

# Undergraduate Texts in Mathematics

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Stephen Abbott

# Understanding Analysis

Second Edition



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# Preface

My primary goal in writing *Understanding Analysis* was to create an elementary one-semester book that exposes students to the rich rewards inherent in taking a mathematically rigorous approach to the study of functions of a real variable. The aim of a course in real analysis should be to challenge and improve mathematical intuition rather than to verify it. There is a tendency, however, to center an introductory course too closely around the familiar theorems of the standard calculus sequence. Producing a rigorous argument that polynomials are continuous is good evidence for a well-chosen definition of continuity, but it is not the reason the subject was created and certainly not the reason it should be required study. By shifting the focus to topics where an untrained intuition is severely disadvantaged (e.g., rearrangements of infinite series, nowhere-differentiable continuous functions, Cantor sets), my intent is to bring an intellectual liveliness to this course by offering the beginning student access to some truly significant achievements of the subject.

## The Main Objectives

Real analysis stands as a beacon of stability in the otherwise unpredictable evolution of the mathematics curriculum. Amid the various pedagogical revolutions in calculus, computing, statistics, and data analysis, nearly every undergraduate program continues to require at least one semester of real analysis. My own department once challenged this norm by creating a mathematical sciences track that allowed students to replace our two core proof-writing classes with electives in departments like physics and computer science. Within a few years, however, we concluded that the pieces did not hold together without a course in analysis. Analysis is, at once, a course in philosophy and applied mathematics. It is abstract and axiomatic in nature, but is engaged with the mathematics used by economists and engineers.

How then do we teach a successful course to students with such diverse interests and expectations? Our desire to make analysis required study for wider audiences must be reconciled with the fact that many students find the subject quite challenging and even a bit intimidating. One unfortunate resolution of this

dilemma is to make the course easier by making it less interesting. The omitted material is inevitably what gives analysis its true flavor. A better solution is to find a way to make the more advanced topics accessible and worth the effort.

I see three essential goals that a semester of real analysis should try to meet:

1. Students need to be confronted with questions that expose the insufficiency of an informal understanding of the objects of calculus. The need for a more rigorous study should be carefully motivated.
2. Having seen mainly intuitive or heuristic arguments, students need to learn what constitutes a rigorous mathematical proof and how to write one.
3. Most importantly, there needs to be significant reward for the difficult work of firming up the logical structure of limits. Specifically, real analysis should not be just an elaborate reworking of standard introductory calculus. Students should be exposed to the tantalizing complexities of the real line, to the subtleties of different flavors of convergence, and to the intellectual delights hidden in the paradoxes of the infinite.

The philosophy of *Understanding Analysis* is to focus attention on questions that give analysis its inherent fascination. Does the Cantor set contain any irrational numbers? Can the set of points where a function is discontinuous be arbitrary? Are derivatives continuous? Are derivatives integrable? Is an infinitely differentiable function necessarily the limit of its Taylor series? In giving these topics center stage, the hard work of a rigorous study is justified by the fact that *they are inaccessible without it*.

## The Audience

This book is an introductory text. The only prerequisite is a robust understanding of the results from single-variable calculus. The theorems of linear algebra are not needed, but the exposure to abstract arguments and proof writing that usually comes with this course would be a valuable asset. Complex numbers are never used.

The proofs in *Understanding Analysis* are written with the beginning student firmly in mind. Brevity and other stylistic concerns are postponed in favor of including a significant level of detail. Most proofs come with a generous amount of discussion about the context of the argument. What should the proof entail? Which definitions are relevant? What is the overall strategy? Whenever there is a choice, efficiency is traded for an opportunity to reinforce some previously learned technique. Especially familiar or predictable arguments are often deferred to the exercises.

The search for recurring ideas exists at the proof-writing level and also on the larger expository level. I have tried to give the course a narrative tone by picking up on the unifying themes of approximation and the transition from the finite to the infinite. Often when we ask a question in analysis the answer is

“sometimes.” Can the order of a double summation be exchanged? Is term-by-term differentiation of an infinite series allowed? By focusing on this recurring pattern, each successive topic builds on the intuition of the previous one. The questions seem more natural, and a coherent story emerges from what might otherwise appear as a long list of theorems and proofs.

