Appendix A
Probability and Random Processes

The theory of probability and random processes is essential in the design and performance analysis of wireless communication systems. This appendix presents a brief review of the basic concepts of probability theory and random processes, with emphasis on the concept needed to understand this book. It is intended that most readers have already had some exposure to probability and random processes, so that this appendix is intended to provide a brief overview. A very thorough treatment of this subject is available in a large number of textbooks, including [201, 254].

This appendix begins in Sect. A.1 with the basic axioms of probability, conditional probability, and Bayes' theorem. It then goes onto means, moments, and generating functions in Sect. A.2. Afterwards, Sect. A.3 presents a variety of discrete probability distributions and continuous probability density functions. Particular emphasis is placed on Gaussian, complex Gaussian, multi-variate Gaussian, multivariate complex Gaussian density functions, and functions of Gaussian random variables. After a brief treatment of upper bounds on probability in Sect. A.4, the appendix then goes onto a treatment of random processes, including means and correlation functions in Sect. A.5.1, cross-correlation and crosscovariance for joint random processes in Sect. A.5.2, complex random processes in Sect. A.5.3, power spectral density in Sect. A.5.4, and filtering of random processes in Sect. A.5.5. The important class of cyclostationary random processes is considered in Sect. A.5.6 and the appendix wraps up with a brief treatment of discrete-time random processes in Sect. A.5.7.

A.1 Conditional Probability and Bayes' Theorem

Let A and B be two events in a sample space S. The conditional probability of A given B is

$$\mathrm{P}[A|B] = \frac{\mathrm{P}[A \bigcap B]}{\mathrm{P}[B]} \tag{A.1}$$

provided that $\mathrm{P}[B] \neq 0$. If $\mathrm{P}[B] = 0$, then $\mathrm{P}[A|B]$ is undefined.

There are several special cases.

- If $A \bigcap B = \emptyset$, then events A and B are mutually exclusive, i.e., if B occurs then A could not have occurred and $\mathrm{P}[A|B] = 0$.
- If $B \subset A$, then knowledge that event B has occurred implies that event A has occurred and so $\mathrm{P}[A|B] = 1$.
- If A and B are statistically independent, then $\mathrm{P}[A \bigcap B] = \mathrm{P}[A]\mathrm{P}[B]$ and so $\mathrm{P}[A|B] = \mathrm{P}[A]$.

There is a strong connection between mutually exclusive and independent events. It may seem that mutually exclusive events are independent, but just the exact opposite is true. Consider two events A and B with $\mathrm{P}[A] > 0$ and $\mathrm{P}[B] > 0$. If A and B are mutually exclusive, then $A \bigcap B = 0$ and $\mathrm{P}[A \bigcap B] = 0 \neq \mathrm{P}[A]\mathrm{P}[B]$. Therefore, mutually exclusive events with non-zero probability cannot be independent. Thus, the disjointness of events is a property of the events themselves, while independence is a property of their probabilities.

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In general, the events $A_i, i = 1, \ldots, n$, are independent if and only if for all collections of k distinct integers $(i_1, i_2, \ldots, i_k)$ chosen from the set $(1, 2, \ldots, n)$,

$$\mathrm{P}\left[A_{i_1} \bigcap A_{i_2} \bigcap \cdots \bigcap A_{i_k}\right] = \mathrm{P}[A_{i_1}]\mathrm{P}[A_{i_2}] \cdots \mathrm{P}[A_{i_k}]$$

for $2 \le k \le n$.

In summary

- If $A_i, i = 1, \ldots, n$ is a sequence of mutually exclusive events, then

$$\mathrm{P}\left[\bigcup_{i=1}^{n}\right] = \sum_{i=1}^{n} \mathrm{P}[A_i]. \tag{A.2}$$

- If $A_i, i = 1, \ldots, n$ is a sequence of independent events, then

$$\mathrm{P}\left[\bigcap_{i=1}^{n}\right] = \prod_{i=1}^{n} \mathrm{P}[A_i]. \tag{A.3}$$

A.1.1 Total Probability

The collection of sets $\{B_i\},\ i = 1, \ldots, n$ forms a *partition* of the sample space S if $B_i \bigcap B_j = \emptyset,\ i \neq j$ and $\bigcup_{i=1}^{n} B_i = S$. For any event $A \subset S$

$$A = \bigcup_{i=1}^{n} (A \bigcap B_i). \tag{A.4}$$

That is, every element of A is contained in one and only one B_i. Since $(A \bigcap B_i) \bigcap (A \bigcap B_j) = \emptyset,\ i \neq j$, the sets $A \bigcap B_i$ are mutually exclusive. Therefore,

$$\begin{aligned} \mathrm{P}[A] &= \sum_{i=1}^{n} \mathrm{P}[A \bigcap B_i] \\ &= \sum_{i=1}^{n} \mathrm{P}[A|B_i]\mathrm{P}[B_i]. \end{aligned} \tag{A.5}$$

This last equation is often referred to as the theorem of total probability.

A.1.2 Bayes' Theorem

Let the events $B_i,\ i = 1, \ldots, n$ be mutually exclusive such that $\bigcup_{i=1}^{n} B_i = S$, where S is the sample space. Let A be an event with non-zero probability. Then as a result of conditional probability and total probability:

$$\begin{aligned} \mathrm{P}[B_i|A] &= \frac{\mathrm{P}[B_i \bigcap A]}{\mathrm{P}[A]} \\ &= \frac{\mathrm{P}[A|B_i]\mathrm{P}[B_i]}{\sum_{i=1}^{n} \mathrm{P}[A|B_i]\mathrm{P}[B_i]}. \end{aligned}$$

a result known as Bayes' theorem.

A.2 Means, Moments, and Moment Generating Functions

The kth moment of a random variable, $\mathrm{E}[X^k]$, is defined as

$$\mathrm{E}[X^k] \triangleq \begin{cases} \sum_{x_i \in R_X} x_i^k p_X(x_i) & \text{if } X \text{ is discrete} \\ \int_{R_X} x^k p_X(x) \mathrm{d}x & \text{if } X \text{ is continuous} \end{cases}, \tag{A.6}$$

where $p_X(x_i) \triangleq \mathrm{P}[X = x_i]$ is the probability distribution function of X, and $p_X(x)$ is the probability density function (pdf) of X. The kth central moment of the random variable X is $\mathrm{E}[(X - \mathrm{E}[X])^k]$. The mean is the first moment

$$\mu_X = \mathrm{E}[X] \tag{A.7}$$

and the variance is the second central moment

$$\sigma_X^2 = \mathrm{E}[(X - \mu_X)^2] = \mathrm{E}[X^2] - \mu_X^2. \tag{A.8}$$

The moment generating function or characteristic function of a random variable X is

$$\psi_X(jv) \triangleq \mathrm{E}[\mathrm{e}^{jvX}] = \begin{cases} \sum_{x_i \in R_X} \mathrm{e}^{jvx_i} p_X(x_i) & \text{if } X \text{ is discrete} \\ \int_{R_X} \mathrm{e}^{jvx} p_X(x) \mathrm{d}x & \text{if } X \text{ is continuous} \end{cases}, \tag{A.9}$$

where $j = \sqrt{-1}$. Note that the continuous version is a Fourier transform, except for the sign in the exponent. Likewise, the discrete version is a z-transform, except for the sign in the exponent.

The probability distribution and probability density functions of discrete and continuous random variables, respectively, can be obtained by taking the inverse transforms of the characteristic functions, i.e.,

$$p_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi_X(jv) \mathrm{e}^{-jvx} \mathrm{d}v \tag{A.10}$$

and

$$p_X(x_k) = \frac{1}{2\pi} \oint_C \psi_X(jv) \mathrm{e}^{-jvx_k} \mathrm{d}v. \tag{A.11}$$

The cumulative distribution function (cdf) of a random variable X is defined as

$$F_X(x) \triangleq \mathrm{P}[X \le x] = \begin{cases} \sum_{x_i \le x} p_X(x_i) & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{x} p_X(x) \mathrm{d}x & \text{if } X \text{ is continuous} \end{cases}, \tag{A.12}$$

and $0 \le F_X(x) \le 1$. The complementary distribution function (cdfc) is defined as

$$F_X^c(x) \triangleq 1 - F_X(x). \tag{A.13}$$

The probability density function of a continuous random variable X is related to the cdf by

$$p_X(x) = \frac{\mathrm{d}F_X(x)}{\mathrm{d}x}. \tag{A.14}$$

A.2.1 Bivariate Random Variables

Consider a pair of random variables X and Y. The joint cdf of X and Y is

$$F_{XY}(x,y) = \mathrm{P}[X \le x, Y \le y]\,, \quad 0 \le F_{XY}(x,y) \le 1, \tag{A.15}$$

and the joint (cdfc) of X and Y is

$$F^c_{XY}(x,y) = \mathrm{P}[X > x, Y > y] = 1 - F_{XY}(x,y)\,, \quad 0 \le F^c_{XY}(x,y) \le 1. \tag{A.16}$$

The joint pdf of X and Y is

$$p_{XY}(x,y) = \frac{\partial^2 F_{XY}(x,y)}{\partial x \partial y}\,, \qquad F_{XY}(x) = \int_{-\infty}^{x}\int_{-\infty}^{y} p_{XY}(x,y)\mathrm{d}x\mathrm{d}y. \tag{A.17}$$

The marginal pdfs of X and Y are

$$p_X(x) = \int_{-\infty}^{\infty} p_{XY}(x,y)\mathrm{d}y \qquad p_Y(x) = \int_{-\infty}^{\infty} p_{XY}(x,y)\mathrm{d}x. \tag{A.18}$$

If X and Y are independent random variables, then the joint pdf has the product form

$$p_{XY}(x,y) = p_X(x)p_Y(x). \tag{A.19}$$

The conditional pdfs of X and Y are

$$p_{X|Y}(x|y) = \frac{p_{XY}(x,y)}{p_Y(y)} \qquad p_{Y|X}(y|x) = \frac{p_{XY}(x,y)}{p_X(x)}. \tag{A.20}$$

The joint moments of X and Y are

$$\mathrm{E}[X^iY^j] = \int_{-\infty}^{\infty} x^iy^j p_{XY}(x,y)\mathrm{d}x\mathrm{d}y. \tag{A.21}$$

The covariance of X and Y is

$$\begin{aligned}\lambda_{XY} &= \mathrm{E}[(X-\mu_X)(Y-\mu_Y)] \\ &= \mathrm{E}[XY - X\mu_Y - Y\mu_X + \mu_X\mu_Y] \\ &= \mathrm{E}[XY] - \mu_X\mu_Y\end{aligned} \tag{A.22}$$

The correlation coefficient of X and Y is

$$\rho_{XY} = \frac{\lambda_{XY}}{\sigma_X\sigma_Y}. \tag{A.23}$$

Two random variables X and Y are *uncorrelated* if and only if $\lambda_{X,Y} = 0$. Two random variables X and Y are *orthogonal* if and only if $\mathrm{E}[XY] = 0$.

