

Appendix A

Analysis Foundations

In this section we will develop some implications of the completeness axiom for \mathbb{R} which are referenced in the text.

The fundamental result from which the others follow is the equivalence of compactness and sequential compactness for subsets of \mathbb{R}^n . Recall from Sect. 11.6 that a set A is sequentially compact if every sequence within A contains a subsequence converging to a limit in A . The equivalence was first proven by Bernard Bolzano in the early 19th century, and later rediscovered by Karl Weierstrass.

Theorem A.1 (Bolzano-Weierstrass) *In \mathbb{R}^n a subset is sequentially compact if and only if it is closed and bounded.*

Proof If $A \subset \mathbb{R}^n$ is unbounded, then there exists a sequence of points $\{\mathbf{x}_j\} \subset A$ with $|\mathbf{x}_j| \rightarrow \infty$. Any subsequence has the same property, so $\{\mathbf{x}_j\}$ has no convergent subsequence. If A is not closed, then there is some $\mathbf{w} \notin A$ which is a boundary point of A . Every neighborhood of \mathbf{w} thus includes points of A , so there exists a sequence $\{\mathbf{x}_j\} \subset A$ converging to \mathbf{w} . All subsequences of $\{\mathbf{x}_j\}$ also converge to \mathbf{w} , and therefore no subsequence converges in A . We conclude that a sequentially compact subset of \mathbb{R}^n is closed and bounded.

For the converse argument, let us first consider the one-dimensional case. Let $\{x_j\}$ be a sequence in a bounded set $A \subset \mathbb{R}$. For each n the real number

$$b_n := \sup \{x_k; k \geq n\} \tag{A.1}$$

exists by the completeness axiom. The sequence $\{b_n\}$ is decreasing, because the supremum is taken over successively smaller sets, and also bounded by the hypothesis on A . Therefore the number

$$\alpha := \inf_{n \in \mathbb{N}} b_n$$

is well-defined in \mathbb{R} . The fact that $\{b_n\}$ is decreasing implies $b_n \geq \alpha$ for all n .

We claim that a subsequence of x_k converges to α . For this purpose it suffices to show that the interval $(\alpha - \varepsilon, \alpha + \varepsilon)$ contains infinitely many x_k for each $\varepsilon > 0$.

If this were not the case, then for some n we would have $x_k \notin (\alpha - \varepsilon, \alpha + \varepsilon)$ for all $k \geq n$. This would imply either $b_n \leq \alpha - \varepsilon$ or $b_n \geq \alpha + \varepsilon$, both of which are impossible by the definition of α .

This proves the existence of a subsequence converging to α . The fact that A is closed implies $\alpha \in A$, so this completes the argument that a closed bounded subset of \mathbb{R} is sequentially compact.

To extend this argument to higher dimensions, consider a sequence $\{\mathbf{x}_k\}$ in a compact subset $A \subset \mathbb{R}^n$. The sequence of first coordinates of the \mathbf{x}_k is a bounded sequence in \mathbb{R} , so the above argument yields a subsequence such that the first coordinates converge. We can then restrict our attention to this subsequence and apply the same reasoning to the second coordinate, and so on. After n steps this procedure produces a subsequence which converges to an element of A . \square

Bolzano used sequential compactness to prove the following result, which serves as the foundation for applications of calculus to optimization problems.

Theorem A.2 (Extreme value theorem) *For a compact set $K \subset \mathbb{R}^n$, a continuous function $K \rightarrow \mathbb{R}$ achieves a maximum and minimum value on K .*

Proof Assume that $f : K \rightarrow \mathbb{R}$ is continuous. We will show first that f is bounded. Suppose there is a sequence $\mathbf{x}_j \in K$ such that $|f(\mathbf{x}_j)| \rightarrow \infty$. By Theorem A.1, after restricting to a subsequence if necessary, we can assume that $\mathbf{x}_j \rightarrow \mathbf{w} \in K$. Continuity implies $f(\mathbf{x}_j) \rightarrow f(\mathbf{w})$, but this is impossible if $|f(\mathbf{x}_j)| \rightarrow \infty$. Therefore a continuous function on K is bounded.

