Appendix A
Stress Analysis

In this appendix the main principles of stress analysis are presented. This includes the graphical method of representing the transformation formulas of the stress tensor by Mohr's circle. The considerations are restricted to two-dimensional states of stress, for reasons of simplicity.

A.1 Transformation Formulas

Suppose that the state of stress in a certain point is described by the stresses τ_{xx}, τ_{xy}, τ_{yx} and τ_{yy}, see Fig. A.1. Following the usual sign convention of applied mechanics a component of stress is considered positive when the force component on a plane whose outward normal vector is directed in a positive coordinate direction, acts in positive direction as well, or when the force component acts in negative direction on a plane whose outward normal vector is directed in negative coordinate direction. For normal stresses this means that tension is considered positive, and pressure is considered negative. In soil mechanics the usual sign convention is just the opposite. The difference is expressed in that in this appendix stresses are denoted by the symbol τ, whereas in the main text of the book stresses are denoted by σ. Formally, the relation is

$$\begin{aligned} \sigma_{xx} &= -\tau_{xx}, \\ \sigma_{xy} &= -\tau_{xy}, \\ \sigma_{yx} &= -\tau_{yx}, \\ \sigma_{yy} &= -\tau_{yy}. \end{aligned} \tag{A.1}$$

The state of stress in a certain point is completely defined by the four stress components τ_{xx}, τ_{xy}, τ_{yx} and τ_{yy}. Of these four stresses the shear stresses are equal, as can be shown by considering equilibrium of moments with respect to the center of the element,

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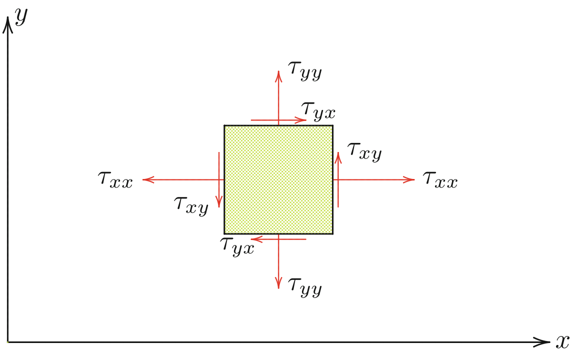


Fig. A.1 Stress in two dimensions

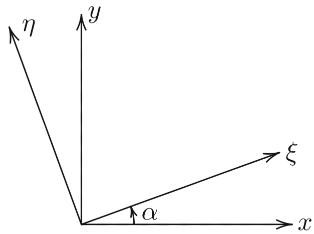


Fig. A.2 Rotation of axes

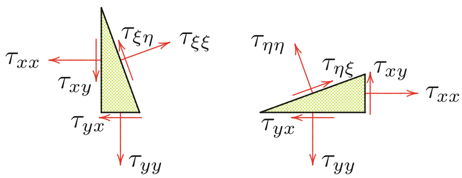


Fig. A.3 Stresses in a rotated system of coordinates

$$\tau_{xy} = \tau_{yx}. \tag{A.2}$$

In many situations it is necessary to describe the stress transfer by also considering planes other than those in the directions of the cartesian coordinates x and y. The stress state should then be described in a rotated set of coordinate axes, denoted by ξ and η, rotated with respect to the original coordinates over an angle α, see Fig. A.2. The transformation formulas can be derived most conveniently by considering equilibrium of a suitably chosen elementary triangle, see Fig. A.3.

In formulating the equilibrium conditions it should be remembered that the basis of this principle is equilibrium of *forces*, not stresses. This means that the magnitude of the various planes on which the various stress components act should be taken into account.

By formulating the conditions of equilibrium in ξ-direction and in η-direction of a suitably chosen triangular element, with only two unknown stress components in the pair of equations, see Fig. A.3, it follows that

$$
\begin{aligned}
\tau_{\xi\xi} &= \left(\frac{\tau_{xx}+\tau_{yy}}{2}\right) + \left(\frac{\tau_{xx}-\tau_{yy}}{2}\right)\cos 2\alpha + \tau_{xy}\sin 2\alpha,\\
\tau_{\eta\eta} &= \left(\frac{\tau_{xx}+\tau_{yy}}{2}\right) - \left(\frac{\tau_{xx}-\tau_{yy}}{2}\right)\cos 2\alpha - \tau_{xy}\sin 2\alpha,\\
\tau_{\xi\eta} &= \tau_{\eta\xi} = \tau_{xy}\cos 2\alpha - \left(\frac{\tau_{xx}-\tau_{yy}}{2}\right)\sin 2\alpha.
\end{aligned}
\tag{A.3}
$$

A.2 Principal Directions

For certain values of the rotation angle α the shear stresses $\tau_{\xi\eta}$ and $\tau_{\eta\xi}$ are zero. This means that there are certain planes on which only a normal stress is acting, and no shear stress. The directions normal to these planes are called the *principal directions* of the stress tensor. The value of α for which the shear stress is zero will be denoted by α_0. Its value can be determined by setting the last equation of (A.3) equal to zero. This gives

$$
\tan 2\alpha_0 = \frac{\tau_{xy}}{\frac{1}{2}(\tau_{xx}-\tau_{yy})}. \tag{A.4}
$$

Because of the periodic property of the function $\tan 2\alpha_0$ it follows that there are two solutions, which differ by a factor $\frac{1}{2}\pi$. The corresponding values of the normal stresses can be found by substitution of this value of α into the first two equations of the system (A.3). These normal stresses are denoted by τ_1 and τ_2, the *principal stresses*. It is assumed that τ_1 is the largest of these two stresses, the *major* principal stress, and τ_2 is the smallest of the two stresses, the *minor* principal stress. Using some trigonometric relations, it can be shown that

$$
\tau_{1,2} = \left(\frac{\tau_{xx}+\tau_{yy}}{2}\right) \pm \sqrt{\left(\frac{\tau_{xx}-\tau_{yy}}{2}\right)^2 + \tau_{xy}^2} \tag{A.5}
$$

The notions of principal stress and principal direction introduced here are special cases of the more general properties of *eigen value* and *eigen vector* of matrices and tensors.

A.3 Mohr's Circle

The formulas derived above can be represented in a simple graphical form, using *Mohr's circle*. For this purpose it is most convenient to use the transformation formulas in the form (A.3), but expressed into the principal stresses. The orientation of the x-axis with respect to the direction of the major principal stress is denoted by γ, see

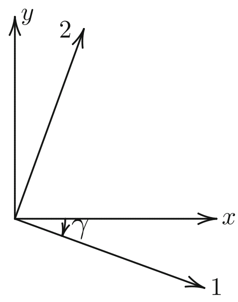


Fig. A.4 Rotation of axes

Fig. A.4. The directions of the major and the minor principal stresses are indicated by 1 and 2. The transformation formulas for the transition from the axes 1 and 2 to the axes x and y can easily be obtained from the formulas (A.3), by replacing x and y by 1 and 2 (with $\tau_{12} = 0$), and replacing ξ and η by x and y, and the angle α by γ. The result is

$$\begin{aligned}
\tau_{xx} &= \left(\frac{\tau_1 + \tau_2}{2}\right) + \left(\frac{\tau_1 - \tau_2}{2}\right) \cos 2\gamma, \\
\tau_{yy} &= \left(\frac{\tau_1 + \tau_2}{2}\right) - \left(\frac{\tau_1 - \tau_2}{2}\right) \cos 2\gamma, \\
\tau_{xy} &= \tau_{yx} = -\left(\frac{\tau_1 - \tau_2}{2}\right) \sin 2\gamma.
\end{aligned} \tag{A.6}$$

These formulas admit a simple graphical interpretation, see Fig. A.5. In this figure, Mohr's diagram, the normal stresses τ_{xx} and τ_{yy} are plotted positive towards the right. The shear stress τ_{yx} is plotted positive in upward direction, and the shear stress τ_{xy}