The joint characteristic function is

$$\Phi_{XY}(v_1,v_2) = \mathrm{E}[\mathrm{e}^{jv_1X+jv_2Y}] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_{XY}(x,y)\mathrm{e}^{jv_1x+jv_2y}\mathrm{d}x\mathrm{d}y. \tag{A.24}$$

If X and Y are independent, then

$$\begin{aligned}\Phi_{XY}(v_1, v_2) &= \mathrm{E}[\mathrm{e}^{jv_1X+jv_2Y}] \\ &= \int_{-\infty}^{\infty} p_X(x)\mathrm{e}^{jv_1x}\mathrm{d}x \int_{-\infty}^{\infty} p_Y(y)\mathrm{e}^{jv_2y}\mathrm{d}y \\ &= \Phi_X(v_1)\Phi_Y(v_2). \end{aligned} \tag{A.25}$$

Moments can be generated according to

$$\mathrm{E}[XY] = -\frac{\partial^2 \Phi_{XY}(v_1, v_2)}{\partial v_1 \partial v_2} \big|_{v_1=v_2=0} \; . \tag{A.26}$$

with higher order moments generated in a straightforward extension.

A.3 Some Useful Probability Distributions

A.3.1 Discrete Distributions

A.3.1.1 Binomial Distribution

Let X be a Bernoulli random variable such that $X = 0$ with probability $1 - p$ and $X = 1$ with probability p. Although X is a discrete random random variable with an associated probability distribution function, it is possible to treat X as a continuous random variable with a pdf by using dirac delta functions. In this case, the pdf of X has the form

$$p_X(x) = (1-p)\delta(x) + p\delta(x-1). \tag{A.27}$$

Let $Y = \sum_{i=1}^{n} X_i$, where the X_i are independent and identically distributed with density $p_X(x)$. Then the random variable Y is an integer from the set $\{0, 1, \ldots, n\}$ and the probability distribution of Y is the binomial distribution

$$p_Y(k) \equiv \mathrm{P}[Y = k] = \binom{n}{k} p^k(1-p)^{n-k} \; , \;\; k = 0, 1, \ldots, n. \tag{A.28}$$

The random variable Y also has the pdf

$$p_Y(y) = \sum_{k=0}^{n} \binom{n}{k} p^k(1-p)^{n-k}\delta(y-k). \tag{A.29}$$

A.3.1.2 Poisson Distribution

The random variable X has a Poisson distribution if

$$p_X(k) = \frac{\lambda^k \mathrm{e}^{-\lambda}}{k!} \; , \;\; k = 0, 1, \ldots, \; \infty \tag{A.30}$$

A.3.1.3 Geometric Distribution

The random variable X has a geometric distribution if

$$p_X(k) = (1-p)^{k-1}p \; , \;\; k = 1, 2, \ldots, \infty. \tag{A.31}$$

A.3.2 Continuous Distributions

Many communication systems are affected by Gaussian random processes. Therefore, Gaussian random variables and various functions of Gaussian random variables play a central role in the characterization and analysis of communication systems.

A.3.2.1 Gaussian Distribution

A real-valued Gaussian or normal random variable X has the pdf

$$p_X(x) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}, \tag{A.32}$$

where $\mu = \mathrm{E}[X]$ is the mean of X and $\sigma^2 = \mathrm{E}[(X-\mu)^2]$ is the variance of X. Sometimes the shorthand notation $X \sim \mathscr{N}(\mu, \sigma^2)$ is used meaning that X is a Gaussian random variable with mean μ and variance σ^2. The random variable X is said to have a standard normal distribution if $X \sim \mathscr{N}(0, 1)$.

The cumulative distribution function (cdf) of a Gaussian random variable X is

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{(y-\mu)^2}{2\sigma^2}\right\} \mathrm{d}y. \tag{A.33}$$

The cdf of a standard normal distribution defines the Gaussian Q function

$$Q(x) \triangleq \int_{x}^{\infty} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-y^2/2} \mathrm{d}y \tag{A.34}$$

and the cdfc defines the Gaussian Φ function

$$\Phi(x) \triangleq 1 - Q(x). \tag{A.35}$$

If X is a non-standard normal random variable, $X \sim \mathscr{N}(\mu, \sigma^2)$, then

$$F_X(x) = \Phi\left(\frac{x-\mu}{\sigma}\right) \tag{A.36}$$

$$F_X^c(x) = Q\left(\frac{x-\mu}{\sigma}\right). \tag{A.37}$$

Sometimes the cumulative distribution function of a Gaussian random variable is described in terms of the complementary error function $\mathrm{erfc}(x)$, defined as

$$\mathrm{erfc}(x) \triangleq \frac{2}{\sqrt{\pi}} \int_{x}^{\infty} \mathrm{e}^{-y^2} \mathrm{d}y. \tag{A.38}$$

The complementary error function and the Gaussian Q function are related as follows:

$$\mathrm{erfc}(x) = 2Q(\sqrt{2}x) \tag{A.39}$$

$$Q(x) = \frac{1}{2}\mathrm{erfc}\left(\frac{x}{\sqrt{2}}\right). \tag{A.40}$$

These identities can be established by using the Gaussian Q function in (A.34). The error function of a Gaussian random variable is defined as

$$\mathrm{erf}(x) \triangleq \frac{2}{\sqrt{\pi}} \int_{0}^{x} \mathrm{e}^{-y^2} \mathrm{d}y. \tag{A.41}$$

Note that $\text{erfc}(x) + \text{erf}(x) \neq 1$. Also,

$$Q(x) = \frac{1}{2} - \frac{1}{2}\text{erf}\left(\frac{x}{\sqrt{2}}\right), \ x \geq 0. \tag{A.42}$$

A.3.2.2 Multivariate Gaussian Distribution

Let $X_i \sim \mathcal{N}(\mu_i, \sigma_i^2), i = 1, \ldots, n$, be a collection of n real-valued Gaussian random variables having means $\mu_i = \text{E}[X_i]$ and covariances

$$\begin{aligned}\lambda_{X_iX_j} &= \text{E}\left[(X_i - \mu_i)(X_j - \mu_j)\right] \\ &= \text{E}\left[X_iX_j\right] - \mu_i\mu_j, \quad 1 \leq i,j \leq n.\end{aligned}$$

Let

$$\begin{aligned}\mathbf{X} &= (X_1, X_2, \ldots, X_n)^T \\ \mathbf{x} &= (x_1, x_2, \ldots, x_n)^T \\ \boldsymbol{\mu}_X &= (\mu_1, \mu_2, \ldots, \mu_n)^T \\ \boldsymbol{\Lambda} &= \begin{bmatrix} \lambda_{X_1X_1} & \cdots & \lambda_{X_1X_n} \\ \vdots & & \vdots \\ \lambda_{X_nX_1} & \cdots & \lambda_{X_nX_n} \end{bmatrix},\end{aligned}$$

where $\mathbf{X}^T$ is the transpose of $\mathbf{X}$. The random vector $\mathbf{X}$ has the multivariate Gaussian distribution

$$p_{\mathbf{X}}(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}|\boldsymbol{\Lambda}|^{1/2}}\exp\left\{-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu}_X)^T\boldsymbol{\Lambda}^{-1}(\mathbf{x} - \boldsymbol{\mu}_X)\right\}, \tag{A.43}$$

where $|\boldsymbol{\Lambda}|$ is the determinant of $\boldsymbol{\Lambda}$.

A.3.2.3 Multivariate Complex Gaussian Distribution

Complex Gaussian distributions often arise in the treatment of fading channels and narrow-band Gaussian noise. Let

$$\begin{aligned}\mathbf{X} &= (X_1, X_2, \ldots, X_n)^T \\ \mathbf{Y} &= (Y_1, Y_2, \ldots, Y_n)^T\end{aligned}$$

be length-n vectors of real-valued Gaussian random variables, such that $X_i \sim \mathcal{N}(\mu_{X_i}, \sigma_{X_i}^2)$, and $Y_i \sim \mathcal{N}(\mu_{Y_i}, \sigma_{Y_i}^2), i = 1, \ldots, n$. The complex random vector $\mathbf{Z} = \mathbf{X} + j\mathbf{Y}$ has a complex Gaussian distribution that can be described with the following three parameters:

$$\begin{aligned}\boldsymbol{\mu}_Z &= \text{E}[\mathbf{Z}] = \boldsymbol{\mu}_X + j\boldsymbol{\mu}_Y \\ \boldsymbol{\Gamma} &= \frac{1}{2}\text{E}[(\mathbf{Z} - \boldsymbol{\mu}_Z)(\mathbf{Z} - \boldsymbol{\mu}_Z)^H] \\ \mathbf{C} &= \frac{1}{2}\text{E}[(\mathbf{Z} - \boldsymbol{\mu}_Z)(\mathbf{Z} - \boldsymbol{\mu}_Z)^T],\end{aligned}$$

