\right].$$

Dividing both sides by 2^{k-1} then rearranging yields us the inequality

$$\tfrac{1}{2^{k-1}} f\left(x+2^{k-1}h\right) - \tfrac{1}{2^k} f\left(x+2^k h\right) \leq \tfrac{1}{2^k} f(x).$$

If we take the sum of these inequalities for $k = 1,\ldots,n$, then the inner terms cancel out on the left-hand side, and we get that

$$f(x+h) - \frac{1}{2^n} f\left(x+2^n h\right) \leq \left(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^n}\right) f(x) = \left(1-\frac{1}{2^n}\right) f(x).$$

A further rearrangement give us the inequality

$$f(x+h) - f(x) \leq \frac{1}{2^n}\cdot\left[f\left(x+2^n h\right)-f(x)\right]. \tag{9}$$

10.101 By Exercise 10.99, f is bounded on the interval $J = (x_0 - \delta, x_0 + \delta)$. Let $|f(x)| \leq M$ for all $x \in J$. Let $\varepsilon > 0$ be fixed, and choose a positive integer n such that $2^n > 2M/\varepsilon$ holds. If $|t| < \delta/2^n$, then $x_0 + 2^n t \in J$, so $|f(x_0 + 2^n t)| \leq M$. Thus by (9), we have

$$f(x_0+t) - f(x_0) \leq \tfrac{1}{2^n}\cdot\left[f\left(x_0+2^n t\right)-f(x_0)\right] \leq \tfrac{1}{2^n}\cdot 2M < \varepsilon.$$

If, however, we apply (9) with the choices $x = x_0 + t$ and $h = -t$, then we get that

$$f(x_0) - f(x_0+t) \leq \tfrac{1}{2^n}\cdot\left[f\left(x_0-(2^n-1)t\right)-f(x_0+t)\right] \leq \tfrac{1}{2^n}\cdot 2M < \varepsilon.$$

Finally, we conclude that $|f(x_0+t) - f(x_0)| < \varepsilon$ for all $|t| < \delta/2^n$, which proves that f is continuous at x_0.

Chapter 11

11.36

(a) First of all, we show that

$$\sin^{-2}(k\pi/2m) + \sin^{-2}\left((m-k)\pi/2m\right) = 4\sin^{-2}(k\pi/m) \tag{10}$$

for all $0 < k < m$. This is because $\sin^{-2}\left((m-k)\pi/2m\right) = \cos^{-2}(k\pi/2m)$, and so the left-hand side of (10) is

$$\sin^{-2}(k\pi/2m)+\cos^{-2}(k\pi/2m)=\frac{\cos^2(k\pi/2m)+\sin^2(k\pi/2m)}{\sin^2(k\pi/2m)\cdot\cos^2(k\pi/2m)}=$$

$$=\frac{1}{\sin^2(k\pi/2m)\cdot\cos^2(k\pi/2m)}=\frac{4}{\sin^2\left(2\cdot(k\pi/2m)\right)}=4\cdot\sin^{-2}(k\pi/m).$$

Applying the equality (10) with $m=2n$, we get that

$$\sin^{-2}(k\pi/4n)+\sin^{-2}\left((2n-k)\pi/4n\right)=4\sin^{-2}(k\pi/2n) \tag{11}$$

for all $0<k<2n$. If we now sum the equalities (11) for $k=1,\dots,n$, then on the left-hand side, we get every term of the sum defining A_{2n}, except that $\sin^{-2}(n\pi/4n)=2$ appears twice, and $\sin^{-2}(2n\pi/4n)=1$ is missing. Thus the sum of the left-hand sides is $A_{2n}+1$. Since the sum of the right-hand sides is $4A_n$, we get that $A_{2n}=4A_n-1$.

(b) The statement is clear for $n=0$, and follows for $n>0$ by induction.

(c) The inequality is clear by Theorem 11.26. Applying this for $x=k\pi/2n$ and then summing the inequality we get for $k=1,\dots,n$ gives us the desired bound.

(d) By the above, the partial sum $S_{2^n}=\sum_{k=1}^{2^n}(1/k^2)$ satisfies

$$\left(\frac{\pi}{2\cdot 2^n}\right)^2\cdot(A_{2^n}-2^n)\le S_{2^n}\le\left(\frac{\pi}{2\cdot 2^n}\right)^2\cdot A_{2^n},$$

and so

$$\frac{\pi^2}{6}-\frac{\pi^2 2^n}{4^{n+1}}\le S_{2^n}\le\frac{\pi^2}{6}+\frac{\pi^2}{3\cdot 4^{n+1}}.$$

Then by the squeeze theorem, $S_{2^n}\to\pi^2/6$. Since the series is convergent by Example 7.11.1, we have $S_n\to\pi^2/6$, that is, $\sum_{n=1}^{\infty}1/k^2=\pi^2/6$.

Chapter 12

12.15 Let

$$m_n=\min\left(\frac{f(x_n)-f(a)}{x_n-a},\frac{f(y_n)-f(a)}{y_n-a}\right)$$

and

$$M_n=\max\left(\frac{f(x_n)-f(a)}{x_n-a},\frac{f(y_n)-f(a)}{y_n-a}\right)$$

for all n. It is clear that $m_n\to f'(a)$ and $M_n\to f'(a)$ if $n\to\infty$.
Let $p_n=(a-x_n)/(y_n-x_n)$ and $q_n=(y_n-a)/(y_n-x_n)$. Then $p_n,q_n>0$ and $p_n+q_n=1$. Since

$$\frac{f(y_n)-f(x_n)}{y_n-x_n}=p_n\cdot\frac{f(a)-f(x_n)}{a-x_n}+q_n\cdot\frac{f(y_n)-f(a)}{y_n-a},$$

we have $m_n \le (f(y_n) - f(x_n))/(y_n - x_n) \le M_n$. Then the statement follows by the squeeze theorem.

12.19 See Example 13.43.

12.23 Let (x_0, y_0) be a common point of the two graphs. Then $\sqrt{4a(a-x_0)} = \sqrt{4b(b+x_0)}$, so $a(a-x_0) = b(b+x_0)$ and $x_0 = a-b$. The slopes of the two graphs at the point (x_0, y_0) are

$$m_1 = -2a/\sqrt{4a(a-x_0)} \text{ and } m_2 = 2b/\sqrt{4b(b+x_0)}.$$

Thus

$$m_1 \cdot m_2 = \frac{-4ab}{\sqrt{4a(a-x_0)} \cdot \sqrt{4b(b+x_0)}} = \frac{-4ab}{\left(\sqrt{4a(a-x_0)}\right)^2} = \frac{-b}{a-x_0} = -1.$$

It is well known (and easy to show) that if the product of the slopes of two lines is -1, then the two lines are perpendicular.