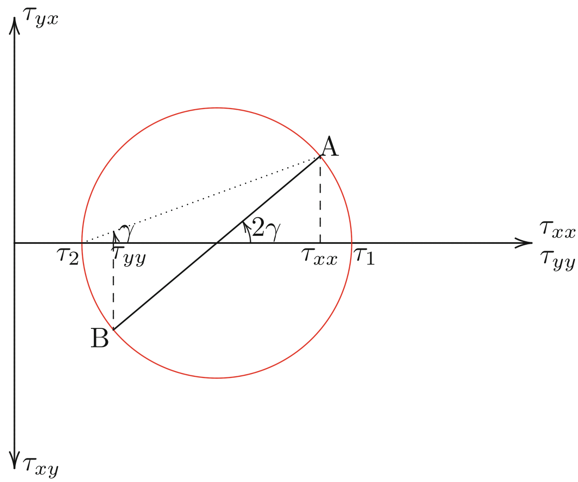


Fig. A.5 Mohr's circle

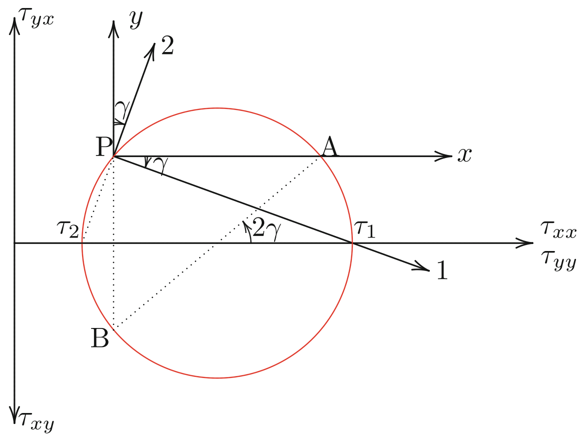
is plotted positive in downward direction. The pair of stresses τ_{xx} and τ_{xy} (i.e. the stresses acting on a plane with its normal in the x-direction) together constitute the point A in the diagram shown in Fig. A.5. The stresses τ_{yy} and τ_{yx} (i.e. the stresses acting on a plane with its normal in the y-direction) together constitute the point B in the figure. The formulas (A.6) indicate that these stress points describe a circle if the orientation angle γ varies. The center of the circle is located in a point of the horizontal axis, at a distance $\frac{1}{2}(\tau_1 + \tau_2)$ to the right of the origin, and the radius of the circle is $\frac{1}{2}(\tau_1 - \tau_2)$. The location of the stress point on the circle is determined by the angle γ, or, more precisely, by the central angle 2γ. If the angle γ increases, the stress points move along the circumference of the circle. In the case shown in the figure both τ_{xy} and τ_{yx} are negative.

A special point can be identified on the circle: the *pole*, the point from which the stresses in any direction can be found by a simple construction (Point P in Fig. A.6). The pole can be found by drawing a line in x-direction from the stress point A, and intersecting this line with a line from the stress point B in the y-direction. The principal directions can now be found by drawing lines from the pole to the rightmost and leftmost points of the circle. The stresses on an arbitrary plane can be found by drawing a line in the direction of the normal vector to that plane, and intersecting it with the circle. The validity of the construction follows from the fact that an angle on the circumference of the circle, spanning a certain arc, is just one-half of the central angle on the same arc.

The graphical constructions described above are very useful in soil mechanics, to determine the directions of the major and the minor principal stresses, and also for the determination of the most critical planes, potential slip planes.

It may be mentioned that the considerations of this appendix apply to any symmetric second order tensor, strain as well as stress, for instance.

Fig. A.6 Pole



Appendix B
Elasticity

In this appendix the basic equations of the theory of elasticity are presented, together with some elementary solutions. The material is supposed to be isotropic, i.e. all properties are independent of the orientation.

B.1 Basic Equations

The basic equations of the theory of elasticity describe the relations between stresses, strain and displacements in an isotropic linear elastic material.

The basic variables are the components of the displacement vector. In a cartesian coordinate system these can be denoted by u_x, u_y and u_z. The components of the strain tensor (or deformation tensor) can be derived from the displacements by differentiation,

$$
\begin{aligned}
\varepsilon_{xx} &= \frac{\partial u_x}{\partial x}, \qquad & \varepsilon_{xy} &= \tfrac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right),\\
\varepsilon_{yy} &= \frac{\partial u_y}{\partial y}, \qquad & \varepsilon_{yz} &= \tfrac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right),\\
\varepsilon_{zz} &= \frac{\partial u_z}{\partial z}, \qquad & \varepsilon_{zx} &= \tfrac{1}{2}\left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}\right).
\end{aligned}
\tag{B.1}
$$

These expressions are illustrated in Fig. B.1. It has been assumed that all the partial derivatives in the system of equations (B.1) are small. The strains ε_{xx}, ε_{yy} and ε_{zz} are a dimensionless measure for the relative change of length in the three coordinate directions. The shear strains ε_{xy}, ε_{yz} and ε_{zx} indicate the angular deformations. The quantity ε_{xy}, for instance, is one half of the reduction of the right angle in the lower left corner of the element shown in Fig. B.1.

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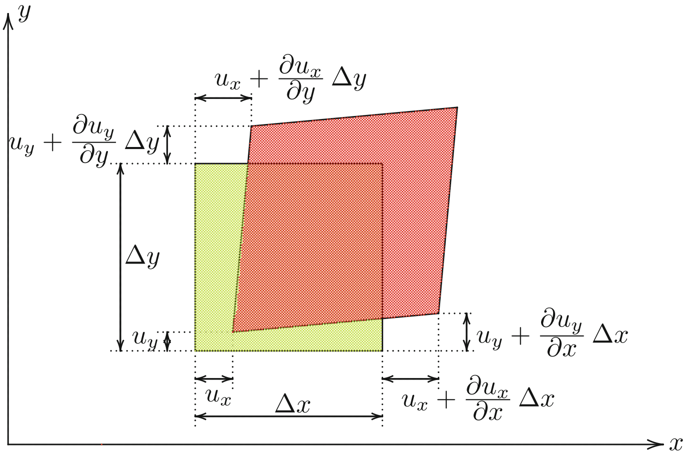


Fig. B.1 Strains

The relative increase of the volume is the *volume strain*, and is denoted by the symbol ε_{vol},

$$\varepsilon_{\text{vol}} = \frac{\Delta V}{V}. \tag{B.2}$$

If the strains are small (compared to 1) this is the sum of the strains in the coordinate directions,

$$\varepsilon_{\text{vol}} = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}. \tag{B.3}$$

For an isotropic linear elastic material the stresses can be expressed into the strains by Hooke's law,

$$\begin{aligned} \tau_{xx} &= \lambda\varepsilon_{\text{vol}} + 2\mu\varepsilon_{xx}, &\quad \tau_{xy} &= 2\mu\varepsilon_{xy}, \\ \tau_{yy} &= \lambda\varepsilon_{\text{vol}} + 2\mu\varepsilon_{yy}, &\quad \tau_{yz} &= 2\mu\varepsilon_{yz}, \\ \tau_{zz} &= \lambda\varepsilon_{\text{vol}} + 2\mu\varepsilon_{zz}, &\quad \tau_{zx} &= 2\mu\varepsilon_{zx}. \end{aligned} \tag{B.4}$$

The parameters λ and μ are Lamé's elastic constants. They are related to Young's modulus E and Poisson's ratio ν by

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \qquad \mu = \frac{E}{2(1+\nu)}. \tag{B.5}$$

The sign convention for the stresses is that a stress component is positive when acting in positive coordinate direction on a plane having its outward normal in positive

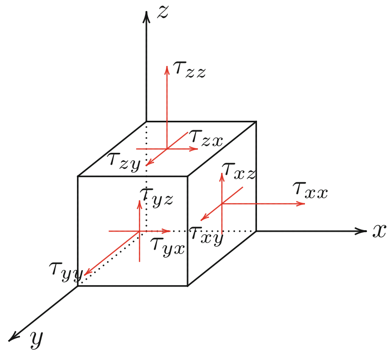


Fig. B.2 Stresses on a small element

coordinate direction. This is the usual sign convention of continuum mechanics. It means that tensile stresses are positive, and compressive stresses are negative.

For a small element the stresses on the three visible faces are shown in Fig. B.2. It may be noted that in soil mechanics the sign convention often is just the opposite, with compressive stresses being considered positive. Compressive stresses σ_{ij} can be related to the stresses τ_{ij} considered here, using the formula $\sigma_{ij} = -\tau_{ij}$.

The stresses should satisfy the equilibrium equations. In the absence of body forces these are

$$\begin{aligned}
\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} &= 0, \qquad \tau_{xy} = \tau_{yx}, \\
\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} &= 0, \qquad \tau_{yz} = \tau_{zy}, \\
\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} &= 0. \qquad \tau_{zx} = \tau_{xz}.
\end{aligned} \tag{B.6}$$

These equations can be derived by considering equilibrium of a small element, in the three coordinate directions, and equilibrium of moments about the three axes.

The stresses, strains and displacements in an isotropic linear elastic material should satisfy all the equations given above, and the appropriate boundary conditions at the surface of the body. Deriving solutions is not an easy matter. There are many books presenting techniques for the solution of elastic problems, for instance the book by Timoshenko and Goodier (1970). In the next sections some special solutions will be presented.

For many solution methods it is convenient to express the equations of equilibrium into the displacement components. If the elastic coefficients λ and μ are constants (i.e. if the material is homogeneous), it follows from Eqs. (B.1), (B.4) and (B.6) that

$$\begin{aligned}
(\lambda+\mu)\frac{\partial\varepsilon_{\text{vol}}}{\partial x}+\mu\nabla^2 u_x&=0,\\
(\lambda+\mu)\frac{\partial\varepsilon_{\text{vol}}}{\partial y}+\mu\nabla^2 u_y&=0,\\
(\lambda+\mu)\frac{\partial\varepsilon_{\text{vol}}}{\partial z}+\mu\nabla^2 u_z&=0.
\end{aligned} \tag{B.7}$$

These equations form a system of three differential equations with three basic variables, the equations of Navier.

