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Stochastic Processes and Calculus

An Elementary Introduction with Applications

Springer

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I do not know what I may appear to the world, but to myself I seem to have been only like a boy playing on the sea-shore, and diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me.

ISAAC NEWTON

Quoted from the novel Beyond Sleep *by Willem Frederik Hermans*

Preface

Over the past decades great importance has been placed on stochastic calculus and processes in mathematics, finance, and econometrics. This book addresses particularly readers from these fields, although students of other subjects as biology, engineering, or physics may find it useful, too.

Scope of the Book

By now there exist a number of books describing stochastic integrals and stochastic calculus in an accessible manner. Such introductory books, however, typically address an audience having previous knowledge about and interest in one of the following three fields exclusively: finance, econometrics, or mathematics. The textbook at hand attempts to provide an introduction into stochastic calculus and processes for students from each of these fields. Obviously, this can on no account be an exhaustive treatment. In the next chapter a survey of the topics covered is given. In particular, the book does neither deal with finance theory nor with statistical methods from the time series econometrician's toolkit; it rather provides a mathematical background for those readers interested in these fields.

The first part of this book is dedicated to discrete-time processes for modeling temporal dependence in time series. We begin with some basic principles of stochastics enabling us to define stochastic processes as families of random variables in general. We discuss models for short memory (so-called ARMA models), for long memory (fractional integration), and for conditional heteroscedasticity (so-called ARCH models) in respective chapters. One further chapter is concerned with the so-called frequency domain or spectral analysis that is often neglected in introductory books. Here, however, we propose an approach that is not technically too demanding. Throughout, we restrict ourselves to the consideration of stochastic properties and interpretation. The statistical issues of parameter estimation, testing, and model specification are not addressed due to space limitations; instead, we refer to, e.g., Mills and Markellos (2008), Kirchgässner, Wolters, and Hassler (2013), or Tsay (2005).

The second part contains an introduction to stochastic integration. We start with elaborations on the Wiener process $W(t)$ as we will define (almost) all integrals in

terms of Wiener processes. In one chapter we consider Riemann integrals of the form $\int f(t)W(t)dt$, where f is a deterministic function. In another chapter Stieltjes integrals are constructed as $\int f(t)dW(t)$. More specifically, stochastic integrals as such result when a stochastic process is integrated with respect to the Wiener process, e.g., the Ito integral $\int W(t)dW(t)$. Solving stochastic differential equations is one task of stochastic integration for which we will need to use Ito's lemma. Our description aims at a similar compromise between concreteness and mathematical rigor as, e.g., Mikosch (1998). If the reader wants to address this matter more rigorously, we recommend Klebaner (2005) or Øksendal (2003).

The third part of the book applies previous results. The chapter on stochastic differential equations consists basically of applications of Ito's lemma. Concrete differential equations, as they are used, e.g., when modeling interest rate dynamics, will be covered in a separate chapter. The second area of application concerns certain limiting distributions of time series econometrics. A separate chapter on the asymptotics of integrated processes covers weak convergence to Wiener processes. The final two chapters contain applications for nonstationary processes without cointegration on the one hand and for the analysis of cointegrated processes on the other. Further details regarding econometric application can be found in the books by Banerjee, Dolado, Galbraith and Hendry (1993), Hamilton (1994), or Tanaka (1996).

The exposition in this book is elementary in the sense that knowledge of measure theory is neither assumed nor used. Consequently, mathematical foundations cannot be treated rigorously which is why, e.g., proofs of existence are omitted. Rather I had two goals in mind when writing this book. On the one hand, I wanted to give a basic and illustrative presentation of the relevant topics without many "troublesome" derivations. On the other hand, in many parts a technically advanced level has been aimed at: procedures are not only presented in form of recipes but are to be understood as far as possible which means they are to be proven. In order to meet both requirements jointly, this book is equipped with a lot of challenging problems at the end of each chapter as well as with the corresponding detailed solutions. Thus the virtual text – augmented with more than 60 basic examples and 45 illustrative figures – is rather easy to read while a part of the technical arguments is transferred to the exercise problems and their solutions. This is why there are at least two possible ways to work with the book. For those who are merely interested in applying the methods introduced, the reading of the text is sufficient. However, for an in-depth knowledge of the theory and its application, the reader necessarily needs to study the problems and their solution extensively.

Note to Students and Instructors

I have taught the material collected here to master students (and diploma students in the old days) of economics and finance or students of mathematics with a minor in those fields. From my personal experience I may say that the material presented here is too vast to be treated in a course comprising 45 contact hours. I used the

textbook at hand for four slightly differing courses corresponding to four slightly differing routes through the parts of the book. Each of these routes consists of three stages: time series models, stochastic integration, and applications. After Part I on time series modeling, the different routes separate.

The finance route: When teaching an audience with an exclusive interest in finance, one may simply drop the final three chapters. The second stage of the course then consists of Chaps. 7, 8, 9, 10, and 11. This Part II on stochastic integration is finally applied to the solution of stochastic differential equations and interest rate modeling in Chaps. 12 and 13, respectively.

The mathematics route: There is a slight variant of the finance route for the mathematically inclined audience with an equal interest in finance or econometrics. One simply replaces Chap. 13 on interest rate modeling by Chap. 14 on weak convergence on function spaces, which is relevant for modern time series asymptotics.

The econometrics route: After Part I on time series modeling, the students from a class on time series econometrics should be exposed to Chaps. 7, 8, 9, and 10 on Wiener processes and stochastic integrals. The three chapters (Chaps. 11, 12, and 13) on Ito's lemma and its applications may be skipped to conclude the course with the last three chapters (Chaps. 14, 15, and 16) culminating in the topic of "cointegration."

The nontechnical route: Finally, the entire content of the textbook at hand can still be covered in one single semester; however, this comes with the cost of omitting technical aspects for the most part. Each chapter contains a rather technical section which in principle can be skipped without leading to a loss in understanding. When omitting these potentially difficult sections, it is possible to go through all the chapters in a single course. The following sections should be skipped for a less technical route:

$$\boxed{3.3}\ \&\ \boxed{4.3}\ \&\ \boxed{5.4}\ \&\ \boxed{6.4}\ \&\ \boxed{7.3}\ \&\ \boxed{8.4}\ \&\ \boxed{9.4}$$
$$\&\ \boxed{10.4}\ \&\ \boxed{11.4}\ \&\ \boxed{12.2}\ \&\ \boxed{13.4}\ \&\ \boxed{14.3}\ \&\ \boxed{15.4}\ \&\ \boxed{16.4}\ .$$

It has been mentioned that each chapter concludes with problems and solutions. Some of them are clearly too hard or lengthy to be dealt with in exams, while others are questions from former exams of my own or are representative of problems to be solved in my exams.

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