where $\mathbf{X}^T$ and $\mathbf{X}^H$ are the transpose and complex conjugate transpose of $\mathbf{X}$, respectively. The covariance matrix $\boldsymbol{\Gamma}$ must be Hermitian ($\boldsymbol{\Gamma} = \boldsymbol{\Gamma}^H$) and the relation matrix $\mathbf{C}$ should be symmetric ($\mathbf{C} = \mathbf{C}^T$). Matrices $\boldsymbol{\Gamma}$ and $\mathbf{C}$ can be related to the covariance matrices of $\mathbf{X}$ and $\mathbf{Y}$ as follows:

$$\mathbf{\Lambda_{XX}} = \mathrm{E}[(\mathbf{X}-\boldsymbol{\mu}_X)(\mathbf{X}-\boldsymbol{\mu}_X)^T] = \mathrm{Re}\{\mathbf{\Gamma}+\mathbf{C}\} \tag{A.44}$$
$$\mathbf{\Lambda_{XY}} = \mathrm{E}[(\mathbf{X}-\boldsymbol{\mu}_X)(\mathbf{Y}-\boldsymbol{\mu}_Y)^T] = \mathrm{Im}\{-\mathbf{\Gamma}+\mathbf{C}\} \tag{A.45}$$
$$\mathbf{\Lambda_{YX}} = \mathrm{E}[(\mathbf{Y}-\boldsymbol{\mu}_Y)(\mathbf{X}-\boldsymbol{\mu}_X)^T] = \mathrm{Im}\{\mathbf{\Gamma}+\mathbf{C}\} \tag{A.46}$$
$$\mathbf{\Lambda_{YY}} = \mathrm{E}[(\mathbf{Y}-\boldsymbol{\mu}_Y)(\mathbf{Y}-\boldsymbol{\mu}_Y)^T] = \mathrm{Re}\{\mathbf{\Gamma}-\mathbf{C}\} \tag{A.47}$$

and, conversely,

$$\begin{aligned}\mathbf{\Gamma} &= \frac{1}{2}\Big(\mathbf{\Lambda_{XX}}+\mathbf{\Lambda_{YY}}+j(\mathbf{\Lambda_{YX}}-\mathbf{\Lambda_{XY}})\Big)\\ \mathbf{C} &= \frac{1}{2}\Big(\mathbf{\Lambda_{XX}}-\mathbf{\Lambda_{YY}}+j(\mathbf{\Lambda_{YX}}+\mathbf{\Lambda_{XY}})\Big).\end{aligned} \tag{A.48}$$

The complex random vector $\mathbf{Z}$ has the complex multivariate Gaussian distribution

$$p_{\mathbf{Z}}(\mathbf{z}) = \frac{1}{(2\pi)^n\sqrt{\det(\mathbf{\Gamma})\det(\mathbf{P})}}\exp\left\{-\frac{1}{4}\left((\mathbf{z}-\boldsymbol{\mu}_Z)^H,(\mathbf{z}-\boldsymbol{\mu}_Z)^T\right)\begin{pmatrix}\mathbf{\Gamma} & \mathbf{C}\\ \mathbf{C}^H & \mathbf{\Gamma}^*\end{pmatrix}^{-1}\begin{pmatrix}(\mathbf{z}-\boldsymbol{\mu}_Z)\\(\mathbf{z}^*-\boldsymbol{\mu}_Z^*)\end{pmatrix}\right\}, \tag{A.49}$$

where

$$\mathbf{P} = \mathbf{\Gamma}^* - \mathbf{C}^H\mathbf{\Gamma}^{-1}\mathbf{C}. \tag{A.50}$$

For a circular-symmetric complex Gaussian distribution $\mathbf{C} = \mathbf{0}$, and the complex multivariate Gaussian distribution simplifies considerably as

$$p_{\mathbf{Z}}(\mathbf{z}) = \frac{1}{(2\pi)^n\det(\mathbf{\Gamma})}\exp\left\{-\frac{1}{2}(\mathbf{z}-\boldsymbol{\mu}_Z)^H\mathbf{\Gamma}^{-1}(\mathbf{z}-\boldsymbol{\mu}_Z)\right\}. \tag{A.51}$$

The circular-symmetric scalar complex Gaussian random variable $Z = X + jY$ has the density

$$p_Z(z) = \frac{1}{2\pi\sigma_Z^2}\exp\left\{-\frac{|z-\mu_Z|^2}{2\sigma_Z^2}\right\}, \tag{A.52}$$

where $\mu_Z = \mathrm{E}[Z]$ and $\sigma_Z^2 = \frac{1}{2}\mathrm{E}[|z-\mu_Z|^2]$. Sometimes this is denoted with the shorthand notation $Z_i \sim \mathscr{C}\mathscr{N}(\mu_Z,\sigma_Z^2)$. The standard complex Gaussian distribution $Z_i \sim \mathscr{C}\mathscr{N}(0,1)$ has the density

$$p_Z(z) = \frac{1}{2\pi}\exp\left\{-\frac{|z|^2}{2}\right\}. \tag{A.53}$$

A.3.2.4 Rayleigh Distribution

Let $X \sim \mathscr{N}(0,\sigma^2)$ and $Y \sim \mathscr{N}(0,\sigma^2)$ be independent real-valued normal random variables. The random variable $R = \sqrt{X^2+Y^2}$ is said to be Rayleigh distributed. To find the pdf and cdf of R first define the auxiliary variable

$$V = \mathrm{Tan}^{-1}(Y/X).$$

Then

$$\begin{aligned}X &= R\cos V\\ Y &= R\sin V.\end{aligned}$$

By using a bivariate transformation of random variables

$$p_{RV}(r,v) = p_{XY}(r\cos v, r\sin v)\,|J(r,v)|$$

where

$$J(r,v) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \cos v & r\sin v \\ \sin v & r\cos v \end{vmatrix} = r(\cos^2 v + \sin^2 v) = r$$

Since

$$p_{XY}(x,y) = \frac{1}{2\pi\sigma^2}\exp\left\{-\frac{x^2+y^2}{2\sigma^2}\right\}$$

it follows that

$$p_{RV}(r,v) = \frac{r}{2\pi\sigma^2}\exp\left\{-\frac{r^2}{2\sigma^2}\right\}. \tag{A.54}$$

The marginal pdf of R has the Rayleigh distribution

$$\begin{aligned} p_R(r) &= \int_0^{2\pi} p_{RV}(r,v)\mathrm{d}v \\ &= \frac{r}{\sigma^2}\exp\left\{-\frac{r^2}{2\sigma^2}\right\}\;,\quad r \ge 0. \end{aligned} \tag{A.55}$$

The cdf of R is

$$F_R(r) = 1 - \exp\left\{-\frac{r^2}{2\sigma^2}\right\}\;,\quad r \ge 0. \tag{A.56}$$

The marginal pdf of V is

$$\begin{aligned} p_V(v) &= \int_0^{\infty} p_{RV}(r,v)\mathrm{d}r \\ &= \frac{1}{2\pi}\;,\quad \pi \le v \le \pi. \end{aligned} \tag{A.57}$$

which is a uniform distribution on the interval $[-\pi,\pi)$.

A.3.2.5 Rice Distribution

Let $X \sim \mathcal{N}(\mu_1,\sigma^2)$ and $Y \sim \mathcal{N}(\mu_2,\sigma^2)$ be independent normal random variables with non-zero means. The random variable $R = \sqrt{X^2+Y^2}$ has a Rice distribution or is said to be Ricean distributed. To find the pdf and cdf of R again define the auxiliary variable $V = \mathrm{Tan}^{-1}(Y/X)$. Then by using a bivariate transformation $J(r,v) = r$ and

$$p_{RV}(r,v) = r \cdot p_{XY}(r\cos v, r\sin v). \tag{A.58}$$

However,

$$\begin{aligned} p_{XY}(x,y) &= \frac{1}{2\pi\sigma^2}\exp\left\{-\frac{(x-\mu_1)^2+(y-\mu_2)^2}{2\sigma^2}\right\} \\ &= \frac{1}{2\pi\sigma^2}\exp\left\{-\frac{x^2+y^2+\mu_1^2+\mu_2^2-2(x\mu_1+y\mu_2)}{2\sigma^2}\right\}. \end{aligned}$$

Hence,

$$p_{RV}(r,v) = \frac{r}{2\pi\sigma^2}\exp\left\{-\frac{r^2+\mu_1^2+\mu_2^2-2r(\mu_1\cos v+\mu_2\sin v)}{2\sigma^2}\right\}.$$

Now define $s \triangleq \sqrt{\mu_1^2+\mu_2^2}$ and $t \triangleq \mathrm{Tan}^{-1}\mu_2/\mu_1,\ -\pi \le t \le \pi$, so that $\mu_1 = s\cos t$ and $\mu_2 = s\sin t$. Then

$$\begin{aligned} p_{RV}(r,v) &= \frac{r}{2\pi\sigma^2}\exp\left\{-\frac{r^2+s^2-2rs(\cos t\cos v+\sin t\sin v)}{2\sigma^2}\right\} \\ &= \frac{r}{2\pi\sigma^2}\exp\left\{-\frac{r^2+s^2-2rs\cos(v-t)}{2\sigma^2}\right\}. \end{aligned}$$

The marginal pdf of R is

$$P_R(r) = \frac{r}{\sigma^2}\exp\left\{-\frac{r^2+s^2}{2\sigma^2}\right\}\frac{1}{2\pi}\int_0^{2\pi}\exp\left\{\frac{rs}{\sigma^2}\cos(v-t)\right\}\mathrm{d}v. \tag{A.59}$$

The zero order modified Bessel function of the first kind is defined as

$$I_0(x) \triangleq \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{x\cos\theta}\mathrm{d}\theta. \tag{A.60}$$

This gives the Rice distribution

$$P_R(r) = \frac{r}{\sigma^2}\exp\left\{-\frac{r^2+s^2}{2\sigma^2}\right\} I_0\left(\frac{rs}{\sigma^2}\right),\quad r \ge 0. \tag{A.61}$$

The cdf of R is

$$\begin{aligned} F_R(r) &= \int_0^r p_R(r)\mathrm{d}r \\ &= 1 - Q\left(\frac{s}{\sigma},\frac{r}{\sigma}\right), \end{aligned}$$

where $Q(a,b)$ is called the Marcum Q-function.