B.2 Boussinesq Problems

For geotechnical engineering the class of problems of an elastic half space ($z > 0$), bounded by the plane $z = 0$, is of great importance. If the surface is loaded by normal stresses only, see Fig. B.3, a solution can be found following methods developed by Boussinesq in (1885).

Problems of this type, with given normal stresses on the boundary, and no shear stresses on the boundary, can be solved relatively easily by introducing a special potential function Φ. The displacements can be expressed into this potential by the equations

$$\begin{aligned}
u_x&=\frac{\partial\Phi}{\partial x}+\frac{\lambda+\mu}{\mu}z\frac{\partial^2\Phi}{\partial x\partial z},\\
u_y&=\frac{\partial\Phi}{\partial y}+\frac{\lambda+\mu}{\mu}z\frac{\partial^2\Phi}{\partial y\partial z},\\
u_z&=-\frac{\lambda+2\mu}{\mu}\frac{\partial\Phi}{\partial z}+\frac{\lambda+\mu}{\mu}z\frac{\partial^2\Phi}{\partial z^2}.
\end{aligned} \tag{B.8}$$

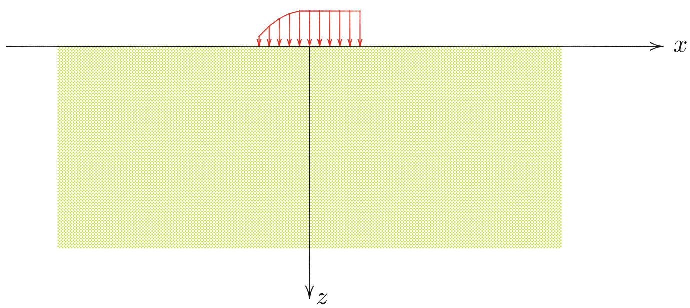


Fig. B.3 Boussinesq problem

Substitution into the Eq. (B.7) shows that all these equations are satisfied, provided that the function Φ satisfies Laplace's differential equation,

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = 0. \tag{B.9}$$

It follows that there is only a single unknown function, Φ, which should satisfy a rather simple differential equation, Laplace's equation. Many solutions of this equation are available.

The applicability of the potential Φ appears when the stresses are expressed in terms of this function. Using (B.1), (B.4) and (B.9), it follows that the normal stresses are

$$\begin{aligned}
\frac{\tau_{xx}}{2\mu} &= \frac{\partial^2\Phi}{\partial x^2} + \frac{\lambda+\mu}{\mu} z \frac{\partial^3\Phi}{\partial x^2 \partial z} - \frac{\lambda}{\mu}\frac{\partial^2\Phi}{\partial z^2},\\
\frac{\tau_{yy}}{2\mu} &= \frac{\partial^2\Phi}{\partial y^2} + \frac{\lambda+\mu}{\mu} z \frac{\partial^3\Phi}{\partial y^2 \partial z} - \frac{\lambda}{\mu}\frac{\partial^2\Phi}{\partial z^2},\\
\frac{\tau_{zz}}{2\mu} &= -\frac{\lambda+\mu}{\mu}\frac{\partial^2\Phi}{\partial z^2} + \frac{\lambda+\mu}{\mu} z \frac{\partial^3\Phi}{\partial z^3}.
\end{aligned} \tag{B.10}$$

And the shear stresses are found to be

$$\begin{aligned}
\frac{\tau_{xy}}{2\mu} &= \frac{\partial^2\Phi}{\partial x \partial y} + \frac{\lambda+\mu}{\mu} z \frac{\partial^3\Phi}{\partial x \partial y \partial z},\\
\frac{\tau_{yz}}{2\mu} &= \frac{\lambda+\mu}{\mu} z \frac{\partial^3\Phi}{\partial y \partial z^2},\\
\frac{\tau_{zx}}{2\mu} &= \frac{\lambda+\mu}{\mu} z \frac{\partial^3\Phi}{\partial x \partial z^2}.
\end{aligned} \tag{B.11}$$

The last two equations show that on the plane $z = 0$ the shear stresses are automatically zero,

$$z = 0 : \quad \tau_{zx} = \tau_{zy} = 0, \tag{B.12}$$

whatever the function Φ is. This means that the potential Φ can be used only for problems in which the surface $z = 0$ is free of shear stresses. That is an important restriction, which limits the use of this potential very severely. On the other hand, the class of problems of a half space loaded by normal stresses is an important class of problems for soil mechanics, and the differential equation is rather simple. On the surface $z = 0$ the normal stress τ_{zz} may be prescribed, or the displacement u_z. Some examples will be given below.

B.3 Point Load

A classical solution, described by Boussinesq, is the problem of a point load P on an elastic half space $z > 0$, see Fig. B.4.

The solution is assumed to be

$$\Phi = -\frac{P}{4\pi(\lambda+\mu)} \ln(z+R), \tag{B.13}$$

in which R is the spherical coordinate,

$$R = \sqrt{x^2+y^2+z^2}. \tag{B.14}$$

That this function satisfies the differential equation (B.9) can easily be verified by substitution into this equation. Next it must be checked that the boundary conditions are satisfied. The shear stresses on the surface $z = 0$ are automatically zero, and the condition for the normal stresses can be verified as follows.

Differentiation of Φ with respect to z gives

$$\frac{\partial \Phi}{\partial z} = -\frac{P}{4\pi(\lambda+\mu)} \frac{1}{R}, \tag{B.15}$$

$$\frac{\partial^2 \Phi}{\partial z^2} = \frac{P}{4\pi(\lambda+\mu)} \frac{z}{R^3}, \tag{B.16}$$

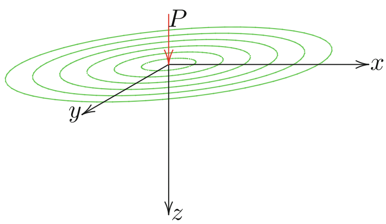
$$\frac{\partial^3 \Phi}{\partial z^3} = \frac{P}{4\pi(\lambda+\mu)} \left(\frac{1}{R^3} - 3\frac{z^2}{R^5}\right). \tag{B.17}$$

The vertical normal stress τ_{zz} now is, with (B.10),

$$\tau_{zz} = -\frac{3P}{2\pi} \frac{z^3}{R^5}. \tag{B.18}$$

On the surface $z = 0$ this stress is zero, except in the origin, where the stress is infinitely large. The resultant force of the stress distribution can be obtained by integrating the vertical normal stress over an entire horizontal plane. This gives

Fig. B.4 Point load on half space



$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tau_{zz}\, dx\, dy = -P. \tag{B.19}$$

Every horizontal plane appears to transfer a force of magnitude P, as required. The solution (B.13) appears to satisfy all necessary conditions, and it can be concluded that it is the correct solution of the problem.

The vertical displacement is, with (B.8),

$$u_z = \frac{P}{4\pi\mu R}\left(\frac{\lambda+2\mu}{\lambda+\mu} + \frac{z^2}{R^2}\right). \tag{B.20}$$

The factor $(\lambda + 2\mu)/(\lambda + \mu)$ can also be written as $2(1-\nu)$. The displacements of the surface $z = 0$ is, when expressed in E and ν,

$$z = 0 \; : \quad u_z = \frac{P(1-\nu^2)}{\pi E R}. \tag{B.21}$$

This is singular in the origin, as might be expected for this case of a concentrated load.

All other stresses and displacements can easily be derived from the solution (B.13).

B.4 Distributed Load

On the basis of the elementary solution (B.13) many other interesting solutions can be derived. As an example the displacement in the center of a circular area, carrying a uniform load will be derived, see Fig. B.5. A load of magnitude pdA at a distance r from the origin leads to a displacement of the origin of magnitude

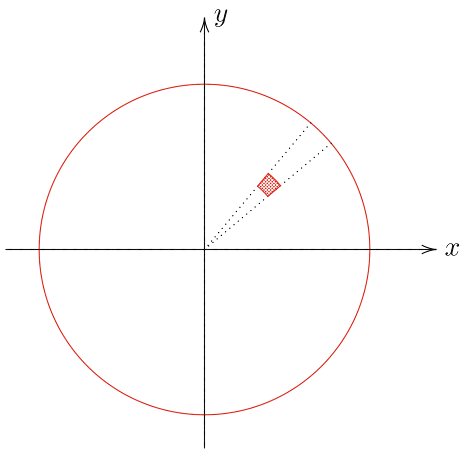


Fig. B.5 Distributed load, on circular area

$$\frac{p\,dA\,(1-\nu^2)}{\pi E r},$$

in agreement with formula (B.21).

The displacement caused by a uniform load over a circular area, with radius a, can be found by integration over that area. Because $dA = r\,dr\,d\theta$, integration over θ from $\theta = 0$ to $\theta = 2\pi$, and integration over r from $r = 0$ to $r = a$ gives

$$r = 0,\ z = 0\ :\quad u = \frac{2pa(1-\nu^2)}{E}. \tag{B.22}$$

This is a well known and useful result.