12.25 Since $\pi - e < 1$, if $x < 0$, then $2^x < 1 < (\pi - e)^x$, and if $x > 0$, then $2^x > 1 > (\pi - e)^x$. Thus the only point of intersection of the two graphs is $(0,1)$. At this point, the slopes of the tangent lines are $\log 2$ and $\log(\pi - e)$. This means that the angle between the tangent line of 2^x at $(0,1)$ and the x-axis is $\operatorname{arc\,tg}(\log 2)$, while the angle between the tangent line of $(\pi - e)^x$ at $(0,1)$ and the x-axis is $\operatorname{arc\,tg}\left(\log(\pi - e)\right)$. Thus the angle between the two tangent lines is $\operatorname{arc\,tg}(\log 2) - \operatorname{arc\,tg}\left(\log(\pi - e)\right)$.

12.44 If $y = \log_a x$, then $y' = 1/(x \cdot \log a)$, so xy' is constant, $(xy')' = 0$, $y' + xy'' = 0$. Thus $-1 = (-x)' = (y'/y'')' = (y''^2 - y'y''')/y''^2$, so $y'y''' - 2(y'')^2 = 0$.

12.48 Let $a \le b$. The volume of the box is

$$K(x) = (a-2x)(b-2x)x = 4x^3 - 2(a+b)x^2 + abx.$$

We need to find the maximum of this function on the interval $[0, a/2]$. Since $K(0) = K(a/2) = 0$, the absolute maximum is in the interior of the interval, so it is a local extremum. The solutions of the equation $K'(x) = 12x^2 - 4(a+b)x + ab = 0$ are

$$x = \frac{a+b \pm \sqrt{a^2+b^2-ab}}{6}.$$

Since

$$\frac{a+b+\sqrt{a^2+b^2-ab}}{6} \ge \frac{a+b+\sqrt{a^2+b^2-b^2}}{6} = \frac{2a+b}{6} \ge \frac{a}{2}$$

and we are looking for extrema inside $(0, a/2)$, $K(x)$ has a local and therefore absolute maximum at the point

$$x = \frac{a+b-\sqrt{a^2+b^2-ab}}{6}.$$

For example, in the case $a = b$, the box has maximum volume if $x = a/6$.

12.53 $f'(x) = 1+4x\cdot\sin(1/x)-2\cos(1/x)$, if $x \neq 0$ and

$$f'(0) = \lim_{x\to 0}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0}(1+2x\sin(1/x)) = 1.$$

Thus $f'(0) > 0$. At the same time, f' takes on negative values in any right- or left-hand neighborhood of 0. This is because

$$f'\left(\frac{\pm 1}{2k\pi}\right) = 1-2 < 0$$

for every positive integer k. It follows that 0 does not have a neighborhood in which f is monotone increasing, since f is strictly locally decreasing at each of the points $1/(2k\pi)$.

12.87 Suppose that $e = p/q$, where p and q are positive integers. Then $q > 1$, since e is not an integer. Let $a_n = 1+1/1!+\cdots+1/n!$. The sequence (a_n) is strictly monotone increasing, and by Exercise 12.86, $e = \lim_{n\to\infty} a_n$. Thus $e > a_n$ for all n. If $n > q$, then

$$\begin{aligned}
q!\cdot(a_n-a_q) &= q!\cdot\left(\frac{1}{(q+1)!}+\cdots+\frac{1}{n!}\right) = \\
&= \frac{1}{(q+1)}+\frac{1}{(q+1)(q+2)}+\cdots+\frac{1}{(q+1)\cdot\ldots\cdot n} \le \\
&\le \frac{1}{(q+1)}+\frac{1}{(q+1)^2}+\cdots+\frac{1}{(q+1)^{n-q}} = \\
&= \frac{1}{(q+1)}\cdot\left(1-\frac{1}{(q+1)^{n-q}}\right)\Big/\left(1-\frac{1}{(q+1)}\right) < \\
&< \frac{1}{(q+1)}\Big/\left(1-\frac{1}{(q+1)}\right) = \frac{1}{q}.
\end{aligned}$$

This holds for all $n > q$, so $0 < q!\cdot(e-a_q) \le 1/q < 1$. On the other hand, since $e = p/q$,

$$q!\cdot(e-a_q) = q!\cdot\left(\frac{p}{q}-1-\frac{1}{1!}-\frac{1}{2!}-\ldots-\frac{1}{q!}\right)$$

is an integer, which is impossible.

12.95 The function f has a strict local minimum at 0, since $f(0) = 0$ and $f(x) > 0$ if $x \neq 0$. Now

$$f'(x) = x^2\left[4x\left(2+\sin\frac{1}{x}\right)-\cos\frac{1}{x}\right]$$

if $x \neq 0$. We can see that f' takes on both negative and positive values in every right-hand neighborhood of 0. For example, if $k \geq 2$ is an integer, then

$$f'\left(\frac{1}{2k\pi}\right) = \frac{1}{(2k\pi)^2} \cdot \left(\frac{8}{2k\pi} - 1\right) < 0$$

and

$$f'\left(\frac{1}{(2k+1)\pi}\right) = \frac{1}{(2k+1)^2\pi^2} \cdot \left(\frac{8}{(2k+1)\pi} + 1\right) > 0.$$

Chapter 13

13.5 The interval $[0,1]$ is mapped to $[-1,1]$ by the function $2x-1$. Thus we first need to determine the nth Bernstein polynomial of the function $f(2x-1)$, which is

$$\sum_{k=0}^{n} f\left(\frac{2k}{n} - 1\right) \cdot \binom{n}{k} x^k (1-x)^{n-k}.$$

We have to transform this function back onto $[-1,1]$, that is, we need to replace x by $(1+x)/2$, which gives us the desired formula.

13.7 $B_1 = 1$, $B_2 = B_3 = (1+x^2)/2$, $B_4 = B_5 = (3+6x^2-x^4)/8$.

13.11

$$\sum_{k=0}^{n} \frac{k}{n} \binom{n}{k} x^k (1-x)^{n-k} = x \cdot \sum_{k=1}^{n} \binom{n-1}{k-1} x^{k-1} (1-x)^{n-k} =$$
$$= x \cdot \left(x + (1-x)\right)^{n-1} = x.$$

13.12

$$\sum_{k=0}^{n} \frac{k^2}{n^2} \binom{n}{k} x^k (1-x)^{n-k} = \sum_{k=1}^{n} \frac{k}{n} \binom{n-1}{k-1} x^k (1-x)^{n-k} =$$
$$= \frac{n-1}{n} \sum_{k=1}^{n} \frac{k-1}{n-1} \binom{n-1}{k-1} x^k (1-x)^{n-k} + \frac{1}{n} \sum_{k=1}^{n} \binom{n-1}{k-1} x^k (1-x)^{n-k} =$$
$$= \frac{n-1}{n} \sum_{k=2}^{n} \binom{n-2}{k-2} x^k (1-x)^{n-k} + \frac{x}{n} =$$
$$= \frac{n-1}{n} x^2 + \frac{x}{n} = x^2 + \frac{x-x^2}{n}.$$

13.32 Let $f(x) = x^2 \sin(1/x)$ if $x \neq 0$ and $f(0) = 0$. We know that f is differentiable everywhere, and $f'(x) = 2x\sin(1/x) - \cos(1/x)$ if $x \neq 0$ and $f'(0) = 0$ (see Example 13.43). The function $f' + h_2$ is continuous everywhere, so by Theorem 15.5, it has a primitive function. If $g' = f' + h_2$, then $(g-f)' = h_2$, so $g - f$ is a primitive function of h_2.

