Appendix: Numerical Tables

A.1 The Gaussian Distribution and the Error Function

The Gaussian distribution (3.11) is defined as

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}. \tag{A.1}$$

The maximum value is obtained at $x = \mu$, and the value of x where the Gaussian is a times the peak value is given by

$$z \equiv \frac{x-\mu}{\sigma} = \sqrt{-2\ln a}. \tag{A.2}$$

Figure A.1 shows a standard Gaussian normalized to its peak value, and values of a times the peak value are tabulated in Table A.1. The Half Width at Half Maximum (HWHM) has a value of approximately 1.18σ.

The error function is defined in (3.13) as

$$\text{erf}\, z = \frac{1}{\sqrt{\pi}} \int_{-z}^{z} e^{-x^2} dx \tag{A.3}$$

and it is related to the integral of the Gaussian distribution defined in (3.12),

$$A(z) = \int_{\mu-z\sigma}^{\mu+z\sigma} f(x)dx = \frac{1}{\sqrt{2\pi}} \int_{-z}^{z} e^{-\frac{x^2}{2}} dx. \tag{A.4}$$

The relationship between the two integrals is given by

$$\text{erf}\left(z/\sqrt{2}\right) = A(z). \tag{A.5}$$

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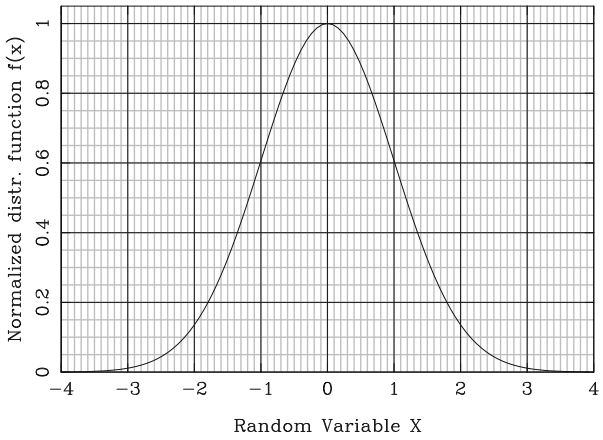

Fig. A.1 Normalized values of the probability distribution function of a standard Gaussian ($\mu = 0$ and $\sigma = 1$)

Table A.1 Values of a times the peak value for a Gaussian distribution

a	z	a	z	a	z	a	z	a	z
0.980	0.201	0.960	0.286	0.940	0.352	0.920	0.408	0.900	0.459
0.880	0.506	0.860	0.549	0.840	0.591	0.820	0.630	0.800	0.668
0.780	0.705	0.760	0.741	0.740	0.776	0.720	0.811	0.700	0.845
0.680	0.878	0.660	0.912	0.640	0.945	0.620	0.978	0.600	1.011
0.580	1.044	0.560	1.077	0.540	1.110	0.520	1.144	0.500	1.177
0.480	1.212	0.460	1.246	0.440	1.281	0.420	1.317	0.400	1.354
0.380	1.391	0.360	1.429	0.340	1.469	0.320	1.510	0.300	1.552
0.280	1.596	0.260	1.641	0.240	1.689	0.220	1.740	0.200	1.794
0.180	1.852	0.160	1.914	0.140	1.983	0.120	2.059	0.100	2.146
0.080	2.248	0.060	2.372	0.040	2.537	0.020	2.797	0.010	3.035

The function $A(z)$ describes the integrated probability of a Gaussian distribution to have values between $\mu - z\sigma$ and $\mu + z\sigma$. The number z therefore represents the number of σ by which the interval extends in each direction. The function $A(z)$ is tabulated in Table A.2, where each number in the table corresponds to a number z given by the number in the left column (e.g., 0.0, 0.1, etc.), and for which the second decimal digit is given by the number in the top column (e.g., the value of 0.007979 corresponds to $z = 0.01$).

The cumulative distribution of a standard Gaussian function was defined in (3.14) as

$$B(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt; \tag{A.6}$$

Table A.2 Values of the integral $A(z)$ as a function of z, the number of standard errors σ

	0	1	2	3	4	5	6	7	8	9
0.0	0.000000	0.007979	0.015957	0.023933	0.031907	0.039878	0.047844	0.055806	0.063763	0.071713
0.1	0.079656	0.087591	0.095517	0.103434	0.111340	0.119235	0.127119	0.134990	0.142847	0.150691
0.2	0.158519	0.166332	0.174129	0.181908	0.189670	0.197413	0.205136	0.212840	0.220523	0.228184
0.3	0.235823	0.243439	0.251032	0.258600	0.266144	0.273661	0.281153	0.288618	0.296055	0.303464
0.4	0.310844	0.318194	0.325515	0.332804	0.340063	0.347290	0.354484	0.361645	0.368773	0.375866
0.5	0.382925	0.389949	0.396937	0.403888	0.410803	0.417681	0.424521	0.431322	0.438086	0.444810
0.6	0.451494	0.458138	0.464742	0.471306	0.477828	0.484308	0.490746	0.497142	0.503496	0.509806
0.7	0.516073	0.522296	0.528475	0.534610	0.540700	0.546746	0.552746	0.558700	0.564609	0.570472
0.8	0.576289	0.582060	0.587784	0.593461	0.599092	0.604675	0.610211	0.615700	0.621141	0.626534
0.9	0.631880	0.637178	0.642428	0.647629	0.652783	0.657888	0.662945	0.667954	0.672914	0.677826
1.0	0.682690	0.687505	0.692272	0.696990	0.701660	0.706282	0.710856	0.715381	0.719858	0.724287
1.1	0.728668	0.733001	0.737287	0.741524	0.745714	0.749856	0.753952	0.757999	0.762000	0.765954
1.2	0.769861	0.773721	0.777535	0.781303	0.785025	0.788701	0.792331	0.795916	0.799455	0.802950
1.3	0.806399	0.809805	0.813165	0.816482	0.819755	0.822984	0.826170	0.829313	0.832414	0.835471
1.4	0.838487	0.841461	0.844393	0.847283	0.850133	0.852942	0.855710	0.858439	0.861127	0.863776
1.5	0.866386	0.868957	0.871489	0.873984	0.876440	0.878859	0.881240	0.883585	0.885894	0.888166
1.6	0.890402	0.892603	0.894768	0.896899	0.898995	0.901057	0.903086	0.905081	0.907043	0.908972
1.7	0.910869	0.912735	0.914568	0.916370	0.918141	0.919882	0.921593	0.923273	0.924924	0.926546
1.8	0.928140	0.929705	0.931241	0.932750	0.934232	0.935687	0.937115	0.938517	0.939892	0.941242
1.9	0.942567	0.943867	0.945142	0.946394	0.947621	0.948824	0.950005	0.951162	0.952297	0.953409
2.0	0.954500	0.955569	0.956617	0.957644	0.958650	0.959636	0.960602	0.961548	0.962475	0.963383
2.1	0.964272	0.965142	0.965994	0.966829	0.967646	0.968445	0.969228	0.969994	0.970743	0.971476
2.2	0.972194	0.972895	0.973582	0.974253	0.974909	0.975551	0.976179	0.976793	0.977393	0.977979
2.3	0.978552	0.979112	0.979660	0.980194	0.980717	0.981227	0.981725	0.982212	0.982688	0.983152
2.4	0.983605	0.984048	0.984480	0.984902	0.985313	0.985715	0.986107	0.986489	0.986862	0.987226

(continued)

Table A.2 (continued)

	0	1	2	3	4	5	6	7	8	9
2.5	0.987581	0.987927	0.988265	0.988594	0.988915	0.989228	0.989533	0.989831	0.990120	0.990403
2.6	0.990678	0.990946	0.991207	0.991462	0.991710	0.991951	0.992186	0.992415	0.992638	0.992855
2.7	0.993066	0.993272	0.993472	0.993667	0.993857	0.994041	0.994220	0.994395	0.994565	0.994730
2.8	0.994890	0.995046	0.995198	0.995346	0.995489	0.995628	0.995764	0.995896	0.996024	0.996148
2.9	0.996269	0.996386	0.996500	0.996611	0.996718	0.996823	0.996924	0.997022	0.997118	0.997211
3.0	0.997301	0.997388	0.997473	0.997555	0.997635	0.997712	0.997787	0.997860	0.997930	0.997999
3.1	0.998065	0.998130	0.998192	0.998252	0.998311	0.998368	0.998423	0.998476	0.998528	0.998578
3.2	0.998626	0.998673	0.998719	0.998763	0.998805	0.998846	0.998886	0.998925	0.998962	0.998999
3.3	0.999034	0.999067	0.999100	0.999132	0.999163	0.999192	0.999221	0.999249	0.999276	0.999301
3.4	0.999327	0.999351	0.999374	0.999397	0.999419	0.999440	0.999460	0.999480	0.999499	0.999517
3.5	0.999535	0.999552	0.999569	0.999585	0.999600	0.999615	0.999630	0.999643	0.999657	0.999670
3.6	0.999682	0.999694	0.999706	0.999717	0.999728	0.999738	0.999748	0.999758	0.999767	0.999776
3.7	0.999785	0.999793	0.999801	0.999809	0.999816	0.999824	0.999831	0.999837	0.999844	0.999850
3.8	0.999856	0.999861	0.999867	0.999872	0.999877	0.999882	0.999887	0.999892	0.999896	0.999900
3.9	0.999904	0.999908	0.999912	0.999915	0.999919	0.999922	0.999925	0.999929	0.999932	0.999934
4.0	0.999937	0.999940	0.999942	0.999945	0.999947	0.999949	0.999951	0.999953	0.999955	0.999957
4.1	0.999959	0.999961	0.999963	0.999964	0.999966	0.999967	0.999969	0.999970	0.999971	0.999973
4.2	0.999974	0.999975	0.999976	0.999977	0.999978	0.999979	0.999980	0.999981	0.999982	0.999983
4.3	0.999983	0.999984	0.999985	0.999986	0.999986	0.999987	0.999987	0.999988	0.999989	0.999989
4.4	0.999990	0.999990	0.999991	0.999991	0.999991	0.999992	0.999992	0.999993	0.999993	0.999993
4.5	0.999994	0.999994	0.999994	0.999995	0.999995	0.999995	0.999995	0.999996	0.999996	0.999996
4.6	0.999996	0.999996	0.999997	0.999997	0.999997	0.999997	0.999997	0.999997	0.999998	0.999998
4.7	0.999998	0.999998	0.999998	0.999998	0.999998	0.999998	0.999998	0.999999	0.999999	0.999999
4.8	0.999999	0.999999	0.999999	0.999999	0.999999	0.999999	0.999999	0.999999	0.999999	0.999999
4.9	0.999999	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000

and it is therefore related to the integral $A(z)$ by

$$B(z) = \frac{1}{2} + \frac{A(z)}{2}. \tag{A.7}$$

The values of $B(z)$ are tabulated in Table A.3. Each number in the table corresponds to a number z given by the number in the left column (e.g., 0.0, 0.1, etc.), and for which the second decimal digit is given by the number in the top column (e.g., the value of 0.503990 corresponds to $z = 0.01$).

Critical values of the standard Gaussian distribution functions corresponding to selected values of the integrals $A(z)$ and $B(z)$ are shown in Table A.4. They indicate the value of the variable z required to include a given probability, and are useful for either two-sided or one-sided rejection regions in hypothesis testing.

A.2 Upper and Lower Limits for a Poisson Distribution

The Gehrels approximation described in [16] can be used to calculate upper and lower limits for a Poisson distribution, when n_{obs} counts are recorded. The confidence level is described by the parameter S, corresponding to the number of standard deviations σ for a Gaussian distribution; for example, $S = 1$ corresponds to an 84.1 % confidence level, $S = 2$ to a 97.7 %, and $S = 3$ corresponds to 99.9 %; see Table 5.2 for correspondence between values of S and probability. The upper and lower limits are described, in the simplest approximation, by

$$\begin{cases} \lambda_{up} = n_{obs} + \dfrac{S^2 + 3}{4} + S\sqrt{n_{obs} + \dfrac{3}{4}} \\ \lambda_{lo} = n_{obs}\left(1 - \dfrac{1}{9n_{obs}} - \dfrac{S}{3\sqrt{n_{obs}}}\right)^3 \end{cases} \tag{A.8}$$

and more accurate approximations are provided in [16] (Tables A.5 and A.6).

