Appendix A
Analysis Foundations

In this section we will develop some implications of the completeness axiom for $\mathbb{R}$ which are referenced in the text.

The fundamental result from which the others follow is the equivalence of compactness and sequential compactness for subsets of $\mathbb{R}^n$. Recall from Sect. 11.6 that a set A is sequentially compact if every sequence within A contains a subsequence converging to a limit in A. The equivalence was first proven by Bernard Bolzano in the early 19th century, and later rediscovered by Karl Weierstrass.

Theorem A.1 (Bolzano-Weierstrass) *In $\mathbb{R}^n$ a subset is sequentially compact if and only if it is closed and bounded.*

Proof If $A \subset \mathbb{R}^n$ is unbounded, then there exists a sequence of points $\{\boldsymbol{x}_j\} \subset A$ with $|\boldsymbol{x}_j| \to \infty$. Any subsequence has the same property, so $\{\boldsymbol{x}_j\}$ has no convergent subsequence. If A is not closed, then there is some $\boldsymbol{w} \notin A$ which is a boundary point of A. Every neighborhood of $\boldsymbol{w}$ thus includes points of A, so there exists a sequence $\{\boldsymbol{x}_j\} \subset A$ converging to $\boldsymbol{w}$. All subsequences of $\{\boldsymbol{x}_j\}$ also converge to $\boldsymbol{w}$, and therefore no subsequence converges in A. We conclude that a sequentially compact subset of $\mathbb{R}^n$ is closed and bounded.

For the converse argument, let us first consider the one-dimensional case. Let $\{x_j\}$ be a sequence in a bounded set $A \subset \mathbb{R}$. For each n the real number

$$b_n := \sup\{x_k;\ k \geq n\} \tag{A.1}$$

exists by the completeness axiom. The sequence $\{b_n\}$ is decreasing, because the supremum is taken over successively smaller sets, and also bounded by the hypothesis on A. Therefore the number

$$\alpha := \inf_{n\in\mathbb{N}} b_n$$

is well-defined in $\mathbb{R}$. The fact that $\{b_n\}$ is decreasing implies $b_n \geq \alpha$ for all n.

We claim that a subsequence of x_k converges to α. For this purpose it suffices to show that the interval $(\alpha - \varepsilon, \alpha + \varepsilon)$ contains infinitely many x_k for each $\varepsilon > 0$.

D. Borthwick, *Introduction to Partial Differential Equations*,
Universitext, DOI 10.1007/978-3-319-48936-0

If this were not the case, then for some n we would have $x_k \notin (\alpha - \varepsilon, \alpha + \varepsilon)$ for all $k \geq n$. This would imply either $b_n \leq \alpha - \varepsilon$ or $b_n \geq \alpha + \varepsilon$, both of which are impossible by the definition of α.

This proves the existence of a subsequence converging to α. The fact that A is closed implies $\alpha \in A$, so this completes the argument that a closed bounded subset of $\mathbb{R}$ is sequentially compact.

To extend this argument to higher dimensions, consider a sequence $\{\boldsymbol{x}_k\}$ in a compact subset $A \subset \mathbb{R}^n$. The sequence of first coordinates of the $\boldsymbol{x}_k$ is a bounded sequence in $\mathbb{R}$, so the above argument yields a subsequence such that the first coordinates converge. We can then restrict our attention to this subsequence and apply the same reasoning to the second coordinate, and so on. After n steps this procedure produces a subsequence which converges to an element of A. □

Bolzano used sequential compactness to prove the following result, which serves as the foundation for applications of calculus to optimization problems.

Theorem A.2 (Extreme value theorem) *For a compact set $K \subset \mathbb{R}^n$, a continuous function $K \to \mathbb{R}$ achieves a maximum and minimum value on K.*

Proof Assume that $f : K \to \mathbb{R}$ is continuous. We will show first that f is bounded. Suppose there is a sequence $\boldsymbol{x}_j \in K$ such that $\left|f(\boldsymbol{x}_j)\right| \to \infty$. By Theorem A.1, after restricting to a subsequence if necessary, we can assume that $\boldsymbol{x}_j \to \boldsymbol{w} \in K$. Continuity implies $f(\boldsymbol{x}_j) \to f(\boldsymbol{w})$, but this is impossible if $\left|f(\boldsymbol{x}_j)\right| \to \infty$. Therefore a continuous function on K is bounded.