If we start with the function $f_1(x) = x^2\cos(1/x)$, $f_1(0) = 0$, then a similar argument gives a primitive function of h_1.

13.33 The function $h_1^2 + h_2^2$ vanishes at zero, and is 1 everywhere else. Thus $h_1^2 + h_2^2$ is not Darboux, so it does not have a primitive function. On the other hand, $h_2^2 - h_1^2 = h_2(x/2)$, so $h_2^2 - h_1^2$ has a primitive function. Now the statement follows.

Chapter 14

14.6 I. The solution is based on the following statement: *Let $g = g_1 + g_2$, where $g_1\colon [c,b] \to \mathbb{R}$ is continuous and $g_2\colon [c,b] \to \mathbb{R}$ is monotone decreasing. If $g(b) = \max g$, then the image of g is an interval.*
Let $c \le d < b$ and $g(d) < y < g(b)$. We will show that if $s = \sup\{x \in [d,b]\colon g(t) \le y \text{ for all } t \in [d,x]\}$, then $g(s) = y$. To see this, suppose that $g(s) < y$. Then $s < b$ but $\lim_{x\to d+0} g_1(x) = g_1(s)$. Now $\lim_{x\to d+0} g_2(x) \le g_2(s)$ implies that $g(x) < y$ in a right-hand neighborhood of the point s, which is impossible. If $f(s) > y$, then $s > d$. Thus $\lim_{x\to d-0} g_1(x) = g_1(s)$ and $\lim_{x\to d-0} g_2(x) \ge g_2(s)$ imply that $g(x) > y$ in a left-hand neighborhood of the point s, which is also impossible. This proves the statement.

II. Suppose first that there is only one base point. We can assume that $f \ge 0$. Let $M = \sup\{f(t)\colon t \in [a,b]\}$. Let $g(x)$ denote the upper sum corresponding to the partition $a < x < b$, and let $g(a) = g(b) = M(b-a)$. We have to show that if $a < c < b$ and $g(c) < y < M(b-a)$, then g takes on the value y. One of the two values $M_1 = \sup\{f(t)\colon t \in [a,c]\}$, $M_2 = \sup\{f(t)\colon t \in [c,b]\}$ must be equal to M. By symmetry, we may assume that $M_1 = M$. If $c \le x \le b$, then the first term appearing in the upper sum of $g(x)$, which is $\sup\{f(t)\colon t \in [a,x]\} \cdot (x-a) = M(x-a)$, is continuous. The second term, which is $\sup\{f(t)\colon t \in [x,b]\} \cdot (b-x)$, is monotone decreasing.

Thus over the interval $[c,b]$, the function g is the sum of a continuous and a monotone decreasing function. Moreover, $g(b) = M(b-a) = \max g$. By the above, it follows that g takes on the value of y, and so the set of upper sums corresponding to partitions with one base point forms an interval.

III. Now let $F\colon a = x_0 < x_2 < \cdots < x_n = b$ be an arbitrary partition, and let $S_F < y < M(b-a)$. We have to show that there is an upper sum that is equal to y. We can assume that n is the smallest number such that there exists a partition into n parts whose upper sum is less than y. Then for the partition $F'\colon a = x_0 < x_2 < \cdots < x_n = b$, we have $S_{F'} \ge y$. If $S_{F'} = y$, then we are done. If $S_{F'} > y$, then we can apply step II for the interval $[a,x_2]$ to find a point $a < x < x_2$ such that the partition $F''\colon a = x_0 < x < x_2 < \cdots < x_n = b$ satisfies $S_{F''} = y$.

14.10 The statement is not true: if f is the Dirichlet function and $F\colon a < b$, then the possible values of σ_F are $b-a$ and 0.

14.12 The statement is true. By Exercise 14.9, there exists a point $x_0 \in [a,b]$ where f is continuous. Since $f(x_0) > 0$, there exist points $a \le c < d \le b$ such that $f(x) > f(x_0)/2$ for all $x \in [c,d]$. It is clear that if the partition F_0 contains the points c and d, then $s_{F_0} \ge (d-c)\cdot f(x_0)/2$. By Lemma 14.4, it then follows that $S_F \ge (d-c)\cdot f(x_0)/2$ for every partition, so $\int_a^b f\,dx \ge (d-c)\cdot f(x_0)/2 > 0$.

14.33 To prove the nontrivial direction, suppose that f is nonnegative, continuous, and not identically zero. If $f(x_0) > 0$, then by a similar argument as in the solution of Exercise 14.12, we get that $\int_a^b f\,dx > 0$.

Chapter 15

15.24 We will use the notation of the proof of Stirling's formula (Theorem 15.15). Since the sequence (a_n) is strictly monotone increasing and tends to 1, we have $a_n < 1$ for all n. This proves the inequality $n! > (n/e)^n \cdot \sqrt{2\pi n}$.

By inequality (15.14),

$$\log a_{N+1} - \log a_n = \sum_{k=n}^{N} (\log a_{k+1} - \log a_k) < \sum_{k=n}^{N} \frac{1}{4k^2} < \frac{1}{4}\sum_{k=n}^{N} \frac{1}{(k-1)k} = \\ = \frac{1}{4(n-1)} - \frac{1}{4N}$$

for all $n < N$. Here letting N go to infinity, we get that $-\log a_n \le \frac{1}{4(n-1)}$, that is, $a_n \ge e^{-1/\left(4(n-1)\right)}$, which is exactly the second inequality we wanted to prove.

15.30 $\int \sqrt{x^2-1}\,dx = \frac{1}{2}x\sqrt{x^2-1} - \frac{1}{2}\operatorname{arch} x + c$.

15.39 Start with the equality

$$\left(x^k \cdot \sqrt{f(x)}\right)' = kx^{k-1}\sqrt{f(x)} + \frac{x^k f'(x)}{2\sqrt{f(x)}} = \frac{k \cdot x^{k-1} \cdot f(x) + \frac{1}{2}x^k f'(x)}{\sqrt{f(x)}}.$$

The numerator of the fraction on the right-hand side is a polynomial of degree exactly $k+n-1$. It follows that I_{k+n-1} can be expressed as a linear combination of an elementary function and the integrals $I_0, I_1, \ldots, I_{k+n-2}$. The statement then follows.

