Appendix A
Algebraic Structures

An algebraic structure is a set X with one or more operations satisfying certain axioms. In this appendix, we will briefly review groups and vector spaces.

A.1 Groups

Group. A *group* is a set G equipped with an operation $m : G \times G \to G$ such that:

1. $\forall x, y, z \in G$, $m(x, m(y, z)) = m(m(x, y), z)$ (associativity);
2. $\forall x \in G$, there exists an element $u \in G$ such that $m(u, x) = m(x, u) = x$ (existence of the identity element);
3. $\forall x \in G$, there exists x^{-1} such that $m(x, x^{-1}) = m(x^{-1}, x) = u$ (existence of the inverse element).

If $m(x, y) = m(y, x)$ $\forall x, y \in G$ then G is an *abelian group*.

Note that in every group the identity element is unique. If we assume there are two identity elements, say u and u', we should have $m(u, u') = u$ and $m(u, u') = u'$, and therefore $u = u'$. Even the inverse element must be unique. If x had two inverse elements, say x^{-1} and x'^{-1}, then

$$\begin{aligned} x^{-1} &= m(u, x^{-1}) = m(m(x'^{-1}, x), x^{-1}) \\ &= m(x'^{-1}, m(x, x^{-1})) = m(x'^{-1}, u) = x'^{-1} \, . \end{aligned} \tag{A.1}$$

The set of real numbers $\mathbb{R}$ with the common operation of sum is an abelian group with the identity element 0; the inverse of the element x is denoted $-x$. The set $\mathbb{R}/\{0\}$ with the common operation of product is an abelian group with the identity element 1; the inverse of the element x is denoted $1/x$.

Let $M(n, \mathbb{R})$ be the set of real matrices $n \times n$. Then

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$$GL(n, \mathbb{R}) = \{A \in M(n, \mathbb{R}) \mid \det A \neq 0\} \ , \tag{A.2}$$

is the subset of invertible matrices in $M(n, \mathbb{R})$. If we introduce the common product of matrices, $GL(n, \mathbb{R})$ is a non-abelian group.

The Galilei group, the Lorentz group, and the Poincaré group introduced in Chaps. 1 and 2 are other examples of groups.

A.2 Vector Spaces

Vector space. A *vector space* over a field K is a set V equipped with two operations, $m : V \times V \to V$ (addition) and $p : K \times V \to V$ (multiplication by an element of K), such that:

1. $\forall \mathbf{x}, \mathbf{y} \in V$, $m(\mathbf{x}, \mathbf{y}) = m(\mathbf{y}, \mathbf{x})$ (commutativity);
2. $\forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in V$, $m(\mathbf{x}, m(\mathbf{y}, \mathbf{z})) = m(m(\mathbf{x}, \mathbf{y}), \mathbf{z})$ (associativity);
3. $\forall \mathbf{x} \in V$, there exists an element $\mathbf{0} \in V$ such that $m(\mathbf{x}, \mathbf{0}) = \mathbf{x}$ (existence of the identity element);
4. $\forall \mathbf{x} \in V$, there exists $-\mathbf{x}$ such that $m(\mathbf{x}, -\mathbf{x}) = \mathbf{0}$ (existence of the inverse element);
5. $\forall \mathbf{x}, \mathbf{y} \in V$ and $\forall a \in K$, $p(a, m(\mathbf{x}, \mathbf{y})) = m(p(a, \mathbf{x}), p(a, \mathbf{y}))$;
6. $\forall \mathbf{x} \in V$ and $\forall a, b \in K$, $p(a + b, \mathbf{x}) = m(p(a, \mathbf{x}), p(b, \mathbf{x}))$;
7. $\forall \mathbf{x} \in V$ and $\forall a, b \in K$, $p(ab, \mathbf{x}) = p(a, p(b, \mathbf{x}))$,
8. $p(1, \mathbf{x}) = \mathbf{x}$.

The elements of V are called *vectors* and the elements of K are called *scalars*.