This book always emphasizes core ideas over generality, and it makes no effort to be a complete, deductive catalog of results. It is designed to capture the intellectual imagination. Those who become interested are then exceptionally well prepared for a second course starting from complex-valued functions on more general spaces, while those content with a single semester come away with a strong sense of the essence and purpose of real analysis.

## The Structure of the Book

Although the book finds its way to some sophisticated results, the main body of each chapter consists of a lean and focused treatment of the core topics that make up the center of most courses in analysis. Fundamental results about completeness, compactness, sequential and functional limits, continuity, uniform convergence, differentiation, and integration are all incorporated.

What is specific here is where the emphasis is placed. In the chapter on integration, for instance, the exposition revolves around deciphering the relationship between continuity and the Riemann integral. Enough properties of the integral are obtained to justify a proof of the Fundamental Theorem of Calculus, but the theme of the chapter is the pursuit of a characterization of integrable functions in terms of continuity. Whether or not Lebesgue’s measure-zero criterion is treated, framing the material in this way is still valuable because it is the questions that are important. Mathematics is not a static discipline. Students should be aware of the historical reasons for the creation of the mathematics they are learning and by extension realize that there is no last word on the subject. In the case of integration, this point is made explicitly by including some relatively modern developments on the generalized Riemann integral in the additional topics of the last chapter.

The structure of the chapters has the following distinctive features.

**Discussion Sections:** Each chapter begins with the discussion of some motivating examples and open questions. The tone in these discussions is intentionally informal, and full use is made of familiar functions and results from calculus. The idea is to freely explore the terrain, providing context for the upcoming definitions and theorems. After these exploratory introductions, the tone of the writing changes, and the treatment becomes rigorously tight but still not overly formal. With the questions in place, the need for the ensuing development of the material is well motivated and the payoff is in sight.

**Project Sections:** The penultimate section of each chapter (the final section is a short epilogue) is written with the exercises incorporated into the exposition. Proofs are outlined but not completed, and additional exercises are included to elucidate the material being discussed. The sections are written as self-guided

tutorials, but they can also be the subject of lectures. I typically use them in place of a final examination, and they work especially well as collaborative assignments that can culminate in a class presentation. The body of each chapter contains the necessary tools, so there is some satisfaction in letting the students use their newly acquired skills to ferret out for themselves answers to questions that have been driving the exposition.

## Building a Course

Although this book was originally designed for a 12–14-week semester, it has been used successfully in any number of formats including independent study. The dependence of the sections follows the natural ordering, but there is some flexibility as to what can be treated and omitted.

- The introductory discussions to each chapter can be the subject of lecture, assigned as reading, omitted, or substituted with something preferable. There are no theorems proved here that show up later in the text. I do develop some important examples in these introductions (the Cantor set, Dirichlet’s nowhere-continuous function) that probably need to find their way into discussions at some point.
- Chapter 3, Basic Topology of  $\mathbf{R}$ , is much longer than it needs to be. All that is required by the ensuing chapters are fundamental results about open and closed sets and a thorough understanding of sequential compactness. The characterization of compactness using open covers as well as the section on perfect and connected sets are included for their own intrinsic interest. They are not, however, crucial to any future proofs. The one exception to this is a presentation of the Intermediate Value Theorem (IVT) as a special case of the preservation of connected sets by continuous functions. To keep connectedness truly optional, I have included two direct proofs of IVT based on completeness results from Chapter 1.
- All the project sections (1.6, 2.8, 3.5, 4.6, 5.4, 6.7, 7.6, 8.1–8.6) are optional in the sense that no results in later chapters depend on material in these sections. The six topics covered in Chapter 8 are also written in this tutorial-style format, where the exercises make up a significant part of the development. The only one of these sections that might benefit from a lecture is the unit on Fourier series, which is a bit longer than the others.

## Changes in the Second Edition

In light of the encouraging feedback—especially from students—I decided not to attempt any major alterations to the central narrative of the text as it was set out in the original edition. Some longer sections have been edited down, or in one case split in two, and the unit on Taylor series is now part of the

core material of Chapter 6 instead of being relegated to the closing project section. In contrast to the main body of the book, significant effort has gone into revising the exercises and projects. There are roughly 150 new exercises in this edition alongside 200 or so of what I feel are the most effective problems from the first edition. Some of these introduce new ideas not covered in the chapters (e.g., Euler's constant, infinite products, inverse functions), but the majority are designed to kindle debates about the major ideas under discussion in what I hope are engaging ways. There are ample propositions to prove but also a good supply of Moore-method type exercises that require assessing the validity of various conjectures, deciphering invented definitions, or searching for examples that may not exist.