Chapter 16

16.4 Let $\varepsilon > 0$ be given. Since A is measurable, we can find rectangles $R_1, \ldots, R_n \subset \mathbb{R}^p$ such that $A \subset \cup_{i=1}^n R_i$ and $\sum_{i=1}^n m_p(R_i) < m_p(A) + \varepsilon$. Similarly, there are rectangles $S_1,\dots,S_k \subset \mathbb{R}^q$ such that $B \subset \cup_{j=1}^k S_j$ and $\sum_{j=1}^k m_q(S_j) < m_q(B)+\varepsilon$. Then $T_{ij} = R_i \times S_j$ is a rectangle in $\mathbb{R}^{p+q}$ and $m_{p+q}(T_{ij}) = m_p(R_i)\cdot m_q(S_j)$ for every $i=1,\dots,n,\ j=1,\dots,k$. Clearly,

$$A\times B \subset \bigcup_{i=1}^{n}\bigcup_{j=1}^{k} T_{ij},$$

and thus

$$\begin{aligned}\overline{m}_{p+q}(A\times B) &\le \sum_{i=1}^{n}\sum_{j=1}^{k} m_{p+q}(T_{ij}) = \sum_{i=1}^{n}\sum_{j=1}^{k} m_p(R_i)\cdot m_q(S_j) = \\ &= \left(\sum_{i=1}^{n} m_p(R_i)\right)\cdot\left(\sum_{j=1}^{k} m_q(T_{ij})\right) < \\ &< (m_p(A)+\varepsilon)\cdot(m_q(B)+\varepsilon).\end{aligned}$$

This is true for every $\varepsilon>0$, and therefore, we obtain $\overline{m}_{p+q}(A\times B) \le m_p(A)\cdot m_q(B)$. A similar argument gives $\underline{m}_{p+q}(A\times B) \ge m_p(A)\cdot m_q(B)$. Then we have $\underline{m}_{p+q}\ (A\times B) = \overline{m}_{p+q}(A\times B) = m_p(A)\cdot m_q(B)$; that is, $A\times B$ is measurable, and its measure equals $m_p(A)\cdot m_q(B)$.

16.8 The function f is nonnegative and continuous on the interval $[0,r]$, so by Corollary 16.10, the sector $B_f = \{(x,y)\colon 0\le x\le r,\ 0\le y\le f(x)\}$ is measurable and has area

$$\begin{aligned}&\int_0^{r\cos\delta} \frac{\sin\delta}{\cos\delta}\cdot x\,dx + r^2\int_{r\cos\delta}^{r}\sqrt{r^2-x^2}\,dx = \\ &= \frac{1}{2}r^2\cos\delta+\sin\delta + r^2\int_{\delta}^{0}\sin t\cdot(-\sin t)\,dt = \\ &= \frac{1}{2}r^2\cos\delta\cdot\sin\delta + \frac{1}{2}r^2\delta - \frac{r^2\sin 2\delta}{4} = \frac{1}{2}r^2\delta.\end{aligned}$$

16.9 The part of the region A_u that falls in the upper half-plane is the difference between the triangle T_u defined by the points $(0,0)$, $(u,0)$, and (u,v), and the region $B_u = \{(x,y)\colon 1\le x\le u,\ 0\le y\le\sqrt{x^2-1}\}$. Thus

$$\frac{1}{2}\cdot t(A_u) = \frac{1}{2}u\sqrt{u^2-1} - \int_1^u \sqrt{x^2-1}\,dx.$$

The value of the integral, by Exercise 15.30, is $(1/2)u\sqrt{u^2-1}-(1/2)\operatorname{arch}u$, so $t(A_u)/2 = (\operatorname{arch}u)/2$ and $t(A_u) = \operatorname{arch}u$.

16.27 We show that if $c>2$, then the graph of f^c is rectifiable. Let F be an arbitrary partition, and let N denote the least common denominator of the rational base points of F. Since adding new base points does not decrease the length of the inscribed polygonal path, we can assume that the points i/N $(i=0,\dots,N)$ are all base points.

Then all the rest of the base points of F are irrational. It is clear that in this case, the length of the inscribed polygonal path corresponding to F is at most

$$\sum_{i=1}^{N}\left(f^c\left((i-1)/N\right)+(1/N)+f^c(i/N)\right)\leq 1+2\cdot\sum_{i=0}^{N}f^c(i/N).$$

If $1<k\leq N$, then among the numbers of the form i/N $(i\leq N)$, there are at most k with denominator k (after simplifying). Here the value of f is $1/k^c$, so

$$\sum_{i=0}^{N}f^c(i/N)\leq 2+\sum_{k=2}^{N}k\cdot(1/k^c)=2+\sum_{k=2}^{N}k^{1-c}.$$

We can now use the fact that if $b>0$, then $\sum_{k=1}^{N}1/\left(k^{b+1}\right)<(b+1)/b$ for all N (see Exercise 7.5). We get that $\sum_{k=2}^{N}k^{1-c}<1/(c-2)$, so the length of every inscribed polygonal path is at most $5+(2/(c-2))$.

Now we show that if $c\leq 2$, then the graph of f^c is not rectifiable. Let $N>1$ be an integer, and consider a partition

$$F_N\colon\, 0=x_0<y_0<x_1<y_1<x_2<\cdots<x_{N-1}<y_{N-1}<x_N=1$$

such that $x_i=i/N$ $(i=0,\ldots,N)$ and y_i is irrational for all $i=0,\ldots,N-1$. If p is a prime divisor of N, then the numbers $1/p,\ldots,(p-1)/p$ appear among the base points, and the length of the segment of the inscribed polygonal path over the interval $[x_i,y_i]$ is at least $1/p^2$. It is then clear that the length of the whole inscribed polygonal path is at least $\sum_{p|N}(p-1)/p^2\geq(1/2)\cdot\sum_{p|N}1/p$.

Now we use the fact that the sum of the reciprocals of the first n primes tends to infinity as $n\to\infty$ (see Corollary 18.16). Thus if N is equal to the product of the first n primes, then the partition F_N above gives us an inscribed polygonal path whose length can be arbitrarily long.

16.29 Let the coordinates of g be g_1 and g_2. By Heine's theorem (Theorem 10.61), there exists a $\delta>0$ such that if $u,v\in[a,b]$, $|u-v|<\delta$, then $|g_i(u)-g_i(v)|<\varepsilon/2$ $(i=1,2)$, and so $|g(u)-g(v)|<\varepsilon$.
Let the arc length of the curve be L, and let $F\colon\, a=t_0<t_1<\cdots<t_n=b$ be a partition with mesh smaller than δ. If $r_i=\sup\{|g(t)-g(t_{i-1})|\colon\, t\in[t_{i-1},t_i]\}$, then by choosing δ, we have $r_i\leq\varepsilon$ for all i. Then the disks $D_i=\{x\in\mathbb{R}^2\colon\, |x-g(t_{i-1})|\leq r_i\}$ $(i=1,\ldots,n)$ cover the set $g([a,b])$.