Note that the definition above of vector space is equivalent to saying that V is an abelian group with the operation $p : K \times V \to V$ satisfying the points 5–8. It is common to employ the symbol $+$ to denote the operation m, i.e. $m(\mathbf{x}, \mathbf{y}) = \mathbf{x} + \mathbf{y}$, and no symbol for the operation p, i.e. $p(a, \mathbf{x}) = a\mathbf{x}$. Vector spaces over $\mathbb{R}$ are called real vector spaces, while those over $\mathbb{C}$ are called complex vector spaces.

Linear operator. Let V and W be two vector spaces over K. The function $f : V \to W$ is a *linear operator* if:

$$f(a\mathbf{v} + b\mathbf{w}) = af(\mathbf{v}) + bf(\mathbf{w}) \ , \tag{A.3}$$

$\forall \mathbf{v}, \mathbf{w} \in V$ and $\forall a, b \in K$.

Let us denote with $L(V; W)$ the set of all linear operators from V into W. We define the operator

$$\tilde{m} : L(V; W) \times L(V; W) \rightarrow L(V; W) \tag{A.4}$$

as the sum at every point of two elements of $L(V; W)$, and the operator

$$\tilde{p} : K \times L(V; W) \rightarrow L(V; W) \tag{A.5}$$

as the product at every point of an element of $L(V; W)$ with an element of K. The operators $\tilde{m}$ and $\tilde{p}$ provide $L(V; W)$ the structure of a vector space. The concept of linear operators can be extended to define bilinear operators and, more in general, n-linear operators. The function $f : V \times W \rightarrow Z$ is called a bilinear operator if $f(\mathbf{v}, \cdot) : W \rightarrow Z$ and $f(\cdot, \mathbf{w}) : V \rightarrow Z$ are linear operators $\forall \mathbf{v} \in V$ and $\forall \mathbf{w} \in W$. Let us indicate with $L(V, W; Z)$ the set of all bilinear operators from $V \times W$ into Z. If we define the operation of sum between two bilinear operators and the operation of product between a bilinear operator and an element of K as done in the case of linear operators, then $L(V, W; Z)$ becomes a vector space. The generalization to n-linear operators from $V_1 \times V_2 \times \cdots \times V_n$ into Z and to the vector space $L(V_1, V_2, \ldots, V_n; Z)$ is straightforward.

Isomorphism. Let V and W be two vector spaces. The function $f : V \rightarrow W$ is an *isomorphism* if it is bijective, linear, and the application inverse is also linear. The vector spaces V and W are said to be *isomorphic* if there exists an isomorphism between V and W.

Dual space. Let V be a vector space. The *dual space* of V is the vector space $L(V; \mathbb{R})$ and is denoted V^*.

Subspace. A *subspace* of a vector space V over K is a subset W of V such that, $\forall \mathbf{w}_1, \mathbf{w}_2 \in W$ and $\forall a, b \in K$, $a\mathbf{w}_1 + b\mathbf{w}_2 \in W$.

Let U be a subset of a vector space V over K. A linear combination of elements of U is an element $\mathbf{v} \in V$ of the form $\mathbf{v} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \cdots + a_n\mathbf{u}_n$, where $\{\mathbf{u}_i\}$ is a finite subset of U and $\{a_i\}$ are elements of K. The subspace generated by U is the set of linear combinations that can be obtained from the elements of U and is denoted $\langle U \rangle$. The elements $\{u_i\}$ are linearly independent if

$$a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \cdots , +a_n\mathbf{u}_n = \mathbf{0} \tag{A.6}$$

if and only if (note the difference between the element $\mathbf{0} \in V$ and the element $0 \in K$)

$$a_1 = a_2 = \cdots = a_n = 0\,. \tag{A.7}$$

Basis. A *basis* of a vector space V is a subset B of V consisting of linearly independent elements and such that $\langle B \rangle = V$. If B has n elements, then we say that the vector space V has dimension n.

Every vector space admits a basis and the number of elements of the basis is independent of the choice of the basis (for the proof, see e.g. Ref. [1]). Let us now consider a vector space V of dimension n and with the basis $B = \{\mathbf{e}_i\}$. If n is a finite number, then V^* has the same dimension as V [1]. We indicate with $\mathbf{e}^i$ the elements of V^* such that $\mathbf{e}^i(\mathbf{e}_j) = \delta^i_j$, where δ^i_j is the Kronecker delta. The set $B^* = \{\mathbf{e}^i\}$ is a basis of V^* and is called the dual basis of B.