A.3 The χ^2 Distribution

The probability distribution function for a χ^2 variable is defined in (7.11) as

$$f_{\chi^2}(z) = \left(\frac{1}{2}\right)^{f/2} \frac{1}{\Gamma(f/2)} e^{-\frac{z}{2}} z^{\frac{f}{2}-1},$$

Table A.3 Values of the integral $B(z)$ as a function of z

	0	1	2	3	4	5	6	7	8	9
0.0	0.500000	0.503990	0.507979	0.511967	0.515954	0.519939	0.523922	0.527903	0.531882	0.535857
0.1	0.539828	0.543796	0.547759	0.551717	0.555670	0.559618	0.563560	0.567495	0.571424	0.575346
0.2	0.579260	0.583166	0.587065	0.590954	0.594835	0.598707	0.602568	0.606420	0.610262	0.614092
0.3	0.617912	0.621720	0.625516	0.629300	0.633072	0.636831	0.640577	0.644309	0.648028	0.651732
0.4	0.655422	0.659097	0.662758	0.666402	0.670032	0.673645	0.677242	0.680823	0.684387	0.687933
0.5	0.691463	0.694975	0.698469	0.701944	0.705402	0.708841	0.712261	0.715661	0.719043	0.722405
0.6	0.725747	0.729069	0.732371	0.735653	0.738914	0.742154	0.745373	0.748571	0.751748	0.754903
0.7	0.758037	0.761148	0.764238	0.767305	0.770350	0.773373	0.776373	0.779350	0.782305	0.785236
0.8	0.788145	0.791030	0.793892	0.796731	0.799546	0.802338	0.805106	0.807850	0.810571	0.813267
0.9	0.815940	0.818589	0.821214	0.823815	0.826392	0.828944	0.831473	0.833977	0.836457	0.838913
1.0	0.841345	0.843753	0.846136	0.848495	0.850830	0.853141	0.855428	0.857691	0.859929	0.862144
1.1	0.864334	0.866501	0.868643	0.870762	0.872857	0.874928	0.876976	0.879000	0.881000	0.882977
1.2	0.884931	0.886861	0.888768	0.890652	0.892513	0.894351	0.896166	0.897958	0.899728	0.901475
1.3	0.903200	0.904902	0.906583	0.908241	0.909878	0.911492	0.913085	0.914657	0.916207	0.917736
1.4	0.919244	0.920731	0.922197	0.923642	0.925067	0.926471	0.927855	0.929220	0.930564	0.931888
1.5	0.933193	0.934479	0.935745	0.936992	0.938220	0.939430	0.940620	0.941793	0.942947	0.944083
1.6	0.945201	0.946301	0.947384	0.948450	0.949498	0.950529	0.951543	0.952541	0.953522	0.954486
1.7	0.955435	0.956367	0.957284	0.958185	0.959071	0.959941	0.960797	0.961637	0.962462	0.963273
1.8	0.964070	0.964853	0.965621	0.966375	0.967116	0.967844	0.968558	0.969259	0.969946	0.970621
1.9	0.971284	0.971934	0.972571	0.973197	0.973811	0.974412	0.975003	0.975581	0.976149	0.976705
2.0	0.977250	0.977785	0.978309	0.978822	0.979325	0.979818	0.980301	0.980774	0.981238	0.981692
2.1	0.982136	0.982571	0.982997	0.983415	0.983823	0.984223	0.984614	0.984997	0.985372	0.985738
2.2	0.986097	0.986448	0.986791	0.987127	0.987455	0.987776	0.988090	0.988397	0.988697	0.988990
2.3	0.989276	0.989556	0.989830	0.990097	0.990359	0.990614	0.990863	0.991106	0.991344	0.991576
2.4	0.991803	0.992024	0.992240	0.992451	0.992657	0.992858	0.993054	0.993245	0.993431	0.993613

2.5	0.993791	0.993964	0.994133	0.994297	0.994458	0.994614	0.994767	0.994915	0.995060	0.995202
2.6	0.995339	0.995473	0.995604	0.995731	0.995855	0.995976	0.996093	0.996208	0.996319	0.996428
2.7	0.996533	0.996636	0.996736	0.996834	0.996928	0.997021	0.997110	0.997198	0.997282	0.997365
2.8	0.997445	0.997523	0.997599	0.997673	0.997745	0.997814	0.997882	0.997948	0.998012	0.998074
2.9	0.998135	0.998193	0.998250	0.998306	0.998359	0.998412	0.998462	0.998511	0.998559	0.998606
3.0	0.998651	0.998694	0.998737	0.998778	0.998818	0.998856	0.998894	0.998930	0.998965	0.999000
3.1	0.999033	0.999065	0.999096	0.999126	0.999156	0.999184	0.999212	0.999238	0.999264	0.999289
3.2	0.999313	0.999337	0.999359	0.999381	0.999403	0.999423	0.999443	0.999463	0.999481	0.999499
3.3	0.999517	0.999534	0.999550	0.999566	0.999582	0.999596	0.999611	0.999625	0.999638	0.999651
3.4	0.999663	0.999676	0.999687	0.999699	0.999710	0.999720	0.999730	0.999740	0.999750	0.999759
3.5	0.999768	0.999776	0.999785	0.999793	0.999800	0.999808	0.999815	0.999822	0.999829	0.999835
3.6	0.999841	0.999847	0.999853	0.999859	0.999864	0.999869	0.999874	0.999879	0.999884	0.999888
3.7	0.999893	0.999897	0.999901	0.999905	0.999908	0.999912	0.999915	0.999919	0.999922	0.999925
3.8	0.999928	0.999931	0.999934	0.999936	0.999939	0.999941	0.999944	0.999946	0.999948	0.999950
3.9	0.999952	0.999954	0.999956	0.999958	0.999960	0.999961	0.999963	0.999964	0.999966	0.999967
4.0	0.999969	0.999970	0.999971	0.999973	0.999974	0.999975	0.999976	0.999977	0.999978	0.999979
4.1	0.999980	0.999981	0.999981	0.999982	0.999983	0.999984	0.999985	0.999985	0.999986	0.999986
4.2	0.999987	0.999988	0.999988	0.999989	0.999989	0.999990	0.999990	0.999991	0.999991	0.999991
4.3	0.999992	0.999992	0.999993	0.999993	0.999993	0.999994	0.999994	0.999994	0.999994	0.999995
4.4	0.999995	0.999995	0.999995	0.999996	0.999996	0.999996	0.999996	0.999997	0.999997	0.999997
4.5	0.999997	0.999997	0.999997	0.999997	0.999998	0.999998	0.999998	0.999998	0.999998	0.999998
4.6	0.999998	0.999998	0.999999	0.999999	0.999999	0.999999	0.999999	0.999999	0.999999	0.999999
4.7	0.999999	0.999999	0.999999	0.999999	0.999999	0.999999	0.999999	1.000000	1.000000	1.000000
4.8	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
4.9	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000

Table A.4 Table of critical values of the standard Gaussian distribution to include a given probability, for two-sided confidence intervals (-z, z) of the integral $A(z)$, and for one-sided intervals (-∞, z) of the integral $B(z)$

Probability	Two-sided z	One-sided z
0.01	0.013	−2.326
0.05	0.063	−1.645
0.10	0.126	−1.282
0.20	0.253	−0.842
0.30	0.385	−0.524
0.40	0.524	−0.253
0.50	0.674	−0.000
0.60	0.842	0.253
0.70	1.036	0.524
0.80	1.282	0.842
0.90	1.645	1.282
0.95	1.960	1.645
0.99	2.576	2.326
0.999	3.290	3.090
0.9999	3.890	3.718

Table A.5 Selected upper limits for a Poisson variable using the Gehrels approximation

	Upper limits		
	Poisson parameter S or confidence level		
	$S = 1$	$S = 2$	$S = 3$
n_{obs}	(1-σ, or 84.1 %)	(2-σ, or 97.7 %)	(3-σ, or 99.9 %)
0	1.87	3.48	5.60
1	3.32	5.40	7.97
2	4.66	7.07	9.97
3	5.94	8.62	11.81
4	7.18	10.11	13.54
5	8.40	11.55	15.19
6	9.60	12.95	16.79
7	10.78	14.32	18.35
8	11.96	15.67	19.87
9	13.12	16.99	21.37
10	14.28	18.31	22.84
20	25.56	30.86	36.67
30	36.55	42.84	49.64
40	47.38	54.52	62.15
50	58.12	66.00	74.37
60	68.79	77.34	86.38
70	79.41	88.57	98.23
80	89.99	99.72	109.96
90	100.53	110.80	121.58
100	111.04	121.82	133.11

Table A.6 Selected lower limits for a Poisson variable using the Gehrels approximation

	Lower limits		
	Poisson parameter S or confidence level		
	$S = 1$	$S = 2$	$S = 3$
n_{obs}	(1-σ, or 84.1 %)	(2-σ, or 97.7 %)	(3-σ, or 99.9 %)
1	0.17	0.01	0.00
2	0.71	0.21	0.03
3	1.37	0.58	0.17
4	2.09	1.04	0.42
5	2.85	1.57	0.75
6	3.63	2.14	1.13
7	4.42	2.75	1.56
8	5.24	3.38	2.02
9	6.06	4.04	2.52
10	6.90	4.71	3.04
20	15.57	12.08	9.16
30	24.56	20.07	16.16
40	33.70	28.37	23.63
50	42.96	36.88	31.40
60	52.28	45.53	39.38
70	61.66	54.28	47.52
80	71.08	63.13	55.79
90	80.53	72.04	64.17
100	90.02	81.01	72.63

where f is the number of degrees of freedom. The critical value or p-quantile of the distribution is given by

$$P_{\chi^2}(z \leq \chi^2_{crit}) = \int_0^{\chi^2_{crit}} f_{\chi^2}(z)dz = p \tag{A.9}$$

or, equivalently,

$$P_{\chi^2}(z \geq \chi^2_{crit}) = \int_{\chi^2_{crit}}^{\infty} f_{\chi^2}(z)dz = 1 - p. \tag{A.10}$$

The critical value is a function of the number of degrees of freedom f and the level of probability p. Normally p is intended as a large number, such as 0.68, 0.90, or 0.99, meaning that there is just a 32, 10, or 1 % probability to have values higher than the critical value χ^2_{crit}.

As described in Sect. 7.2, the χ^2 distribution has the following mean and variance:

$$\begin{cases} \mu = f \\ \sigma^2 = 2f. \end{cases}$$

It is convenient to tabulate the value of reduced χ^2, or χ^2_{crit}/f, that corresponds to a given probability level, as function of the number of degrees of freedom. Selected critical values of the χ^2 distribution are reported in Table A.7. When using this table, remember to multiply the tabulated reduced χ^2 by the number of defrees of freedom f to obtain the value of χ^2.

If Z is a χ^2-distributed variable with f degrees of freedoms,

$$\lim_{f\to\infty} \frac{Z-f}{\sqrt{2f}} = N(0,1). \tag{A.11}$$

In fact, a χ^2 variable is obtained as the sum of independent distributions (Sect. 7.2), to which the central theorem limit applies (Sect. 4.3). For a large number of degrees of freedom, the standard Gaussian distribution can be used to supplement Table A.7 according to (A.11). For example, for $p = 0.99$, the one-sided critical value of the standard Gaussian is approximately 2.326, according to Table A.4. Using this value into (A.11) for $f = 200$ would give a critical value for the χ^2 distribution of 1.2326 (compare to 1.247 from Table A.7). The values of $f = \infty$ in Table A.7 is obtained using the Gaussian approximation, according to (A.11).

A.4 The *F* Distribution

The F distribution with f_1, f_2 degrees of freedom is defined in (7.22) as

$$f_F(z) = \frac{\Gamma\left(\frac{f_1+f_2}{2}\right)}{\Gamma\left(\frac{f_1}{2}\right)\Gamma\left(\frac{f_2}{2}\right)} \left(\frac{f_1}{f_2}\right)^{\frac{f_1}{2}} \frac{z^{\frac{f_1}{2}-1}}{\left(1+z\frac{f_1}{f_2}\right)^{\frac{f_1+f_2}{2}}}.$$

The critical value F_{crit} that includes a probability p is given by

$$P(z \geq F_{crit}) = \int_{F_{crit}}^{\infty} f_F(z)dz = 1-p, \tag{A.12}$$

and it is a function of the degrees of freedom f_1 and f_2. In Table A.8 are reported the critical values for various probability levels p, for a fixed value $f_1 = 1$, and as function of f_2. Tables A.9, A.10, A.11, A.12, A.13, A.14, and A.15 have the critical values as function of both f_1 and f_2.

Asymptotic values when f_1 and f_2 approach infinity can be found using (7.25), reported here for convenience:

$$\begin{cases} \lim_{f_2\to\infty} f_F(z,f_1,f_2) = f_{\chi^2}(x,f_1) \text{ where } x = f_1 z \\ \lim_{f_1\to\infty} f_F(z,f_1,f_2) = f_{\chi^2}(x,f_2) \text{ where } x = f_2/z. \end{cases}$$

Table A.7 Critical values of the χ^2 distribution

	Probability p to have a value of reduced χ^2 below the critical value												
f	0.01	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.95	0.99
1	0.00016	0.00390	0.01580	0.0642	0.1485	0.2750	0.4549	0.7083	1.0742	1.6424	2.7055	3.8415	6.6349
2	0.0101	0.0513	0.1054	0.2231	0.3567	0.5108	0.6931	0.9163	1.2040	1.6094	2.3026	2.9957	4.6052
3	0.0383	0.1173	0.1948	0.3351	0.4746	0.6231	0.7887	0.9821	1.2216	1.5472	2.0838	2.6049	3.7816
4	0.0743	0.1777	0.2659	0.4122	0.5487	0.6882	0.8392	1.0112	1.2196	1.4972	1.9449	2.3719	3.3192
5	0.1109	0.2291	0.3221	0.4685	0.6000	0.7311	0.8703	1.0264	1.2129	1.4578	1.8473	2.2141	3.0172
6	0.1454	0.2726	0.3674	0.5117	0.6379	0.7617	0.8914	1.0351	1.2052	1.4263	1.7741	2.0986	2.8020
7	0.1770	0.3096	0.4047	0.5460	0.6673	0.7847	0.9065	1.0405	1.1976	1.4005	1.7167	2.0096	2.6393
8	0.2058	0.3416	0.4362	0.5742	0.6909	0.8028	0.9180	1.0438	1.1906	1.3788	1.6702	1.9384	2.5113
9	0.2320	0.3695	0.4631	0.5978	0.7104	0.8174	0.9270	1.0460	1.1841	1.3602	1.6315	1.8799	2.4073
10	0.256	0.394	0.487	0.618	0.727	0.830	0.934	1.047	1.178	1.344	1.599	1.831	2.321
11	0.278	0.416	0.507	0.635	0.741	0.840	0.940	1.048	1.173	1.330	1.570	1.789	2.248
12	0.298	0.436	0.525	0.651	0.753	0.848	0.945	1.049	1.168	1.318	1.546	1.752	2.185
13	0.316	0.453	0.542	0.664	0.764	0.856	0.949	1.049	1.163	1.307	1.524	1.720	2.130
14	0.333	0.469	0.556	0.676	0.773	0.863	0.953	1.049	1.159	1.296	1.505	1.692	2.082
15	0.349	0.484	0.570	0.687	0.781	0.869	0.956	1.049	1.155	1.287	1.487	1.666	2.039
16	0.363	0.498	0.582	0.697	0.789	0.874	0.959	1.049	1.151	1.279	1.471	1.643	2.000
17	0.377	0.510	0.593	0.706	0.796	0.879	0.961	1.048	1.148	1.271	1.457	1.623	1.965
18	0.390	0.522	0.604	0.714	0.802	0.883	0.963	1.048	1.145	1.264	1.444	1.604	1.934
19	0.402	0.532	0.613	0.722	0.808	0.887	0.965	1.048	1.142	1.258	1.432	1.587	1.905
20	0.413	0.543	0.622	0.729	0.813	0.890	0.967	1.048	1.139	1.252	1.421	1.571	1.878
30	0.498	0.616	0.687	0.779	0.850	0.915	0.978	1.044	1.118	1.208	1.342	1.459	1.696
40	0.554	0.663	0.726	0.809	0.872	0.928	0.983	1.041	1.104	1.182	1.295	1.394	1.592
50	0.594	0.695	0.754	0.829	0.886	0.937	0.987	1.038	1.094	1.163	1.263	1.350	1.523

