

Appendix A1

The Monotone Class Lemma

The monotone class lemma is a tool of measure theory which is very useful in several arguments of probability theory. We give here the version of this lemma that is used in several places in this volume.

Let E be an arbitrary set, and let $\mathcal{P}(E)$ denote the set of all subsets of E . If $\mathcal{C} \subset \mathcal{P}(E)$, $\sigma(\mathcal{C})$ stands for the smallest σ -field on E containing \mathcal{C} (it is also the intersection of all σ -fields containing \mathcal{C}).

Definition A subset \mathcal{M} of $\mathcal{P}(E)$ is called a *monotone class* if the following properties hold:

- (i) $E \in \mathcal{M}$.
- (ii) If $A, B \in \mathcal{M}$ and $A \subset B$, then $B \setminus A \in \mathcal{M}$.
- (iii) If $(A_n)_{n \geq 0}$ is an increasing sequence of subsets of E such that $A_n \in \mathcal{M}$ for every $n \geq 0$, then $\bigcup_{n \geq 0} A_n \in \mathcal{M}$.

A σ -field is a monotone class. As in the case of σ -fields, one immediately checks that the intersection of an arbitrary collection of monotone classes is again a monotone class. If \mathcal{C} is an arbitrary subset of $\mathcal{P}(E)$, the monotone class generated by \mathcal{C} , which is denoted by $\mathcal{M}(\mathcal{C})$, is defined by setting

$$\mathcal{M}(\mathcal{C}) = \bigcap_{\substack{\mathcal{M} \text{ monotone class,} \\ \mathcal{C} \subset \mathcal{M}}} \mathcal{M}.$$

Monotone class lemma If $\mathcal{C} \subset \mathcal{P}(E)$ is stable under finite intersections, then $\mathcal{M}(\mathcal{C}) = \sigma(\mathcal{C})$.

Proof Since any σ -field is a monotone class, it is clear that $\mathcal{M}(\mathcal{C}) \subset \sigma(\mathcal{C})$. To prove the reverse inclusion, it is enough to verify that $\mathcal{M}(\mathcal{C})$ is a σ -field. However, a monotone class is a σ -field if and only if it is stable under finite intersections (indeed, considering the complementary sets shows that it is then stable under finite

unions, and via an increasing passage to the limit one gets that it is also stable under countable unions). Let us show that $\mathcal{M}(\mathcal{C})$ is stable under finite intersections.

For every $A \in \mathcal{P}(E)$, set

$$\mathcal{M}_A = \{B \in \mathcal{M}(\mathcal{C}) : A \cap B \in \mathcal{M}(\mathcal{C})\}.$$

Fix $A \in \mathcal{C}$. Since \mathcal{C} is stable under finite intersections, we have $\mathcal{C} \subset \mathcal{M}_A$. Let us verify that \mathcal{M}_A is a monotone class:

- $E \in \mathcal{M}_A$ is trivial.
- If $B, B' \in \mathcal{M}_A$ and $B \subset B'$, we have $A \cap (B' \setminus B) = (A \cap B') \setminus (A \cap B) \in \mathcal{M}(\mathcal{C})$ and thus $B' \setminus B \in \mathcal{M}_A$.
- If $B_n \in \mathcal{M}_A$ for every $n \geq 0$ and the sequence $(B_n)_{n \geq 0}$ is increasing, we have $A \cap (\cup B_n) = \cup(A \cap B_n) \in \mathcal{M}(\mathcal{C})$ and therefore $\cup B_n \in \mathcal{M}_A$.

Since \mathcal{M}_A is a monotone class that contains \mathcal{C} , \mathcal{M}_A also contains $\mathcal{M}(\mathcal{C})$. We have thus obtained that

$$\forall A \in \mathcal{C}, \forall B \in \mathcal{M}(\mathcal{C}), A \cap B \in \mathcal{M}(\mathcal{C}).$$

This is not yet the desired result, but we can use the same idea another time. Precisely, we now fix $A \in \mathcal{M}(\mathcal{C})$. According to the first part of the proof, $\mathcal{C} \subset \mathcal{M}_A$. By exactly the same arguments as in the first part of the proof, we get that \mathcal{M}_A is a monotone class. It follows that $\mathcal{M}(\mathcal{C}) \subset \mathcal{M}_A$, which shows that $\mathcal{M}(\mathcal{C})$ is stable under finite intersections, and completes the proof. \square

Here are a few consequences of the monotone class lemma that are used above.