Choose points $u_i\in[t_{i-1},t_i]$ such that $|g(u_i)-g(t_{i-1})|\geq r_i/2$ $(i=1,\ldots,n)$. Since the partition with base points t_i and u_i has an inscribed polygonal path of length at most L, we have $\sum_{i=1}^{n}r_i\leq 2\cdot\sum_{i=1}^{n}|g(u_i)-g(t_{i-1})|\leq 2L$. Thus the area of the union of the discs D_i is at most

$$\sum_{i=1}^{n}\pi r_i^2\leq\pi\cdot\sum_{i=1}^{n}\varepsilon\cdot r_i\leq 2L\pi\cdot\varepsilon.$$

16.40 Cover the disk D by a sphere G of the same radius. For every planar strip S, consider the (nonplanar) strip S' in space whose projection onto the plane is S. If the strips S_i cover the disk, then the corresponding strips S_i' cover the sphere in space. Each strip S_i' cuts out a strip of the sphere with surface area $2\pi r \cdot d_i$ from G, where d_i is the width of the strips S_i and S_i'. Since these strips of the sphere cover G, we have $\sum 2\pi r \cdot d_i \geq 4r^2\pi$, so $\sum d_i \geq 2r$.

Chapter 17

17.1 Let $F\colon a = x_0 < x_1 < \cdots < x_n = b$. By the Lipschitz condition, we know that $\omega_i = \omega(f;[x_{i-1},x_i]) \leq K \cdot (x_i - x_{i-1}) \leq K \cdot \delta(F)$ for all i. Thus

$$\Omega_F(f) = \sum_{i=1}^{n} \omega_i \cdot (x_i - x_{i-1}) \leq K \cdot \delta(F) \cdot \sum_{i=1}^{n} (x_i - x_{i-1}) = K \cdot (b-a) \cdot \delta(F).$$

17.12 Since every integrable function is already bounded, there exists a K such that $|f'(x)| \leq K$ for all $x \in [a,b]$. By the mean value theorem, it follows that f is Lipschitz, so by statement (ii) of Theorem 17.3, f is of bounded variation.

Let $F\colon a = x_0 < \cdots < x_n = b$ be an arbitrary partition of the interval $[a,b]$. By the mean value theorem, there exist points $c_i \in (x_{i-1}, x_i)$ such that $|f(x_i) - f(x_{i-1})| = |f'(c_i)|(x_i - x_{i-1})$ for all $i = 1, \ldots, n$, and so $V_F(f)$ is equal to the Riemann sum $\sigma_F\big(|f'|\colon (c_i)\big)$.

Let $\int_a^b |f'|\,dx = I$. For each $\varepsilon > 0$, there exists a partition F such that every Riemann sum of the function $|f'|$ corresponding to the partition F differs from I by at most ε. Thus $V_F(f)$ also differs from I by less than ε, so it follows that $V(f;[a,b]) \geq I$.

On the other hand, an arbitrary partition F has a refinement F' such that every Riemann sum of the function $|f'|$ corresponding to F' differs from I by less than ε. Then $V_{F'}(f)$ also differs from I by less than ε, so $V_{F'}(f) < I + \varepsilon$. Since $V_F(f) \leq V_{F'}(f)$, we have $V_F(f) < I + \varepsilon$. This holds for every partition F and every $\varepsilon > 0$, so $V(f;[a,b]) \leq I$.

17.13 Let $0 \leq x < y \leq 1$. If $\alpha \geq 1$, then by the mean value theorem, for a suitable $z \in (x,y)$, we have

$$y^\alpha - x^\alpha = \alpha \cdot z^{\alpha-1} \cdot (y-x) \leq \alpha \cdot (y-x) = \alpha \cdot (y-x)^\beta.$$

If $\alpha < 1$, then $y^\alpha - x^\alpha \leq (y-x)^\alpha = (y-x)^\beta$.

17.15 If $\alpha \geq \beta+1$, then it is easy to check that f' is bounded in $(0,1]$. Then f is Lipschitz, that is, Hölder 1 on the interval $[0,1]$. We can then suppose that $\alpha < \beta+1$ and $\gamma = \alpha/(\beta+1) < 1$. Let $0 \leq x < y \leq 1$ be fixed. Then

$$\begin{aligned}|f(x)-f(y)| &= \left|x^{\alpha}\cdot\sin x^{-\beta} - y^{\alpha}\cdot\sin y^{-\beta}\right| \leq \\ &\leq \left|x^{\alpha}\cdot\sin x^{-\beta} - y^{\alpha}\cdot\sin x^{-\beta}\right| + y^{\alpha}\cdot\left|\sin x^{-\beta} - \sin y^{-\beta}\right| \leq \\ &\leq |x^{\alpha}-y^{\alpha}| + y^{\alpha}\cdot\min\left(2, \left|x^{-\beta}-y^{-\beta}\right|\right),\end{aligned}$$

so it suffices to show that $y^{\alpha}-x^{\alpha} \leq C\cdot(y-x)^{\gamma}$ and

$$y^{\alpha}\cdot\min\left(2,(x^{-\beta}-y^{-\beta})\right) \leq C\cdot(y-x)^{\gamma} \tag{12}$$

with a suitable constant C. By the previous exercise, there is a constant C depending only on α such that $y^{\alpha}-x^{\alpha} \leq C\cdot(y-x)^{\min(1,\alpha)} \leq C\cdot(y-x)^{\gamma}$, since $\gamma < \min(1,\alpha)$.

We distinguish two cases in the proof of the inequality (12). If $y-x \geq y^{\beta+1}/2$, then

$$y^{\alpha}\cdot 2 = 2\cdot\left(y^{\beta+1}\right)^{\alpha/(\beta+1)} \leq 2\cdot 2^{\alpha/(\beta+1)}\cdot(y-x)^{\alpha/(\beta+1)} < 4\cdot(y-x)^{\gamma}.$$

Now suppose that $y-x < y^{\beta+1}/2$. Then on the one hand, $x > y/2$, and on the other hand, $y > \left(2(y-x)\right)^{1/(\beta+1)}$. By the mean value theorem, for a suitable $u \in (x,y)$, we have

$$\begin{aligned}y^{\alpha}\cdot(x^{-\beta}-y^{-\beta}) &= y^{\alpha}\cdot(-\beta)\cdot u^{-\beta-1}\cdot(x-y) = \\ &= y^{\alpha}\cdot\beta\cdot u^{-\beta-1}\cdot(y-x) < \\ &< y^{\alpha}\cdot\beta\cdot(y/2)^{-\beta-1}\cdot(y-x) \leq C\cdot y^{\alpha-\beta-1}\cdot(y-x) < \\ &< C\cdot\left(2(y-x)\right)^{(\alpha-\beta-1)/(\beta+1)}\cdot(y-x) < C\cdot(y-x)^{\gamma},\end{aligned}$$

where $C = \beta\cdot 2^{\beta+1}$.

Chapter 18

18.4 Let c be a shared point of discontinuity. We can assume that $c < b$, and f is discontinuous from the right at c (since the proof is similar when $c > a$ and f is discontinuous from the left at c). We distinguish two cases. First, we suppose that g is discontinuous from the right at c. Then there exists an $\varepsilon > 0$ such that every right-hand neighborhood of c contains points x and y such that $|f(x)-f(c)| \geq \varepsilon$ and $|g(y)-g(c)| \geq \varepsilon$. It follows that for every $\delta > 0$, we can find a partition $a = x_0 < x_1 < \cdots < x_n = b$ with mesh smaller than δ such that c is one of the base points, say $c = x_{k-1}$, and $|g(x_k)-g(c)| \geq \varepsilon$.