(continued)

Table A.7 (continued)

	Probability p to have a value of reduced χ^2 below the critical value												
f	0.01	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.95	0.99
60	0.625	0.720	0.774	0.844	0.897	0.944	0.989	1.036	1.087	1.150	1.240	1.318	1.473
70	0.649	0.739	0.790	0.856	0.905	0.949	0.990	1.034	1.081	1.139	1.222	1.293	1.435
80	0.669	0.755	0.803	0.865	0.911	0.952	0.992	1.032	1.076	1.130	1.207	1.274	1.404
90	0.686	0.768	0.814	0.873	0.917	0.955	0.993	1.031	1.073	1.123	1.195	1.257	1.379
100	0.700	0.780	0.823	0.880	0.921	0.958	0.993	1.030	1.069	1.117	1.185	1.243	1.358
200	0.782	0.841	0.874	0.915	0.945	0.971	0.997	1.022	1.050	1.083	1.130	1.170	1.247
300	0.820	0.870	0.897	0.931	0.956	0.977	0.998	1.019	1.041	1.068	1.106	1.138	1.200
400	0.843	0.887	0.910	0.940	0.962	0.980	0.998	1.016	1.036	1.059	1.092	1.119	1.172
500	0.86	0.90	0.92	0.95	0.97	0.98	1.00	1.01	1.03	1.05	1.08	1.10	1.15
1000	0.90	0.93	0.94	0.96	0.98	0.99	1.00	1.01	1.02	1.04	1.06	1.07	1.11
∞	0.90	0.93	0.94	0.96	0.98	0.99	1.00	1.01	1.02	1.04	1.06	1.07	1.10

Table A.8 Critical values of F statistics for $f_1 = 1$ degrees of freedom

	Probability p to have a value of F below the critical value						
f_2	0.50	0.60	0.70	0.80	0.90	0.95	0.99
1	1.000	1.894	3.852	9.472	39.863	161.448	4052.182
2	0.667	1.125	1.922	3.556	8.526	18.513	98.503
3	0.585	0.957	1.562	2.682	5.538	10.128	34.116
4	0.549	0.885	1.415	2.351	4.545	7.709	21.198
5	0.528	0.846	1.336	2.178	4.060	6.608	16.258
6	0.515	0.820	1.286	2.073	3.776	5.987	13.745
7	0.506	0.803	1.253	2.002	3.589	5.591	12.246
8	0.499	0.790	1.228	1.951	3.458	5.318	11.259
9	0.494	0.780	1.209	1.913	3.360	5.117	10.561
10	0.490	0.773	1.195	1.883	3.285	4.965	10.044
20	0.472	0.740	1.132	1.757	2.975	4.351	8.096
30	0.466	0.729	1.112	1.717	2.881	4.171	7.562
40	0.463	0.724	1.103	1.698	2.835	4.085	7.314
50	0.462	0.721	1.097	1.687	2.809	4.034	7.171
60	0.461	0.719	1.093	1.679	2.791	4.001	7.077
70	0.460	0.717	1.090	1.674	2.779	3.978	7.011
80	0.459	0.716	1.088	1.670	2.769	3.960	6.963
90	0.459	0.715	1.087	1.667	2.762	3.947	6.925
100	0.458	0.714	1.085	1.664	2.756	3.936	6.895
200	0.457	0.711	1.080	1.653	2.731	3.888	6.763
∞	0.455	0.708	1.074	1.642	2.706	3.842	6.635

For example, the critical values of the F distribution for $f_1 = 1$ and in the limit of large f_2 are obtained from the first row of Table A.7.

A.5 The Student's t Distribution

The Student t distribution is given by (7.34),

$$f_T(t) = \frac{1}{\sqrt{f\pi}} \frac{\Gamma((f+1)/2)}{\Gamma(f/2)} \times \left(1 + \frac{t^2}{f}\right)^{-\frac{1}{2}(f+1)},$$

where f is the number of degrees of freedom. The probability p that the absolute value of a t variable exceeds a critical value T_{crit} is given by

$$P(|t| \leq T_{crit}) = P(|\bar{x} - \mu| \leq T_{crit} \cdot s/\sqrt{n}) = \int_{-T_{crit}}^{T_{crit}} f_T(t)dt = 1 - p. \qquad \text{(A.13)}$$

Table A.9 Critical values of F statistic that include $p = 0.50$ probability

	f_1									
f_2	2	4	6	8	10	20	40	60	80	100
1	1.500	1.823	1.942	2.004	2.042	2.119	2.158	2.172	2.178	2.182
2	1.000	1.207	1.282	1.321	1.345	1.393	1.418	1.426	1.430	1.433
3	0.881	1.063	1.129	1.163	1.183	1.225	1.246	1.254	1.257	1.259
4	0.828	1.000	1.062	1.093	1.113	1.152	1.172	1.178	1.182	1.184
5	0.799	0.965	1.024	1.055	1.073	1.111	1.130	1.136	1.139	1.141
6	0.780	0.942	1.000	1.030	1.048	1.084	1.103	1.109	1.113	1.114
7	0.767	0.926	0.983	1.013	1.030	1.066	1.085	1.091	1.094	1.096
8	0.757	0.915	0.971	1.000	1.017	1.053	1.071	1.077	1.080	1.082
9	0.749	0.906	0.962	0.990	1.008	1.043	1.061	1.067	1.070	1.072
10	0.743	0.899	0.954	0.983	1.000	1.035	1.053	1.059	1.062	1.063
20	0.718	0.868	0.922	0.950	0.966	1.000	1.017	1.023	1.026	1.027
30	0.709	0.858	0.912	0.939	0.955	0.989	1.006	1.011	1.014	1.016
40	0.705	0.854	0.907	0.934	0.950	0.983	1.000	1.006	1.008	1.010
50	0.703	0.851	0.903	0.930	0.947	0.980	0.997	1.002	1.005	1.007
60	0.701	0.849	0.901	0.928	0.945	0.978	0.994	1.000	1.003	1.004
70	0.700	0.847	0.900	0.927	0.943	0.976	0.993	0.998	1.001	1.003
80	0.699	0.846	0.899	0.926	0.942	0.975	0.992	0.997	1.000	1.002
90	0.699	0.845	0.898	0.925	0.941	0.974	0.991	0.996	0.999	1.001
100	0.698	0.845	0.897	0.924	0.940	0.973	0.990	0.996	0.998	1.000
200	0.696	0.842	0.894	0.921	0.937	0.970	0.987	0.992	0.995	0.997
∞	0.693	0.839	0.891	0.918	0.934	0.967	0.983	0.989	0.992	0.993

These two-sided critical values are tabulated in Tables A.16, A.17, A.18, A.19, A.20, A.21, and A.22 for selected values of f, as function of the critical value T_{crit}. In these tables, the left column indicates the value of T_{crit} to the first decimal digit, and the values on the top column are the second decimal digit.

Table A.23 provides a comparison of the probability p for five critical values, $T_{crit} = 1$ through 5, as function of f. The case of $f = \infty$ corresponds to a standard Gaussian.

A.6 The Linear Correlation Coefficient *r*

The linear correlation coefficient is defined as

$$r^2 = \frac{\left(N\sum x_i y_i - \sum x_i \sum y_i\right)^2}{\left(N\sum x_i^2 - \left(\sum x_i\right)^2\right)\left(N\sum y_i^2 - \left(\sum y_i\right)^2\right)} \tag{A.14}$$

Table A.10 Critical values of F statistic that include $p = 0.60$ probability

	f_1									
f_2	2	4	6	8	10	20	40	60	80	100
1	2.625	3.093	3.266	3.355	3.410	3.522	3.579	3.598	3.608	3.613
2	1.500	1.718	1.796	1.835	1.859	1.908	1.933	1.941	1.945	1.948
3	1.263	1.432	1.489	1.518	1.535	1.570	1.588	1.593	1.596	1.598
4	1.162	1.310	1.359	1.383	1.397	1.425	1.439	1.444	1.446	1.448
5	1.107	1.243	1.287	1.308	1.320	1.345	1.356	1.360	1.362	1.363
6	1.072	1.200	1.241	1.260	1.272	1.293	1.303	1.307	1.308	1.309
7	1.047	1.171	1.209	1.227	1.238	1.257	1.266	1.269	1.270	1.271
8	1.030	1.150	1.186	1.203	1.213	1.231	1.239	1.241	1.242	1.243
9	1.016	1.133	1.168	1.185	1.194	1.210	1.217	1.219	1.220	1.221
10	1.006	1.120	1.154	1.170	1.179	1.194	1.200	1.202	1.203	1.204
20	0.960	1.064	1.093	1.106	1.112	1.122	1.124	1.124	1.124	1.124
30	0.945	1.046	1.074	1.085	1.090	1.097	1.097	1.097	1.096	1.096
40	0.938	1.037	1.064	1.075	1.080	1.085	1.084	1.083	1.082	1.081
50	0.933	1.032	1.058	1.068	1.073	1.078	1.076	1.074	1.073	1.072
60	0.930	1.029	1.054	1.064	1.069	1.073	1.070	1.068	1.066	1.065
70	0.928	1.026	1.052	1.061	1.066	1.069	1.066	1.064	1.062	1.061
80	0.927	1.024	1.049	1.059	1.064	1.067	1.063	1.060	1.059	1.057
90	0.926	1.023	1.048	1.057	1.062	1.065	1.061	1.058	1.056	1.054
100	0.925	1.021	1.047	1.056	1.060	1.063	1.059	1.056	1.054	1.052
200	0.921	1.016	1.041	1.050	1.054	1.055	1.050	1.046	1.043	1.041
∞	0.916	1.011	1.035	1.044	1.047	1.048	1.041	1.036	1.032	1.029

and it is equal to the product bb', where b is the best-fit slope of the linear regression of Y on X, and b' is the slope of the linear regression of X on Y. The probability distribution function of r, under the hypothesis that the variables X and Y are not correlated, is given by

$$f_r(r) = \frac{1}{\sqrt{\pi}} \frac{\Gamma(\frac{f+1}{2})}{\Gamma(\frac{f}{2})} \left(\frac{1}{1-r^2}\right)^{-\frac{f-2}{2}} \tag{A.15}$$

where N is the size of the sample, and $f = N- 2$ is the effective number of degrees of freedom of the dataset.

In Table A.24 we report the critical values of r calculated from the following equation,

$$1 - p = \int_{-r_{crit}}^{r_{crit}} f_r(r) dr \tag{A.16}$$

Table A.11 Critical values of F statistic that include $p = 0.70$ probability

	f_1									
f_2	2	4	6	8	10	20	40	60	80	100
1	5.056	5.830	6.117	6.267	6.358	6.544	6.639	6.671	6.687	6.697
2	2.333	2.561	2.640	2.681	2.705	2.754	2.779	2.787	2.791	2.794
3	1.847	1.985	2.028	2.048	2.061	2.084	2.096	2.100	2.102	2.103
4	1.651	1.753	1.781	1.793	1.800	1.812	1.818	1.819	1.820	1.821
5	1.547	1.629	1.648	1.656	1.659	1.665	1.666	1.666	1.667	1.667
6	1.481	1.551	1.565	1.570	1.571	1.572	1.570	1.570	1.569	1.569
7	1.437	1.499	1.509	1.511	1.511	1.507	1.504	1.502	1.501	1.501
8	1.405	1.460	1.467	1.468	1.466	1.460	1.455	1.452	1.451	1.450
9	1.380	1.431	1.436	1.435	1.433	1.424	1.417	1.414	1.413	1.412
10	1.361	1.408	1.412	1.409	1.406	1.395	1.387	1.384	1.382	1.381
20	1.279	1.311	1.305	1.297	1.290	1.268	1.252	1.245	1.242	1.240
30	1.254	1.280	1.271	1.261	1.253	1.226	1.206	1.197	1.192	1.189
40	1.241	1.264	1.255	1.243	1.234	1.205	1.182	1.172	1.167	1.163
50	1.233	1.255	1.245	1.233	1.223	1.192	1.167	1.156	1.150	1.146
60	1.228	1.249	1.238	1.226	1.215	1.183	1.157	1.146	1.139	1.135
70	1.225	1.245	1.233	1.221	1.210	1.177	1.150	1.138	1.131	1.127
80	1.222	1.242	1.230	1.217	1.206	1.172	1.144	1.132	1.125	1.120
90	1.220	1.239	1.227	1.214	1.203	1.168	1.140	1.127	1.120	1.115
100	1.219	1.237	1.225	1.212	1.200	1.165	1.137	1.123	1.116	1.111
200	1.211	1.228	1.215	1.201	1.189	1.152	1.121	1.106	1.097	1.091
∞	1.204	1.220	1.205	1.191	1.178	1.139	1.104	1.087	1.076	1.069

where p is the probability for a given value of the correlation coefficient to exceed, in absolute value, the critical value r_{crit}. The critical values are function of the number of degrees of freedom, and of the probability p.