A.3.2.6 Central Chi-Square Distribution

Let $X \sim \mathcal{N}(0,\sigma^2)$ and $Y = X^2$. Then it can be shown that

$$\begin{aligned} p_Y(y) &= \frac{p_X(\sqrt{y})+p_X(-\sqrt{y})}{2\sqrt{y}} \\ &= \frac{1}{\sqrt{2\pi y}\sigma}\exp\left\{-\frac{y}{2\sigma^2}\right\},\quad y \ge 0. \end{aligned}$$

The characteristic function of Y is

$$\begin{aligned} \psi_Y(jv) &= \int_{-\infty}^{\infty} \mathrm{e}^{jvy}p_Y(y)\mathrm{d}y \\ &= \frac{1}{\sqrt{1-j2v\sigma^2}}. \end{aligned} \tag{A.62}$$

Now define the random variable $Y = \sum_{i=1}^{n} X_i^2$, where the X_i are independent and $X_i \sim \mathcal{N}(0, \sigma^2)$. Then

$$\psi_Y(jv) = \frac{1}{(1 - j2v\sigma^2)^{n/2}}. \tag{A.63}$$

Taking the inverse transform gives

$$\begin{aligned} p_Y(y) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi_Y(jv) \mathrm{e}^{-jvy} \mathrm{d}v \\ &= \frac{1}{(2\sigma^2)^{n/2} \Gamma(n/2)} y^{n/2-1} \exp\left\{-\frac{y}{2\sigma^2}\right\}, \quad y \geq 0. \end{aligned}$$

where $\Gamma(k)$ is the Gamma function and

$$\Gamma(k) = \int_0^{\infty} u^{k-1} \mathrm{e}^{-u} \mathrm{d}u = (k-1)!$$

if k is a positive integer. If n is even (which is usually the case in practice) and $m = n/2$, then the pdf of Y defines the central chi-square distribution with $2m$ degrees of freedom

$$p_Y(y) = \frac{1}{(2\sigma^2)^m (m-1)!} y^{m-1} \exp\left\{-\frac{y}{2\sigma^2}\right\}, \quad y \geq 0. \tag{A.64}$$

The cdf of Y is

$$F_Y(y) = 1 - \exp\left\{-\frac{y}{2\sigma^2}\right\} \sum_{k=0}^{m-1} \frac{1}{k!} \left(\frac{y}{2\sigma^2}\right)^k, \quad y \geq 0. \tag{A.65}$$

The exponential distribution is a special case of the central chi-square distribution with $m = 1$ (2 degrees of freedom). In this case

$$\begin{aligned} p_Y(y) &= \frac{1}{2\sigma^2} \exp\left\{-\frac{y}{2\sigma^2}\right\}, \quad y \geq 0 \\ F_Y(y) &= 1 - \exp\left\{-\frac{y}{2\sigma^2}\right\}, \quad y \geq 0. \end{aligned} \tag{A.66}$$

A.3.2.7 Non-central Chi-Square Distribution

Let $X \sim \mathcal{N}(\mu, \sigma^2)$ and $Y = X^2$. Then

$$\begin{aligned} p_Y(y) &= \frac{p_X(\sqrt{y}) + p_X(-\sqrt{y})}{2\sqrt{y}} \\ &= \frac{1}{\sqrt{2\pi y}\sigma} \exp\left\{-\frac{(y + \mu^2)}{2\sigma^2}\right\} \cosh\left(\frac{\sqrt{y}\mu}{\sigma^2}\right), \quad y \geq 0. \end{aligned}$$

The characteristic function of Y is

$$\begin{aligned} \psi_Y(jv) &= \int_{-\infty}^{\infty} \mathrm{e}^{jvy} p_Y(y) \mathrm{d}y \\ &= \frac{1}{\sqrt{1 - j2v\sigma^2}} \exp\left\{\frac{jv\mu^2}{1 - j2v\sigma^2}\right\}. \end{aligned}$$

Now define the random variable $Y = \sum_{i=1}^{n} X_i^2$, where the X_i are independent normal random variables and $X_i \sim \mathcal{N}(\mu_i, \sigma^2)$. Then

$$\psi_Y(jv) = \frac{1}{(1-j2v\sigma^2)^{n/2}} \exp\left\{ \frac{jv \sum_{i=1}^{n} \mu_i^2}{1-j2v\sigma^2} \right\} .$$

Taking the inverse transform gives

$$p_Y(y) = \frac{1}{2\sigma^2} \left(\frac{y}{s^2} \right)^{\frac{n-2}{4}} \exp\left\{ -\frac{(s^2+y)}{2\sigma^2} \right\} I_{n/2-1}\left(\sqrt{y}\frac{s}{\sigma^2} \right) , \quad y \geq 0,$$

where

$$s^2 = \sum_{i=1}^{n} \mu_i^2$$

and $I_k(x)$ is the modified Bessel function of the first kind and order k, defined by

$$I_k(x) \triangleq \frac{1}{2\pi} \int_0^{2\pi} \mathrm{e}^{x\cos\theta} \cos(k\theta)\mathrm{d}\theta.$$

If n is even (which is usually the case in practice) and $m = n/2$, then the pdf of Y defines the non-central chi-square distribution with $2m$ degrees of freedom

$$p_Y(y) = \frac{1}{2\sigma^2} \left(\frac{y}{s^2} \right)^{\frac{m-1}{2}} \exp\left\{ -\frac{(s^2+y)}{2\sigma^2} \right\} I_{m-1}\left(\sqrt{y}\frac{s}{\sigma^2} \right) , \quad y \geq 0 \tag{A.67}$$

and the cdf of Y is

$$F_Y(y) = 1 - Q_m\left(\frac{s}{\sigma}, \frac{\sqrt{y}}{\sigma} \right) , \quad y \geq 0, \tag{A.68}$$

where $Q_m(a, b)$ is called the generalized Q-function.

A.4 Upper Bounds on the cdfc

Several different approaches can be used to upper bound the tail area under a probability density function including the Chebyshev and Chernoff bounds.

A.4.1 Chebyshev Bound

The Chebyshev bound is derived as follows. Let X be a random variable with mean μ_X, variance σ_X^2, and pdf $p_X(x)$. Then the variance of X is

$$\begin{aligned}
\sigma_X^2 &= \int_{-\infty}^{\infty} (x-\mu_X)^2 p_X(x)\mathrm{d}x \\
&\geq \int_{|x-\mu_X|\geq\delta} (x-\mu_X)^2 p_X(x)\mathrm{d}x \\
&\geq \delta^2 \int_{|x-\mu_X|\geq\delta} p_X(x)\mathrm{d}x \\
&= \delta^2 \mathrm{P}[|X-\mu_X| \geq \delta].
\end{aligned}$$

Hence,

$$P[|X - \mu_X| \geq \delta] \leq \frac{\sigma_X^2}{\delta^2}. \tag{A.69}$$

The Chebyshev bound is straightforward to apply but it tends to be quite loose.

A.4.2 Chernoff Bound

The Chernoff bound is more difficult to compute but is much tighter than the Chebyshev bound. To derive the Chernoff bound the following inequality is used

$$u(x) \leq \mathrm{e}^{\lambda x}, \qquad \forall\, x \text{ and } \forall\, \lambda \geq 0,$$

where $u(x)$ is the unit step function. Then,

$$\begin{aligned} P[X \geq 0] &= \int_0^{\infty} p_X(x)\mathrm{d}x \\ &= \int_{-\infty}^{\infty} u(x)p_X(x)\mathrm{d}x \\ &\leq \int_{-\infty}^{\infty} \mathrm{e}^{\lambda x}p_X(x)\mathrm{d}x \\ &= \mathrm{E}[\mathrm{e}^{\lambda x}]. \end{aligned}$$

The Chernoff bound parameter, $\lambda, \lambda > 0$, can be optimized to give the tightest upper bound. This can be accomplished by setting the derivative to zero

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\mathrm{E}[\mathrm{e}^{\lambda x}] = \mathrm{E}\left[\frac{\mathrm{d}}{\mathrm{d}\lambda}\mathrm{e}^{\lambda x}\right] = \mathrm{E}[x\mathrm{e}^{\lambda x}] = 0.$$

Let $\lambda^* = \arg\min_{\lambda \geq 0} \mathrm{E}[\mathrm{e}^{\lambda x}]$ be the solution to the above equation. Then

$$P[X \geq 0] \leq \mathrm{E}[\mathrm{e}^{\lambda^* x}]. \tag{A.70}$$

Example A.1: Let $X_i,\ i = 1, \ldots, n$ be independent and identically distributed random variables with density

$$p_X(x) = p\delta(x-1) + (1-p)\delta(x+1).$$

Let

$$Y = \sum_{i=1}^{n} X_i.$$

Consider the quantity $P[Y \geq 0]$. To compute this probability exactly,

$$\begin{aligned} P[Y \geq 0] &= P\,[\lceil n/2 \rceil \text{ or more of the } X_i \text{ are ones }] \\ &= \sum_{k=\lceil n/2 \rceil}^{n} \binom{n}{k} p^k (1-p)^{n-k}. \end{aligned}$$

(continued)

Example A.1 (continued)
For $n = 10$ and $p = 0.1$

$$P[Y \geq 0] = 0.0016349. \tag{A.71}$$

Chebyshev Bound

To compute the Chebyshev bound, first determine the mean and variance of Y.

$$\begin{aligned}\mu_Y &= n\mathrm{E}[X_i] \\ &= n[p - 1 + p] \\ &= n(2p - 1).\end{aligned}$$

$$\begin{aligned}\sigma_Y^2 &= n\sigma_X^2 \\ &= n\left(\mathrm{E}[X_i^2] - \mathrm{E}^2[X_i]\right) \\ &= n\left(1 - (2p-1)^2\right) \\ &= n\left(1 - 4p^2 + 4p - 1\right) \\ &= 4np(1-p).\end{aligned}$$

Hence,

$$P[|Y - \mu_Y| \geq \mu_Y] \leq \frac{\sigma_Y^2}{\mu_Y^2} = \frac{4np(1-p)}{n^2(2p-1)^2}.$$