Let $c_i = x_{i-1}$ for all $i = 1, \dots, n$, let $d_i = x_{i-1}$ for all $i \neq k$, and let $d_k \in (x_{k-1}, x_k]$ be a point such that $|f(d_k) - f(c_k)| = |f(d_k) - f(c)| \geq \varepsilon$. Then the sums $S_1 = \sum_{i=1}^n f(c_i) \cdot \big(g(x_i) - g(x_{i-1})\big)$ and $S_2 = \sum_{i=1}^n f(d_i) \cdot \big(g(x_i) - g(x_{i-1})\big)$ differ only in the kth term, and

$$|S_1 - S_2| = \big|\big(f(c_k) - f(d_k)\big) \cdot \big(g(x_k) - g(x_{k-1})\big)\big| \geq \varepsilon^2.$$

This means that we cannot find a δ for ε^2 such that the condition in Definition 18.3 is satisfied, that is, the Stieltjes integral $\int_a^b f\,dg$ does not exist.

Now suppose that g is continuous from the right at c. Then $c \in (a,b)$, and g is discontinuous from the left at c. It follows that for every $\delta > 0$, we can find a partition $a = x_0 < x_1 < \cdots < x_n = b$ with mesh smaller than δ such that c is not a base point, say $x_{k-1} < c < x_k$, and $|g(x_k) - g(x_{k-1})| \geq \varepsilon$. Let $c_i = d_i = x_{i-1}$ for all $i \neq k$. Also, let $c_k = c$ and $d_k \in (c, x_k]$ be a point such that $|f(d_k) - f(c_k)| = |f(d_k) - f(c)| \geq \varepsilon$. Then (just as in the previous case) the sums S_1 and S_2 differ by at least ε^2 from each other, and so the Stieltjes integral $\int_a^b f\,dg$ does not exist.

18.5 Let $x_i = 2/\big((2i+1)\pi\big)$ for all $i = 0, 1 \dots$. Then $f(x_i) = (-1)^i \cdot \sqrt{2/\big((2i+1)\pi\big)}$ $(i \in \mathbb{N})$. Let $\delta > 0$ be given. Fix an integer $N > 1/\delta$, and let $x_N = y_0 < y_1 < \cdots < y_n = 1$ be a partition of $[x_N, 1]$ with mesh smaller than δ. Then

$$F_M\colon\ 0 < x_M < x_{M-1} < \cdots < x_N < y_1 < \cdots < y_n = 1$$

is a partition of $[0,1]$ with mesh smaller than δ for all $M > N$. We show that if M is sufficiently large, then there exists a approximating sum that is greater than 1, and there also exists one that is less than -1.

In each of the intervals $[0, x_M]$ and $[y_{j-1}, y_j]$ $(j = 1, \dots, n)$, let the inner point be the left endpoint of the interval. Let $c_i = x_i$ $(N \leq i \leq M-1)$. Then $f(c_i) = f(x_i) = (-1)^i \cdot \sqrt{2/\big((2i+1)\pi\big)}$ for all $N \leq i \leq M-1$, so

$$\sum_{i=N}^{M-1} f(c_i)\big(f(x_i) - f(x_{i+1})\big) = \sum_{i=N}^{M-1} \sqrt{\frac{2}{(2i+1)\pi}} \cdot \left(\sqrt{\frac{2}{(2i+1)\pi}} + \sqrt{\frac{2}{(2i+3)\pi}}\ \right) >$$

$$> \sum_{i=N}^{M-1} \frac{2}{(2i+2)\pi} = \frac{1}{\pi} \cdot \sum_{i=N+1}^{M} \frac{1}{i}.$$

We get the approximating sum corresponding to the partition F_M by taking the above sum and adding the terms corresponding to $[y_{j-1}, y_j]$ $(j = 1, \dots, n)$. Note that the sum of these new terms does not depend on M. Since $\lim_{M\to\infty} \sum_{i=N+1}^{M} (1/i) = \infty$, choosing M sufficiently large gives us a approximating sum that is arbitrarily large. Similarly, we can show that with the choice $c_i = x_{i+1}$ $(N \leq i \leq M-1)$, we can get arbitrarily small approximating sums for sufficiently large M.

Chapter 19

19.7 By Cauchy's criterion, $\lim_{x\to\infty}\int_x^{2x} f\,dt = 0$. Since if $t \in [x,2x]$ then $f(t) \geq f(2x)$, we have that

$$0 \leq x \cdot f(2x) \leq \lim_{x\to\infty}\int_x^{2x} f\,dt,$$

and so by the squeeze theorem, $x \cdot f(2x) \to 0$ if $x \to \infty$.

19.9

(i) We can suppose that f is monotone decreasing and nonnegative.
Let $\varepsilon > 0$ be fixed, and choose a $0 < \delta < 1$ such that $\left|\int_x^1 f\,dt - I\right| < \varepsilon$ holds for all $0 < x \leq \delta$, where $I = \int_0^1 f\,dx$. Then $\int_0^x f\,dt < \varepsilon$ also holds for all $0 < x \leq \delta$. Fix an integer $n > 1/\delta$. If $k/n \leq \delta < (k+1)/n$, then the partition $F\colon k/n < \cdots < n/n = 1$ gives us intervals of length $1/n$, so by (14.19), the lower sum

$$s_F = \frac{1}{n} \cdot \sum_{i=k+1}^{n} f\left(\frac{i}{n}\right)$$

corresponding to F differs from the integral $\int_{k/n}^1 f\,dx$ by less than $\big(f(k/n) - f(1)\big)/n$, so it differs from the integral I by less than $\varepsilon + \big(f(k/n) - f(1)\big)/n$. Now by $k/n \leq \delta$, it follows that

$$\frac{1}{n} \cdot \sum_{i=1}^{k} f\left(\frac{i}{n}\right) \leq \int_0^{k/n} f\,dx < \varepsilon,$$

and thus $f(1/n)/n < \varepsilon$. Therefore,

$$\left|\frac{1}{n} \cdot \sum_{i=1}^{n} f\left(\frac{i}{n}\right) - I\right| \leq \varepsilon + |s_F - I| \leq$$
$$\leq \varepsilon + \varepsilon + \tfrac{1}{n} \cdot \big(f(k/n) - f(1)\big) \leq 2\varepsilon + \tfrac{1}{n} \cdot f(1/n) < 3\varepsilon.$$

This holds for all $n > 1/\delta$, which concludes the solution of the first part of the exercise.

(ii) If the function is not monotone, the statement is not true. The function $f(1/n) = n^2$ $(n = 1,2,\ldots)$, $f(x) = 0$ $(x \neq 1/n)$ is a simple counterexample.