Since $f(K)$ is a bounded subset of \mathbb{R} , $b := \sup f(K)$ exists in \mathbb{R} by the completeness axiom. To prove that f achieves a maximum, we need to show $b \in f(K)$. If $b \notin f(K)$ then the function

$$h(\mathbf{x}) := \frac{1}{b - f(\mathbf{x})}$$

is continuous on K , and therefore bounded by the above argument. However, $h(\mathbf{x}) \leq M$ for $\mathbf{x} \in K$ would imply that $\sup f(K) \leq b - 1/M$, contradicting the definition of b . Therefore $b \in f(K)$, so f achieves a maximum. A similar argument applies to the minimum. \square

The final result is the completeness of \mathbb{R}^n as a normed vector space, as noted in Sect. 7.4.

Theorem A.3 *In \mathbb{R}^n a sequence converges if and only if it is Cauchy.*

Proof We have already noted that a convergent sequence is Cauchy in a normed vector space. Suppose that $\{\mathbf{x}_k\}$ is a Cauchy sequence in \mathbb{R}^n . This implies in particular that the sequence is bounded. Therefore, by Theorem A.1, there exists a subsequence converging to some $\mathbf{w} \in \mathbb{R}^n$.

By the definition of Cauchy, for $\varepsilon > 0$ there exists N sufficiently large such that

$$|\mathbf{x}_j - \mathbf{x}_k| < \varepsilon$$

for all $j, k \geq N$. We can also choose an element \mathbf{x}_l in the subsequence such that $l \geq N$ and

$$|\mathbf{x}_l - \mathbf{w}| < \varepsilon.$$

The triangle inequality then gives

$$\begin{aligned} |\mathbf{x}_j - \mathbf{w}| &\leq |\mathbf{x}_j - \mathbf{x}_l| + |\mathbf{x}_l - \mathbf{w}| \\ &< 2\varepsilon, \end{aligned}$$

for all $j \geq N$. Since the choice of ε was arbitrary, this shows that the full sequence converges to \mathbf{w} . \square

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Index

A

acoustic waves, 59
action functional, 234
advection, 27, 97
almost everywhere, 115

B

ball, 12
Banach space, 121
Bessel
 functions, 82
 inequality, 123, 135
 zeros, 86
Bolzano-Weierstrass theorem, 277
boundary conditions
 Dirichlet, 4
 Neumann, 4
 self-adjoint, 126
boundary point, 12
Burger's equation, 43

C

calculus of variations, 234
Cauchy-Schwarz inequality, 113
Cauchy sequence, 120
characteristic, 27, 33
characteristic function, 115
classical solution, 1
closed, 12, 121
closure, 13, 188
coercive, 211
compact, 13
 sequential, 224
 sequentially, 224, 277
 support, 13

completeness, 9, 121, 278
conduction, 97
conjugate, 10
connected, 12
continuity equation, 27, 33
convection, 97
convergence
 pointwise, 137
 uniform, 137, 141
convolution, 103, 139, 249
Coulomb, 239

D

Darboux, 61
delta function, 241, 243
dense, 119
Dirichlet, 138
 eigenvalues, 217, 228
 energy, 205
 kernel, 139
Dirichlet's principle, 207
dispersive estimate, 118
distribution, 239, 242
 tempered, 267
distributional derivative, 245
divergence-free, 33
divergence theorem, 20
domain, 12
domain of dependence, 55
Duhamel's method, 54

E

eigenfunction, 77
eigenvalue, 77
elliptic, 3, 155, 167, 194, 237

regularity, 214
 uniformly, 167, 172
 energy, 67, 73, 97, 166
 equilibrium, 155
 essential supremum, 118
 Euler, 60, 234
 formula, 11
 Euler-Lagrange equation, 235
 evolution equations, 3

F

finite element method, 6, 232
 flux, 25, 32
 forcing term, 42, 54
 Fourier, 97
 series, 134, 191, 215
 transform, 261
 functional, 205
 fundamental solution, 248

