Epilogue

In this book we have developed the basic theory of quantum mechanics, explored some of its consequences for physics, chemistry and technology, and studied what the theory says about the nature of reality. I hope that I have managed to convey some of the beauty and elegance of the theory and its applications, and at the same time made you feel a little uncomfortable about the things that quantum mechanics has to say about nature. Fundamentally, quantum mechanics tells us how we should reason about physical systems without falling into the trap of assuming that things behave the way we expect even when we are not looking at them. In this sense, quantum theory is a refinement of our understanding of probabilities. As such, the theory has a much broader application than it is traditionally afforded in atomic, solid state, and particle physics.

What I presented here is just the tip of the iceberg, and if you continue your studies in quantum mechanics, what can you expect to learn further? First of all, you will learn how to solve the Schrödinger equation for a range or problems in one, two, and three dimensions. We discussed here the particle in a one-dimensional box, a one-dimensional pendulum, and we sketched roughly the solutions for the hydrogen atom in three dimensions, but there is a whole family of interesting potentials $V(x, y, z)$ that we can substitute into the Hamiltonian. One important example is when we have a periodic potential, describing for example the (attractive) nuclei in a crystal. An electron with a certain energy in such a crystal can be bound to a particular nucleus, or it may be able to move freely through the crystal. The available energy levels for the electrons then determine the physical properties of the crystal, from insulators, to metals and semiconductors, and they determine properties like heat capacity and conductivity.

Solving partial differential equations is generally hard, and given a particular potential $V(x, y, z)$ it is unlikely that you can find solutions in closed form. However, the potential may be somewhat close to a different, simpler potential for which we do have solutions. In that case the solutions to the complicated problem will be close to the solutions of a simple problem, and by studying how the potential changes between the two we can find approximations to the quantum state and the eigenvalues

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of various observables of interest, usually energy. This is an example of perturbation theory. For example, our system may have a potential in the Hamiltonian that has a quadratic term $\frac{1}{2}m\omega^2x^2$ like the pendulum, but it may also contain a small term proportional to x^3 or x^4, etc. The motion of the system will be almost harmonic, but not quite. Perturbation theory can tell us how the motion will be different from the ideal pendulum.

There are many different perturbation theory techniques: first and second order, time-dependent, the variational method (which describes the above example), and the Wentzel-Kramers-Brillouin method that is suitable when the system behaves almost classically. In particle physics, perturbation theory takes the form of Feynman diagrams: all possible ways for the particles to interact are organised in an infinite series of increasingly complex—and unlikely—processes, represented graphically by a Feynman diagram. The more of these processes we calculate, the more precise the answer of our calculation will be. Problems in chemistry and solid state physics also almost always require us to use one or more methods from perturbation theory. While this is typically a very technical topic it is nevertheless essential, since it allows us to get meaningful answers from quantum mechanics in almost any application.

Sometimes perturbations are caused by external classical fields, such as electric and magnetic fields. Since electrons are both electrically charged and have a magnetic moment (their spin), we would expect that such fields will have an influence on the energy levels of the electron. Indeed, this is the case. You will learn about so-called Stark shifts of energy levels that are created by electric fields, and Zeeman splittings of energy levels that are due to magnetic fields. Typically, these effects cannot be calculated exactly, and you will again have to use perturbation theory.

The next topic you may encounter is the complete description of angular momentum. We encountered angular momentum mostly in the form of the electron spin in this book, but there is also the orbital angular momentum associated with particles in a rotational motion. Angular momentum is important any time there is a central potential in a physics problem, such as the Coulomb potential in Sect. 8.7. Classically, the angular momentum is defined as the cross product $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, where $\mathbf{r}$ is the position vector of the particle, and $\mathbf{p}$ is its momentum vector. Since $\mathbf{r}$ and $\mathbf{p}$ do not commute in quantum mechanics, the three components of $\mathbf{L}$ do not commute and cannot be measured simultaneously (just like we saw in chapter three that S_x, S_y and S_z cannot be measured simultaneously). The commutation relations between L_x, L_y, and L_z are ultimately responsible for the quantisation of angular momentum. We already encountered this in Sect. 8.7, Eq. (8.91) where we introduced two new integer quantum numbers, l and m, connected to angular momentum. In a more advanced course in quantum mechanics you will learn how to combine spin and orbital angular momentum into a combined total angular momentum, and in quantum optics you will encounter a very deep connection between angular momentum and the polarisation of light, but also, more surprisingly, with optical networks of beam splitters and phase shifters.

