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Introduction to Partial Differential Equations

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Dedicated to my parents

Preface

Partial differential equations (PDE) first appeared over 300 years ago, and the vast scope of the theory and applications that have since developed makes it challenging to give a reasonable introduction in a single semester. The modern mathematical approach to the subject requires considerable background in analysis, including topics such as metric space topology, measure theory, and functional analysis.

This book is intended for an introductory course for students who do not necessarily have this analysis background. Courses taught at this level traditionally focus on some of the more elementary topics, such as Fourier series and simple boundary value problems. This approach risks giving students a somewhat narrow and outdated view of the subject.

My goal here is to give a balanced presentation that includes modern methods, without requiring prerequisites beyond vector calculus and linear algebra. To allow for some of the more advanced methods to be reached within a single semester, the treatment is necessarily streamlined in certain ways. Concepts and definitions from analysis are introduced only as they will be needed in the text, and the reader is asked to accept certain fundamental results without justification. The emphasis is not on the rigorous development of analysis in its own right, but rather on the role that tools from analysis play in PDE applications.

The text generally focuses on the most important classical PDE, which are the wave, heat, and Laplace equations. Nonlinear equations are discussed to some extent, but this coverage is limited. (Even at a very introductory level, the nonlinear theory merits a full course to itself.)

I have tried to stress the interplay between modeling and mathematical analysis wherever possible. These connections are vital to the subject, both as a source of problems and as an inspiration for the development of methods.

I owe a great debt of gratitude to my colleague Alessandro Veneziani, with whom I collaborated on this project originally. The philosophy of the book and choice of topics were heavily influenced by our discussions, and I am grateful for his support throughout the process. I would also like to thank former student Dallas Albritton, for offering comments and suggestions on an early draft. Thanks also to the series editors at Springer, for comments that helped improve the writing and presentation.

Atlanta, USA David Borthwick
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The original version of the book was revised: Belated corrections from author have been incorporated. The erratum to the book is available at https://doi.org/10.1007/978-3-319-48936-0_14

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Notations

$:=$	Equal by definition
$\equiv$	Equal except on a set of measure zero
$\sim$	Asymptotic to (ratio approaches 1)
$\asymp$	Comparable to (ratio bounded above and below)
$\lVert\cdot\rVert$	Norm
$\lVert\cdot\rVert_p$	L^p norm
$\langle\cdot,\cdot\rangle$	Inner product (L^2 by default)
$(\cdot,\cdot)$	Distributional pairing
A_n	Volume of the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$
$\mathbb{B}^n$	Open unit ball $\{\lvert\boldsymbol{x}\rvert < 1\} \subset \mathbb{R}^n$
$B(\boldsymbol{x}_0; R)$	Open ball $\{\lvert\boldsymbol{x} - \boldsymbol{x}_0\rvert < R\} \subset \mathbb{R}^n$
$c_k[\cdot]$	Fourier coefficient
$\mathbb{C}$	Complex numbers
$C^m(\Omega)$	m-times continuously differentiable functions $\Omega \to \mathbb{C}$
$C^m(\Omega; \mathbb{R})$	Real valued C^m functions
$C^m(\overline{\Omega})$	C^m functions that admit extension across $\partial\Omega$
$C^m_{cpt}(\Omega)$	C^m functions with compact support in Ω
χ_A	Characteristic (or indicator) function of a set A
$\mathbb{D}$	Unit disk in $\mathbb{R}^2$
D^α	Multivariable derivative
$\delta_{\boldsymbol{x}}$	Dirac delta function
$\partial\Omega$	Boundary of Ω
$\mathcal{D}'(\Omega)$	Distributions on Ω
$\mathcal{D}_f[\cdot]$	Dirichlet energy
dS	Surface integral element
$\mathcal{E}$	Energy of a solution
$\mathcal{F}$	Fourier transform
Γ	Gamma function
∇	Gradient operator
$H^m(\Omega)$	Sobolev space, functions with weak derivatives in L^2 to order m

$H^m_{loc}(\Omega)$	Local Sobolev functions
$H^1_0(\Omega)$	Closure of $C^\infty_{cpt}(\Omega)$ in $H^1(\Omega)$
H_t	Heat kernel
Δ	Laplacian operator on $\mathbb{R}^n$
$L^p(\Omega)$	p-th power integrable functions $\Omega \to \mathbb{C}$
$L^1_{loc}(\Omega)$	Locally integrable functions $\Omega \to \mathbb{C}$
ℓ^p	Discrete L_p space
$\mathbb{N}$	Natural numbers $\{1, 2, 3, \ldots\}$
$\mathbb{N}_0$	Non-negative integers $\{0, 1, 2, 3, \ldots\}$
ν	Outward unit normal
Ω	Domain (open, connected set) in $\mathbb{R}^n$
$\bar{\Omega}$	Closure of Ω $(\Omega \cup \partial\Omega)$
Φ	Fundamental solution
r	$\lvert \boldsymbol{x} \rvert$ in $\mathbb{R}^n$
$\mathbb{R}$	Real numbers
$\mathcal{R}[\cdot]$	Rayleigh quotient
$\mathbb{S}^{n-1}$	Unit sphere $\{\lvert \boldsymbol{x} \rvert = 1\} \subset \mathbb{R}^n$
$\mathcal{S}(\mathbb{R}^n)$	Schwartz functions (smooth functions with rapid decay)
$\mathcal{S}'(\mathbb{R}^n)$	Tempered distributions
$S_n[\cdot]$	Partial sum of Fourier series
$\mathbb{T}$	$\mathbb{R}/2\pi\mathbb{Z}$
W_t	Wave kernel
$\mathbb{Z}$	Integers $\{\ldots, -1, 0, 1, \ldots\}$