To evaluate the probability distribution function in the case of large f, a convenient approximation can be given using the asymptotic expansion for the Gamma function (see [1]):

$$\Gamma(az + b) \simeq \sqrt{2\pi} e^{-az} (az)^{az+b-1/2}. \tag{A.17}$$

For large values of f, the ratio of the Gamma functions can therefore be approximated as

$$\frac{\Gamma\left(\frac{f+1}{2}\right)}{\Gamma\left(\frac{f}{2}\right)} \simeq \sqrt{\frac{f}{2}}.$$

Table A.12 Critical values of F statistic that include $p = 0.80$ probability

	f_1									
f_2	2	4	6	8	10	20	40	60	80	100
1	12.000	13.644	14.258	14.577	14.772	15.171	15.374	15.442	15.477	15.497
2	4.000	4.236	4.317	4.358	4.382	4.432	4.456	4.465	4.469	4.471
3	2.886	2.956	2.971	2.976	2.979	2.983	2.984	2.984	2.984	2.984
4	2.472	2.483	2.473	2.465	2.460	2.445	2.436	2.433	2.431	2.430
5	2.259	2.240	2.217	2.202	2.191	2.166	2.151	2.146	2.143	2.141
6	2.130	2.092	2.062	2.042	2.028	1.995	1.976	1.969	1.965	1.963
7	2.043	1.994	1.957	1.934	1.918	1.879	1.857	1.849	1.844	1.842
8	1.981	1.923	1.883	1.856	1.838	1.796	1.770	1.761	1.756	1.753
9	1.935	1.870	1.826	1.798	1.778	1.732	1.704	1.694	1.689	1.686
10	1.899	1.829	1.782	1.752	1.732	1.682	1.653	1.642	1.636	1.633
20	1.746	1.654	1.596	1.558	1.531	1.466	1.424	1.408	1.399	1.394
30	1.699	1.600	1.538	1.497	1.468	1.395	1.347	1.328	1.318	1.312
40	1.676	1.574	1.509	1.467	1.437	1.360	1.308	1.287	1.276	1.269
50	1.662	1.558	1.492	1.449	1.418	1.338	1.284	1.262	1.249	1.241
60	1.653	1.548	1.481	1.437	1.406	1.324	1.268	1.244	1.231	1.223
70	1.647	1.540	1.473	1.429	1.397	1.314	1.256	1.231	1.218	1.209
80	1.642	1.535	1.467	1.422	1.390	1.306	1.247	1.222	1.208	1.199
90	1.639	1.531	1.463	1.418	1.385	1.300	1.240	1.214	1.200	1.191
100	1.636	1.527	1.459	1.414	1.381	1.296	1.234	1.208	1.193	1.184
200	1.622	1.512	1.443	1.396	1.363	1.274	1.209	1.180	1.163	1.152
∞	1.609	1.497	1.426	1.379	1.344	1.252	1.182	1.150	1.130	1.117

A.7 The Kolmogorov–Smirnov Test

The one-sample Kolmogorov–Smirnov statistic D_N is defined in (13.7) as

$$D_N = \max_x |F_N(x) - F(x)|,$$

where $F(x)$ is the parent distribution, and $F_N(x)$ the sample distribution.

The cumulative distribution of the test statistic can be approximated by

$$P(D_N < z/(\sqrt{N} + 0.12 + 0.11/\sqrt{N})) \simeq \Phi(z).$$

where

$$\Phi(z) = \sum_{r=-\infty}^{\infty} (-1)^r e^{-2r^2z^2}.$$

Table A.13 Critical values of F statistic that include $p = 0.90$ probability

	f_1									
f_2	2	4	6	8	10	20	40	60	80	100
1	49.500	55.833	58.204	59.439	60.195	61.740	62.529	62.794	62.927	63.007
2	9.000	9.243	9.326	9.367	9.392	9.441	9.466	9.475	9.479	9.481
3	5.462	5.343	5.285	5.252	5.230	5.184	5.160	5.151	5.147	5.144
4	4.325	4.107	4.010	3.955	3.920	3.844	3.804	3.790	3.782	3.778
5	3.780	3.520	3.404	3.339	3.297	3.207	3.157	3.140	3.132	3.126
6	3.463	3.181	3.055	2.983	2.937	2.836	2.781	2.762	2.752	2.746
7	3.257	2.960	2.827	2.752	2.703	2.595	2.535	2.514	2.504	2.497
8	3.113	2.806	2.668	2.589	2.538	2.425	2.361	2.339	2.328	2.321
9	3.006	2.693	2.551	2.469	2.416	2.298	2.232	2.208	2.196	2.189
10	2.924	2.605	2.461	2.377	2.323	2.201	2.132	2.107	2.095	2.087
20	2.589	2.249	2.091	1.999	1.937	1.794	1.708	1.677	1.660	1.650
30	2.489	2.142	1.980	1.884	1.820	1.667	1.573	1.538	1.519	1.507
40	2.440	2.091	1.927	1.829	1.763	1.605	1.506	1.467	1.447	1.434
50	2.412	2.061	1.895	1.796	1.729	1.568	1.465	1.424	1.402	1.389
60	2.393	2.041	1.875	1.775	1.707	1.544	1.437	1.395	1.372	1.358
70	2.380	2.027	1.860	1.760	1.691	1.526	1.418	1.374	1.350	1.335
80	2.370	2.016	1.849	1.748	1.680	1.513	1.403	1.358	1.334	1.318
90	2.362	2.008	1.841	1.739	1.671	1.503	1.391	1.346	1.320	1.304
100	2.356	2.002	1.834	1.732	1.663	1.494	1.382	1.336	1.310	1.293
200	2.329	1.973	1.804	1.701	1.631	1.458	1.339	1.289	1.261	1.242
∞	2.303	1.945	1.774	1.670	1.599	1.421	1.295	1.240	1.207	1.185

and it is independent of the form of the parent distribution $F(x)$. For large values of N, we can use the asymptotic equation

$$P(D_N < z/\sqrt{N}) = \Phi(z).$$

In Table A.25 are listed the critical values of $\sqrt{N}D_N$ for various levels of probability. Values of the Kolmogorov–Smirnov statistic above the critical value indicate a rejection of the null hypothesis that the data are drawn from the parent model.

The two-sample Kolmogorov–Smirnov statistic is

$$D_{NM} = \max_x |F_M(x) - G_N(x)|$$

where $F_M(x)$ and $G_N(x)$ are the sample cumulative distribution of two independent sets of observations of size M and N. This statistic has the same distribution as the one-sample Kolmogorov-Smirnov statistic, with the substitution of $MN/(M+N)$ in place of N, and in the limit of large M and N, (13.12).

Table A.14 Critical values of F statistic that include $p = 0.95$ probability

	f_1									
f_2	2	4	6	8	10	20	40	60	80	100
1	199.500	224.583	233.986	238.883	241.882	248.013	251.143	252.196	252.724	253.041
2	19.000	19.247	19.330	19.371	19.396	19.446	19.471	19.479	19.483	19.486
3	9.552	9.117	8.941	8.845	8.786	8.660	8.594	8.572	8.561	8.554
4	6.944	6.388	6.163	6.041	5.964	5.803	5.717	5.688	5.673	5.664
5	5.786	5.192	4.950	4.818	4.735	4.558	4.464	4.431	4.415	4.405
6	5.143	4.534	4.284	4.147	4.060	3.874	3.774	3.740	3.722	3.712
7	4.737	4.120	3.866	3.726	3.636	3.444	3.340	3.304	3.286	3.275
8	4.459	3.838	3.581	3.438	3.347	3.150	3.043	3.005	2.986	2.975
9	4.256	3.633	3.374	3.230	3.137	2.936	2.826	2.787	2.768	2.756
10	4.103	3.478	3.217	3.072	2.978	2.774	2.661	2.621	2.601	2.588
20	3.493	2.866	2.599	2.447	2.348	2.124	1.994	1.946	1.922	1.907
30	3.316	2.690	2.421	2.266	2.165	1.932	1.792	1.740	1.712	1.695
40	3.232	2.606	2.336	2.180	2.077	1.839	1.693	1.637	1.608	1.589
50	3.183	2.557	2.286	2.130	2.026	1.784	1.634	1.576	1.544	1.525
60	3.150	2.525	2.254	2.097	1.992	1.748	1.594	1.534	1.502	1.481
70	3.128	2.503	2.231	2.074	1.969	1.722	1.566	1.504	1.471	1.450
80	3.111	2.486	2.214	2.056	1.951	1.703	1.545	1.482	1.448	1.426
90	3.098	2.473	2.201	2.043	1.938	1.688	1.528	1.464	1.429	1.407
100	3.087	2.463	2.191	2.032	1.927	1.677	1.515	1.450	1.414	1.392
200	3.041	2.417	2.144	1.985	1.878	1.623	1.455	1.385	1.346	1.321
∞	2.996	2.372	2.099	1.938	1.831	1.571	1.394	1.318	1.273	1.243

Table A.15 Critical values of F statistic that include $p = 0.99$ probability

	f_1									
f_2	2	4	6	8	10	20	40	60	80	100
1	4999.500	5624.583	5858.986	5981.070	6055.847	6208.730	6286.782	6313.030	6326.197	6334.110
2	99.000	99.249	99.333	99.374	99.399	99.449	99.474	99.482	99.487	99.489
3	30.817	28.710	27.911	27.489	27.229	26.690	26.411	26.316	26.269	26.240
4	18.000	15.977	15.207	14.799	14.546	14.020	13.745	13.652	13.605	13.577
5	13.274	11.392	10.672	10.289	10.051	9.553	9.291	9.202	9.157	9.130
6	10.925	9.148	8.466	8.102	7.874	7.396	7.143	7.057	7.013	6.987
7	9.547	7.847	7.191	6.840	6.620	6.155	5.908	5.824	5.781	5.755
8	8.649	7.006	6.371	6.029	5.814	5.359	5.116	5.032	4.989	4.963
9	8.022	6.422	5.802	5.467	5.257	4.808	4.567	4.483	4.441	4.415
10	7.559	5.994	5.386	5.057	4.849	4.405	4.165	4.082	4.039	4.014
20	5.849	4.431	3.871	3.564	3.368	2.938	2.695	2.608	2.563	2.535
30	5.390	4.018	3.473	3.173	2.979	2.549	2.299	2.208	2.160	2.131
40	5.178	3.828	3.291	2.993	2.801	2.369	2.114	2.019	1.969	1.938
50	5.057	3.720	3.186	2.890	2.698	2.265	2.006	1.909	1.857	1.825
60	4.977	3.649	3.119	2.823	2.632	2.198	1.936	1.836	1.783	1.749
70	4.922	3.600	3.071	2.777	2.585	2.150	1.886	1.784	1.730	1.696
80	4.881	3.563	3.036	2.742	2.551	2.115	1.849	1.746	1.690	1.655
90	4.849	3.535	3.009	2.715	2.524	2.088	1.820	1.716	1.659	1.623
100	4.824	3.513	2.988	2.694	2.503	2.067	1.797	1.692	1.634	1.598
200	4.713	3.414	2.893	2.601	2.411	1.971	1.695	1.584	1.521	1.481
∞	4.605	3.319	2.802	2.511	2.321	1.878	1.592	1.473	1.404	1.358

Table A.16 Integral of Student's function ($f = 1$), or probability p, as function of critical value T_{crit}