Another key topic in the study of quantum mechanics, particularly in the context of particle physics, is scattering. In a particle accelerator we prepare particles in

states that are close to momentum eigenstates, typically with high energy. When the particles collide, this means that they will be deflected into a superposition of different momentum eigenstates. For each input momentum p there will be an amplitude for the scattering probability into an output momentum q. If we call this amplitude $S(p, q)$, then S is the so-called scattering matrix. The form of the scattering matrix tells us about the structure of the particles, and this is how, for example, we discovered that protons and neutrons are made up of three quarks. A simple one-dimensional example of scattering is the tunnelling example in Sect. 8.6. A particle in a momentum eigenstate gets scattered back from the barrier or gets transmitted (note that we assumed the particle was in a momentum eigenstate), and by measuring the reflected and transmitted parts for different initial momentum states we can infer the height and width of the potential barrier. You can also think of the beam splitter matrix in chapter two as a scattering matrix, because it sends a photon with a specific momentum either into the reflected beam or into the transmitted beam. The scattering matrix is therefore basically the unitary transformation that changes the input state of a particle to the output state, written in the momentum basis. Another interesting example is Compton scattering, in which a photon scatters off an electron and changes frequency due to energy transfer between the electron and the photon. This was discovered in 1923 by Arthur Compton, and it played a key role (alongside the photo-electric effect) in establishing that light cannot be thought of as a simple wave phenomenon.

We started this book by considering the definition of a photon. We could get quite some way towards our description of quantum theory, but a full description of the photon must take into account that two photons can be fundamentally indistinguishable from each other: Nothing can tell these photons apart, not their position, nor their polarisation, frequency, or shape of the wave packet. We cannot individually label the photons, and yet there are two of them. Similarly, two electrons can also be indistinguishable, but whereas photons can all happily sit together in the same quantum state, electrons do not want to share their quantum state with any other electron. This is the difference between bosons (like photons) and fermions (like electrons). The concept of identical particles plays a central role when we wish to extend quantum mechanics to the special relativistic case (excluding gravity; we still do not know how to reconcile that with quantum theory). In relativistic quantum mechanics you will learn how the spin of a particle determines whether it is a boson or a fermion. This is the so-called spin-statistics theorem: particles with integer spin (0, 1, 2, …) are bosons, while particles with half-integer spin ($\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$, …) are fermions. This will lead to the relativistic Klein-Gordon equation for massive bosons and the Dirac equation for electrons, which are the relativistic analogs of the Schrödinger equation. These equations must allow negative energy solutions, which was a big puzzle until Dirac identified these solutions with anti-particles.

A comprehensive description of relativistic quantum particles requires quantum field theory, which is the basis for the Standard Model of particle physics. However, quantum field theory also has important applications in optical physics and solid state physics. A fully quantum mechanical treatment of an atom interacting with quantum light needs a quantum theory for electric and magnetic fields. The standard tool for this type of problems is the Jaynes–Cummings model, which describes the

interaction of an atom in a cavity with the quantised electromagnetic field in the cavity. In condensed matter, we can treat crystal lattice vibrations as quantum fields at finite temperatures. This requires the concept of the density matrix that we encountered in chapter seven, but now for a very large number of identical particles. Generally, you will need non-relativistic quantum field theory to study the discipline of quantum statistical physics, and it will lead to such beautiful behaviour as Bose–Einstein condensation, superconductivity, and superfluidity, which have all been observed experimentally (in the case of superconductivity even before quantum theory was developed).