Then by symmetry

$$\begin{aligned}P[Y \geq 0] &= \frac{1}{2}P[|Y - \mu_Y| \geq \mu_Y] \\ &\leq \frac{2p(1-p)}{n(2p-1)^2}.\end{aligned}$$

For $n = 10$ and $p = 0.1$

$$P[Y \geq 0] \leq 0.028125. \tag{A.72}$$

Chernoff Bound

The Chernoff bound is given by

$$\begin{aligned}P[Y \geq 0] &\leq \mathrm{E}[\mathrm{e}^{\lambda y}] \\ &= \left(\mathrm{E}[\mathrm{e}^{\lambda x}]\right)^n.\end{aligned}$$

However,

$$\mathrm{E}[\mathrm{e}^{\lambda x}] = p\mathrm{e}^{\lambda} + (1-p)\mathrm{e}^{-\lambda}.$$

(continued)

Example A.1 (continued)

To find the optimal Chernoff bound parameter, solve

$$\frac{d}{d\lambda}\mathrm{E}[\mathrm{e}^{\lambda x}] = p\mathrm{e}^{\lambda} - (1-p)\mathrm{e}^{-\lambda} = 0$$

giving

$$\lambda^* = \ln\left(\sqrt{\frac{1-p}{p}}\right).$$

Hence,

$$\begin{aligned}\mathrm{P}[Y \geq 0] &\leq \left(\mathrm{E}[\mathrm{e}^{\lambda^* x}]\right)^n \\ &= (4p(1-p))^{n/2}.\end{aligned}$$

For $n = 10$ and $p = 0.1$

$$\mathrm{P}[Y \geq 0)] \;\leq\; 0.0060466.$$

Notice that the Chernoff bound is much tighter that the Chebyshev bound in this case.

A.5 Random Processes

A random process, or stochastic process, $X(t)$, is an ensemble of sample functions $\{X_1(t), X_2(t), \ldots, X_\xi(t)\}$ together with a probability rule which assigns a probability to any meaningful event associated with the observation of these sample functions. Consider the set of sample functions shown in Fig. A.1. The sample function x_i corresponds to the sample point s_1 in the sample space and occurs with probability $\mathrm{P}[s_1]$. The number of sample functions, ξ, in the ensemble may be finite or infinite. The function $X_i(t)$ is deterministic once the index i is known. Sample functions may be defined at discrete or

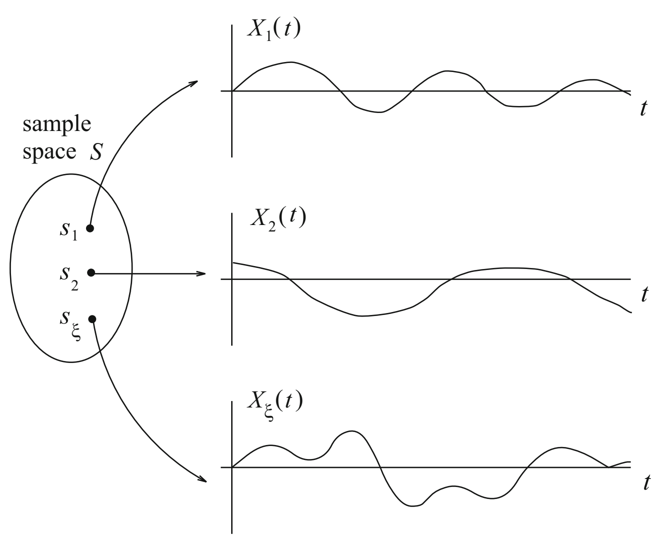

Fig. A.1 Ensemble of sample functions for a random process continuous instants in time, which define discrete- or continuous-time random processes, respectively. Furthermore, their values (or parameters) at these time instants may be either discrete or continuous valued as well, which defines a discrete- or continuous-parameter random process, respectively. Hence, random processes may be discrete-time discrete-parameter, discrete-time continuous-parameter, continuous-time discrete-parameter, or continuous-time continuous-parameter.

Suppose that all sample functions of a random process are observed at some time instant t_1. Their values form the set of numbers $\{X_i(t_1)\}, i = 1, 2, \ldots, \xi$. Since $X_i(t_1)$ occurs with probability $\mathrm{P}[s_i]$, the collection of numbers $\{X_i(t_1)\}, i = 1, 2, \ldots, \xi$, forms a random variable, denoted by $X(t_1)$. By observing the set of waveforms at another time instant t_2 a different random variable $X(t_2)$ is obtained. The collection of n such random variables, $X(t_1),\ X(t_2), \ldots,\ X(t_n)$, has the joint cdf

$$F_{X(t_1),\ldots,\ X(t_n)}(x_1, \ldots,\ x_n) = \mathrm{P}[X(t_1) < x_1, \ldots, X(t_n) < x_n].$$

A more compact notation can be obtained by defining the vectors

$$\mathbf{x} \triangleq (x_1, x_2, \ldots, x_n)^T$$
$$\mathbf{X}(t) \triangleq (X(t_1), X(t_2), \ldots, X(t_n))^T.$$

Then the joint cdf and joint pdf of the random vector $\mathbf{X}(t)$ are, respectively,

$$F_{\mathbf{X}(t)}(\mathbf{x}) = \mathrm{P}(\mathbf{X}(t) \leq \mathbf{x}) \tag{A.73}$$

$$p_{\mathbf{X}(t)}(\mathbf{x}) = \frac{\partial^n F_{\mathbf{X}(t)}(\mathbf{x})}{\partial x_1 \partial x_2 \cdots \partial x_n}. \tag{A.74}$$

A random process is strictly stationary if and only if the joint density function $p_{\mathbf{X}(t)}(\mathbf{x})$ is invariant under shifts of the time origin. In this case, the equality

$$p_{\mathbf{X}(t)}(\mathbf{x}) = p_{\mathbf{X}(t+\tau)}(\mathbf{x}) \tag{A.75}$$

holds for all sets of time instants $\{t_1, t_2, \ldots, t_n\}$ and all time shifts τ. Some important random processes that are encountered in practice are strictly stationary, while many are not.

A.5.1 Moments and Correlation Functions

To describe the moments and correlation functions of a random process, it is useful to define the following two operators:

$$\mathrm{E}[\,\cdot\,] \triangleq \text{ensemble average}$$
$$\langle\,\cdot\,\rangle \triangleq \text{time average.}$$

The ensemble average of a random process at time t is

$$\mu_X(t) = \mathrm{E}[X(t)] = \int_{-\infty}^{\infty} x p_{X(t)}(x)\mathrm{d}x. \tag{A.76}$$

Note that the ensemble average is generally a function of time. However, if the ensemble average changes with time, then the process is not strictly stationary. The time average of a random process is

$$\langle X(t)\rangle = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} X(t)\mathrm{d}t. \tag{A.77}$$

In general, the time average $\langle X(t)\rangle$ is also a random variable, because it depends on the particular sample function that is selected for time averaging.

The autocorrelation of a random process $X(t)$ is defined as

$$\phi_{XX}(t_1,t_2) = \mathrm{E}\left[X(t_1)X(t_2)\right]. \tag{A.78}$$

The autocovariance of a random process $X(t)$ is defined as

$$\begin{aligned}\lambda_{XX}(t_1,t_2) &= \mathrm{E}\left[(X(t_1)-\mu_X(t_1))\,(X(t_2)-\mu_X(t_2))\right]\\ &= \phi_{XX}(t_1,t_2)-\mu_X(t_1)\mu_X(t_2).\end{aligned} \tag{A.79}$$

A random process that is strictly stationary must have

$$\mathrm{E}[X^n(t)] = \mathrm{E}[X^n] \quad \forall\ t, n.$$

Hence, strictly stationary random process must have

$$\begin{aligned}\mu_X(t) &= \mu\\ \sigma_X^2(t) &= \sigma_X^2\\ \phi_{XX}(t_1,t_2) &= \phi_{XX}(t_2-t_1) \equiv \phi_{XX}(\tau)\\ \lambda_{XX}(t_1,t_2) &= \lambda_{XX}(t_2-t_1) \equiv \lambda_{XX}(\tau),\end{aligned}$$

where $\tau = t_2 - t_1$.

If a random process satisfies the following two conditions

$$\begin{aligned}\mu_X(t) &= \mu_X\\ \phi_{XX}(t_1,t_2) &= \phi_{XX}(\tau)\,, \quad \tau = t_2 - t_1\,,\end{aligned}$$

then it is said to be wide-sense stationary. Note that if a random process is strictly stationary, then it is wide-sense stationary; however, the converse may not be true. A notable exception is the Gaussian random process which is strictly stationary if and only if it is wide-sense stationary. The reason is that a joint Gaussian density of the vector $\mathbf{X}(t) = (X(t_1), X(t_2), \ldots, X(t_n))$ is completely described by the means and covariances of the $X(t_i)$.

A.5.1.1 Properties of $\phi_{XX}(\tau)$

The autocorrelation function, $\phi_{XX}(\tau)$, of a stationary random process satisfies the following properties.

- $\phi_{XX}(0) = \mathrm{E}[X^2(t)]$. This is the total power in the random process.
- $\phi_{XX}(\tau) = \phi_{XX}(-\tau)$. The autocorrelation function must be an even function.
- $|\phi_{XX}(\tau)| \le \phi_{XX}(0)$. This is a variant of the Cauchy-Schwartz inequality.
- $\phi_{XX}(\infty) = \mathrm{E}^2[X(t)] = \mu_X^2$. This holds if $X(t)$ contains no periodic components and is equal to the d.c. power.