1. Let \mathcal{A} be a σ -field on E , and let μ and ν be two probability measures on (E, \mathcal{A}) . Assume that there exists a class $\mathcal{C} \subset \mathcal{A}$, which is stable under finite intersections, such that $\sigma(\mathcal{C}) = \mathcal{A}$ and $\mu(A) = \nu(A)$ for every $A \in \mathcal{C}$. Then $\mu = \nu$. (Use the fact that $\mathcal{G} := \{A \in \mathcal{A} : \mu(A) = \nu(A)\}$ is a monotone class.)
2. Let $(X_i)_{i \in I}$ be an arbitrary collection of random variables, and let \mathcal{G} be a σ -field on the same probability space. In order to show that the σ -fields $\sigma(X_i, i \in I)$ and \mathcal{G} are independent, it is enough to verify that $(X_{i_1}, \dots, X_{i_p})$ is independent of \mathcal{G} , for any choice of the finite set $\{i_1, \dots, i_p\} \subset I$. (Observe that the class of all events that depend on a finite number of the variables $X_i, i \in I$, is stable under finite intersections and generates $\sigma(X_i, i \in I)$.)
3. Let $(X_i)_{i \in I}$ be an arbitrary collection of random variables, and let Z be a bounded real variable. Let $i_0 \in I$. In order to verify that $E[Z | X_i, i \in I] = E[Z | X_{i_0}]$, it is enough to show that $E[Z | X_{i_0}, X_{i_1}, \dots, X_{i_p}] = E[Z | X_{i_0}]$ for any choice of the finite collection $\{i_1, \dots, i_p\} \subset I$. (Observe that the class of all events A such that $E[\mathbf{1}_A Z] = E[\mathbf{1}_A E[Z | Y_{i_0}]]$ is a monotone class.)

This last consequence of the monotone class lemma is useful in the theory of Markov processes.

Appendix A2

Discrete Martingales

In this appendix, we recall without proof the results about discrete time martingales and supermartingales that are used in Chap.3. The proof of the subsequent statements can be found in Neveu’s book [65], and in many other books dealing with discrete time martingales (see in particular Williams [82] and Chapter XII in Grimmett and Stirzaker [30]).

We use the notation $\mathbb{N} = \{0, 1, 2, \dots\}$. Let us start by recalling the basing definitions. We consider a probability space (Ω, \mathcal{F}, P) , and we fix a discrete filtration, that is, an increasing sequence $(\mathcal{G}_n)_{n \in \mathbb{N}}$ of sub- σ -fields of \mathcal{F} . We also let

$$\mathcal{G}_\infty = \bigvee_{n=0}^{\infty} \mathcal{G}_n$$

be the smallest σ -field that contains all the σ -fields \mathcal{G}_n .

Definition A sequence $(Y_n)_{n \in \mathbb{N}}$ of integrable random variables, such that Y_n is \mathcal{G}_n -measurable for every $n \in \mathbb{N}$, is called

- a *martingale* if, whenever $0 \leq m < n$, $E[Y_n | \mathcal{G}_m] = Y_m$;
- a *supermartingale* if, whenever $0 \leq m < n$, $E[Y_n | \mathcal{G}_m] \leq Y_m$;
- a *submartingale* if, whenever $0 \leq m < n$, $E[Y_n | \mathcal{G}_m] \geq Y_m$.

All these notions obviously depend on the choice of the filtration $(\mathcal{G}_n)_{n \in \mathbb{N}}$, which is fixed in what follows.

Maximal inequality *If $(Y_n)_{n \in \mathbb{N}}$ is a supermartingale, then, for every $\lambda > 0$ and every $k \in \mathbb{N}$,*

$$\lambda P\left(\sup_{n \leq k} |Y_n| > \lambda\right) \leq E[|Y_0|] + 2E[|Y_k|].$$

Doob's inequality in L^p If $(Y_n)_{n \in \mathbb{N}}$ is a martingale, then, for every $k \in \mathbb{N}$ and every $p > 1$,

$$E\left[\sup_{0 \leq n \leq k} |Y_n|^p\right] \leq \left(\frac{p}{p-1}\right)^p E[|Y_k|^p].$$

Remark This inequality is interesting only if $E[|Y_k|^p] < \infty$, since otherwise both sides are infinite.

If $y = (y_n)_{n \in \mathbb{N}}$ is a sequence of real numbers, and $a < b$, the upcrossing number of this sequence along $[a, b]$ before time n , denoted by $M_{ab}^y(n)$, is the largest integer k such that there exists a strictly increasing finite sequence

$$m_1 < n_1 < m_2 < n_2 < \dots < m_k < n_k$$

of nonnegative integers smaller than or equal to n with the properties $y_{m_i} \leq a$ and $y_{n_i} \geq b$, for every $i \in \{1, \dots, k\}$. In what follows we consider a sequence $Y = (Y_n)_{n \in \mathbb{N}}$ of real random variables, and the associated upcrossing number $M_{ab}^Y(n)$ is then an integer-valued random variable.