	0	1	2	3	4	5	6	7	8	9
0.0	0.000000	0.006366	0.012731	0.019093	0.025451	0.031805	0.038151	0.044491	0.050821	0.057142
0.1	0.063451	0.069748	0.076031	0.082299	0.088551	0.094786	0.101003	0.107201	0.113378	0.119533
0.2	0.125666	0.131775	0.137860	0.143920	0.149953	0.155958	0.161936	0.167884	0.173803	0.179691
0.3	0.185547	0.191372	0.197163	0.202921	0.208645	0.214334	0.219988	0.225605	0.231187	0.236731
0.4	0.242238	0.247707	0.253138	0.258530	0.263883	0.269197	0.274472	0.279706	0.284900	0.290054
0.5	0.295167	0.300240	0.305272	0.310262	0.315212	0.320120	0.324987	0.329813	0.334597	0.339340
0.6	0.344042	0.348702	0.353321	0.357899	0.362436	0.366932	0.371387	0.375801	0.380175	0.384508
0.7	0.388800	0.393053	0.397266	0.401438	0.405572	0.409666	0.413721	0.417737	0.421714	0.425653
0.8	0.429554	0.433417	0.437242	0.441030	0.444781	0.448495	0.452173	0.455814	0.459420	0.462990
0.9	0.466525	0.470025	0.473490	0.476920	0.480317	0.483680	0.487010	0.490306	0.493570	0.496801
1.0	0.500000	0.503167	0.506303	0.509408	0.512481	0.515524	0.518537	0.521520	0.524474	0.527398
1.1	0.530293	0.533159	0.535997	0.538807	0.541589	0.544344	0.547071	0.549772	0.552446	0.555094
1.2	0.557716	0.560312	0.562883	0.565429	0.567950	0.570447	0.572919	0.575367	0.577792	0.580193
1.3	0.582571	0.584927	0.587259	0.589570	0.591858	0.594124	0.596369	0.598592	0.600795	0.602976
1.4	0.605137	0.607278	0.609398	0.611499	0.613580	0.615641	0.617684	0.619707	0.621712	0.623698
1.5	0.625666	0.627616	0.629548	0.631462	0.633359	0.635239	0.637101	0.638947	0.640776	0.642589
1.6	0.644385	0.646165	0.647930	0.649678	0.651411	0.653129	0.654832	0.656519	0.658192	0.659850
1.7	0.661494	0.663124	0.664739	0.666340	0.667928	0.669502	0.671062	0.672609	0.674143	0.675663
1.8	0.677171	0.678666	0.680149	0.681619	0.683077	0.684522	0.685956	0.687377	0.688787	0.690185
1.9	0.691572	0.692947	0.694311	0.695664	0.697006	0.698337	0.699657	0.700967	0.702266	0.703555
2.0	0.704833	0.706101	0.707359	0.708607	0.709846	0.711074	0.712293	0.713502	0.714702	0.715893
2.1	0.717074	0.718246	0.719410	0.720564	0.721709	0.722846	0.723974	0.725093	0.726204	0.727307
2.2	0.728401	0.729487	0.730565	0.731635	0.732696	0.733750	0.734797	0.735835	0.736866	0.737889
2.3	0.738905	0.739914	0.740915	0.741909	0.742895	0.743875	0.744847	0.745813	0.746772	0.747723
2.4	0.748668	0.749607	0.750539	0.751464	0.752383	0.753295	0.754201	0.755101	0.755994	0.756881

(continued)

Table A.16 (continued)

	0	1	2	3	4	5	6	7	8	9
2.5	0.757762	0.758638	0.759507	0.760370	0.761227	0.762078	0.762924	0.763764	0.764598	0.765427
2.6	0.766250	0.767068	0.767880	0.768687	0.769488	0.770284	0.771075	0.771861	0.772642	0.773417
2.7	0.774188	0.774953	0.775714	0.776469	0.777220	0.777966	0.778707	0.779443	0.780175	0.780902
2.8	0.781625	0.782342	0.783056	0.783765	0.784469	0.785169	0.785865	0.786556	0.787243	0.787926
2.9	0.788605	0.789279	0.789949	0.790616	0.791278	0.791936	0.792590	0.793240	0.793887	0.794529
3.0	0.795168	0.795802	0.796433	0.797060	0.797684	0.798304	0.798920	0.799532	0.800141	0.800746
3.1	0.801348	0.801946	0.802541	0.803133	0.803720	0.804305	0.804886	0.805464	0.806039	0.806610
3.2	0.807178	0.807743	0.808304	0.808863	0.809418	0.809970	0.810519	0.811065	0.811608	0.812148
3.3	0.812685	0.813219	0.813750	0.814278	0.814803	0.815325	0.815845	0.816361	0.816875	0.817386
3.4	0.817894	0.818400	0.818903	0.819403	0.819900	0.820395	0.820887	0.821376	0.821863	0.822348
3.5	0.822829	0.823308	0.823785	0.824259	0.824731	0.825200	0.825667	0.826131	0.826593	0.827053
3.6	0.827510	0.827965	0.828418	0.828868	0.829316	0.829761	0.830205	0.830646	0.831085	0.831521
3.7	0.831956	0.832388	0.832818	0.833246	0.833672	0.834096	0.834517	0.834937	0.835354	0.835770
3.8	0.836183	0.836594	0.837004	0.837411	0.837816	0.838220	0.838621	0.839020	0.839418	0.839813
3.9	0.840207	0.840599	0.840989	0.841377	0.841763	0.842147	0.842530	0.842911	0.843290	0.843667
4.0	0.844042	0.844416	0.844788	0.845158	0.845526	0.845893	0.846258	0.846621	0.846983	0.847343
4.1	0.847701	0.848057	0.848412	0.848766	0.849118	0.849468	0.849816	0.850163	0.850509	0.850853
4.2	0.851195	0.851536	0.851875	0.852213	0.852549	0.852883	0.853217	0.853548	0.853879	0.854208
4.3	0.854535	0.854861	0.855185	0.855508	0.855830	0.856150	0.856469	0.856787	0.857103	0.857417
4.4	0.857731	0.858043	0.858353	0.858663	0.858971	0.859277	0.859583	0.859887	0.860190	0.860491
4.5	0.860791	0.861090	0.861388	0.861684	0.861980	0.862274	0.862566	0.862858	0.863148	0.863437
4.6	0.863725	0.864012	0.864297	0.864582	0.864865	0.865147	0.865428	0.865707	0.865986	0.866263
4.7	0.866539	0.866815	0.867089	0.867362	0.867633	0.867904	0.868174	0.868442	0.868710	0.868976
4.8	0.869242	0.869506	0.869769	0.870031	0.870292	0.870553	0.870812	0.871070	0.871327	0.871583
4.9	0.871838	0.872092	0.872345	0.872597	0.872848	0.873098	0.873347	0.873596	0.873843	0.874089

Table A.17 Integral of Student's function (f = 2), or probability p, as function of critical value T_{crit}

	0	1	2	3	4	5	6	7	8	9
0.0	0.000000	0.007071	0.014141	0.021208	0.028273	0.035333	0.042388	0.049437	0.056478	0.063511
0.1	0.070535	0.077548	0.084549	0.091538	0.098513	0.105474	0.112420	0.119349	0.126261	0.133154
0.2	0.140028	0.146882	0.153715	0.160526	0.167313	0.174078	0.180817	0.187532	0.194220	0.200881
0.3	0.207514	0.214119	0.220695	0.227241	0.233756	0.240239	0.246691	0.253110	0.259496	0.265848
0.4	0.272166	0.278448	0.284695	0.290906	0.297080	0.303218	0.309318	0.315380	0.321403	0.327388
0.5	0.333333	0.339240	0.345106	0.350932	0.356718	0.362462	0.368166	0.373829	0.379450	0.385029
0.6	0.390567	0.396062	0.401516	0.406926	0.412295	0.417620	0.422903	0.428144	0.433341	0.438496
0.7	0.443607	0.448676	0.453702	0.458684	0.463624	0.468521	0.473376	0.478187	0.482956	0.487682
0.8	0.492366	0.497008	0.501607	0.506164	0.510679	0.515152	0.519583	0.523973	0.528322	0.532629
0.9	0.536895	0.541121	0.545306	0.549450	0.553555	0.557619	0.561644	0.565629	0.569575	0.573482
1.0	0.577351	0.581180	0.584972	0.588725	0.592441	0.596120	0.599761	0.603366	0.606933	0.610465
1.1	0.613960	0.617420	0.620844	0.624233	0.627588	0.630907	0.634193	0.637444	0.640662	0.643846
1.2	0.646997	0.650115	0.653201	0.656255	0.659276	0.662266	0.665225	0.668153	0.671050	0.673917
1.3	0.676753	0.679560	0.682337	0.685085	0.687804	0.690494	0.693156	0.695790	0.698397	0.700975
1.4	0.703527	0.706051	0.708549	0.711021	0.713466	0.715886	0.718280	0.720649	0.722993	0.725312
1.5	0.727607	0.729878	0.732125	0.734348	0.736547	0.738724	0.740878	0.743009	0.745118	0.747204
1.6	0.749269	0.751312	0.753334	0.755335	0.757314	0.759273	0.761212	0.763130	0.765029	0.766908
1.7	0.768767	0.770607	0.772428	0.774230	0.776013	0.777778	0.779525	0.781254	0.782965	0.784658
1.8	0.786334	0.787993	0.789635	0.791260	0.792868	0.794460	0.796036	0.797596	0.799140	0.800668
1.9	0.802181	0.803679	0.805161	0.806628	0.808081	0.809519	0.810943	0.812352	0.813748	0.815129
2.0	0.816497	0.817851	0.819192	0.820519	0.821833	0.823134	0.824423	0.825698	0.826961	0.828212
2.1	0.829450	0.830677	0.831891	0.833094	0.834285	0.835464	0.836632	0.837788	0.838934	0.840068
2.2	0.841191	0.842304	0.843406	0.844497	0.845578	0.846649	0.847710	0.848760	0.849801	0.850831
2.3	0.851852	0.852864	0.853865	0.854858	0.855841	0.856815	0.857780	0.858735	0.859682	0.860621
2.4	0.861550	0.862471	0.863384	0.864288	0.865183	0.866071	0.866950	0.867822	0.868685	0.869541

(continued)

Table A.17 (continued)

	0	1	2	3	4	5	6	7	8	9
2.5	0.870389	0.871229	0.872061	0.872887	0.873704	0.874515	0.875318	0.876114	0.876902	0.877684
2.6	0.878459	0.879227	0.879988	0.880743	0.881490	0.882232	0.882966	0.883695	0.884417	0.885132
2.7	0.885842	0.886545	0.887242	0.887933	0.888618	0.889298	0.889971	0.890639	0.891301	0.891957
2.8	0.892608	0.893253	0.893893	0.894527	0.895156	0.895780	0.896398	0.897011	0.897619	0.898222
2.9	0.898820	0.899413	0.900001	0.900584	0.901163	0.901736	0.902305	0.902869	0.903429	0.903984
3.0	0.904534	0.905080	0.905622	0.906159	0.906692	0.907220	0.907745	0.908265	0.908780	0.909292
3.1	0.909800	0.910303	0.910803	0.911298	0.911790	0.912278	0.912762	0.913242	0.913718	0.914191
3.2	0.914660	0.915125	0.915586	0.916044	0.916499	0.916950	0.917397	0.917841	0.918282	0.918719
3.3	0.919153	0.919583	0.920010	0.920434	0.920855	0.921273	0.921687	0.922098	0.922507	0.922912
3.4	0.923314	0.923713	0.924109	0.924502	0.924892	0.925279	0.925664	0.926045	0.926424	0.926800
3.5	0.927173	0.927543	0.927911	0.928276	0.928639	0.928998	0.929355	0.929710	0.930062	0.930411
3.6	0.930758	0.931103	0.931445	0.931784	0.932121	0.932456	0.932788	0.933118	0.933445	0.933771
3.7	0.934094	0.934414	0.934733	0.935049	0.935363	0.935675	0.935984	0.936292	0.936597	0.936900
3.8	0.937201	0.937500	0.937797	0.938092	0.938385	0.938676	0.938965	0.939252	0.939537	0.939820
3.9	0.940101	0.940380	0.940657	0.940933	0.941206	0.941478	0.941748	0.942016	0.942282	0.942547
4.0	0.942809	0.943070	0.943330	0.943587	0.943843	0.944097	0.944350	0.944601	0.944850	0.945098
4.1	0.945343	0.945588	0.945831	0.946072	0.946311	0.946550	0.946786	0.947021	0.947255	0.947487
4.2	0.947717	0.947946	0.948174	0.948400	0.948625	0.948848	0.949070	0.949290	0.949509	0.949727
4.3	0.949943	0.950158	0.950372	0.950584	0.950795	0.951005	0.951213	0.951420	0.951626	0.951830
4.4	0.952034	0.952235	0.952436	0.952636	0.952834	0.953031	0.953227	0.953422	0.953615	0.953807
4.5	0.953998	0.954188	0.954377	0.954565	0.954752	0.954937	0.955121	0.955305	0.955487	0.955668
4.6	0.955848	0.956027	0.956205	0.956381	0.956557	0.956732	0.956905	0.957078	0.957250	0.957420
4.7	0.957590	0.957759	0.957926	0.958093	0.958259	0.958424	0.958587	0.958750	0.958912	0.959073
4.8	0.959233	0.959392	0.959551	0.959708	0.959865	0.960020	0.960175	0.960329	0.960481	0.960634
4.9	0.960785	0.960935	0.961085	0.961233	0.961381	0.961528	0.961674	0.961820	0.961964	0.962108

Table A.18 Integral of Student's function (f = 3), or probability p, as function of critical value T_{crit}