Example A.2: This example shows that $|\phi_{XX}(\tau)| \le \phi_{XX}(0)$. This inequality can be established through the following steps.

$$\begin{aligned}0 &\le \mathrm{E}[X(t) \pm (X(t+\tau))^2]\\ &= \mathrm{E}[X^2(t) + X^2(t+\tau) \pm 2X(t)X(t+\tau)]\\ &= \mathrm{E}[X^2(t)] + \mathrm{E}[X^2(t+\tau)] \pm 2\mathrm{E}[X(t)X(t+\tau)]\\ &= 2\mathrm{E}[X^2(t)] \pm 2\mathrm{E}[X(t)X(t+\tau)]\\ &= 2\phi_{XX}(0) \pm 2\phi_{XX}(\tau).\end{aligned}$$

(continued)

Example A.2 (continued)

Therefore,

$$\pm\phi_{XX}(\tau) \leq \phi_{XX}(0)$$
$$|\phi_{XX}(\tau)| \leq \phi_{XX}(0).$$

A.5.1.2 Ergodic Random Processes

A random process is ergodic if for all $g(\mathbf{X})$ and $\mathbf{X}$

$$\begin{aligned} \mathrm{E}[g(\mathbf{X})] &= \int_{-\infty}^{\infty} g(\mathbf{X}) p_{\mathbf{X}(t)}(\mathbf{x}) \mathrm{d}\mathbf{x} \\ &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} g[\mathbf{X}(t)] \mathrm{d}t \\ &= \langle g[\mathbf{X}(t)] \rangle. \end{aligned} \tag{A.80}$$

For a random process to be ergodic, it must be strictly stationary. However, not all strictly stationary random processes are ergodic. A random process is ergodic in the mean if $\langle X(t) \rangle = \mu_X$ and ergodic in the autocorrelation if $\langle X(t)X(t+\tau) \rangle = \phi_{XX}(\tau)$.

Example A.3: Consider the random process

$$X(t) = A\cos(2\pi f_c t + \Theta)$$

where A and f_c are constants, and

$$p_\Theta(\theta) = \begin{cases} 1/(2\pi)\,, & 0 \leq \theta \leq 2\pi \\ 0\,, & \text{elsewhere} \end{cases} .$$

The mean of $X(t)$ is

$$\mu_X(t) = \mathrm{E}_\Theta[A\cos(2\pi f_c t + \theta)] = 0 = \mu_X$$

and autocorrelation of $X(t)$ is

$$\begin{aligned} \phi_{XX}(t_1, t_2) &= \mathrm{E}_\Theta[X(t_1)X(t_2)] \\ &= \mathrm{E}_\Theta[A^2\cos(2\pi f_c t_1 + \theta)\cos(2\pi f_c t_2 + \theta)] \\ &= \frac{A^2}{2}\mathrm{E}_\Theta[\cos(2\pi f_c t_1 + 2\pi f_c t_2 + 2\theta)] + \frac{A^2}{2}\mathrm{E}_\Theta[\cos(2\pi f_c(t_2 - t_1))] \\ &= \frac{A^2}{2}\cos(2\pi f_c(t_2 - t_1)) \\ &= \frac{A^2}{2}\cos(2\pi f_c\tau), \quad \tau = t_2 - t_1. \end{aligned}$$

It is clear that this random process is wide-sense stationary.

(continued)

Example A.3 (continued)

The time average mean of $X(t)$ is

$$\langle X(t)\rangle = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} A\cos(2\pi f_c t+\theta)\mathrm{d}t = 0$$

and the time average autocorrelation of $X(t)$ is

$$\begin{aligned}
&\langle X(t+\tau)X(t)\rangle \\
&= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} A^2\cos(2\pi f_c t+\theta)\cos(2\pi f_c t+2\pi f_c\tau+\theta)\mathrm{d}t \\
&= \lim_{T\to\infty}\frac{A^2}{4T}\int_{-T}^{T} A^2\left[\cos(2\pi f_c\tau)+\cos(4\pi f_c t+2\pi f_c\tau+2\theta)\right]\mathrm{d}t \\
&= \frac{A^2}{2}\cos(2\pi f_c\tau).
\end{aligned}$$

By comparing the ensemble and time average mean and autocorrelation, it can be concluded that this random process is ergodic in the mean and ergodic in the autocorrelation.

Example A.4: Consider the random process

$$Y(t) = X\cos t, \quad X\sim\mathcal{N}(0,1).$$

Find the probability density function of $Y(0)$, the joint probability density function of $Y(0)$ and $Y(\pi)$, and determine whether or not $Y(t)$ is strictly stationary.

1. To find the probability density function of $Y(0)$,

$$Y(0) = X\cos 0 = X.$$

Therefore,

$$p_{Y(0)}(y_0) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-y_0^2/2}.$$

2. To find the joint density of $Y(0)$ and $Y(\pi)$,

$$Y(0) = X = -Y(\pi).$$

Therefore,

$$p_{Y(0)|Y(\pi)}(y_0|y_\pi) = \delta(y_0+y_\pi)$$

and

$$\begin{aligned}
p_{Y(0)Y(\pi)}(y_0,y_\pi) &= p_{Y(0)|Y(\pi)}(y_0|y_\pi)p_{Y(\pi)}(y_\pi) \\
&= \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-y_\pi^2/2}\delta(y_0+y_\pi).
\end{aligned}$$

(continued)

Example A.4 (continued)

3. To determine whether or not $Y(t)$ is strictly stationary, note that

$$\mathrm{E}[Y(t)] = \mathrm{E}[X]\cos t = 0$$
$$\mathrm{E}[Y^2(t)] = \mathrm{E}[X^2]\cos^2 t.$$

Since the second moment and, hence, the pdf of this random process varies with time, the random process is not strictly stationary.

A.5.2 Cross-correlation and Crosscovariance

Consider two random processes $X(t)$ and $Y(t)$. The cross-correlation of $X(t)$ and $Y(t)$ is

$$\phi_{XY}(t_1,t_2) = \mathrm{E}[X(t_1)Y(t_2)] \tag{A.81}$$
$$\phi_{YX}(t_1,t_2) = \mathrm{E}[Y(t_1)X(t_2)]. \tag{A.82}$$

The correlation matrix of $X(t)$ and $Y(t)$ is

$$\mathbf{\Phi}(t_1,t_2) = \begin{bmatrix} \phi_{XX}(t_1,t_2) & \phi_{XY}(t_1,t_2) \\ \phi_{YX}(t_1,t_2) & \phi_{YY}(t_1,t_2) \end{bmatrix}. \tag{A.83}$$

The crosscovariance of $X(t)$ and $Y(t)$ is

$$\begin{aligned}\lambda_{XY}(t_1,t_2) &= \mathrm{E}\left[(X(t_1)-\mu_X(t_1))\,(X(t_2)-\mu_X(t_2))\right] \\ &= \phi_{XY}(t_1,t_2) - \mu_X(t_1)\mu_X(t_2).\end{aligned} \tag{A.84}$$

The covariance matrix of $X(t)$ and $Y(t)$ is

$$\mathbf{\Lambda}(t_1,t_2) = \begin{bmatrix} \lambda_{XX}(t_1,t_2) & \lambda_{XY}(t_1,t_2) \\ \lambda_{YX}(t_1,t_2) & \lambda_{YY}(t_1,t_2) \end{bmatrix}. \tag{A.85}$$

If $X(t)$ and $Y(t)$ are each wide-sense stationary and jointly wide-sense stationary, then

$$\mathbf{\Phi}(t_1,t_2) = \mathbf{\Phi}(t_2-t_1) = \mathbf{\Phi}(\tau) \tag{A.86}$$
$$\mathbf{\Lambda}(t_1,t_2) = \mathbf{\Lambda}(t_2-t_1) = \mathbf{\Lambda}(\tau), \tag{A.87}$$

where $\tau = t_2 - t_1$.

A.5.2.1 Properties of $\phi_{XY}(\tau)$

Consider two random processes $X(t)$ and $Y(t)$ are each wide-sense stationary and jointly wide-sense stationary. The cross-correlation function $\phi_{XY}(\tau)$ has the following properties.

- $\phi_{XY}(\tau) = \phi_{YX}(-\tau)$
- $|\phi_{XY}(\tau)| \leq \frac{1}{2}[\phi_{XX}(0) + \phi_{YY}(0)]$
- $|\phi_{XY}(\tau)|^2 \leq \phi_{XX}(0)\phi_{YY}(0)$ if $X(t)$ and $Y(t)$ have zero mean.

A.5.2.2 Classifications of Random Processes

Two random processes $X(t)$ and $Y(t)$ are said to be

- uncorrelated if and only if $\lambda_{XY}(\tau) = 0$.
- orthogonal if and only if $\phi_{XY}(\tau) = 0$.
- statistically independent if and only if

$$p_{\mathbf{X}(t)\mathbf{Y}(t+\tau)}(\mathbf{x}, \mathbf{y}) = p_{\mathbf{X}(t)}(\mathbf{x}) p_{\mathbf{Y}(t+\tau)}(\mathbf{y}).$$

Furthermore, if $\mu_X = 0$ or $\mu_Y = 0$, then the random processes are also orthogonal if they are uncorrelated. Statistically independent random processes are always uncorrelated, however, not all uncorrelated random processes are statistically independent. In the special case of Gaussian random processes, if the processes are uncorrelated, then they are also statistically independent.

Example A.5: Find the autocorrelation function of the random process

$$Z(t) = X(t) + Y(t)$$

where $X(t)$ and $Y(t)$ are wide-sense stationary random processes.