Doob's upcrossing inequality If $(Y_n)_{n \in \mathbb{N}}$ is a supermartingale, then, for every $n \in \mathbb{N}$ and every $a < b$,

$$E[M_{ab}^Y(n)] \leq \frac{1}{b-a} E[(Y_n - a)^-].$$

This inequality is a crucial tool for proving the convergence theorems for discrete-time martingales and supermartingales. Let us recall two important instances of these theorems.

Convergence theorem for discrete-time supermartingales If $(Y_n)_{n \in \mathbb{N}}$ is a supermartingale, and if the sequence $(Y_n)_{n \in \mathbb{N}}$ is bounded in L^1 , then there exists a random variable $Y_\infty \in L^1$ such that

$$Y_n \xrightarrow[n \rightarrow \infty]{\text{a.s.}} Y_\infty.$$

Convergence theorem for uniformly integrable discrete-time martingales Let $(Y_n)_{n \in \mathbb{N}}$ be a martingale. The following are equivalent:

- (i) The martingale $(Y_n)_{n \in \mathbb{N}}$ is closed, in the sense that there exists a random variable $Z \in L^1(\Omega, \mathcal{F}, P)$ such that $Y_n = E[Z | \mathcal{G}_n]$ for every $n \in \mathbb{N}$.
- (ii) The sequence $(Y_n)_{n \in \mathbb{N}}$ converges a.s. and in L^1 .
- (iii) The sequence $(Y_n)_{n \in \mathbb{N}}$ is uniformly integrable.

If these properties hold, the a.s. limit of the sequence $(Y_n)_{n \in \mathbb{N}}$ is $Y_\infty = E[Z | \mathcal{G}_\infty]$.

We now recall two versions of the optional stopping theorem in discrete time. A (discrete) stopping time is a random variable T with values in $\mathbb{N} \cup \{\infty\}$, such that $\{T = n\} \in \mathcal{G}_n$ for every $n \in \mathbb{N}$. The σ -field of the past before T is then $\mathcal{G}_T = \{A \in \mathcal{G}_\infty : A \cap \{T = n\} \in \mathcal{G}_n, \text{ for every } n \in \mathbb{N}\}$.

Optional stopping theorem for uniformly integrable discrete-time martingales

Let $(Y_n)_{n \in \mathbb{N}}$ be a uniformly integrable martingale, and let Y_∞ be the a.s. limit of Y_n when $n \rightarrow \infty$. Then, for every choice of the stopping times S and T such that $S \leq T$, we have $Y_T \in L^1$ and

$$Y_S = E[Y_T \mid \mathcal{G}_S]$$

with the convention that $Y_T = Y_\infty$ on the event $\{T = \infty\}$, and similarly for Y_S .

Optional stopping theorem for discrete-time supermartingales (bounded case)

If $(Y_n)_{n \in \mathbb{N}}$ is a supermartingale, then, for every choice of the bounded stopping times S and T such that $S \leq T$, we have

$$Y_S \geq E[Y_T \mid \mathcal{G}_S].$$

We conclude with a variant of the convergence theorem for supermartingales in the backward case. We consider a backward filtration, that is, an increasing sequence of filtrations $(\mathcal{H}_n)_{n \in -\mathbb{N}}$ indexed by negative integers (in such a way that the σ -field \mathcal{H}_n is “smaller and smaller” when $n \rightarrow -\infty$). A sequence $(Y_n)_{n \in -\mathbb{N}}$ of integrable random variables indexed by negative integers is called a backward supermartingale if, for every $n \in -\mathbb{N}$, Y_n is \mathcal{H}_n -measurable and, for every $m \leq n \leq 0$, $E[Y_n \mid \mathcal{H}_m] \leq Y_m$.

Convergence theorem for backward discrete-time supermartingales *If $(Y_n)_{n \in -\mathbb{N}}$ is a backward supermartingale, and if the sequence $(Y_n)_{n \in -\mathbb{N}}$ is bounded in L^1 , then the sequence $(Y_n)_{n \in -\mathbb{N}}$ converges a.s. and in L^1 when $n \rightarrow -\infty$.*

It is crucial for the applications developed in Chap. 3 that the convergence also holds in L^1 in the backward case (compare with the analogous result in the “forward” case).

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