These are the more traditional topics in advanced textbooks on quantum mechanics. However, in recent years great progress has been made in controlling individual quantum systems. We can now control the interaction between a single atom and a single photon quite precisely (Serge Haroche and David Wineland won the Nobel prize for this in 2012), we can control single electrons in very small electrical circuits, and we are getting to the point where we can fully control electrons locked in small semiconductor boxes, called quantum dots. To make experimental progress on this front we need a theory that can cope with the imperfections that inevitably arise in the setup. Detectors will not always give the "right" answer, and we must develop a method that includes such errors. We can take a similar approach to the way we derived the density matrix: we construct a probability distribution that governs how likely it is that quantum state $|a_j\rangle$ triggers a measurement outcome m_k instead of m_j. This is enough to construct the most general measurement operators, which are no longer projectors but probability distributions over projectors called POVMs. This leads to a new definition of "generalised observables", where instead of a spectral decomposition in terms of projectors, the observable is given in terms of the POVM.

Similarly, it will be very difficult in most experiments to ensure that our system is completely isolated from the environment. As a result, the evolution of the system over time is no longer perfectly reversible since some information about the state of the system may leak into the environment. That means the evolution is no longer unitary. Generally, this is how a pure state turns into a mixed state, and the general method was described in Chap. 7 via the tracing out of the environment. However, in practice, this will be a very difficult calculation for all but the simplest systems, and we need additional techniques to describe non-unitary evolution of quantum systems. We can again consider probability distributions over various possible unitary transformations, but this also has limited use. Instead we will have to introduce a replacement for the Schrödinger equation, called the Lindblad equation. It is an equation for density matrices instead of vectors, and in the case of unitary evolution it reduces to the Schrödinger equation, as required. The Lindblad equation is a very powerful tool, and requires sophisticated mathematical techniques to set up and solve.

Finally, you may study quantum mechanics in a completely different context, namely that of quantum information theory. This includes quantum computing and quantum communication, and a large effort is currently underway to come up with ways to implement error correction that allows us to actually build quantum computers and communication systems even in the presence of practical imperfections. The theory of quantum error correction is less a physical theory and more a quantum

computer science research area, even though the errors will be physical in origin. Similarly, quantum mechanics can be used to study the capacity of communication channels for transmitting information. It asks questions like "how much information can Alice transmit to Bob if they have access to shared entanglement?" Quantum information theory is very much built upon the mathematics of density matrices and entropy that we encountered in chapter seven.

In all these topics, the basic principles described in this book still apply, even though the mathematics will get much more complicated. Hopefully, this book, along with its interactive figures will have given you a solid conceptual foundation of quantum mechanics so that you can focus on the technical aspects of the more advanced subjects.

Further Reading

The following is a list of recommended reading for anyone who wants to learn quantum mechanics in more detail. It is by no means an exhaustive list of all the excellent books on the topic, but these are the ones that are close to my heart.

Matrix Operations for Engineers and Scientists *Alan Jeffrey, Springer* (2010). Any serious study of quantum mechanics requires a good grounding in linear algebra, and especially the theory of matrices. Jeffrey takes a pedagogical approach similar to what I adopted here, and develops the material from example problems. I highly recommend *Matrix Operations* alongside this book.

Matrix Analysis *R. A. Horn and C. R. Johnson, Cambridge University Press* (2013). This book is a comprehensive textbook that covers everything you need to know and more. It is written in a mature mathematical style, and is a good follow-up from Jeffrey's book. It has a wealth of practice problems.

Quantum Mechanics *A. Rae, Fifth Edition, Taylor and Francis* (2008). This is an introductory textbook on quantum mechanics based on a more traditional approach. The book starts with the photoelectric effect, the Compton effect and wave-particle duality, and proceeds via the Schrödinger equation in one and three dimensions to angular momentum, perturbation theory, scattering theory and many-particle systems. It assumes a good working knowledge of calculus and complex numbers.

Quantum Systems, Processes and Information *B. Schumacher and M. Westmoreland, Cambridge University Press* (2010). This textbook is the natural next step from the present book, in that it approaches quantum mechanics as a theory of information and processes. After a brief introduction to the smallest physical systems—qubits—the book develops the mathematical formalism of quantum theory in terms of Hilbert spaces. It covers the topics in this book with more generality, and includes more dedicated chapters on quantum information processing, entropy, and error correction.