Since $f(K)$ is a bounded subset of $\mathbb{R}$, $b := \sup f(K)$ exists in $\mathbb{R}$ by the completeness axiom. To prove that f achieves a maximum, we need to show $b \in f(K)$. If $b \notin f(K)$ then the function

$$h(\boldsymbol{x}) := \frac{1}{b - f(\boldsymbol{x})}$$

is continuous on K, and therefore bounded by the above argument. However, $h(\boldsymbol{x}) \leq M$ for $\boldsymbol{x} \in K$ would imply that $\sup f(K) \leq b - 1/M$, contradicting the definition of b. Therefore $b \in f(K)$, so f achieves a maximum. A similar argument applies to the minimum. □

The final result is the completeness of $\mathbb{R}^n$ as a normed vector space, as noted in Sect. 7.4.

Theorem A.3 *In $\mathbb{R}^n$ a sequence converges if and only if it is Cauchy.*

Proof We have already noted that a convergent sequence is Cauchy in a normed vector space. Suppose that $\{\boldsymbol{x}_k\}$ is a Cauchy sequence in $\mathbb{R}^n$. This implies in particular that the sequence is bounded. Therefore, by Theorem A.1, there exists a subsequence converging to some $\boldsymbol{w} \in \mathbb{R}^n$.

By the definition of Cauchy, for $\varepsilon > 0$ there exists N sufficiently large such that

$$\left|\boldsymbol{x}_j - \boldsymbol{x}_k\right| < \varepsilon$$

for all $j, k \geq N$. We can also choose an element $\boldsymbol{x}_l$ in the subsequence such that $l \geq N$ and

$$|\boldsymbol{x}_l - \boldsymbol{w}| < \varepsilon.$$

The triangle inequality then gives

$$\begin{aligned}\left|\boldsymbol{x}_j - \boldsymbol{w}\right| &\leq \left|\boldsymbol{x}_j - \boldsymbol{x}_l\right| + |\boldsymbol{x}_l - \boldsymbol{w}| \\ &< 2\varepsilon,\end{aligned}$$

for all $j \geq N$. Since the choice of ε was arbitrary, this shows that the full sequence converges to $\boldsymbol{w}$. □

References

For additional analysis background

Apostol, T. M. (1974). Mathematical analysis (2nd ed.). Addison-Wesley.

Edwards, R. E. (1979). Fourier series. A modern introduction (2nd ed.). Graduate Texts in Mathematics 64. Springer.

Folland, G. B. (1999). Real analysis (2nd ed.). Pure and Applied Mathematics. Wiley.

Rudin, W. (1976). Principles of mathematical analysis (3rd ed.). McGraw-Hill.

Stein, E. M., & Shakarchi, R. (2003). *Fourier analysis*. Princeton Lectures in Analysis I: Princeton.

Stein, E. M., & Shakarchi, R. (2005). *Real analysis*. Princeton Lectures in Analysis III: Princeton.

Stein, E. M., & Shakarchi, R. (2011). *Functional analysis*. Princeton Lectures in Analysis IV: Princeton.

Strichartz, R.S. (2003). *A Guide to Distribution Theory and Fourier Transforms*. World Scientific.

For further study of PDE

DiBenedetto, E. (2010). Partial differential equations (3nd ed.). Birkhuser.

Evans, L. C. (2010). Partial differential equations (2nd ed.). Graduate Studies in Mathematics 19. American Mathematical Society.

Gilbarg, D., & Trudinger, N. S. (2001). *Elliptic partial differential equations of second order*. Classics in Mathematics: Springer.

Jost, J. (2013). Partial differential equations (3rd ed.). Graduate Texts in Mathematics 214. Springer.

Strauss, W. A. (2008). Partial differential equations. An introduction (2nd ed.). Wiley.

Taylor, M. E. (2011). Partial differential equations I. Basic theory, Applied Mathematical Sciences 115. Springer.

Vasy, A. (2015). Partial differential equations, Graduate Studies in Mathematics 169. American Mathematical Society.

D. Borthwick, *Introduction to Partial Differential Equations*,
Universitext, DOI 10.1007/978-3-319-48936-0

Index

D. Borthwick, *Introduction to Partial Differential Equations*,
Universitext, DOI 10.1007/978-3-319-48936-0