G

gamma function, 24, 270
 Gauss, 206
 Gaussian function, 262
 gradient, 18
 Green's function, 254
 Green's identities, 23

H

Hamilton-Jacobi equation, 43
 harmonic function, 155
 harmonic ODE, 17
 harmonic oscillator, 78
 heat equation, 93, 97, 104, 131, 148, 150, 170
 heat kernel, 104, 273
 Heaviside step function, 102
 Helmholtz equation, 76, 132, 194
 Hermite polynomials, 94
 Hilbert space, 121
 Hölder continuity, 248
 Hopf's lemma, 175
 Huygens' principle, 50, 64
 hyperbolic, 4

I

infimum, 9
 infinite propagation speed, 106
 inflow boundary, 42
 initial conditions, 4

inner product, 111
 integrable, 116
 integral kernel, 104
 interior point, 12
 irrotational, 156

K

Kirchhoff, 63
 Klein-Gordon equation, 72

L

Lagrange, 28, 234
 Lagrangian derivative, 28, 34, 60
 Lagrangian function, 234
 Laplace equation, 155
 Laplacian, 3, 18, 59
 eigenvalue equation, 77
 polar coordinate, 82
 spherical, 88
 Legendre functions, 88
 linear conservation equation, 27, 33
 linear equation, 3
 Liouville's theorem, 174
 local integrability, 178

M

material derivative, 28
 maximum, 10
 measurable function, 115
 measure, 114
 Mersenne, 76, 79
 method of characteristics, 29, 47
 method of descent, 65
 method of images, 255
 minimal surface equation, 236
 minimax principle, 228
 minimum, 10
 Minkowski inequality, 117
 multi-index, 180

N

Navier-Stokes, 6
 neighborhood, 12
 Neumann
 boundary conditions, 4
 Newton, 207
 nodes, 79
 norm, 112
 normal vector, 19

O

open set, 12
 order, 1
 order notation, 145
 orthogonal, 112
 complement, 221
 orthonormal, 122
 basis, 122
 overtones, 76

P

parabolic, 4, 155, 172
 parallelogram law, 128
 piecewise C^1 boundary, 14
 piecewise linear, 187
 plane wave, 72, 261
 Plateau problem, 235
 Poincaré constant, 211
 Poisson
 equation, 206, 214
 integral formula, 65
 kernel, 157, 273
 summation formula, 274
 polar coordinates, 23, 81
 potential function, 156
 principal value, 247
 propagation speed, 47, 49
 finite, 67

Q

quasilinear equation, 36

R

range of influence, 57, 64
 Rankine-Hugoniot condition, 184
 rapidly, 262
 rarefaction wave, 203
 Rayleigh
 principle, 218
 quotient, 218
 Rayleigh-Ritz method, 231
 reaction-diffusion equations, 109
 reaction term, 42
 regularity
 elliptic, 214
 Rellich's theorem, 219, 224
 resonance, 58

S

Schrödinger equation, 71

Schwartz functions, 262
 separation of variables, 75
 sequentially compactness, 224
 sequentially compact, 224, 277
 sesquilinear, 112
 sesquilinearity, 112
 shock, 41, 43
 curve, 183
 smooth, 13
 Sobolev
 embedding theorem, 190
 regularity, 190
 spaces, 187
 solenoidal, 33, 156
 spectral
 analysis, 131
 theorem, 125, 217
 spectrum, 79
 spherical harmonics, 89
 stability, 5
 subharmonic, 164
 sup norm, 114
 superharmonic, 164
 superposition principle, 3
 support, 13
 supremum, 9
 surface integral, 19

T

telegraph equation, 70
 tempered distribution, 267
 test function, 177
 traffic equation, 37
 transport equation, 27

V

variation of parameters, 54
 vector space, 111

W

wave equation, 2, 47, 85, 271
 acoustic, 61
 damped, 93
 wave kernel, 259, 271
 weak derivative, 178
 weak solution, 1, 51, 177
 Weierstrass, 277
 well posed, 5, 166