The introductory discussion to Chapter 6 is new and tells the story of how Euler's deft and audacious manipulations of power series led to a computation of  $\sum 1/n^2$ . Providing a proper proof for Euler's sum is the topic of one of three new project sections. The other two are a treatment of the Weierstrass Approximation Theorem and an exploration of how to best extend the domain of the factorial function to all of  $\mathbf{R}$ . Each of these three topics represents a seminal achievement in the history of analysis, but my decision to include them has as much to do with the associated ideas that accompany the main proofs. For the Weierstrass Approximation Theorem, the particular argument that I chose relies on Taylor series and a deep understanding of uniform convergence, making it an ideal project to conclude Chapter 6. The journey to a proper definition of  $x!$  allowed me to include a short unit on improper integrals and a proof of Leibniz's rule for differentiating under the integral sign. The accompanying topics for the project on Euler's sum are an analysis of the integral remainder formula for Taylor series and a proof of Wallis's famous product formula for  $\pi$ . Yes these are challenging arguments but they are also beautiful ideas. Returning to the thesis of this text, it is my conviction that encounters with results like these make the task of learning analysis less daunting and more meaningful. They make the epsilons matter.

## Acknowledgements

I never met Robert Bartle, although it seems like I did. As a student and a young professor, I spent many hours learning and teaching analysis from his books. I did eventually correspond with him back in 2000 while working on the first edition of this text because I wanted to include a project based on his article, "Return to the Riemann Integral." Professor Bartle was gracious and helpful, even though he was editing his own competing text to include the same material. In September 2003, Robert Bartle died following a long battle with cancer at the age of 76. The section his article inspired on the Generalized Riemann integral continues to be one of my favorite projects to assign, but it is fair to say that Professor Bartle's lucid mathematical writing has been a source of inspiration for the entire text.

My long and winding journey to find an elegant proof of Euler's sum constructed only from theorems in the first seven chapters in this text came to a happy conclusion in Peter Duren's recently published *Invitation to Classical Analysis*. A treasure trove of fascinating topics that have been largely excised from the undergraduate curriculum, Duren's book is remarkable in part for how much he accomplishes without the use of Lebesgue's theory. T.W. Körner's wonderfully opinionated *A Companion to Analysis* is another engaging read that inspired a few of the new exercises in this edition. *Analysis by Its History*, by E. Hairer and G. Wanner, and *A Radical Approach to Real Analysis*, by David Bressoud, were both cited in the acknowledgements of the first edition as sources for many of the historical anecdotes that permeate the text. Since then, Professor Bressoud has published a sequel, *A Radical Approach to Lebesgue's Theory of Integration*, which I heartily recommend.

The significant contributions of Benjamin Lotto, Loren Pitt, and Paul Humke to the content of the first edition warrant a second nod in these acknowledgements. As for the new edition, Dan Velleman taught from a draft of the text and provided much helpful feedback. Whatever problems still remain are likely places where I stubbornly did not follow Dan's advice. Back in 2001, Steve Kennedy penned a review of *Understanding Analysis* which I am sure enhanced the audience for this book. His kind assessment nevertheless included a number of worthy suggestions for improvement, most of which I have incorporated. I should also acknowledge Fernando Gouvea as the one who suggested that the series of articles by David Fowler on the factorial function would fit well with the themes of this book. The result is Section 8.4.

I would like to express my continued appreciation to the staff at Springer, and in particular to Marc Strauss and Eugene Ha for their support and unwavering faith in the merits of this project. The large email file of thoughtful suggestions from users of the book is too long to enumerate, but perhaps this is the place to say that I continue to welcome comments from readers, even moderately disgruntled ones. The most gratifying aspect of authoring the first edition is the sense of being connected to the larger mathematical community and of being an active participant in it.

The margins of my original copy of *Understanding Analysis* are filled with vestiges of my internal debates about what to revise, what to preserve, and what to discard. The final decisions I made are the result of 15 years of classroom experiments with the text, and it is comforting to report that the main body of the book has weathered the test of time with only a modest tune-up. On a similarly positive note, the original dedication of this book to my wife Katy is another feature of the first edition that has required no additional editing.

Middlebury, VT, USA  
March 2015

Stephen Abbott

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