Quantum Computer Science *D. Mermin, Cambridge University Press* (2007). This is a textbook aimed at readers from different backgrounds, including

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mathematics and computer science. Written in his inimitable style, Mermin explores quantum mechanics with a focus on computation, rather than physics. It requires some basic knowledge of matrices and complex numbers, but is generally very accessible and heartily recommended to readers who enjoyed Chap. 6 and want to know more about the information theoretic applications of quantum mechanics.

Q is for Quantum *Terry Rudolph, Amazon* (2017). In this highly accessible book Rudolph explores the fundamental puzzles that quantum mechanics poses to our classical world view. The style of the book is much like Chap. 1, but it penetrates deeply into the topics we covered in Chap. 10. Recommended reading for anyone who wants to explore the foundations of quantum mechanics further.

The Principles of Quantum Mechanics *P. A. M. Dirac, Oxford University Press* (1988). Not often in science would one recommend reading the earlier textbooks on a topic, but this book by Paul Dirac is a notable exception. It is still one of the clearest expositions of quantum mechanics (including his famous treatment of the relativistic electron), and it is the original source for the bracket notation that almost everybody uses today.

Useful Formulas

Complex numbers:

$$(x+iy)^* = x - iy$$
$$(x+iy)(u+iv) = (xu - yv) + i(xv + yu)$$
$$(r_1 e^{i\phi_1})(r_2 e^{i\phi_2}) = r_1 r_2 e^{i(\phi_1+\phi_2)}$$

Series expansion and trigonometry:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$
$$e^{i\phi} = \cos\phi + i\sin\phi$$
$$\cos\phi = \frac{e^{i\phi} + e^{-i\phi}}{2}$$
$$\sin\phi = \frac{e^{i\phi} - e^{-i\phi}}{2i}$$

Dirac delta function:

$$\delta(x - x_0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i(x-x_0)y}\, dy$$

When $a \le x_0 \le b$ we have

$$\int_a^b f(x)\delta(x - x_0)\, dx = f(x_0)$$

Fourier transform of complex functions:

$$\Psi(p) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{+\infty} dx\ \psi(x)e^{-ipx/\hbar}$$

Gaussian function with peak at x_0 and width σ:

$$f(x) = \frac{1}{\sqrt[4]{\pi\sigma^2}}\exp\left[-\frac{(x-x_0)^2}{2\sigma^2}\right].$$

Operator definitions:

Hermitian: $A^\dagger = A$
Unitary: $U^\dagger = U^{-1}$
Projector: $P^2 = P$
Commutator: $[A, B] = AB - BA$

Determinant and trace of 2×2 matrices:

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$
$$\mathrm{Tr}\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] = a + d$$

Cyclic property of the trace:

$$\mathrm{Tr}[ABC] = \mathrm{Tr}[CAB]$$

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Eigenvalue equation:

$$\det(A - \lambda \mathbb{I}) = 0$$

The Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$
$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Anti-commutation relations of the Pauli matrices:

$$\sigma_x \sigma_y + \sigma_y \sigma_x = 0$$
$$\sigma_x \sigma_z + \sigma_z \sigma_x = 0$$
$$\sigma_y \sigma_z + \sigma_z \sigma_y = 0$$

Born rule:

$$\Pr(\text{outcome}) = |\langle \text{outcome} | \psi \rangle|^2$$
$$\Pr([\text{a}, \text{b}]) = \int_a^b dx \; |\psi(x)|^2$$

Expectation value and variance:

$$\langle A \rangle = \langle \psi | A | \psi \rangle$$
$$(\Delta A)^2 = \langle \psi | A^2 | \psi \rangle - \langle \psi | A | \psi \rangle^2$$

Uncertainty relation for operators:

$$\Delta A \, \Delta B \geq \frac{1}{2} |\langle [A, B] \rangle|$$

Error propagation formula:

$$\delta\theta = \left| \frac{d\langle A \rangle}{d\theta} \right|^{-1} \Delta A$$