The autocorrelation function of $Z(t)$ is

$$\begin{aligned} \phi_{ZZ}(\tau) &= \mathrm{E}[Z(t)Z(t+\tau)] \\ &= \mathrm{E}\left[(X(t) + Y(t))\,(X(t+\tau) + Y(t+\tau))\right] \\ &= \phi_{XX}(\tau) + \phi_{YX}(\tau) + \phi_{XY}(\tau) + \phi_{YY}(\tau). \end{aligned}$$

If $X(t)$ and $Y(t)$ are uncorrelated, then

$$\phi_{YX}(\tau) = \phi_{XY}(\tau) = \mu_X \mu_Y$$

and

$$\phi_{ZZ}(\tau) = \phi_{XX}(\tau) + \phi_{YY}(\tau) + 2\mu_X \mu_Y.$$

If $X(t)$ and $Y(t)$ are uncorrelated and at least one has zero-mean, then

$$\phi_{ZZ}(\tau) = \phi_{XX}(\tau) + \phi_{YY}(\tau).$$

Example A.6: Can the following be a correlation matrix for two jointly wide-sense stationary zero-mean random processes?

$$\boldsymbol{\Phi}(\tau) = \begin{bmatrix} \phi_{XX}(\tau) & \phi_{XY}(\tau) \\ \phi_{YX}(\tau) & \phi_{YY}(\tau) \end{bmatrix} = \begin{bmatrix} A^2 \cos(\tau) & 2A^2 \cos(3\tau/2) \\ 2A^2 \cos(3\tau/2) & A^2 \sin(2\tau) \end{bmatrix}.$$

(continued)

Example A.6 (continued)
The answer is no, because the following two conditions are violated:

1. $|\phi_{XY}(\tau)| \leq \frac{1}{2}[\phi_{XX}(0) + \phi_{YY}(0)]$
2. $|\phi_{XY}(\tau)|^2 \leq \phi_{XX}(0)\phi_{YY}(0)$ if $X(t)$ and $Y(t)$ have zero mean.

A.5.3 Complex-Valued Random Processes

A complex-valued random process is given by

$$Z(t) = X(t) \; + \; jY(t)$$

where $X(t)$ and $Y(t)$ are real-valued random processes.

A.5.3.1 Autocorrelation Function

The autocorrelation function of a complex-valued random process is

$$\begin{aligned}\phi_{ZZ}(t_1,t_2) &= \frac{1}{2}\mathrm{E}[Z^*(t_1)Z(t_2)] \\ &= \frac{1}{2}\mathrm{E}\left[(X(t_1) - jY(t_1))\,(X(t_2) + jY(t_2))\right] \\ &= \frac{1}{2}\Big(\phi_{XX}(t_1,t_2) + \phi_{YY}(t_1,t_2) + j(\phi_{XY}(t_1,t_2) - \phi_{YX}(t_1,t_2))\Big). \end{aligned} \tag{A.88}$$

The factor of $1/2$ in included for convenience, when $Z(t)$ is a complex-valued Gaussian random process. If $Z(t)$ is wide-sense stationary, then

$$\phi_{ZZ}(t_1,t_2) = \phi_{ZZ}(t_2 - t_1) = \phi_{ZZ}(\tau)\;, \quad \tau = t_2 - t_1 .$$

A.5.3.2 Cross-correlation Function

Consider two complex-valued random processes

$$\begin{aligned} Z(t) &= X(t) + jY(t) \\ W(t) &= U(t) + jV(t). \end{aligned}$$

The cross-correlation function of $Z(t)$ and $W(t)$ is

$$\begin{aligned}\phi_{ZW}(t_1,t_2) &= \frac{1}{2}\mathrm{E}[Z^*(t_1)W(t_2)] \\ &= \frac{1}{2}\Big(\phi_{XU}(t_1,t_2) + \phi_{YV}(t_1,t_2) + j(\phi_{XV}(t_1,t_2) - \phi_{YU}(t_1,t_2))\Big). \end{aligned} \tag{A.89}$$

If $X(t)$, $Y(t)$, $U(t)$ and $V(t)$ are pairwise wide-sense stationary random processes, then

$$\phi_{ZW}(t_1,t_2) = \phi_{ZW}(t_2 - t_1) = \phi_{ZW}(\tau). \tag{A.90}$$

The cross-correlation of a complex wide-sense stationary random process satisfies the following property

$$\begin{aligned}\phi^*_{ZW}(\tau) &= \frac{1}{2}\mathrm{E}[Z(t)W^*(t+\tau)] \\ &= \frac{1}{2}\mathrm{E}[Z(\hat{t}-\tau)W^*(\hat{t})] \\ &= \frac{1}{2}\mathrm{E}[W^*(\hat{t})Z(\hat{t}-\tau)] \\ &= \phi_{WZ}(-\tau),\end{aligned} \tag{A.91}$$

where the second line uses the change of variable $\hat{t} = t + \tau$. For a complex-valued random process $Z(t)$, it also follows that

$$\phi^*_{ZZ}(\tau) = \phi_{ZZ}(-\tau). \tag{A.92}$$

A.5.4 Power Spectral Density

The power spectral density (psd) of a wide-sense stationary random process $X(t)$ is the Fourier transform of the autocorrelation function, i.e.,

$$S_{XX}(f) = = \int_{-\infty}^{\infty} \phi_{XX}(\tau)\mathrm{e}^{-j2\pi f\tau}\mathrm{d}\tau \tag{A.93}$$

$$\phi_{XX}(\tau) = \int_{-\infty}^{\infty} S_{XX}(f)\mathrm{e}^{j2\pi f\tau}\mathrm{d}f. \tag{A.94}$$

If $X(t)$ is a real-valued wide-sense stationary random process, then its autocorrelation function $\phi_{XX}(\tau)$ is real and even. Therefore, $S_{XX}(-f) = S_{XX}(f)$ meaning that the power spectrum $S_{XX}(f)$ is also real and even. If $Z(t)$ is a complex-valued wide-sense stationary random process, then $\phi_{ZZ}(\tau) = \phi^*_{ZZ}(-\tau)$, and $S^*_{ZZ}(f) = S_{ZZ}(f)$ meaning that the power spectrum $S_{ZZ}(f)$ is real but not necessarily even.

The power, P, in a wide-sense stationary random process $X(t)$ is

$$\begin{aligned}P &= \mathrm{E}[X^2(t)] \\ &= \phi_{XX}(0) \\ &= \int_{-\infty}^{\infty} S_{XX}(f)\mathrm{d}f\end{aligned}$$

a result known as Parseval's theorem.

The cross power spectral density between two random processes $X(t)$ and $Y(t)$ is

$$S_{XY}(f) = \int_{-\infty}^{\infty} \phi_{XY}(\tau)\mathrm{e}^{-j2\pi f\tau}\mathrm{d}\tau. \tag{A.95}$$

If $X(t)$ and $Y(t)$ are both real-valued random processes, then

$$\phi_{XY}(\tau) = \phi_{YX}(-\tau)$$

and

$$S_{XY}(f) = S_{YX}(-f).$$

If $X(t)$ and $Y(t)$ are complex-valued random processes, then

$$\phi^*_{XY}(\tau) = \phi_{YX}(-\tau)$$

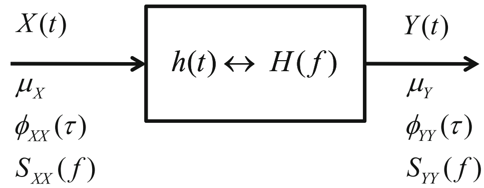

Fig. A.2 Random process through a linear system

and

$$S^*_{XY}(f) = S_{YX}(f).$$

A.5.5 Random Processes Filtered by Linear Systems

Consider the linear system with impulse response $h(t)$, shown in Fig. A.2. Suppose that the input to the linear system is a real-valued wide-sense stationary random process $X(t)$, with mean μ_X and autocorrelation $\phi_{XX}(\tau)$. The input and output are related by the convolution integral

$$Y(t) = \int_{-\infty}^{\infty} h(\tau)X(t-\tau)\mathrm{d}\tau.$$

Hence,

$$Y(f) = H(f)X(f).$$

The output mean is

$$\mu_Y = \int_{-\infty}^{\infty} h(\tau)\mathrm{E}[X(t-\tau)]\mathrm{d}\tau = \mu_X \int_{-\infty}^{\infty} h(\tau)\mathrm{d}\tau = \mu_X H(0),$$

which is equal to the input mean multiplied by the d.c. gain of the filter.

The output autocorrelation function is

$$\begin{aligned}
\phi_{YY}(\tau) &= \mathrm{E}[Y(t)Y(t+\tau)] \\
&= \mathrm{E}\left[\int_{-\infty}^{\infty} h(\beta)X(t-\beta)\mathrm{d}\beta \int_{-\infty}^{\infty} h(\alpha)X(t+\tau-\alpha)\mathrm{d}\alpha\right] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\beta)h(\alpha)\phi_{XX}(\tau-\alpha+\beta)\mathrm{d}\beta\mathrm{d}\alpha \\
&= \int_{-\infty}^{\infty} h(\alpha)\int_{-\infty}^{\infty} h(\beta)\phi_{XX}(\tau+\beta-\alpha)\mathrm{d}\beta\mathrm{d}\alpha \\
&= \left(\int_{-\infty}^{\infty} h(\beta)\phi_{XX}(\tau+\beta)\mathrm{d}\beta\right) * h(\tau) \\
&= h(-\tau) * \phi_{XX}(\tau) * h(\tau).
\end{aligned}$$

Taking the Fourier transform of both sides, the power density spectrum of the output process $Y(t)$ is

$$\begin{aligned}
S_{YY}(f) &= H(f)H^*(f)S_{XX}(f) \\
&= |H(f)|^2\, S_{XX}(f).
\end{aligned}$$

Example A.7: Consider the linear system shown in Fig. A.2. In this example we will find the cross-correlation between the input process $X(t)$ and the output $Y(t)$. The cross-correlation $\phi_{XY}(\tau)$ is given by

$$\begin{aligned}\phi_{XY}(\tau) &= \mathrm{E}[X(t)Y(t+\tau)]\\ &= \mathrm{E}\left[X(t)\int_{-\infty}^{\infty} h(\alpha)X(t+\tau-\alpha)\mathrm{d}\alpha\right]\\ &= \int_{-\infty}^{\infty} h(\alpha)\mathrm{E}[X(t)X(t+\tau-\alpha)]\mathrm{d}\alpha\\ &= \int_{-\infty}^{\infty} h(\alpha)\phi_{XX}(\tau-\alpha)\mathrm{d}\alpha\\ &= h(\tau)*\phi_{XX}(\tau).\end{aligned}$$