19.11 We can assume that f is differentiable on $[a,b)$ and that f' is integrable on $[a,x]$ for all $a < x < b$.

We show that the length of every inscribed polygonal path is $\leq I$. Let $F\colon a = x_0 < x_1 < \ldots < x_n = b$ be a partition, and let $\varepsilon > 0$ be given. By the continuity of f and the convergence of the improper integral, there exists $0 < \delta < \varepsilon$ such that $|f(x) - f(b)| < \varepsilon$ and $|\int_a^x f\,dt - I| < \varepsilon$ for all $b - \delta < x < b$. Since adding new base points does not decrease the length of the inscribed polygonal path, we can assume that $x_{n-1} > b-\delta$. By Remark 16.21, the graph of f over the interval $[a, x_{n-1}]$ is rectifiable, and its arc length is $\int_a^{x_{n-1}} f\,dx < I+\varepsilon$. Thus the inscribed polygonal path corresponding to the partition $F_1:\ a = x_0 < x_1 < \cdots < x_{n-1}$ has length $< I+\varepsilon$, and so the inscribed polygonal path corresponding to the partition F has length less than

$$I+\varepsilon+\sqrt{(b-x_{n-1})^2+\big(f(b)-f(x_{n-1})\big)^2} \le$$
$$\le I+\varepsilon+|b-x_{n-1}|+|f(b)-f(x_{n-1})| \le I+3\varepsilon.$$

This is true for every ε, which shows that the graph of f is rectifiable, and its arc length is at most I. On the other hand, for every $a<x<b$, there exists a partition of $[a,x]$ such that the corresponding inscribed polygonal path gets arbitrarily close to the value of the integral $\int_a^x f\,dt$. Thus it is clear that the arc length of the graph of f is not less than I.

19.20

(a) Since $\sin x$ is concave on $[0,\pi/2]$, we have that $\sin x \ge 2x/\pi$ for all $x\in[0,\pi/2]$. Thus $|\log\sin x| = -\log\sin x \le -\log(2x/\pi) = |\log x| + \log(\pi/2)$. By Example 19.5, $\int_0^1 |\log x|\,dx$ is convergent (and its value is 1), so applying the majorization principle, we get that $\int_0^{\pi/2}\log\sin x\,dx$ is convergent. The substitution $x=\pi-t$ gives that $\int_{\pi/2}^{\pi}\log\sin x\,dx$ is also convergent.

(b) Let $\int_0^{\pi/2}\log\sin x\,dx = \int_{\pi/2}^{\pi}\log\sin x\,dx = I$. Applying the substitution $x=(\pi/2)-t$ gives us that $\int_0^{\pi/2}\log\cos x\,dx = I$. Now apply the substitution $x=2t$:

$$2I = \int_0^{\pi}\log\sin x\,dx = \int_0^{\pi/2}\log\sin(2t)\cdot 2\,dt =$$
$$= 2\cdot\int_0^{\pi/2}(\log 2+\log\sin t+\log\cos t)\,dt =$$
$$= \log 2\cdot\pi + 4I,$$

so $I = -\log 2\cdot\pi/2$ and $\int_0^{\pi}\log\sin x\,dx = -\log 2\cdot\pi$.

19.21 The function $\log\sin x$ is monotone on the intervals $[0,\pi/2]$ and $[\pi/2,\pi]$. Thus by the statement of Exercise 19.9,

$$\lim_{n\to\infty}\frac{\pi}{n}\cdot\sum_{i=1}^{n-1}\log\sin\frac{i\pi}{n} = \int_0^{\pi}\log\sin x\,dx = -\log 2\cdot\pi,$$

so

$$\lim_{n\to\infty}\frac{1}{n-1}\cdot\sum_{i=1}^{n-1}\log\sin\frac{i\pi}{n} = -\log 2.$$

If we raise e to the power of the expressions present on each side, we get that if $n\to\infty$, then the geometric mean of the numbers $\sin\pi/n$, $\sin 2\pi/n, \ldots, \sin(n-1)\pi/n$ tends to $e^{-\log 2} = 1/2$.

The arithmetic mean clearly tends to $\frac{1}{\pi}\cdot\int_0^\pi \sin x\,dx = 2/\pi$. The inequality $(1/2) < 2/\pi$ is obviously true.

19.28 For every $n \in \mathbb{N}^+$, let $f_n\colon [n-1,n] \to \mathbb{R}$ be a nonnegative continuous function such that $f_n(n-1) = f_n(n) = 0$, $\max f_n \geq 1$, and $\int_{n-1}^n f_n\,dx \leq 1/2^n$. (We may take the function for which $f_n(x) = 0$ if $|x-a_n| \geq \varepsilon_n$, $f_n(a_n) = 1$, and f_n is linear in the intervals $[a_n - \varepsilon_n, a_n]$ and $[a_n, a_n + \varepsilon_n]$, where $a_n = (2n-1)/2$ and $\varepsilon_n = 2^{-n}$.)

Let $f(x) = f_n(x)$ if $x \in [n-1,n]$ and $n \in \mathbb{N}^+$. Clearly, f is continuous. Since f is also nonnegative, the function $x \mapsto \int_0^x f\,dt$ is monotone increasing, and so the limit $\lim_{x\to\infty} \int_0^x f\,dt$ exists. On the other hand, $\int_0^n f\,dt \leq 2^{-1} + \cdots + 2^{-n} < 1$ for all n, so the limit is finite, and the improper integral $\int_0^\infty f\,dt$ is convergent.

19.29 Let $\varepsilon > 0$ be given. By the uniform continuity of f, there exists a $\delta > 0$ such that $|x-y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$. By Cauchy's criterion, for the convergence of the integral, we can pick a number $K > 0$ such that if $K < x < y$, then $|\int_x^y f\,dt| < \varepsilon\cdot\delta$. We show that $|f(x)| < 2\varepsilon$ for all $x > K$. Suppose that $x > K$ and $f(x) \geq 2\varepsilon$. Then by the choice of δ, we have $f(t) \geq \varepsilon$ for all $t \in (x, x+\delta)$, and so $\int_x^{x+\delta} f\,dt \geq \varepsilon\cdot\delta$, which contradicts the choice of K. Thus $f(x) < 2\varepsilon$ for all $x > K$, and we can similarly show that $f(x) > -2\varepsilon$ for all $x > K$.

19.33

(a) Let $a_0 = a$. If $n > 0$ and we have already chosen the number $a_{n-1} > a$, then let $a_n > \max(n, a_{n-1})$ be such that $\int_{a_n}^\infty f\,dx < 1/(n\cdot 2^n)$. Thus we have chosen numbers a_n for all $n = 0,1,\ldots$. Let $g(x) = n-1$ for all $x \in (a_{n-1}, a_n)$ and $n = 1,2,\ldots$. Then $\lim_{x\to\infty} g(x) = \infty$. Now

$$\int_a^{a_n} g\cdot f\,dx = \sum_{i=0}^{n-1}\int_{a_i}^{a_{i+1}} g\cdot f\,dx = \sum_{i=1}^{n-1} i\cdot\int_{a_i}^{a_{i+1}} f\,dx < \sum_{i=1}^{n-1} i\cdot\frac{1}{i\cdot 2^i} < 1,$$

and so the function $x \mapsto \int_a^x g\cdot f\,dt$ is bounded. Since it is also monotone, its limit is finite, and so the improper integral is convergent.