The Schrödinger equation:

$$i\hbar |\psi(t)\rangle = H |\psi(t)\rangle$$

Time evolution:

$$|\psi t\rangle = U(t) |\psi(0)\rangle$$
$$= \exp\left(-\frac{i}{\hbar} H t \right) |\psi(0)\rangle$$

Ladder operators:

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}} \, \hat{x} + i \frac{1}{\sqrt{2m\hbar\omega}} \, \hat{p}$$
$$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}} \, \hat{x} - i \frac{1}{\sqrt{2m\hbar\omega}} \, \hat{p}$$

Bell states for qubits:

$$|\Phi^\pm\rangle = \frac{|00\rangle \pm |11\rangle}{\sqrt{2}}$$
$$|\Psi^\pm\rangle = \frac{|01\rangle \pm |10\rangle}{\sqrt{2}}$$

Von Neuman entropy:

$$S(\rho) = -\text{Tr}\big[\rho \log_2 \rho\big]$$
$$= -\sum_j p_j \log_2 p_j$$

where the p_j are eigenvalues of ρ.

Answers to Selected Problems

1.2 $\Pr(\mathrm{D}_1) = 0.3$ and $\Pr(\mathrm{D}_2) = 0.7$.

1.5 A partial memory of the photon path will partially destroy the interference. So instead of detector D_1 being always silent, it now occasionally fires (but not as often as D_2).

2.1

$$AB = \begin{pmatrix} 27 & 60 \\ 3 & 30 \end{pmatrix}, \; BA = \begin{pmatrix} 54 & 39 \\ -12 & 3 \end{pmatrix}$$

2.2

$$z_1 = 5\, e^{i \arctan(4/3)}$$
$$z_2 = 4\sqrt{13}\, e^{-i \arctan(2/3)}$$
$$z_3 = 3\sqrt{41}\, e^{-i \arctan(4/5)}.$$

2.6 $\langle\psi|\phi\rangle = -(21 + 15i)$ and $\langle\phi|\psi\rangle = -(21 - 15i)$.

2.9 $\Pr(\text{right}) = 9/25$.

2.11 $\phi = \pi/3$.

2.15 For any state $e^{i\phi}|\psi\rangle$ and event a, the probability of a is given by $\Pr(\mathrm{a}) = |e^{i\phi}\langle a|\psi\rangle|^2 = e^{i\phi}e^{-i\phi}|\langle a|\psi\rangle|^2$. But $e^{i\phi}e^{-i\phi} = 1$, so the probability does not depend on ϕ.

3.1 $\Pr(\downarrow)=2/3$, $\Pr(+)=1/2+\sqrt{2}/3$.

3.3 The normalisation constant is $\sqrt{13}$, and $\Pr(\uparrow) = 2/13$, $\Pr(+) = 1/2$. The expectation value in the z-direction is $\langle S_z\rangle = -5\hbar/26$.

3.11 $|\psi\rangle = 0.500|\uparrow\rangle + (0.433 + 0.750\, i)|\downarrow\rangle$.

3.13 $\langle S_x\rangle = 12\hbar/25$, and $\Delta S_x = 7\hbar/50$.

4.2 $|\psi(t)\rangle = (|g\rangle + \sqrt{2}e^{i\omega t}|e\rangle)/\sqrt{3}$, and $\Pr(+) = 1/2 + \frac{1}{3}\sqrt{2}\cos\omega\mathrm{t}$.

4.4 (a)

$$H = -\frac{e\hbar B}{2\sqrt{2}mc}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

(b) Use that $1/\sqrt{2} = \cos(\pi/4) = \sin(\pi/4)$ and use the addition formulas from trigonometry.

(d) No.

5.1 (a) -3, (b) $-5 + 14i$, (c) -69.

5.4 Both, neither, Hermitian.

5.6 (a) $\cos\left(\frac{eBT}{2m}\right)|\uparrow\rangle + \sin\left(\frac{eBT}{2m}\right)|\downarrow\rangle$.

(b) $\Pr(+) = \frac{1}{2} + \frac{1}{2}\sin\left(\frac{eBT}{m}\right)$.