	0	1	2	3	4	5	6	7	8	9
0.0	0.000000	0.007351	0.014701	0.022049	0.029394	0.036735	0.044071	0.051401	0.058725	0.066041
0.1	0.073348	0.080645	0.087932	0.095207	0.102469	0.109718	0.116953	0.124172	0.131375	0.138562
0.2	0.145730	0.152879	0.160009	0.167118	0.174205	0.181271	0.188313	0.195332	0.202326	0.209294
0.3	0.216237	0.223152	0.230040	0.236900	0.243730	0.250531	0.257301	0.264040	0.270748	0.277423
0.4	0.284065	0.290674	0.297248	0.303788	0.310293	0.316761	0.323194	0.329590	0.335948	0.342269
0.5	0.348552	0.354796	0.361002	0.367168	0.373294	0.379380	0.385426	0.391431	0.397395	0.403318
0.6	0.409199	0.415038	0.420835	0.426590	0.432302	0.437972	0.443599	0.449182	0.454723	0.460220
0.7	0.465673	0.471083	0.476449	0.481772	0.487051	0.492286	0.497477	0.502624	0.507727	0.512786
0.8	0.517801	0.522773	0.527700	0.532584	0.537424	0.542220	0.546973	0.551682	0.556348	0.560970
0.9	0.565549	0.570085	0.574579	0.579029	0.583437	0.587802	0.592125	0.596406	0.600645	0.604842
1.0	0.608998	0.613112	0.617186	0.621218	0.625209	0.629160	0.633071	0.636942	0.640773	0.644565
1.1	0.648317	0.652030	0.655705	0.659341	0.662939	0.666499	0.670021	0.673506	0.676953	0.680364
1.2	0.683738	0.687076	0.690378	0.693644	0.696875	0.700071	0.703232	0.706358	0.709450	0.712508
1.3	0.715533	0.718524	0.721482	0.724407	0.727300	0.730161	0.732990	0.735787	0.738553	0.741289
1.4	0.743993	0.746667	0.749311	0.751925	0.754510	0.757066	0.759593	0.762091	0.764561	0.767002
1.5	0.769416	0.771803	0.774163	0.776495	0.778801	0.781081	0.783335	0.785563	0.787766	0.789943
1.6	0.792096	0.794223	0.796327	0.798406	0.800462	0.802494	0.804503	0.806488	0.808451	0.810392
1.7	0.812310	0.814206	0.816080	0.817933	0.819764	0.821575	0.823365	0.825134	0.826883	0.828611
1.8	0.830320	0.832010	0.833680	0.835331	0.836962	0.838576	0.840170	0.841747	0.843305	0.844846
1.9	0.846369	0.847874	0.849363	0.850834	0.852289	0.853727	0.855148	0.856554	0.857943	0.859316
2.0	0.860674	0.862017	0.863344	0.864656	0.865954	0.867236	0.868504	0.869758	0.870998	0.872223
2.1	0.873435	0.874633	0.875818	0.876989	0.878147	0.879292	0.880425	0.881544	0.882651	0.883746
2.2	0.884828	0.885899	0.886957	0.888004	0.889039	0.890062	0.891074	0.892075	0.893065	0.894044
2.3	0.895012	0.895969	0.896916	0.897853	0.898779	0.899695	0.900600	0.901496	0.902383	0.903259
2.4	0.904126	0.904983	0.905831	0.906670	0.907500	0.908321	0.909133	0.909936	0.910730	0.911516

(continued)

Table A.18 (continued)

	0	1	2	3	4	5	6	7	8	9
2.5	0.912294	0.913063	0.913824	0.914576	0.915321	0.916057	0.916786	0.917507	0.918221	0.918926
2.6	0.919625	0.920315	0.920999	0.921675	0.922344	0.923007	0.923662	0.924310	0.924951	0.925586
2.7	0.926214	0.926836	0.927451	0.928060	0.928662	0.929258	0.929848	0.930432	0.931010	0.931582
2.8	0.932147	0.932708	0.933262	0.933811	0.934354	0.934891	0.935423	0.935950	0.936471	0.936987
2.9	0.937498	0.938004	0.938504	0.939000	0.939490	0.939976	0.940456	0.940932	0.941403	0.941870
3.0	0.942332	0.942789	0.943241	0.943689	0.944133	0.944572	0.945007	0.945438	0.945864	0.946287
3.1	0.946705	0.947119	0.947529	0.947935	0.948337	0.948735	0.949129	0.949520	0.949906	0.950289
3.2	0.950669	0.951044	0.951416	0.951785	0.952149	0.952511	0.952869	0.953223	0.953575	0.953922
3.3	0.954267	0.954608	0.954946	0.955281	0.955613	0.955942	0.956267	0.956590	0.956909	0.957226
3.4	0.957539	0.957850	0.958157	0.958462	0.958764	0.959064	0.959360	0.959654	0.959945	0.960234
3.5	0.960519	0.960803	0.961083	0.961361	0.961637	0.961910	0.962180	0.962448	0.962714	0.962977
3.6	0.963238	0.963497	0.963753	0.964007	0.964259	0.964508	0.964755	0.965000	0.965243	0.965484
3.7	0.965722	0.965959	0.966193	0.966426	0.966656	0.966884	0.967110	0.967335	0.967557	0.967777
3.8	0.967996	0.968212	0.968427	0.968640	0.968851	0.969060	0.969268	0.969473	0.969677	0.969879
3.9	0.970079	0.970278	0.970475	0.970670	0.970864	0.971056	0.971246	0.971435	0.971622	0.971808
4.0	0.971992	0.972174	0.972355	0.972535	0.972713	0.972889	0.973064	0.973238	0.973410	0.973581
4.1	0.973750	0.973918	0.974084	0.974250	0.974413	0.974576	0.974737	0.974897	0.975055	0.975212
4.2	0.975368	0.975523	0.975676	0.975829	0.975980	0.976129	0.976278	0.976425	0.976571	0.976716
4.3	0.976860	0.977003	0.977144	0.977285	0.977424	0.977562	0.977699	0.977835	0.977970	0.978104
4.4	0.978237	0.978369	0.978500	0.978630	0.978758	0.978886	0.979013	0.979139	0.979263	0.979387
4.5	0.979510	0.979632	0.979753	0.979873	0.979992	0.980110	0.980228	0.980344	0.980460	0.980574
4.6	0.980688	0.980801	0.980913	0.981024	0.981135	0.981244	0.981353	0.981461	0.981568	0.981674
4.7	0.981780	0.981884	0.981988	0.982091	0.982194	0.982295	0.982396	0.982496	0.982596	0.982694
4.8	0.982792	0.982890	0.982986	0.983082	0.983177	0.983271	0.983365	0.983458	0.983550	0.983642
4.9	0.983733	0.983823	0.983913	0.984002	0.984091	0.984178	0.984266	0.984352	0.984438	0.984523

Table A.19 Integral of Student's function (f = 5), or probability p, as function of critical value T_{crit}

	0	1	2	3	4	5	6	7	8	9
0.0	0.000000	0.007592	0.015183	0.022772	0.030359	0.037942	0.045520	0.053093	0.060659	0.068219
0.1	0.075770	0.083312	0.090844	0.098366	0.105875	0.113372	0.120856	0.128325	0.135780	0.143218
0.2	0.150640	0.158043	0.165429	0.172795	0.180141	0.187466	0.194769	0.202050	0.209308	0.216542
0.3	0.223751	0.230935	0.238092	0.245223	0.252326	0.259401	0.266446	0.273463	0.280448	0.287403
0.4	0.294327	0.301218	0.308077	0.314902	0.321694	0.328451	0.335173	0.341860	0.348510	0.355124
0.5	0.361701	0.368241	0.374742	0.381206	0.387630	0.394015	0.400361	0.406667	0.412932	0.419156
0.6	0.425340	0.431482	0.437582	0.443641	0.449657	0.455630	0.461561	0.467449	0.473293	0.479094
0.7	0.484851	0.490565	0.496234	0.501859	0.507440	0.512976	0.518468	0.523915	0.529317	0.534674
0.8	0.539986	0.545253	0.550476	0.555653	0.560785	0.565871	0.570913	0.575910	0.580861	0.585768
0.9	0.590629	0.595445	0.600217	0.604944	0.609626	0.614263	0.618855	0.623404	0.627908	0.632367
1.0	0.636783	0.641154	0.645482	0.649767	0.654007	0.658205	0.662359	0.666471	0.670539	0.674566
1.1	0.678549	0.682491	0.686391	0.690249	0.694066	0.697841	0.701576	0.705270	0.708923	0.712536
1.2	0.716109	0.719643	0.723136	0.726591	0.730007	0.733384	0.736723	0.740023	0.743286	0.746512
1.3	0.749700	0.752851	0.755965	0.759043	0.762085	0.765092	0.768062	0.770998	0.773899	0.776765
1.4	0.779596	0.782394	0.785158	0.787889	0.790587	0.793252	0.795885	0.798485	0.801054	0.803591
1.5	0.806097	0.808572	0.811016	0.813430	0.815814	0.818168	0.820493	0.822789	0.825056	0.827295
1.6	0.829505	0.831688	0.833843	0.835970	0.838071	0.840145	0.842192	0.844213	0.846209	0.848179
1.7	0.850124	0.852043	0.853938	0.855809	0.857655	0.859478	0.861277	0.863053	0.864805	0.866535
1.8	0.868243	0.869928	0.871591	0.873233	0.874853	0.876452	0.878030	0.879587	0.881124	0.882640
1.9	0.884137	0.885614	0.887072	0.888510	0.889930	0.891330	0.892712	0.894076	0.895422	0.896750
2.0	0.898061	0.899354	0.900630	0.901889	0.903132	0.904358	0.905567	0.906761	0.907938	0.909101
2.1	0.910247	0.911378	0.912495	0.913596	0.914683	0.915755	0.916813	0.917857	0.918887	0.919904
2.2	0.920906	0.921896	0.922872	0.923835	0.924786	0.925723	0.926649	0.927562	0.928462	0.929351
2.3	0.930228	0.931093	0.931947	0.932789	0.933620	0.934440	0.935249	0.936048	0.936835	0.937613
2.4	0.938380	0.939136	0.939883	0.940620	0.941347	0.942064	0.942772	0.943470	0.944159	0.944839

(continued)

Table A.19 (continued)

	0	1	2	3	4	5	6	7	8	9
2.5	0.945510	0.946172	0.946826	0.947470	0.948107	0.948734	0.949354	0.949965	0.950568	0.951164
2.6	0.951751	0.952331	0.952903	0.953467	0.954024	0.954574	0.955116	0.955652	0.956180	0.956702
2.7	0.957216	0.957724	0.958225	0.958720	0.959208	0.959690	0.960166	0.960635	0.961098	0.961556
2.8	0.962007	0.962452	0.962892	0.963326	0.963754	0.964177	0.964594	0.965006	0.965412	0.965814
2.9	0.966210	0.966601	0.966987	0.967368	0.967744	0.968115	0.968482	0.968843	0.969200	0.969553
3.0	0.969901	0.970245	0.970584	0.970919	0.971250	0.971576	0.971898	0.972217	0.972531	0.972841
3.1	0.973147	0.973450	0.973748	0.974043	0.974334	0.974621	0.974905	0.975185	0.975462	0.975735
3.2	0.976005	0.976272	0.976535	0.976795	0.977051	0.977305	0.977555	0.977802	0.978046	0.978287
3.3	0.978525	0.978760	0.978992	0.979221	0.979448	0.979672	0.979893	0.980111	0.980326	0.980539
3.4	0.980749	0.980957	0.981162	0.981365	0.981565	0.981763	0.981958	0.982151	0.982342	0.982530
3.5	0.982716	0.982900	0.983081	0.983261	0.983438	0.983613	0.983786	0.983957	0.984126	0.984292
3.6	0.984457	0.984620	0.984781	0.984940	0.985097	0.985252	0.985406	0.985557	0.985707	0.985855
3.7	0.986001	0.986146	0.986288	0.986429	0.986569	0.986707	0.986843	0.986977	0.987111	0.987242
3.8	0.987372	0.987500	0.987627	0.987753	0.987877	0.987999	0.988120	0.988240	0.988359	0.988475
3.9	0.988591	0.988705	0.988818	0.988930	0.989041	0.989150	0.989258	0.989364	0.989470	0.989574
4.0	0.989677	0.989779	0.989880	0.989979	0.990078	0.990175	0.990271	0.990366	0.990461	0.990554
4.1	0.990646	0.990737	0.990826	0.990915	0.991003	0.991090	0.991176	0.991261	0.991345	0.991429
4.2	0.991511	0.991592	0.991673	0.991752	0.991831	0.991909	0.991986	0.992062	0.992137	0.992211
4.3	0.992285	0.992358	0.992430	0.992501	0.992572	0.992641	0.992710	0.992778	0.992846	0.992913
4.4	0.992979	0.993044	0.993108	0.993172	0.993236	0.993298	0.993360	0.993421	0.993482	0.993542
4.5	0.993601	0.993660	0.993718	0.993775	0.993832	0.993888	0.993944	0.993999	0.994053	0.994107
4.6	0.994160	0.994213	0.994265	0.994317	0.994368	0.994418	0.994468	0.994518	0.994567	0.994615
4.7	0.994663	0.994711	0.994758	0.994804	0.994850	0.994896	0.994941	0.994986	0.995030	0.995073
4.8	0.995117	0.995160	0.995202	0.995244	0.995285	0.995327	0.995367	0.995408	0.995447	0.995487
4.9	0.995526	0.995565	0.995603	0.995641	0.995678	0.995715	0.995752	0.995789	0.995825	0.995860

Table A.20 Integral of Student's function (f = 10), or probability p, as function of critical value T_{crit}