B.5 Fourier Transforms

A general class of solutions can be found by using Fourier transforms (Sneddon 1951). As an example some a problem of plane strain deformations (for which $u_y = 0$) will be considered here.

The solution is assumed to be

$$\Phi = \int_0^\infty \{f(\alpha)\cos(\alpha x) + g(\alpha)\sin(\alpha x)\}\exp(-\alpha z)\,d\alpha, \tag{B.23}$$

in which $f(\alpha)$ and $g(\alpha)$ are undetermined functions in this stage.

That the expression (B.23) is indeed a solution follows immediately from substitution of the elementary solutions $\cos(\alpha x)\exp(-\alpha z)$ and $\sin(\alpha x)\exp(-\alpha z)$ into the differential equation (B.9). For $z \to \infty$ the solution tends towards zero, which suggests that these solutions may be used for problems in which the stresses should vanish for $z \to \infty$.

The normal stress at the surface $z = 0$ is, with (B.10) and (B.23),

$$z = 0\ :\ \frac{\tau_{zz}}{2\mu} = -\left(\frac{\lambda+\mu}{\mu}\right)\int_0^\infty \{\alpha^2 f(\alpha)\cos(\alpha x) + \alpha^2 g(\alpha)\sin(\alpha x)\}\,d\alpha. \tag{B.24}$$

It is assumed that the boundary condition is

$$z = 0,\ -\infty < x < \infty\ :\quad \tau_{zz} = q(x), \tag{B.25}$$

in which $q(x)$ is a given function. Then the condition is

$$\int_0^\infty \{A(\alpha)\cos(\alpha x) + B(\alpha)\sin(\alpha x)\}\,d\alpha = q(x), \tag{B.26}$$

in which

$$A(\alpha) = -2(\lambda + \mu)\,\alpha^2 f(\alpha), \tag{B.27}$$

and

$$B(\alpha) = -2(\lambda + \mu)\,\alpha^2 g(\alpha). \tag{B.28}$$

The problem of determining the functions $A(\alpha)$ and $B(\alpha)$ from (B.26) is the standard problem from the theory of Fourier transforms. The solution is provided by the *inversion theorem*. The derivation of this theorem will not be given here, see any book on Fourier analysis. The final result is

$$A(\alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty} q(t)\cos(\alpha t)\,dt, \tag{B.29}$$

and

$$B(\alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty} q(t)\sin(\alpha t)\,dt. \tag{B.30}$$

This is the solution of the problem, for an arbitrary load distribution $q(x)$ on the surface. The solution expresses that first the integrals (B.29) and (B.30) must be calculated, and then the results must be substituted into the general solution (B.23). The actual analysis may be quite complicated, depending upon the complexity of the load function $q(x)$. The procedure will be elaborated in the next section, for a simple example.

B.6 Line Load

As an example the case of a line load will be elaborated, see Fig. B.6. In this case the load can be described by the function

$$q(x) = \begin{cases} -F/(2\epsilon) & \text{if } |x| < \epsilon, \\ 0 & \text{if } |x| > \epsilon, \end{cases} \tag{B.31}$$

where ϵ is a small length, with $\epsilon \to 0$. From (B.29) and (B.30) it now follows that

$$A(\alpha) = -\frac{F}{\pi\epsilon}\,\frac{\sin(\alpha\epsilon)}{\alpha},$$

$$B(\alpha) = 0.$$

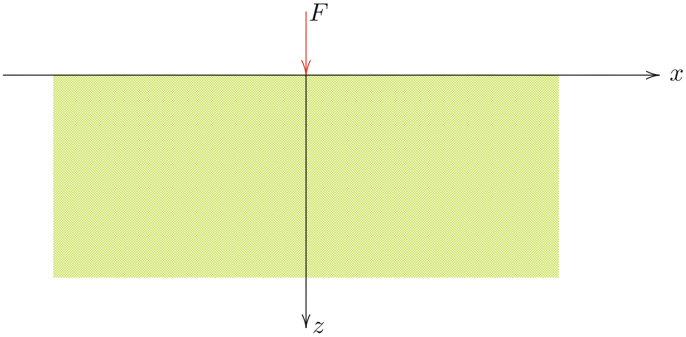


Fig. B.6 Line load on half space

If $\epsilon \to 0$ this reduces to

$$A(\alpha) = -F/\pi, \tag{B.32}$$

$$B(\alpha) = 0. \tag{B.33}$$

With (B.27) and (B.28) the original functions are

$$f(\alpha) = \frac{F}{2\pi(\lambda+\mu)\alpha^2}, \tag{B.34}$$

$$g(\alpha) = 0. \tag{B.35}$$

The final solution of the problem is

$$\Phi = \frac{F}{2\pi(\lambda+\mu)} \int_0^\infty \frac{\cos(\alpha x)\exp(-\alpha z)}{\alpha^2}\,d\alpha. \tag{B.36}$$

Even though this integral does not converge, because of the behavior of the factor α^2 in the denominator for $\alpha \to 0$, the result can be used to determine the stresses, for which the potential must be differentiated. For instance,

$$\frac{\partial^2\Phi}{\partial x^2} = -\frac{F}{2\pi(\lambda+\mu)} \int_0^\infty \cos(\alpha x)\exp(-\alpha z)\,d\alpha,$$

and this integral converges. The result is

$$\frac{\partial^2\Phi}{\partial x^2} = -\frac{F}{2\pi(\lambda+\mu)}\frac{z}{x^2+z^2}. \tag{B.37}$$

In a similar way it can be shown that

$$\frac{\partial^2 \Phi}{\partial z^2} = \frac{F}{2\pi(\lambda+\mu)} \frac{z}{x^2+z^2}. \tag{B.38}$$

Continuing the differentiation gives

$$\frac{\partial^3 \Phi}{\partial z^3} = \frac{F}{2\pi(\lambda+\mu)} \frac{x^2-z^2}{(x^2+z^2)^2}, \tag{B.39}$$

$$\frac{\partial^3 \Phi}{\partial x^2 \partial z} = -\frac{F}{2\pi(\lambda+\mu)} \frac{x^2-z^2}{(x^2+z^2)^2}. \tag{B.40}$$

The stresses finally are, with (B.10) and (B.11),

$$\tau_{xx} = -\frac{2F}{\pi} \frac{x^2 z}{(x^2+z^2)^2}, \tag{B.41}$$

$$\tau_{zz} = -\frac{2F}{\pi} \frac{z^3}{(x^2+z^2)^2}, \tag{B.42}$$

$$\tau_{xz} = -\frac{2F}{\pi} \frac{xz^2}{(x^2+z^2)^2}. \tag{B.43}$$

These formulas were first derived by Flamant, in (1892).

Many more solutions of elastic problems have been found, for instance for layered systems, and for bodies of more complex form than a half plane or a half space, for instance a plane with a row of circular holes (a problem of great interest to aeronautical engineers). Many of these solutions are very complex. A large number of solutions of interest for geotechnical engineering can be found in the book by Poulos and Davis (1974).

Appendix C
Plasticity

In this appendix the main theorems of plasticity theory are presented. These are the *limit theorems*, which enable to determine upper bounds and lower bounds of the failure load of a body.

C.1 Yield Surface

The simplest description of plastic deformations is by considering a *perfectly plastic* material. This is a material that exhibits plastic deformations if (and only if) the stresses satisfy the *yield condition*. For a perfectly plastic material this yield condition is a function of the stresses only (and not of the deformations, or of the time). This yield condition is written in the form

$$f(\sigma_{ij}) = 0. \tag{C.1}$$

Plastic deformations can occur only if $f(\sigma_{ij}) = 0$. Stress states for which $f(\sigma_{ij}) > 0$ are impossible, and if $f(\sigma_{ij}) < 0$ there are no plastic deformations, but such states of stress are perfectly possible. The deformations then are elastic only.

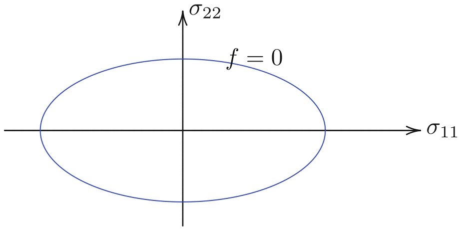
The yield condition can be considered as a relation between the nine stresses σ_{ij}, with $i, j = 1, 2, 3$, in a 9-dimensional space. In such a space the yield condition (C.1) is an 8-dimensional part of space. It is usually called the *yield surface*. If the state of stress can be described by three stresses (for instance the three principal stresses), the yield condition can be written as

$$f(\sigma_1, \sigma_2, \sigma_3) = 0. \tag{C.2}$$

In the 3-dimensional space with axes σ_1, σ_2 and σ_3 this is a surface. For that reason the condition (C.1) in a higher dimensional space is also called the yield surface. In a 2-dimensional space, if there are only two parameters that determine yielding, the yield surface reduces to a (curved) line.