5.11 (a) $P_a = |a\rangle\langle a|$, $P_b = |b\rangle\langle b|$, and $P_c = |c\rangle\langle c|$. $\Pr(\mathrm{a}) = \langle\psi|\mathrm{P_a}|\psi\rangle$, and so on.

(b) $\Pr(\text{not a}) = \langle\psi|\mathrm{P_{not\,a}}|\psi\rangle$, with $P_{\text{not}\,a} = \mathbb{I} - P_a$.

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(c) $P_{a\text{ or }b} = P_a + P_b - P_a P_b$.

6.1 (a) and (b).

6.3 $X = \sigma_x, \lambda_1 = +1$ with $(1,1)/\sqrt{2}$, and $\lambda_2 = -1$ with $(1,-1)/\sqrt{2}$. $\langle X \rangle = 0$.

6.5 For all possible pairs of states calculate that their scalar product is zero. So

$$\langle \phi_1 | \phi_2 \rangle = \frac{1}{2}(\langle 0,1|0,-\rangle + \langle 1,0|1,+\rangle) = \frac{1}{2}\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) = 0\,.$$

6.7 $U|0\rangle = |0\rangle$ and $U|1\rangle = e^{i\omega t}|1\rangle$. Apply U to both qubits: $U_1U_2|00\rangle = |00\rangle$ and $U_1U_2|11\rangle = e^{2i\omega t}|11\rangle$. The phase $e^{2i\omega t}$ returns to 1 twice as fast as $e^{i\omega t}$.

7.1 Purity is $\frac{7}{9} < 1$, so a mixed state.

7.2 (a) We can consider the Hamiltonian as a conditional operation that acts on qubit two only if the state of qubit 1 is $|1\rangle$. We then have $|\Psi(t)\rangle = \frac{1}{\sqrt{2}}(|00\rangle + \cos\omega T|10\rangle - i\sin\omega T|11\rangle)$.

(b) Take the partial trace of the state in (a):

$$\rho_1 = \frac{1}{2}\begin{pmatrix} 1 & \cos\omega T \\ \cos\omega T & 1 \end{pmatrix},$$

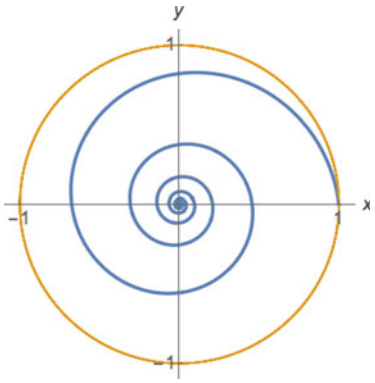
$$\rho_2 = \frac{1}{1}\begin{pmatrix} 1+\cos^2\omega T & -i\sin 2\omega T \\ i\sin 2\omega T & \sin^2\omega T \end{pmatrix}.$$

(c) The entropy of ρ_1 and ρ_2 are the same. Eigenvalues of ρ_1 are $(1 \pm \cos\omega T)/2$, and the entropy becomes

$$S = \cos\omega T \log\sqrt{\frac{1-\cos\omega T}{1+\cos\omega T}} + \log\frac{2}{|\sin\omega T|}.$$

(d) $T = \pi/2\omega$. This is when the entropy of the individual qubits is maximal.

7.8 Top view of equatorial plane of the Bloch sphere:



ω is the angular (rotation) frequency, while γ is the rate that determined how quickly the state moves to the origin.

7.10

$$\begin{aligned} S &= -\sum_j p_j \log p_j \\ &= -\sum_j p_j \log\frac{e^{-\beta E_j}}{Z} \\ &= -\sum_j p_j\left(-\beta E_j - \log Z\right) \\ &= \sum_j p_j \log Z + \beta\sum_j p_j E_j \\ &= \log Z + \beta\langle E\rangle\,. \end{aligned} \tag{A.1}$$

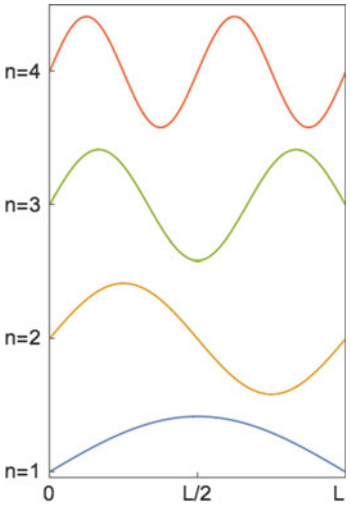
7.14 $(\Delta A)^2 = \mathrm{Tr}\left[A^2\rho\right] - \mathrm{Tr}[A\rho]^2$.