	0	1	2	3	4	5	6	7	8	9
0.0	0.000000	0.007782	0.015563	0.023343	0.031120	0.038893	0.046662	0.054426	0.062184	0.069936
0.1	0.077679	0.085414	0.093140	0.100856	0.108560	0.116253	0.123933	0.131600	0.139252	0.146889
0.2	0.154511	0.162116	0.169703	0.177272	0.184822	0.192352	0.199861	0.207350	0.214816	0.222260
0.3	0.229679	0.237075	0.244446	0.251791	0.259110	0.266402	0.273666	0.280902	0.288109	0.295286
0.4	0.302433	0.309549	0.316633	0.323686	0.330706	0.337692	0.344645	0.351563	0.358447	0.365295
0.5	0.372107	0.378882	0.385621	0.392322	0.398985	0.405610	0.412197	0.418744	0.425251	0.431718
0.6	0.438145	0.444531	0.450876	0.457179	0.463441	0.469660	0.475837	0.481971	0.488061	0.494109
0.7	0.500113	0.506072	0.511988	0.517860	0.523686	0.529468	0.535205	0.540897	0.546543	0.552144
0.8	0.557700	0.563209	0.568673	0.574091	0.579463	0.584788	0.590067	0.595300	0.600487	0.605627
0.9	0.610721	0.615768	0.620769	0.625724	0.630632	0.635494	0.640309	0.645078	0.649800	0.654477
1.0	0.659107	0.663691	0.668230	0.672722	0.677169	0.681569	0.685925	0.690235	0.694499	0.698719
1.1	0.702893	0.707023	0.711108	0.715149	0.719145	0.723097	0.727006	0.730870	0.734691	0.738469
1.2	0.742204	0.745896	0.749545	0.753152	0.756717	0.760240	0.763721	0.767161	0.770559	0.773917
1.3	0.777235	0.780511	0.783748	0.786945	0.790103	0.793221	0.796301	0.799341	0.802344	0.805308
1.4	0.808235	0.811124	0.813976	0.816792	0.819570	0.822313	0.825019	0.827690	0.830326	0.832927
1.5	0.835493	0.838025	0.840523	0.842987	0.845417	0.847815	0.850180	0.852512	0.854813	0.857081
1.6	0.859319	0.861525	0.863700	0.865845	0.867959	0.870044	0.872099	0.874125	0.876122	0.878091
1.7	0.880031	0.881943	0.883828	0.885685	0.887516	0.889319	0.891097	0.892848	0.894573	0.896273
1.8	0.897948	0.899598	0.901223	0.902825	0.904402	0.905955	0.907485	0.908992	0.910477	0.911938
1.9	0.913378	0.914796	0.916191	0.917566	0.918919	0.920252	0.921564	0.922856	0.924128	0.925380
2.0	0.926612	0.927826	0.929020	0.930196	0.931353	0.932492	0.933613	0.934717	0.935803	0.936871
2.1	0.937923	0.938958	0.939977	0.940979	0.941965	0.942936	0.943891	0.944830	0.945755	0.946664
2.2	0.947559	0.948440	0.949306	0.950158	0.950996	0.951821	0.952632	0.953430	0.954215	0.954987
2.3	0.955746	0.956493	0.957228	0.957950	0.958661	0.959360	0.960047	0.960723	0.961388	0.962042
2.4	0.962685	0.963317	0.963939	0.964550	0.965151	0.965743	0.966324	0.966896	0.967458	0.968010

(continued)

Table A.20 (continued)

	0	1	2	3	4	5	6	7	8	9
2.5	0.968554	0.969088	0.969613	0.970130	0.970637	0.971136	0.971627	0.972110	0.972584	0.973050
2.6	0.973509	0.973960	0.974403	0.974838	0.975266	0.975687	0.976101	0.976508	0.976908	0.977301
2.7	0.977687	0.978067	0.978440	0.978807	0.979168	0.979522	0.979871	0.980213	0.980550	0.980881
2.8	0.981206	0.981526	0.981840	0.982149	0.982452	0.982750	0.983044	0.983332	0.983615	0.983893
2.9	0.984167	0.984436	0.984700	0.984960	0.985215	0.985466	0.985712	0.985955	0.986193	0.986427
3.0	0.986657	0.986883	0.987105	0.987323	0.987538	0.987749	0.987956	0.988160	0.988360	0.988556
3.1	0.988750	0.988940	0.989126	0.989310	0.989490	0.989667	0.989842	0.990013	0.990181	0.990346
3.2	0.990509	0.990668	0.990825	0.990979	0.991131	0.991280	0.991426	0.991570	0.991711	0.991850
3.3	0.991987	0.992121	0.992253	0.992383	0.992510	0.992635	0.992758	0.992879	0.992998	0.993115
3.4	0.993229	0.993342	0.993453	0.993562	0.993669	0.993774	0.993878	0.993979	0.994079	0.994177
3.5	0.994274	0.994369	0.994462	0.994554	0.994644	0.994732	0.994819	0.994905	0.994989	0.995071
3.6	0.995153	0.995232	0.995311	0.995388	0.995464	0.995538	0.995611	0.995683	0.995754	0.995824
3.7	0.995892	0.995959	0.996025	0.996090	0.996154	0.996217	0.996278	0.996339	0.996398	0.996457
3.8	0.996515	0.996571	0.996627	0.996682	0.996735	0.996788	0.996840	0.996891	0.996941	0.996991
3.9	0.997039	0.997087	0.997134	0.997180	0.997226	0.997270	0.997314	0.997357	0.997399	0.997441
4.0	0.997482	0.997522	0.997562	0.997601	0.997639	0.997677	0.997714	0.997750	0.997786	0.997821
4.1	0.997856	0.997890	0.997923	0.997956	0.997989	0.998020	0.998052	0.998082	0.998113	0.998142
4.2	0.998172	0.998201	0.998229	0.998257	0.998284	0.998311	0.998337	0.998363	0.998389	0.998414
4.3	0.998439	0.998463	0.998487	0.998511	0.998534	0.998557	0.998579	0.998601	0.998623	0.998644
4.4	0.998665	0.998686	0.998706	0.998726	0.998746	0.998765	0.998784	0.998803	0.998821	0.998840
4.5	0.998857	0.998875	0.998892	0.998909	0.998926	0.998942	0.998958	0.998974	0.998990	0.999005
4.6	0.999020	0.999035	0.999050	0.999064	0.999078	0.999092	0.999106	0.999120	0.999133	0.999146
4.7	0.999159	0.999172	0.999184	0.999196	0.999208	0.999220	0.999232	0.999243	0.999255	0.999266
4.8	0.999277	0.999288	0.999298	0.999309	0.999319	0.999329	0.999339	0.999349	0.999358	0.999368
4.9	0.999377	0.999387	0.999396	0.999404	0.999413	0.999422	0.999430	0.999439	0.999447	0.999455

Table A.21 Integral of Student's function (f = 20), or probability p, as function of critical value T_{crit}

	0	1	2	3	4	5	6	7	8	9
0.0	0.000000	0.007880	0.015758	0.023636	0.031510	0.039382	0.047249	0.055111	0.062968	0.070818
0.1	0.078660	0.086494	0.094320	0.102135	0.109940	0.117733	0.125514	0.133282	0.141036	0.148776
0.2	0.156500	0.164208	0.171899	0.179572	0.187227	0.194863	0.202478	0.210073	0.217647	0.225199
0.3	0.232727	0.240232	0.247713	0.255169	0.262599	0.270003	0.277379	0.284728	0.292049	0.299341
0.4	0.306604	0.313836	0.321037	0.328207	0.335345	0.342450	0.349522	0.356561	0.363565	0.370534
0.5	0.377468	0.384366	0.391228	0.398053	0.404841	0.411591	0.418302	0.424975	0.431608	0.438202
0.6	0.444756	0.451269	0.457742	0.464174	0.470563	0.476911	0.483217	0.489480	0.495700	0.501877
0.7	0.508010	0.514099	0.520145	0.526146	0.532102	0.538013	0.543879	0.549700	0.555476	0.561206
0.8	0.566889	0.572527	0.578119	0.583664	0.589162	0.594614	0.600019	0.605378	0.610689	0.615953
0.9	0.621171	0.626341	0.631463	0.636539	0.641567	0.646548	0.651482	0.656368	0.661207	0.665999
1.0	0.670744	0.675441	0.680091	0.684694	0.689250	0.693759	0.698222	0.702637	0.707006	0.711328
1.1	0.715604	0.719833	0.724016	0.728154	0.732245	0.736291	0.740291	0.744245	0.748155	0.752019
1.2	0.755839	0.759614	0.763344	0.767031	0.770673	0.774271	0.777826	0.781338	0.784807	0.788233
1.3	0.791616	0.794957	0.798256	0.801513	0.804728	0.807903	0.811036	0.814129	0.817181	0.820193
1.4	0.823165	0.826098	0.828992	0.831846	0.834662	0.837440	0.840180	0.842882	0.845546	0.848174
1.5	0.850765	0.853319	0.855837	0.858320	0.860767	0.863179	0.865556	0.867899	0.870207	0.872482
1.6	0.874723	0.876932	0.879107	0.881250	0.883361	0.885440	0.887487	0.889503	0.891489	0.893444
1.7	0.895369	0.897264	0.899130	0.900967	0.902775	0.904554	0.906305	0.908029	0.909725	0.911394
1.8	0.913036	0.914651	0.916241	0.917804	0.919342	0.920855	0.922343	0.923806	0.925245	0.926660
1.9	0.928052	0.929420	0.930765	0.932088	0.933388	0.934666	0.935922	0.937157	0.938370	0.939563
2.0	0.940735	0.941886	0.943018	0.944130	0.945222	0.946295	0.947350	0.948385	0.949402	0.950402
2.1	0.951383	0.952347	0.953293	0.954222	0.955135	0.956031	0.956911	0.957774	0.958622	0.959455
2.2	0.960272	0.961074	0.961861	0.962634	0.963392	0.964136	0.964866	0.965583	0.966286	0.966976
2.3	0.967653	0.968317	0.968969	0.969608	0.970235	0.970850	0.971453	0.972044	0.972624	0.973194
2.4	0.973752	0.974299	0.974835	0.975361	0.975877	0.976383	0.976879	0.977365	0.977842	0.978309

(continued)

Table A.21 (continued)

	0	1	2	3	4	5	6	7	8	9
2.5	0.978767	0.979216	0.979656	0.980087	0.980510	0.980924	0.981330	0.981727	0.982117	0.982499
2.6	0.982873	0.983240	0.983599	0.983951	0.984296	0.984634	0.984965	0.985289	0.985607	0.985918
2.7	0.986222	0.986521	0.986813	0.987099	0.987380	0.987654	0.987923	0.988186	0.988444	0.988696
2.8	0.988943	0.989185	0.989422	0.989654	0.989881	0.990103	0.990321	0.990534	0.990743	0.990947
2.9	0.991146	0.991342	0.991533	0.991721	0.991904	0.992083	0.992259	0.992431	0.992599	0.992764
3.0	0.992925	0.993082	0.993236	0.993387	0.993535	0.993679	0.993820	0.993959	0.994094	0.994226
3.1	0.994356	0.994482	0.994606	0.994727	0.994846	0.994962	0.995075	0.995186	0.995294	0.995400
3.2	0.995504	0.995606	0.995705	0.995802	0.995897	0.995990	0.996081	0.996169	0.996256	0.996341
3.3	0.996424	0.996505	0.996585	0.996662	0.996738	0.996812	0.996885	0.996956	0.997025	0.997093
3.4	0.997159	0.997224	0.997287	0.997349	0.997410	0.997469	0.997527	0.997583	0.997638	0.997692
3.5	0.997745	0.997797	0.997847	0.997897	0.997945	0.997992	0.998038	0.998083	0.998127	0.998170
3.6	0.998212	0.998253	0.998293	0.998333	0.998371	0.998408	0.998445	0.998481	0.998516	0.998550
3.7	0.998583	0.998616	0.998648	0.998679	0.998709	0.998739	0.998768	0.998797	0.998824	0.998851
3.8	0.998878	0.998904	0.998929	0.998954	0.998978	0.999002	0.999025	0.999047	0.999069	0.999091
3.9	0.999112	0.999132	0.999152	0.999172	0.999191	0.999210	0.999228	0.999246	0.999263	0.999280
4.0	0.999297	0.999313	0.999329	0.999345	0.999360	0.999375	0.999389	0.999403	0.999417	0.999430
4.1	0.999444	0.999456	0.999469	0.999481	0.999493	0.999505	0.999516	0.999528	0.999539	0.999549
4.2	0.999560	0.999570	0.999580	0.999590	0.999599	0.999608	0.999617	0.999626	0.999635	0.999643
4.3	0.999652	0.999660	0.999667	0.999675	0.999683	0.999690	0.999697	0.999704	0.999711	0.999718
4.4	0.999724	0.999731	0.999737	0.999743	0.999749	0.999755	0.999760	0.999766	0.999771	0.999776
4.5	0.999782	0.999787	0.999792	0.999796	0.999801	0.999806	0.999810	0.999815	0.999819	0.999823
4.6	0.999827	0.999831	0.999835	0.999839	0.999842	0.999846	0.999850	0.999853	0.999856	0.999860
4.7	0.999863	0.999866	0.999869	0.999872	0.999875	0.999878	0.999881	0.999884	0.999886	0.999889
4.8	0.999891	0.999894	0.999896	0.999899	0.999901	0.999903	0.999906	0.999908	0.999910	0.999912
4.9	0.999914	0.999916	0.999918	0.999920	0.999922	0.999923	0.999925	0.999927	0.999928	0.999930

Table A.22 Integral of Student's function (f = 50), or probability p, as function of critical value T_{crit}