A. Verruijt, *An Introduction to Soil Mechanics*, Theory and Applications of Transport in Porous Media 30, DOI 10.1007/978-3-319-61185-3

Fig. C.1 Yield surface



It is assumed that the origin $\sigma_{ij} = 0$, that is the state of stress in which all stresses are zero, is located inside the yield surface. Furthermore, it is assumed that if a certain point σ^e_{ij} is located inside the yield surface, then $\alpha\sigma^e_{ij}$, with $\alpha < 1$, is also inside the yield surface. In topology it is said that the yield surface is *star-shaped*. Later it will also be assumed that the yield surface is *convex*, see Fig. C.1. That is a more severe restriction than the assumption that it is star-shaped. These assumptions are essential for the derivations to be presented in this chapter.

To simplify the analysis it will be assumed that the material can deform only if $f(\sigma_{ij}) = 0$. This means that all elastic deformations are disregarded. Such a material is called *rigid plastic*.

C.2 Some Geometrical Definitions

Before presenting the mechanics of plastic deformations it is useful to first derive some important geometrical relations, for the expression of a plane tangent to the yield surface, and for a line perpendicular to that surface.

In a 9-dimensional space a plane tangent to the surface $f(\sigma_{ij})$ can be defined as

$$\left(\frac{\partial f}{\partial \sigma_{ij}}\right)_1 (\sigma_{ij} - \sigma^1_{ij}) = 0. \tag{C.3}$$

Here $(\partial f/\partial \sigma_{ij})_1$ denotes the partial derivative of the function f with respect to the variable σ_{ij} in the point σ^1_{ij}. In Eq. (C.3) summation over the indices i and j is implied by the repetition of these indices. This is the summation convention of Einstein,

$$a_i b_i = \sum_{i=1}^{n} a_i b_i, \tag{C.4}$$

in which n is the dimension of space, usually 3, but in this case $n = 9$. The definition (C.3) is a generalization to 9-dimensional space of the usual definition in 3-dimensional space.

The significance of the definition (C.3) can be clarified as follows. On the yield surface the value of f is constant ($f = 0$). Suppose that σ^1_{ij} is a point on that surface, and consider a small increment of the stress from that value, such that both σ^1_{ij} and $\sigma^1_{ij} + d\sigma_{ij}$ are located on the yield surface. The difference df of the functional values in these two points is zero, i.e.

$$df = \left(\frac{\partial f}{\partial \sigma_{ij}}\right)_1 d\sigma_{ij} = 0. \tag{C.5}$$

Equation (C.3) is the generalization of (C.5) for arbitrary points, at an arbitrary distance form σ^1_{ij}, which is also linear in σ_{ij}. It follows that is indeed natural to denote (C.3) as the definition of the tangent plane.

Next the definition of a line perpendicular to the yield surface will be considered. For this purpose it may be noted that the general equation of a plane passing through the point σ^1_{ij} is

$$A_{ij}\,(\sigma_{ij} - \sigma^1_{ij}) = 0, \tag{C.6}$$

in which the constants A_{ij} are given numbers, that define the slopes of the plane in the various directions. A straight line in this plane, through the point σ^1_{ij}, can be written as

$$\sigma_{ij} - \sigma^1_{ij} = a\,(\sigma^2_{ij} - \sigma^1_{ij}), \tag{C.7}$$

in which a is a variable parameter, and σ^2_{ij} is a second point in the plane (C.6), which means that

$$A_{ij}\,(\sigma^2_{ij} - \sigma^1_{ij}) = 0. \tag{C.8}$$

An arbitrary straight line through the point σ^1_{ij}, not necessarily in the plane considered, can be described by the equation

$$\sigma_{ij} - \sigma^1_{ij} = c\,b_{ij}, \tag{C.9}$$

in which b_{ij} are constants, and c a variable parameter.

In general two straight lines

$$\sigma_{ij} - \sigma^a_{ij} = a\,c_{ij}, \tag{C.10}$$

$$\sigma_{ij} - \sigma^b_{ij} = b\,d_{ij}, \tag{C.11}$$

are considered to be perpendicular if the inner product of the directional vectors is zero,

$$c_{ij}\,d_{ij} = 0. \tag{C.12}$$

This is in agreement with the usual definition of orthogonality, by requiring that the inner product of two vectors is zero.

If Eq. (C.7) is now written as

$$\sigma_{ij} - \sigma_{ij}^1 = a\, c_{ij} = a\, (\sigma_{ij}^2 - \sigma_{ij}^1), \tag{C.13}$$

it follows that the line

$$\sigma_{ij} - \sigma_{ij}^0 = b\, A_{ij}, \tag{C.14}$$

is perpendicular to each line of the set (C.13), because $A_{ij}\, c_{ij}$ is always zero, see (C.8). The conclusion must be that the line (C.14) is perpendicular to the plane (C.6). The point σ_{ij}^0 needs not to be located on the yield surface, but this is not forbidden either, and the point may even coincide with the point σ_{ij}^1. It follows that the line

$$\sigma_{ij} - \sigma_{ij}^1 = bA_{ij}, \tag{C.15}$$

passes through the point σ_{ij}^1, and is perpendicular to the plane (C.6).

If this property is applied to the tangent plane of the yield surface, as defined by Eq. (C.3), it follows that a line defined by

$$\sigma_{ij} - \sigma_{ij}^1 = b \left(\frac{\partial f}{\partial \sigma_{ij}} \right)_1, \tag{C.16}$$

is perpendicular to the yield surface, in the point σ_{ij}^1.

As an example consider a yield surface in the form of an ellipse, see Fig. C.1, with axes $2a$ and a,

$$f = \frac{\sigma_{11}^2}{4a^2} + \frac{\sigma_{22}^2}{a^2} = 0. \tag{C.17}$$

In this case the equation of the tangent plane (in this two-dimensional case this is a tangent line) is, following (C.3),

$$\sigma_{11}^1(\sigma_{11} - \sigma_{11}^1) + 4\sigma_{22}^1(\sigma_{22} - \sigma_{22}^1) = 0, \tag{C.18}$$

in which the superscript ^{1} indicates that the point is located on the yield surface.

In the rightmost point of the yield surface $\sigma_{11}^1 = 2a$ and $\sigma_{22}^1 = 0$. Equation (C.18) then defines the tangent as: $\sigma_{11} = 2a$. In the topmost point of the yield surface $\sigma_{11}^1 = 0$ and $\sigma_{22}^1 = a$. In that case Eq. (C.18) defines the tangent as: $\sigma_{22} = a$. These two tangents are shown as dotted lines in Fig. C.2. These two lines are indeed tangent to the yield surface.

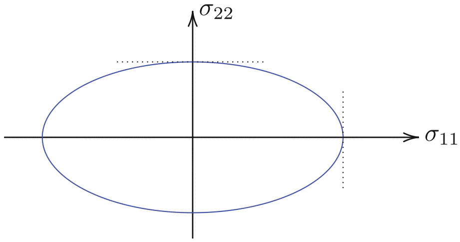


Fig. C.2 Examples of tangents

C.3 Convex Yield Surface

After the definition of some geometrical concepts we now return to the mechanics of plastic materials. As stated before, plastic deformation is governed by the location of the stress point σ_{ij} with respect to the yield surface $f(\sigma_{ij}) = 0$, in a 9-dimensional space. It is now assumed that the yield surface is convex. This is supposed to be defined by the requirement that

$$(\sigma^1_{ij} - \sigma^e_{ij}) \left(\frac{\partial f}{\partial \sigma_{ij}}\right)_1 > 0, \tag{C.19}$$

in which σ^1_{ij} is a point of the yield surface, and σ^e_{ij} is an arbitrary point inside the yield surface. This means that $f(\sigma^1_{ij}) = 0$ and $f(\sigma^e_{ij}) < 0$. Equation (C.19) states that the inner product of the vector from σ^e_{ij} to σ^1_{ij}, and the vector $(\partial f/\partial \sigma_{ij})_1$, which is directed perpendicular to the yield surface, is positive. This means that the angle between these two vectors is smaller than $\pi/2$, which corresponds to the statement that the yield surface is convex., see Fig. C.3. Only if the yield surface would have concave parts it would be possible that a vector from a point inside the yield surface to a point on that surface makes an angle greater than $\pi/2$ with the vector normal to the yield surface, in outward direction. This possibility is excluded here, by assuming that the yield surface is convex. This property will be used in later proofs.

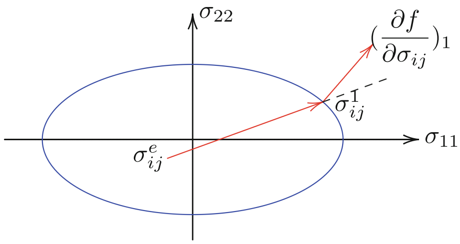


Fig. C.3 Convex yield surface

C.4 Plastic Deformations

It is assumed that the plastic deformations can be described by the *deformation rates* $\dot{\varepsilon}_{ij}$. It follows that

$$\begin{aligned} f(\sigma_{ij}) < 0 : \dot{\varepsilon}_{ij} = 0, \\ f(\sigma_{ij}) = 0 : \dot{\varepsilon}_{ij} \neq 0. \end{aligned} \tag{C.20}$$

This means that plastic deformations, whenever they occur, will continue forever, at a certain rate. If time progresses, the deformations will increase indefinitely.