8.1 (a) Differentiating ψ_n twice gives

$$\psi_n'' = -\frac{n^2\pi^2}{L^2}\psi_n\,.$$

In the region where $V = 0$, the energy eigenvalue equation becomes

$$\frac{n^2\pi^2\hbar^2}{2mL^2}\psi_n = E_n\psi_n\,.$$



(b) From (a) we see the result for E_n immediately.

(c) Since the derivative of ψ_n is not proportional to ψ_n, it is not an eigenfunction of the momentum operator.

(d) Differentiate and find $p = \pm\hbar k$.

(e) $\psi_n(x) \propto \phi_k(x) + \phi_{-k}(x)$, which means two counter-propagating momentum functions. The position wave function ψ_n is a standing wave, and $k = n\pi/L$.

8.3 Both Δx for the state $|p\rangle$ and Δp for the state $|x\rangle$ diverge. When we measure the position of a particle in a momentum eigenstate, the wave function collapses to the position eigenstate corresponding to the measured position.

8.6 (a) The probability is

$$\frac{\exp[-(x-x_0)^2/\sigma^2]}{\sqrt{\pi\sigma^2}}dx\,.$$

The probability density function looks like a bell curve with width σ.

(b)

$$\Psi(p) = \frac{\exp\left[-\frac{p^2}{2\hbar^2/\sigma^2} - \frac{ipx_0}{\hbar}\right]}{\sqrt[4]{\pi\hbar^2/\sigma^2}}.$$

The probability density function looks like a bell curve with width $\hbar/\sigma$.

(c) The widths of the bell curves are inverses of each other (up to a factor $\hbar$).

8.8

$$|\psi\rangle = \int \delta(p-p')|p'\rangle = |p\rangle.$$

9.1

$$\langle A\rangle = \sum_j a_j^2|\langle a_j|\psi\rangle|^2 - \left(\sum_j a_j|\langle a_j|\psi\rangle|^2\right)^2$$

when $|\psi\rangle = |a_k\rangle$, $\langle A\rangle$ becomes

$$\langle A\rangle = \sum_j a_j^2\delta_{jk} - \left(\sum_j a_j\delta_{jk}\right)^2 = a_k^2 - (a_k)^2 = 0$$

9.2 The precision of a grandfather clock is measured in milliseconds per cycle, whereas quantum mechanics puts the ultimate limit at

$$\delta t \geq \sqrt{\frac{\hbar}{2m\omega^3A^2}} \approx 10^{-17}\ \text{s}.$$

9.7

$$\langle x \rangle = A\cos(\omega t + \phi)$$
$$\langle x \rangle = -m\omega A\sin(\omega t + \phi)$$

where ϕ is the phase of the pendulum at $t = 0$.

10.3 (a) For general angles θ for Alice and θ' for Bob:

$$\Pr(++) = \Pr(--) = \frac{\sin^2[(\theta' - \theta)/2]}{2}$$
$$\Pr(+-) = \Pr(-+) = \frac{\cos^2[(\theta' - \theta)/2]}{2}$$

(b) Since $\cos 2x = \cos^2 x - \sin^2$, the range of values for P is between +1 and −1.

(c) The CHSH inequality becomes

$$|\cos\frac{\pi}{4}| + |\cos\frac{\pi}{4}| + |\cos\frac{3\pi}{4}| - |\cos\frac{\pi}{4}| \leq 2.$$

However, the left-hand side is $2\sqrt{2}$, which is greater than 2. Any hidden variable model must admit superluminal signalling.

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