Also,

$$S_{XY}(f) = H(f)S_{XX}(f).$$

Example A.8: Suppose that a real-valued Gaussian random process $X(t)$ with mean μ_X and covariance function $\lambda_{XX}(\tau)$ is passed through the linear filter shown in Fig. A.2. In this example, the joint density of the random variables $X_1 = X(t_1)$ and $X_2 = Y(t_2)$ is of interest. If a Gaussian random process is passed through a linear filter, then the output process will also be a Gaussian random process. This is due to the fact that a sum of Gaussian random variables will yield another Gaussian random variable. Hence, X_1 and X_2 have a joint Gaussian density function as defined in (A.43) that is completely described in terms of their means and covariances.
Step 1: Obtain the mean and covariance matrix of X_1 and X_2.
The crosscovariance of X_1 and X_2 is

$$\begin{aligned}\lambda_{X_1X_2}(\tau) &= \mathrm{E}\left[(X(t)-\mu_X)(Y(t+\tau)-\mu_Y)\right]\\ &= \mathrm{E}\left[X(t)Y(t+\tau)\right]-\mu_Y\mu_X.\end{aligned}$$

Now $\mu_Y = H(0)\mu_X$. Also, from the previous example

$$\begin{aligned}\mathrm{E}[X(t)Y(t+\tau)] &= \int_{-\infty}^{\infty} h(\alpha)\phi_{XX}(\tau-\alpha)\mathrm{d}\alpha\\ &= \int_{-\infty}^{\infty} h(\alpha)[\lambda_{XX}(\tau-\alpha)+\mu_X^2]\mathrm{d}\alpha\\ &= \int_{-\infty}^{\infty} h(\alpha)\lambda_{XX}(\tau-\alpha)\mathrm{d}\alpha + H(0)\mu_X^2.\end{aligned}$$

Therefore,

$$\lambda_{X_1X_2}(\tau) = \int_{-\infty}^{\infty} h(\alpha)\lambda_{XX}(\tau-\alpha)\mathrm{d}\alpha = h(\tau)*\lambda_{XX}(\tau).$$

Also

$$\begin{aligned}&\lambda_{X_2X_1}(\tau) = \lambda_{X_1X_2}(-\tau) = \lambda_{X_1X_2}(\tau)\\ &\lambda_{X_1X_1}(\tau) = \lambda_{XX}(\tau)\end{aligned}$$

(continued)

Example A.8 (continued)

$$\lambda_{X_2X_2}(\tau) = h(\tau) * h(-\tau) * \lambda_{XX}(\tau),$$

where the first line follows from the even property of the autocovariance function. Hence, the covariance matrix is

$$\boldsymbol{\Lambda} = \begin{bmatrix} \lambda_{X_1X_1}(0) & \lambda_{X_1X_2}(\tau) \\ \lambda_{X_2X_1}(\tau) & \lambda_{X_2X_2}(0) \end{bmatrix} = \begin{bmatrix} \lambda_{XX}(0) & h(\tau) * \lambda_{XX}(\tau) \\ h(\tau) * \lambda_{XX}(\tau) & h(\tau) * h(-\tau) * \lambda_{XX}(\tau)\,|_{\tau=0} \end{bmatrix}$$

Step 2: Write the joint density function of X_1 and X_2.
Let

$$\mathbf{X} = (X_1, X_2)^T$$
$$\mathbf{x} = (x_1, x_2)^T$$
$$\boldsymbol{\mu}_X = (\mu_X, \mu_Y)^T = (\mu_X, H(0)\mu_X)^T.$$

Then

$$P_{\mathbf{X}}(\mathbf{x}) = \frac{1}{2\pi|\boldsymbol{\Lambda}|^{1/2}} \exp\left\{-\frac{1}{2}(\mathbf{z} - \boldsymbol{\mu}_X)^T \boldsymbol{\Lambda}^{-1}(\mathbf{z} - \boldsymbol{\mu}_X)\right\}.$$

A.5.6 Cyclostationary Random Processes

Consider the random process

$$X(t) = \sum_{n=-\infty}^{\infty} a_n \psi(t - nT),$$

where $\{a_n\}$ is a sequence of complex random variables with mean μ_a and autocorrelation $\phi_{aa}(n) = \frac{1}{2}\mathrm{E}[a_k^* a_{k+n}]$, and $\psi(t)$ is a real-valued pulse having finite energy. Note that the mean of $X(t)$

$$\mu_X(t) = \mu_a \sum_{n=-\infty}^{\infty} \psi(t - nT)$$

is periodic in t with period T. The autocorrelation function of $X(t)$ is

$$\begin{aligned}\phi_{XX}(t, t+\tau) &= \frac{1}{2}\mathrm{E}[X^*(t)X(t+\tau)] \\ &= \frac{1}{2}\mathrm{E}\left[\sum_{n=-\infty}^{\infty} a_n^* \psi(t - nT) \sum_{m=-\infty}^{\infty} a_m \psi(t + \tau - mT)\right] \\ &= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \phi_{aa}(m-n)\psi(t - nT)\psi(t + \tau - mT).\end{aligned}$$

It is relatively straightforward to show that

$$\phi_{XX}(t + kT, t + \tau + kT) = \phi_{XX}(t, t + \tau).$$

Therefore, the autocorrelation function $\phi_{XX}(t, t+\tau)$ is periodic in t with period T. Such a process with a periodic mean and autocorrelation function is said to be cyclostationary or periodic wide-sense stationary.

The power spectrum of a cyclostationary random process $X(t)$ can be computed by first determining the time-average autocorrelation

$$\phi_{XX}(\tau) = \langle \phi_{XX}(t, t+\tau) \rangle = \frac{1}{T} \int_T \phi_{XX}(t, t+\tau) \mathrm{d}t$$

and then taking the Fourier transform in (A.93).

A.5.7 Discrete-Time Random Processes

Let $X_n \equiv X(n)$, where n is an integer time variable, be a complex-valued discrete-time random process. Then the mth moment of X_n is

$$\mathrm{E}[X_n^m] = \int_{-\infty}^{\infty} x_n^m p_X(x_n) \mathrm{d}x_n. \tag{A.96}$$

The autocorrelation function of X_n is

$$\phi_{XX}(n,k) = \frac{1}{2}\mathrm{E}[X_n^* X_k] = \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_n^* x_k p_{X_n,X_k}(x_n, x_k) \mathrm{d}x_n \mathrm{d}x_k \tag{A.97}$$

and the autocovariance function is

$$\lambda_{XX}(n,k) = \phi(n,k) - \frac{1}{2}\mathrm{E}[X_n^*]\mathrm{E}[X_k]. \tag{A.98}$$

If X_n is a wide-sense stationary discrete-time random process, then

$$\phi_{XX}(n,k) = \phi_{XX}(k-n) \tag{A.99}$$

$$\lambda_{XX}(n,k) = \lambda_{XX}(k-n) = \phi_{XX}(k-n) - \frac{1}{2}|\mu_X|^2. \tag{A.100}$$

From Parseval's theorem, the total power in the process X_n is

$$P = \frac{1}{2}\mathrm{E}[|X_n|^2] = \phi_{XX}(0). \tag{A.101}$$

The power spectrum of a discrete-time random process X_n is the discrete-time Fourier transform of the autocorrelation function

$$S_{XX}(f) = \sum_{n=-\infty}^{\infty} \phi_{XX}(n) \mathrm{e}^{-j2\pi fn} \tag{A.102}$$

and

$$\phi_{XX}(n) = \int_{-1/2}^{1/2} S_{XX}(f) \mathrm{e}^{j2\pi fn} \mathrm{d}f. \tag{A.103}$$

Note that $S_{XX}(f)$ is periodic in f with a period of unity, i.e., $S_{XX}(f) = S_{XX}(f+k)$ for any integer k. This is a characteristic of any discrete-time random process. For example, one obtained by sampling a continuous time random process $X_n = x(nT)$, where T is the sample period.

Suppose that a wide-sense stationary complex-valued discrete-time random process X_n is input to a discrete-time linear time-invariant system with impulse response h_n. The process is assumed to have mean μ_X and autocorrelation function $\phi_{XX}(n)$ The transfer function of the filter is

$$H(f) = \sum_{n=-\infty}^{\infty} h_n \mathrm{e}^{-j2\pi fn}. \tag{A.104}$$

The input, X_n, and output, Y_n, are related by the convolution sum

$$Y_n = \sum_{k=-\infty}^{\infty} h_k X_{n-k}. \tag{A.105}$$

The output mean is

$$\mu_Y = \mathrm{E}[Y_n] = \sum_{k=-\infty}^{\infty} h_k \mathrm{E}[X_{n-k}] = \mu_X \sum_{k=-\infty}^{\infty} h_k = \mu_X H(0). \tag{A.106}$$

The output autocorrelation is

$$\begin{aligned}
\phi_{YY}(k) &= \frac{1}{2}\mathrm{E}[Y_n^* Y_{n+k}] \\
&= \frac{1}{2}\mathrm{E}\left[\sum_{\ell=-\infty}^{\infty} h_\ell^* X_{n-\ell}^* \sum_{m=-\infty}^{\infty} h_m X_{n+k-m}\right] \\
&= \sum_{\ell=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} h_\ell^* h_m \frac{1}{2}\mathrm{E}[X_{n-\ell}^* X_{n+k-m}] \\
&= \sum_{\ell=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} h_\ell^* h_m \phi_{XX}(k+\ell-m) \\
&= \sum_{m=-\infty}^{\infty} h_m \sum_{\ell=-\infty}^{\infty} h_\ell^* \phi_{XX}(k+\ell-m) \\
&= h_k * \left\{\sum_{\ell=-\infty}^{\infty} h_\ell^* \phi_{XX}(k+\ell)\right\} \\
&= h_k * \phi_{XX}(k) * h_{-k}^*.
\end{aligned} \tag{A.107}$$

where the convolution operation is understood to be a discrete-time convolution. The output psd can be obtained by taking the discrete-time Fourier transform of the autocorrelation function, resulting in

$$\begin{aligned}
S_{YY}(f) &= H(f) S_{XX}(f) H^*(f) \\
&= |H(f)|^2 S_{XX}(f).
\end{aligned} \tag{A.108}$$

Once again, $S_{YY}(f)$ is periodic in f with a period of unity, i.e., $S_{YY}(f) = S_{YY}(f+k)$ for any integer k.

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