	0	1	2	3	4	5	6	7	8	9
0.0	0.000000	0.007939	0.015877	0.023814	0.031748	0.039678	0.047605	0.055527	0.063443	0.071353
0.1	0.079256	0.087150	0.095036	0.102912	0.110778	0.118632	0.126474	0.134304	0.142120	0.149922
0.2	0.157708	0.165479	0.173233	0.180970	0.188689	0.196388	0.204069	0.211729	0.219367	0.226985
0.3	0.234579	0.242151	0.249698	0.257221	0.264719	0.272191	0.279637	0.287055	0.294445	0.301807
0.4	0.309140	0.316443	0.323715	0.330957	0.338167	0.345345	0.352490	0.359602	0.366680	0.373723
0.5	0.380732	0.387705	0.394642	0.401542	0.408406	0.415232	0.422020	0.428770	0.435481	0.442152
0.6	0.448784	0.455376	0.461927	0.468437	0.474906	0.481333	0.487717	0.494060	0.500359	0.506616
0.7	0.512829	0.518998	0.525123	0.531204	0.537240	0.543231	0.549177	0.555077	0.560932	0.566742
0.8	0.572505	0.578222	0.583892	0.589516	0.595093	0.600623	0.606106	0.611542	0.616931	0.622272
0.9	0.627566	0.632812	0.638010	0.643161	0.648263	0.653318	0.658326	0.663285	0.668196	0.673059
1.0	0.677875	0.682642	0.687362	0.692033	0.696657	0.701233	0.705762	0.710243	0.714676	0.719062
1.1	0.723400	0.727691	0.731935	0.736132	0.740282	0.744386	0.748442	0.752452	0.756416	0.760334
1.2	0.764206	0.768032	0.771812	0.775547	0.779237	0.782882	0.786482	0.790037	0.793548	0.797015
1.3	0.800438	0.803818	0.807154	0.810447	0.813697	0.816904	0.820070	0.823193	0.826274	0.829314
1.4	0.832312	0.835270	0.838187	0.841064	0.843901	0.846698	0.849455	0.852174	0.854853	0.857495
1.5	0.860098	0.862663	0.865190	0.867681	0.870135	0.872552	0.874933	0.877278	0.879587	0.881862
1.6	0.884101	0.886306	0.888477	0.890614	0.892718	0.894788	0.896826	0.898831	0.900805	0.902746
1.7	0.904656	0.906535	0.908383	0.910201	0.911989	0.913747	0.915476	0.917176	0.918847	0.920490
1.8	0.922105	0.923693	0.925253	0.926786	0.928293	0.929773	0.931227	0.932656	0.934060	0.935439
1.9	0.936793	0.938123	0.939429	0.940711	0.941970	0.943206	0.944420	0.945611	0.946780	0.947927
2.0	0.949053	0.950158	0.951243	0.952306	0.953350	0.954374	0.955378	0.956363	0.957329	0.958276
2.1	0.959205	0.960116	0.961009	0.961884	0.962742	0.963584	0.964408	0.965216	0.966008	0.966784
2.2	0.967544	0.968289	0.969019	0.969734	0.970434	0.971120	0.971791	0.972449	0.973093	0.973724
2.3	0.974341	0.974946	0.975538	0.976117	0.976684	0.977239	0.977782	0.978313	0.978833	0.979342
2.4	0.979840	0.980327	0.980803	0.981269	0.981725	0.982171	0.982607	0.983033	0.983450	0.983857

(continued)

Table A.22 (continued)

	0	1	2	3	4	5	6	7	8	9
2.5	0.984255	0.984645	0.985026	0.985398	0.985761	0.986117	0.986464	0.986804	0.987135	0.987459
2.6	0.987776	0.988085	0.988387	0.988683	0.988971	0.989252	0.989527	0.989796	0.990058	0.990314
2.7	0.990564	0.990807	0.991046	0.991278	0.991505	0.991726	0.991942	0.992153	0.992359	0.992560
2.8	0.992756	0.992947	0.993133	0.993315	0.993493	0.993666	0.993835	0.993999	0.994160	0.994316
2.9	0.994469	0.994618	0.994763	0.994904	0.995042	0.995176	0.995307	0.995435	0.995559	0.995681
3.0	0.995799	0.995914	0.996026	0.996135	0.996242	0.996345	0.996447	0.996545	0.996641	0.996734
3.1	0.996825	0.996914	0.997000	0.997084	0.997165	0.997245	0.997323	0.997398	0.997471	0.997543
3.2	0.997612	0.997680	0.997746	0.997810	0.997872	0.997933	0.997992	0.998050	0.998106	0.998160
3.3	0.998213	0.998264	0.998314	0.998363	0.998411	0.998457	0.998501	0.998545	0.998587	0.998628
3.4	0.998669	0.998707	0.998745	0.998782	0.998818	0.998853	0.998886	0.998919	0.998951	0.998982
3.5	0.999012	0.999042	0.999070	0.999098	0.999125	0.999151	0.999176	0.999201	0.999225	0.999248
3.6	0.999270	0.999292	0.999314	0.999334	0.999354	0.999374	0.999393	0.999411	0.999429	0.999447
3.7	0.999463	0.999480	0.999496	0.999511	0.999526	0.999540	0.999554	0.999568	0.999581	0.999594
3.8	0.999607	0.999619	0.999631	0.999642	0.999653	0.999664	0.999674	0.999684	0.999694	0.999704
3.9	0.999713	0.999722	0.999731	0.999739	0.999747	0.999755	0.999763	0.999770	0.999777	0.999784
4.0	0.999791	0.999798	0.999804	0.999810	0.999816	0.999822	0.999828	0.999833	0.999839	0.999844
4.1	0.999849	0.999854	0.999858	0.999863	0.999867	0.999871	0.999875	0.999879	0.999883	0.999887
4.2	0.999891	0.999894	0.999898	0.999901	0.999904	0.999907	0.999910	0.999913	0.999916	0.999919
4.3	0.999921	0.999924	0.999926	0.999929	0.999931	0.999933	0.999936	0.999938	0.999940	0.999942
4.4	0.999944	0.999945	0.999947	0.999949	0.999951	0.999952	0.999954	0.999955	0.999957	0.999958
4.5	0.999960	0.999961	0.999962	0.999964	0.999965	0.999966	0.999967	0.999968	0.999969	0.999970
4.6	0.999971	0.999972	0.999973	0.999974	0.999975	0.999976	0.999977	0.999977	0.999978	0.999979
4.7	0.999980	0.999980	0.999981	0.999982	0.999982	0.999983	0.999983	0.999984	0.999985	0.999985
4.8	0.999986	0.999986	0.999987	0.999987	0.999988	0.999988	0.999988	0.999989	0.999989	0.999990
4.9	0.999990	0.999990	0.999991	0.999991	0.999991	0.999992	0.999992	0.999992	0.999992	0.999993

Table A.23 Comparison of integrals of Student's function at different critical values.

	Critical value				
f	$T = 1$	$T = 2$	$T = 3$	$T = 4$	$T = 5$
1	0.500000	0.704833	0.795168	0.844042	0.874334
2	0.577351	0.816497	0.904534	0.942809	0.962251
3	0.608998	0.860674	0.942332	0.971992	0.984608
4	0.626099	0.883884	0.960058	0.983870	0.992510
5	0.636783	0.898061	0.969901	0.989677	0.995896
6	0.644083	0.907574	0.975992	0.992881	0.997548
7	0.649384	0.914381	0.980058	0.994811	0.998435
8	0.653407	0.919484	0.982929	0.996051	0.998948
9	0.656564	0.923448	0.985044	0.996890	0.999261
10	0.659107	0.926612	0.986657	0.997482	0.999463
11	0.661200	0.929196	0.987921	0.997914	0.999598
12	0.662951	0.931345	0.988934	0.998239	0.999691
13	0.664439	0.933160	0.989762	0.998488	0.999757
14	0.665718	0.934712	.990449	0.998684	0.999806
15	0.666830	0.936055	0.991028	0.998841	0.999842
16	0.667805	0.937228	0.991521	0.998968	0.999870
17	0.668668	0.938262	0.991946	0.999073	0.999891
18	0.669435	0.939179	0.992315	0.999161	0.999908
19	0.670123	0.939998	0.992639	0.999234	0.999921
20	0.670744	0.940735	0.992925	0.999297	0.999932
21	0.671306	0.941400	0.993179	0.999351	0.999940
22	0.671817	0.942005	0.993406	0.999397	0.999948
23	0.672284	0.942556	0.993610	0.999438	0.999954
24	0.672713	0.943061	0.993795	0.999474	0.999959
25	0.673108	0.943524	0.993962	0.999505	0.999963
26	0.673473	0.943952	0.994115	0.999533	0.999967
27	0.673811	0.944348	0.994255	0.999558	0.999970
28	0.674126	0.944715	0.994383	0.999580	0.999973
29	0.674418	0.945057	0.994501	0.999600	0.999975
30	0.674692	0.945375	0.994610	0.999619	0.999977
31	0.674948	0.945673	0.994712	0.999635	0.999979
32	0.675188	0.945952	0.994806	0.999650	0.999981
33	0.675413	0.946214	0.994893	0.999664	0.999982
34	0.675626	0.946461	0.994975	0.999677	0.999983
35	0.675826	0.946693	0.995052	0.999688	0.999984
36	0.676015	0.946912	0.995123	0.999699	0.999985
37	0.676194	0.947119	0.995191	0.999709	0.999986
38	0.676364	0.947315	0.995254	0.999718	0.999987
39	0.676525	0.947501	0.995314	0.999727	0.999988
40	0.676678	0.947678	0.995370	0.999735	0.999989

(continued)

Table A.23 (continued)

	Critical value				
f	$T = 1$	$T = 2$	$T = 3$	$T = 4$	$T = 5$
41	0.676824	0.947846	0.995424	0.999742	0.999989
42	0.676963	0.948006	0.995474	0.999749	0.999990
43	0.677095	0.948158	0.995522	0.999755	0.999990
44	0.677222	0.948304	0.995568	0.999761	0.999991
45	0.677343	0.948443	0.995611	0.999767	0.999991
46	0.677458	0.948576	0.995652	0.999773	0.999992
47	0.677569	0.948703	0.995691	0.999778	0.999992
48	0.677675	0.948824	0.995729	0.999782	0.999992
49	0.677777	0.948941	0.995765	0.999787	0.999993
50	0.677875	0.949053	0.995799	0.999791	0.999993
∞	0.682690	0.954500	0.997301	0.999937	1.000000

Table A.24 Critical values of the linear correlation coefficient

	Probability p to have an absolute value of r below the critical value						
f	0.50	0.60	0.70	0.80	0.90	0.95	0.99
2	0.500	0.600	0.700	0.800	0.900	0.950	0.990
3	0.404	0.492	0.585	0.687	0.805	0.878	0.959
4	0.347	0.426	0.511	0.608	0.729	0.811	0.917
5	0.309	0.380	0.459	0.551	0.669	0.754	0.875
6	0.281	0.347	0.420	0.507	0.621	0.707	0.834
7	0.260	0.321	0.390	0.472	0.582	0.666	0.798
8	0.242	0.300	0.365	0.443	0.549	0.632	0.765
9	0.228	0.282	0.344	0.419	0.521	0.602	0.735
10	0.216	0.268	0.327	0.398	0.497	0.576	0.708
20	0.152	0.189	0.231	0.284	0.360	0.423	0.537
30	0.124	0.154	0.189	0.233	0.296	0.349	0.449
40	0.107	0.133	0.164	0.202	0.257	0.304	0.393
50	0.096	0.119	0.147	0.181	0.231	0.273	0.354
60	0.087	0.109	0.134	0.165	0.211	0.250	0.325
70	0.081	0.101	0.124	0.153	0.195	0.232	0.302
80	0.076	0.094	0.116	0.143	0.183	0.217	0.283
90	0.071	0.089	0.109	0.135	0.173	0.205	0.267
100	0.068	0.084	0.104	0.128	0.164	0.195	0.254
200	0.048	0.060	0.073	0.091	0.116	0.138	0.181
300	0.039	0.049	0.060	0.074	0.095	0.113	0.148
500	0.030	0.038	0.046	0.057	0.073	0.087	0.114
1000	0.021	0.027	0.033	0.041	0.052	0.062	0.081

Table A.25 Critical values of the Kolmogorov–Smirnov statistic D_N

	Probability p to have $D_N \times \sqrt{N}$ below the critical value						
N	0.50	0.60	0.70	0.80	0.90	0.95	0.99
1	0.750	0.800	0.850	0.900	0.950	0.975	0.995
2	0.707	0.782	0.866	0.967	1.098	1.191	1.314
3	0.753	0.819	0.891	0.978	1.102	1.226	1.436
4	0.762	0.824	0.894	0.985	1.130	1.248	1.468
5	0.765	0.827	0.902	0.999	1.139	1.260	1.495
6	0.767	0.833	0.910	1.005	1.146	1.272	1.510
7	0.772	0.838	0.914	1.009	1.154	1.279	1.523
8	0.776	0.842	0.917	1.013	1.159	1.285	1.532
9	0.779	0.844	0.920	1.017	1.162	1.290	1.540
10	0.781	0.846	0.923	1.020	1.166	1.294	1.546
15	0.788	0.855	0.932	1.030	1.177	1.308	1.565
20	0.793	0.860	0.937	1.035	1.184	1.315	1.576
25	0.796	0.863	0.941	1.039	1.188	1.320	1.583
30	0.799	0.866	0.943	1.042	1.192	1.324	1.588
35	0.801	0.868	0.946	1.045	1.194	1.327	1.591
40	0.803	0.869	0.947	1.046	1.196	1.329	1.594
45	0.804	0.871	0.949	1.048	1.198	1.331	1.596
50	0.805	0.872	0.950	1.049	1.199	1.332	1.598
60	0.807	0.874	0.952	1.051	1.201	1.335	1.601
70	0.808	0.875	0.953	1.053	1.203	1.337	1.604
80	0.810	0.877	0.955	1.054	1.205	1.338	1.605
90	0.811	0.878	0.956	1.055	1.206	1.339	1.607
100	0.811	0.879	0.957	1.056	1.207	1.340	1.608
200	0.816	0.883	0.961	1.061	1.212	1.346	1.614
300	0.818	0.885	0.964	1.063	1.214	1.348	1.617
500	0.820	0.887	0.966	1.065	1.216	1.350	1.620
1000	0.822	0.890	0.968	1.067	1.218	1.353	1.622
∞	0.828	0.895	0.973	1.073	1.224	1.358	1.628

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