The plastic deformation rates $\dot{\varepsilon}_{ij}$ can also be plotted in a 9-dimensional space, and this can be done such that the axes coincide with the axes of stress space. The vectors σ_{ij} and $\dot{\varepsilon}_{ij}$ may then be represented in the same space.

C.5 Plastic Potential

It is postulated that the plastic strain rates can be derived from a *plastic potential* g, that depends on the stresses only, i.e. $g = g(\sigma_{ij})$, in such a way that the strain rates can be obtained by

$$f(\sigma_{ij}) < 0 : \dot{\varepsilon}_{ij} = 0, \tag{C.21}$$

$$f(\sigma_{ij}) = 0 : \dot{\varepsilon}_{ij} = \lambda \frac{\partial g}{\partial \sigma_{ij}}. \tag{C.22}$$

Here λ is an undetermined constant. The essential assumption is that such a function $g(\sigma_{ij})$, from which the strain rates can be determined by differentiation with respect to the corresponding stresses, see (C.22), exists.

From the geometrical considerations presented above, it follows that the vector of strain rates $\dot{\varepsilon}_{ij}$ in 9-dimensional space is perpendicular to the surface of the plastic

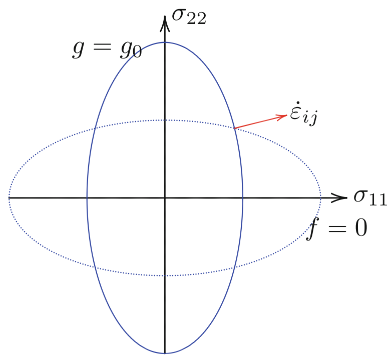


Fig. C.4 Plastic potential

potential g, see Fig. C.4. In this figure the yield condition is shown by a dotted curve. The plastic potential passing through a certain point of the yield surface has been indicated by a fully drawn curve. The vector of strain rates is perpendicular to the plastic potential. Through each point of the yield surface a surface of constant values of g can be drawn, each with its own value of that constant. The shape of the plastic potential surfaces is unknown at this stage. It may be star-shaped, or convex, or perhaps not.

C.6 Drucker's Postulate

It has been found, by comparing theoretical results with experimental data, that for metals very good agreement is obtained if the plastic potential g is identified with the yield function f. This is often called *Drucker's postulate*,

$$\text{Drucker} \; : f = g. \tag{C.23}$$

It has been attempted to find a theoretical derivation of this property, for instance on the basis of some thermodynamical principle. It has been found later, however, that there is no physical necessity for the validity of Drucker's postulate, other than that it provides a reasonable prediction for the plasticity behavior of metals. For other materials, especially frictional materials such as sand, it is very unlikely that Drucker's postulate is valid, as it leads to unrealistic predictions. It is usually concluded that it may be applicable for materials without friction ($\phi = 0$), but is inapplicable if $\phi > 0$.

Notwithstanding the theoretical objections against Drucker's postulate, it may well be used for clays, especially in undrained conditions. For this reason its validity will be assumed in the sequel, until further notice. This will enable to derive limit theorems for clays.

If the plastic potential is identified with the yield condition, Eq. (C.22) can be written as

$$f(\sigma_{ij}) < 0 : \dot{\varepsilon}_{ij} = 0, \tag{C.24}$$

$$f(\sigma_{ij}) = 0 : \dot{\varepsilon}_{ij} = \lambda \frac{\partial f}{\partial \sigma_{ij}}. \tag{C.25}$$

The direction of the vector of plastic deformations now is normal to the yield surface.

In the next sections the limit theorems will be derived, using the assumptions made before. The first step is the formulation and derivation of the virtual work principle.

C.7 Virtual Work

Let there be considered a body in equilibrium. If the volume of the body is V the equilibrium conditions are that in the volume V the following equations are satisfied,

$$\sigma_{ij,i} + F_j = 0, \tag{C.26}$$

and

$$\sigma_{ij} = \sigma_{ji}, \tag{C.27}$$

where F_j is a given volume force. The comma indicates partial differentiation,

$$a_{,i} = \frac{\partial a}{\partial x_i}. \tag{C.28}$$

It is assumed that the boundary conditions are that on a part (S_1) of the boundary the stresses are prescribed, and that on the remaining part of the boundary (S_2) the displacements are prescribed,

$$\text{on } S_1 : \quad \sigma_{ij} n_i = t_j, \tag{C.29}$$

$$\text{on } S_2 : \quad u_i = f_i, \tag{C.30}$$

where t_j is given on S_1 and f_i is given on S_2.

In the sequel the following definitions are needed. A field of stresses that satisfies Eqs. (C.26), (C.27) and (C.29) is a *statically admissible* stress field, or an *equilibrium system*. A field of displacements that satisfies certain regularity conditions (meaning that the material should retain its integrity, and that no overlaps or gaps may be created in the deformation, but that allows sliding of one part with respect to the rest of the body), and that satisfies Eq. (C.30), is a *kinematically admissible* displacement field, or a *mechanism*. To such a field a displacement field can be associated by

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}). \tag{C.31}$$

Now consider an arbitrary statically admissible stress field σ_{ij}, and an arbitrary kinematically admissible displacement field u_i. These fields need not have any relation, except that they must be defined in the same volume V. In general one may write

$$\int_V \sigma_{ij,i} u_j \, dV = \int_V [(\sigma_{ij} u_j)_{,i} - \sigma_{ij} u_{j,i}] \, dV.$$

Using Gauss' divergence theorem and Eq. (C.27) it follows that

$$\int_V \sigma_{ij,i} u_j \, dV = \int_S \sigma_{ij} u_j n_i \, dS - \int_V \frac{1}{2} \sigma_{ij} (u_{i,j} + u_{j,i}) \, dV.$$

With (C.31), (C.29), (C.30) and (C.27) it now follows that

$$\int_V \sigma_{ij} \varepsilon_{ij} \, dV = \int_{S_1} t_i u_i \, dS + \int_{S_2} \sigma_{ij} n_i f_j \, dS + \int_V F_i u_i \, dV. \tag{C.32}$$

Equation (C.32) is valid for any combination of an arbitrary statically admissible field and an arbitrary kinematically admissible displacement field, defined in the same body.

Equation (C.32) must also be valid for the combination of the statically admissible stress field σ_{ij} and the kinematically admissible displacement field $u_i + \dot{u}_i \, dt$. Because this field should also satisfy the boundary condition (C.30), in order to be kinematically admissible, it follows that

$$\text{on } S_2 \; : \quad \dot{u}_i = 0. \tag{C.33}$$

The small increments of the displacement field $\dot{u}_i \, dt$, that satisfies (C.33) constitutes a *virtual displacement*. Similar to Eq. (C.32) the following equation must be satisfied

$$\begin{aligned} \int_V \sigma_{ij} \varepsilon_{ij} \, dV + dt \int_V \sigma_{ij} \dot{\varepsilon}_{ij} \, dV &= \int_{S_1} t_i u_i \, dS + dt \int_{S_1} t_i \dot{u}_i \, dS \\ &+ \int_{S_2} \sigma_{ij} n_i f_j \, dS + \int_V F_i u_i \, dV + dt \int_V F_i \dot{u}_i \, dV. \end{aligned} \tag{C.34}$$

If Eq. (C.32) is subtracted from this equation, the result is, after division by dt,

$$\int_V \sigma_{ij} \dot{\varepsilon}_{ij} \, dV = \int_{S_1} t_i \dot{u}_i \, dS + \int_V F_i \dot{u}_i \, dV. \tag{C.35}$$

This is the *virtual work theorem*. It is valid for any combination of a statically admissible stress field, and a variation of a kinematically admissible displacement field. These fields need not be related at all.

The integral in the left hand side is the (virtual) work by the stresses on the given incremental deformations. The terms in the right hand side can be considered as the (virtual) work by the volume forces and the surface load during the virtual displacement. This virtual work appears to be equal to the work done by the stresses on the incremental strains.

C.8 Lower Bound Theorem

The *lower bound theorem* states that a lower bound for the failure load can be found by considering an equilibrium field. It can be proved in the following way.

Consider a body consisting of a perfectly plastic material, having a convex yield surface, and satisfying Drucker's postulate. Let the body be loaded by a surface load t_i on the part S_1 of the boundary, and by a volume force F_i. It is assumed that failure will occur for a certain combination of loads, say t_i^c and F_i^c. From now on only combinations of loads are considered that are proportional to the failure load, i.e.

$$t_i = \alpha t_i^c, \quad F_i = \alpha F_i^c, \tag{C.36}$$

where α is a constant.