(b) Let $a_0 = a$. If $n > 0$ and we have already chosen the number $a_{n-1} > a$, then let $a_n > \max(n, a_{n-1})$ be such that $\int_{a_{n-1}}^{a_n} f\,dx > n$. Thus we have chosen numbers a_n for all $n = 0,1,\ldots$. Let $g(x) = 1/n$ for all $x \in (a_{n-1}, a_n)$ and $n = 1,2,\ldots$. Then $\lim_{x\to\infty} g(x) = 0$. Now

$$\int_a^{a_n} g\cdot f\,dx = \sum_{i=1}^{n}\int_{a_{i-1}}^{a_i} g\cdot f\,dx = \sum_{i=1}^{n}\frac{1}{i}\cdot\int_{a_{i-1}}^{a_i} f\,dx > \sum_{i=1}^{n}\frac{1}{i}\cdot i = n,$$

so $\int_a^\infty g\cdot f\,dx$ is divergent.

19.35 We will apply the majorization principle. Let $x \geq 3$ be fixed. If we have $0 \leq f(x) < 1/x^2$, then

$$\log\left(f(x)^{1-1/\log x}\right) = (1-1/\log x)\cdot\log f(x) \le$$
$$\le \left(1-\frac{1}{\log x}\right)\cdot(-2\log x) = 2-2\log x,$$

so

$$f(x)^{1-1/\log x} \le e^{2-2\log x} = e^2/x^2.$$

If, however, $f(x) \ge 1/x^2$, then

$$f(x)^{1-1/\log x} = f(x)\cdot f(x)^{-1/\log x} \le f(x)\cdot x^{2/\log x} = e^2\cdot f(x).$$

We get that $f(x)^{1-1/\log x} \le \max\left(e^2/x^2, e^2\cdot f(x)\right) \le e^2\cdot\left((1/x^2)+f(x)\right)$ for all $x \ge 3$. Since the integral $\int_3^\infty \left((1/x^2)+f(x)\right)dx$ is convergent, $\int_3^\infty f(x)^{1-1/\log x}\,dx$ must also be convergent.

19.37 Since f is decreasing and convex, f' is increasing and nonpositive, so $|f'|$ is bounded. It is then clear that the integrals $\int_a^\infty f(x)\cdot\sqrt{1+\left(f'(x)\right)^2}\,dx$ and $\int_a^\infty f(x)\,dx$ are either both convergent or both divergent. By Exercise 19.31, if $\int_a^\infty f\,dx$ is convergent, then $\int_a^\infty f^2\,dx$ is also convergent. Thus only three configurations are possible: All three integrals are convergent, $\int_a^\infty f(x)\cdot\sqrt{1+\left(f'(x)\right)^2}\,dx$ and $\int_a^\infty f(x)\,dx$ are divergent while $\int_a^\infty f^2\,dx$ is convergent, or all three integrals are divergent. Examples for each three cases are given by the functions $1/x^2$, $1/x$, and $1/\sqrt{x}$ over the interval $[1,\infty)$.

19.38 Applying integration by parts gives

$$\Gamma(c+1) = \int_0^\infty x^c\cdot e^{-x}\,dx = \left[x^c\cdot(-e^{-x})\right]_0^\infty + \int_0^\infty c\cdot x^{c-1}\cdot e^{-x}\,dx = 0 + c\cdot\Gamma(c).$$

19.41 Using the substitution $x^3 = t$ gives us that

$$\int_0^\infty e^{-x^3}\,dx = \int_0^\infty e^{-t}\cdot\frac{1}{3}\cdot t^{-2/3}\,dt = \frac{1}{3}\cdot\Gamma(1/3).$$

We similarly get that $\int_0^\infty e^{-x^s}\,dx = \Gamma(1/s)/s$ for all $s > 0$.

19.43 Use the substitution $x = t/n$ in (19.8). We get that

$$\int_0^n \left(1-\frac{t}{n}\right)^n\cdot t^{c-1}\,dt = \frac{n^c\cdot n!}{c(c+1)\cdots(c+n)} \tag{13}$$

for all $n = 1,2,\ldots$. Since $(1-t/n)^n \le e^{-t}$ for all $0 < t \le n$, by (13) we get that $\Gamma(c) > \left(n^c\cdot n!\right)/\left(c(c+1)\cdots(c+n)\right)$.

On the other hand, for a given n, the function $e^t \cdot (1-t/n)^n$ is monotone decreasing on the interval $[0,n]$, since its derivative there is

$$e^t \cdot \left(1-\frac{t}{n}\right)^n - e^t \cdot \left(1-\frac{t}{n}\right)^{n-1} \le 0.$$

Let $\varepsilon > 0$ be fixed. Choose a number $K > 0$ such that $\int_K^\infty e^{-t} \cdot t^{c-1}\,dt < \varepsilon$ holds, and n_0 such that

$$e^K \cdot \left(1-\frac{K}{n}\right)^n > \frac{1}{1+\varepsilon}$$

holds for all $n > n_0$. If $n \ge \max(n_0, K)$ and $0 < t < K$, then

$$e^t \cdot \left(1-\frac{t}{n}\right)^n \ge e^K \cdot \left(1-\frac{K}{n}\right)^n \ge \frac{1}{1+\varepsilon},$$

so

$$e^{-t} \le (1+\varepsilon) \cdot \left(1-\frac{t}{n}\right)^n,$$

and thus

$$\begin{aligned} \Gamma(c) &< \int_K^\infty e^{-t} \cdot t^{c-1} + (1+\varepsilon) \int_0^K \left(1-\frac{t}{n}\right)^n \cdot t^{c-1}\,dt < \\ &< \varepsilon + (1+\varepsilon) \frac{n^c \cdot n!}{c(c+1)\cdots(c+n)}. \end{aligned} \tag{14}$$

Since $\varepsilon > 0$ was arbitrary and (14) holds for every sufficiently large n, (19.9) also holds.

Notation

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References

1. Davidson, K.R., Dosig, A.P.: Real Analysis and Applications. Theory in Practice. Springer, New York (2010)
2. Erdős, P., Surányi, J.: Topics in the Theory of Numbers. Springer, New York (2003)
3. Euclid: The Thirteen Books of the Elements [Translated with introduction and commentary by Sir Thomas Heath]. Second Edition Unabridged. Dover, New York (1956)
4. Hewitt, E., Stromberg, K.: Real and Abstract Analysis. Springer, New York (1975)
5. Niven, I., Zuckerman, H.S., Montgomery, H.L.: An Introduction to the Theory of Numbers, 5th edn. Wiley, New York (1991)
6. Rademacher, H., Toeplitz, O.: Von Zahlen und Figuren. Springer, Berlin (1933) [English translation: The Enjoyment of Mathematics]. Dover, New York (1990)
7. Rudin, W.: Principles of Mathematical Analysis, 3rd edn. McGraw-Hill, New York (1976)
8. Zaidman, S.: Advanced Calculus. An Introduction to Mathematical Analysis. World Scientific, Singapore (1997)

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