The stresses at failure are assumed to be σ_{ij}^c, and the corresponding velocities are supposed to be $\dot{u}_i^c$. The virtual work theorem now gives

$$\int_V \sigma_{ij}^c \dot{\varepsilon}_{ij}^c \, dV = \int_{S_1} t_i^c \dot{u}_i^c \, dS + \int_V F_i^c \dot{u}_i^c \, dV. \tag{C.37}$$

Now assume that for a load $t_i^e = \alpha t_i^c$ and $F_i^e = \alpha F_i^c$ a statically admissible stress field σ_{ij}^e has been found, and that all these stresses are inside the yield criterion. Then this load is smaller than the failure load, i.e.

$$\alpha < 1. \tag{C.38}$$

The proof (ad absurdum) of this theorem can be given as follows. Let it be assumed that the theorem is false, i.e. assume that $\alpha > 1$. From the virtual work theorem it follows that

$$\int_V \sigma_{ij}^e \dot{\varepsilon}_{ij}^c \, dV = \int_{S_1} t_i^e \dot{u}_i^c \, dS + \int_V F_i^e \dot{u}_i^c \, dV,$$

or, with $t_i^e = \alpha t_i^c$ and $F_i^e = \alpha F_i^c$,

$$\int_V \frac{1}{\alpha} \sigma_{ij}^e \dot{\varepsilon}_{ij}^c \, dV = \int_{S_1} t_i^c \dot{u}_i^c \, dS + \int_V F_i^c \dot{u}_i^c \, dV. \tag{C.39}$$

From (C.37) and (C.39) it follows that

$$\int_V \left(\sigma_{ij}^c - \frac{1}{\alpha} \sigma_{ij}^e \right) \dot{\varepsilon}_{ij}^c \, dV = 0. \tag{C.40}$$

Using Drucker's postulate, which has been assumed to be valid, the strain rates at failure are

$$\dot{\varepsilon}_{ij}^{c} = \lambda \left(\frac{\partial f}{\partial \sigma_{ij}} \right)_c . \tag{C.41}$$

Substitution into (C.40) gives

$$\lambda \int_V \left(\sigma_{ij}^c - \frac{1}{\alpha} \sigma_{ij}^e \right) \left(\frac{\partial f}{\partial \sigma_{ij}} \right)_c dV = 0. \tag{C.42}$$

If $\alpha > 1$, and σ_{ij}^e is inside the yield surface (as had been assumed), then σ_{ij}^e/α is certainly inside the yield surface. Because of (C.19), i.e. because of the convexity of the yield surface, it now follows that

$$\left(\sigma_{ij}^c - \frac{1}{\alpha} \sigma_{ij}^e \right) \left(\frac{\partial f}{\partial \sigma_{ij}} \right)_c > 0. \tag{C.43}$$

The integral of this quantity can not be zero, as Eq. (C.42) states. This means that the assumption $\alpha > 1$ must be false. Therefore $\alpha < 1$, and this is just what had to be proved.

The theorem means that a statically admissible stress field that does not violate the yield criterion, constitutes a lower bound for the failure load. The real failure load is always larger than the load for that equilibrium system. The load is on the safe side.

C.9 Upper Bound Theorem

The failure load can also be approached from above. This is expressed by the *upper bound theorem*, which can be derived as follows.

Consider a body consisting of a perfectly plastic material, satisfying Drucker's postulate. The failure load again is t_i^c (on S_1) and F_i^c (in V). The corresponding stresses are σ_{ij}^c. These stresses are located on the yield surface, or partly inside it.

Suppose that a kinematically admissible velocity field $\dot{u}_i^k$ has been chosen, with the corresponding strain rates $\dot{\varepsilon}_{ij}^k$. The plastic strain rates can be derived from the yield function by the relations

$$\dot{\varepsilon}_{ij} = \lambda \frac{\partial f}{\partial \sigma_{ij}}.$$

Using these relations it is possible, at least in principle, to determine the stresses σ_{ij}^k in all points where $\dot{\varepsilon}_{ij}^k \neq 0$. Because the yield surface is convex, and the plastic strain rates are known, there is just one point where the vector of plastic strain rates is perpendicular to the yield surface. This point determines the stress state. Next the following integral can be calculated,

$$D = \int_V \sigma^k_{ij} \dot{\varepsilon}^k_{ij} \, dV. \tag{C.44}$$

This is the energy that would be dissipated by the assumed kinematic field, if it would occur. A load proportional to the failure load, $t^k_i = \beta t^c_i$ and $F^k_i = \beta F^c_i$, can now be calculated such that

$$\int_{S_1} t^k_i \dot{u}^k_i \, dS + \int_V F^k_i \dot{u}^k_i \, dV = D = \int_V \sigma^k_{ij} \dot{\varepsilon}^k_{ij} \, dV. \tag{C.45}$$

Although this formula has the same form as the virtual work principle, it does not follow from that theorem, because the stress field σ^k_{ij} in general is not an equilibrium system, and it need not satisfy the boundary condition for the stresses. Equation (C.45) is simply a procedure to determine the fictitious loads t^k_i and F^k_i.

The upper bound theorem is that the load t^k_i and F^k_i is larger than the failure load, or, in other words, that

$$\beta > 1. \tag{C.46}$$

The proof (ad absurdum) of this theorem is as follows. Let it be assumed that the theorem is false, i.e. assume that

$$\beta = t^k_i / t^c_i = F^k_i / F^c_i < 1.$$

From (C.45) it follows that

$$\int_V \sigma^k_{ij} \dot{\varepsilon}^k_{ij} \, dV = \beta \int_{S_1} t^c_i \dot{u}^k_i \, dS + \beta \int_V F^c_i \dot{u}^k_i \, dV. \tag{C.47}$$

Using the virtual work theorem the following equality can be formulated

$$\beta \int_V \sigma^c_{ij} \dot{\varepsilon}^k_{ij} \, dV = \beta \int_{S_1} t^c_i \dot{u}^k_i \, dS + \beta \int_V F^c_i \dot{u}^k_i \, dV. \tag{C.48}$$

From (C.47) and (C.48) it follows that

$$\int_V (\sigma^k_{ij} - \beta \sigma^c_{ij}) \dot{\varepsilon}^k_{ij} \, dV = 0. \tag{C.49}$$

In all points where $\dot{\varepsilon}^k_{ij} \neq 0$, so that there are contributions to the integral, the point σ^k_{ij} is located on the failure surface. The stress $\beta \sigma^c_{ij}$ is located inside the yield surface, because σ^c_{ij} is a point of the convex yield surface, and $\beta < 1$, by supposition. It then follows from (C.19) that

$$\dot{\varepsilon}^k_{ij} \neq 0 \; : \qquad (\sigma^k_{ij} - \beta \sigma^c_{ij}) \left(\frac{\partial f}{\partial \sigma_{ij}} \right)_k > 0.$$

The integral of this quantity can not be zero, as required by (C.49). This means that a contradiction has been obtained. The conclusion must be that the assumption that $\beta < 1$ must be false, at least if it is assumed that the other assumptions (validity of Drucker's postulate, convex yield surface) are true. Therefore $\beta > 1$, and this is what had to be proved.

The theorem means that a kinematically admissible velocity field, constitutes an upper bound for the failure load. The real failure load is always smaller than the load for that mechanism. The load is on the unsafe side.

C.10 Frictional Materials

For a frictional material, such as most soils, in particular sands, the Mohr–Coulomb criterion is a good representation of the yield condition. For the case that the cohesion $c = 0$ this is shown in Fig. C.5. It is assumed that yielding of the material is determined by the stresses σ_{xx}, σ_{yy}, and $\sigma_{xy} = \sigma_{yx}$ only. The stresses are effective stresses, but as there are no pore pressures (by assumption) they are total stresses as well. The yield condition is that the radius of Mohr's circle equals $\sin\phi$ times the distance of the center of the circle to the origin. This can be expressed as

$$\frac{1}{2}(\sigma_1 - \sigma_3) = \frac{1}{2}(\sigma_1 + \sigma_3)\ \sin\phi, \tag{C.50}$$

or, if the principal stresses are expressed in terms of the stress components in an arbitrary coordinate system of axes x and y,

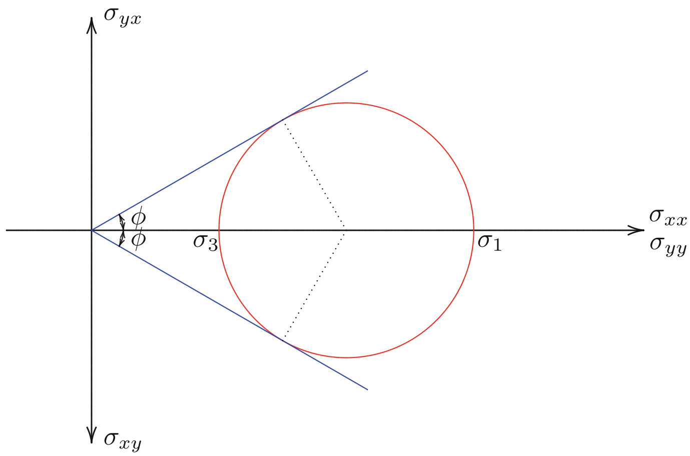


Fig. C.5 Mohr–Coulomb criterion

$$f = \left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \frac{1}{2}\sigma_{xy}^2 + \frac{1}{2}\sigma_{yx}^2 - \left(\frac{\sigma_{xx} + \sigma_{yy}}{2}\right)^2 \sin^2\phi = 0. \tag{C.51}$$

The circumstance that this yield condition depends upon the isotropic stress implies that Drucker's postulate will automatically lead to a deformation corresponding to that stress, i.e. a volume strain. This can be seen formally by calculating the strain rates using Drucker's postulate. This gives

$$\dot{\varepsilon}_{xx} = \lambda\left(\frac{\partial f}{\sigma_{xx}}\right) = \lambda\left\{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right) - \left(\frac{\sigma_{xx} + \sigma_{yy}}{2}\right)\sin^2\phi\right\}, \tag{C.52}$$

$$\dot{\varepsilon}_{yy} = \lambda\left(\frac{\partial f}{\sigma_{yy}}\right) = \lambda\left\{\left(\frac{\sigma_{yy} - \sigma_{xx}}{2}\right) - \left(\frac{\sigma_{xx} + \sigma_{yy}}{2}\right)\sin^2\phi\right\}, \tag{C.53}$$

$$\dot{\varepsilon}_{xy} = \lambda\left(\frac{\partial f}{\sigma_{xy}}\right) = \lambda\sigma_{xy}. \tag{C.54}$$

These strain rates can also be represented graphically in a Mohr diagram. If the radius of that circle is denoted by $\frac{1}{2}\dot{\gamma}$, it follows that

$$\left(\frac{\dot{\gamma}}{2}\right)^2 = \left(\frac{\dot{\varepsilon}_{xx} - \dot{\varepsilon}_{yy}}{2}\right)^2 + \dot{\varepsilon}_{xy}^2. \tag{C.55}$$

Using the expressions (C.52)–(C.54) this can also be written as

$$\left(\frac{\dot{\gamma}}{2}\right)^2 = \lambda^2\left\{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \sigma_{xy}^2\right\}, \tag{C.56}$$

or, because these stresses satisfy the yield criterion (C.51),

$$\left(\frac{\dot{\gamma}}{2}\right)^2 = \lambda^2\left(\frac{\sigma_{xx} + \sigma_{yy}}{2}\right)^2 \sin^2\phi. \tag{C.57}$$

It follows that

$$\frac{\dot{\gamma}}{2} = \lambda\left(\frac{\sigma_{xx} + \sigma_{yy}}{2}\right)\sin\phi. \tag{C.58}$$

On the other hand the volume strain rate is

$$\dot{\varepsilon}_{\text{vol}} = \dot{\varepsilon}_{xx} + \dot{\varepsilon}_{yy} = -2\lambda\left(\frac{\sigma_{xx} + \sigma_{yy}}{2}\right)\sin^2\phi. \tag{C.59}$$

From (C.58) and (C.59) it follows that

$$\dot{\varepsilon}_{\text{vol}} = -\dot{\gamma} \sin \phi. \tag{C.60}$$

Any plastic shear strain γ will be accompanied by a simultaneous volume strain ε_{vol}, in a ratio of $\sin \phi$. The minus sign indicates that this is a volume expansion. That the shear strains in a sand that is failing are accompanied by a continuous volume increase is not what is observed in experiments. It can also not be imagined very well that a sand in failure would continuously increase in volume, as long as it shears. The conclusion must be that Drucker's postulate is not valid for frictional materials. Plasticity theory for such materials must be considerably more complicated, and the proofs of the limit theorems, which heavily rely on the validity of Drucker's postulate, do not apply to frictional materials.

Answers to Problems

1.1 Yes.
1.2 Outer slope.
1.3 Small.
1.4 Preloading by ice.
1.5 At the lower side.
1.6 At the higher side.
1.7 Tower close to canal.
3.1 Mass: 3000 kg. Volumetric weight: 15 $\mathrm{kN/m^3}$.
3.2 $n = 0.42$, $e = 0.73$.
3.3 0.846 $\mathrm{m^3}$, $\gamma = 1923\ \mathrm{kg/m^3}$.
3.4 Settlement: 0.83 m.
3.5 No influence.
3.6 $n = 0.42$.
3.7 $\rho_k = 2636\ \mathrm{kg/m^3}$.
4.1 Total stress unchanged, effective stress increase 5 kPa.
4.2 In the space ship artificial air pressure. Effective stress equals air pressure. On the moon there is no atmospheric pressure. Effective stress zero.
4.3 Yes, if it sinks.
4.4 No, effective stresses unchanged.
4.5 No.
5.1 After reclamation, at 2 m depth: $\sigma = 36$ kPa, $p = 0$, $\sigma' = 36$ kPa.
At 10 m depth: $\sigma = 180$ kPa, $p = 80$ kPa, $\sigma' = 100$ kPa.
5.2 $\sigma = 125$ kPa, $\sigma' = 125$ kPa.
5.3 $\sigma = 125$ kPa, $p = 50$ kPa, $\sigma' = 75$ kPa.
5.4 Water level 10 m: $\sigma = 125$ kPa, $p = 100$ kPa, $\sigma' = 25$ kPa.
Water level 150 m: $\sigma' = 25$ kPa.
5.5 $\sigma' = 86.6$ kPa.
5.6 $\sigma' = 62$ kPa.
5.7 $\Delta\sigma' = 32$ kPa.
6.1 1 m/d $= 1.16 \times 10^{-5}$ m/s. Normal: 1 m/d.
6.2 1 gpd/sqft $= 0.5 \times 10^{-6}$ m/s. Normal: 20 gpd/sqft.

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6.3 $k = 3.33$ m/d.
7.1 $k = 1.48 \times 10^{-4}$ m/s.
7.2 $Q = 0.0628$ cm^3/s.
7.3 To prevent leakage along the top of the sample.
7.4 $k = 0.5$ m/d.
8.1 $\sigma = 152$ kPa, $p = 100$ kPa, $\sigma' = 52$ kPa.
8.2 $\sigma = 144$ kPa, $p = 90$ kPa, $\sigma' = 54$ kPa.
8.3 $\sigma = 184$ kPa, $p = 90$ kPa, $\sigma' = 94$ kPa.
8.4 5 m.
9.1 0.10 kN.
9.2 0.12 kN.
9.3 6.25 m.
9.4 1.40 m.
10.1 $Q = 0.4\,kHB$.
10.2 $i = 0.17$.
10.3 Yes, in case of holes in the clay layer.
11.1 No.
11.2 0.50 m.
11.3 $h \to -\infty$.
11.4 Not forever, if there is no supply.
12.1 Smaller.
12.2 More than 2 cm.
12.3 Yes, because it is so dense.
12.4 To the waist.
13.1 3300 kPa.
13.2 Very small, $\nu \approx 0.5$.
14.1 2.5 cm.
14.2 $E = 50$ à 100 MPa.
14.3 $C_{10} = 4$. Just OK.
16.1 379 s.
16.2 Factor 4 larger.
16.3 650 d.
16.4 0.04 mm.
16.5 0.004.
16.6 Stop if `JJ>100`.
20.1 $\sigma_{xx} = 2\,p$, $\sigma_{xy} = p$.
20.2 $\sigma_{nn} = 1.500\,p$, $\sigma_{nt} = 0.867\,p$, $\alpha = 30°$.
21.1 $F = 340$ N.
21.2 Yes.
28.1 $\sigma_{zz} = 1.23$ kPa.
28.2 $\sigma_{zz} = 3.75$ kPa, in A: $\sigma_{zz} = 0$, at 8000 m depth: $\sigma_{zz} = 0$.
28.3 3.40 kPa, 1.72 kPa, 2.32 kPa.
29.1 No.
29.2 $\sigma_{rr} = (2P/\pi r)\cos\theta$, $\sigma_{r\theta} = 0$, $\sigma_{\theta\theta} = 0$.
30.2 0.213 m.

30.3 0.070 m.
32.1 Cambridge K_0 meter.
32.2 96 kN.
32.3 67 kN/m.
33.1 No.
34.1 OK.
34.2 OK.
34.3 Slope too steep for stability.
34.4 57.6 kN/m.
34.5 71.8 kN/m.
34.6 192 kN.
35.1 OK.
35.2 OK.
35.3 1.90 m.
35.4 11.507 m.
36.1 OK.
36.2 12.67 m.
36.3 10.20 m.
36.4 $d/h = 0.650$.
37.1 8.02 m.
37.2 8.22 m.
37.4 $F = T \times a$.
42.1 OK.
42.2 A large effect.
43.1 9 m.
45.1 No.
45.2 Yes.
46.1 No, q_c is total stress.
46.2 $q_c \approx 8$ MPa.
47.1 Yes. Yes.
47.2 No. Yes.
47.3 3.56 Revolutions per second, $v = 134$ m/s.
47.4 $v = 3$ m/s $= 10.8$ km/h.
48.2 Yes.

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