Let us consider another basis of V, say $B' = \{\mathbf{e}'_i\}$, and the transformation of the change of basis M defined as (note that we use the Einstein convention of summation over repeated indices)

$$\mathbf{e}'_i = M^j_i \mathbf{e}_j\,. \tag{A.8}$$

Every element $\mathbf{v} \in V$ can be written in terms of the basis B, $\mathbf{v} = v^i \mathbf{e}_i$, as well as in terms of the basis B', $\mathbf{v} = v'^i \mathbf{e}'_i$. v^i and v'^i are the components of the vector $\mathbf{v}$ with respect to the bases B and B', respectively. It is easy to see that the components of a vector must change with the inverse transformation with respect to the basis vectors

$$v'^i = (M^{-1})^i_j v^j\,. \tag{A.9}$$

We say that the basis vectors *transform covariantly* under a change of basis and we employ lower indices. The components of a vector *transform contravariantly* and are written with upper indices. In a similar way, we can see that the vectors of the dual basis transform contravariantly, while the components of an element of V^* transform covariantly. If $B'^* = \{\mathbf{e}'^i\}$ is the dual basis of B' and N is the transformation of the change of basis from B^* and B'^* defined as

$$\mathbf{e}'^i = N^i_j \mathbf{e}^j\,, \tag{A.10}$$

then

$$\delta^i_j = \mathbf{e}'^i(\mathbf{e}'_j) = N^i_k \mathbf{e}^k(M^m_j \mathbf{e}_m) = N^i_k M^m_j \mathbf{e}^k(\mathbf{e}_m) = N^i_k M^m_j \delta^k_m = N^i_k M^k_j\,, \tag{A.11}$$

and therefore $N^i_k = (M^{-1})^i_k$. Since these transformations map quantities that transform covariantly (contravariantly) into quantities that transform in the same way, indices under summation are placed as upper (lower) indices, while free indices are placed as lower (upper) indices. A deeper investigation can show that M and

N are matrices in which the indices of row and column transform covariantly or contravariantly, depending on the cases.

If V is of finite dimension, V and V^* are isomorphic. However, in general there is no preferred isomorphism between the two vector spaces. The situation is different if V is a real vector space and is provided with a non-degenerate symmetric bilinear form.

Bilinear form. A *bilinear form* on a real vector space V is an operator $g \in L(V, V; \mathbb{R})$. The bilinear form g is called *symmetric* if $g(\mathbf{v}, \mathbf{w}) = g(\mathbf{w}, \mathbf{v})$ $\forall \mathbf{v}, \mathbf{w} \in V$.

If g is a symmetric bilinear form on a real vector space V, then an element $v \in V$ is called (with respect to g):

1. *Time-like* if $g(\mathbf{v}, \mathbf{v}) < 0$;
2. *Light-like* if $g(\mathbf{v}, \mathbf{v}) = 0$;
3. *Space-like* if $g(\mathbf{v}, \mathbf{v}) > 0$.

As already pointed out in Sect. 2.2, there are two opposite conventions in the literature. The definition above is more popular in the gravity community. In the particle physics community, it is more common to say that $\mathbf{v}$ is a time-like vector if $g(\mathbf{v}, \mathbf{v}) > 0$ and a space-like vector if $g(\mathbf{v}, \mathbf{v}) < 0$.

If g is a symmetric bilinear form on a real vector space V and $\mathbf{v}$ and $\mathbf{w}$ are two elements of V, $\mathbf{v}$ and $\mathbf{w}$ are said to be *orthogonal* if $g(\mathbf{v}, \mathbf{w}) = 0$.

A symmetric bilinear form g on a real vector space V is called *non-degenerate* if the function $\tilde{g} : V \to V^*$ that transforms an element $\mathbf{v} \in V$ into the element $g(\mathbf{v}, \cdot) \in V^*$ is injective.

If the vector space V is of finite dimension, then the form g is represented, with respect to a basis $B = \{\mathbf{e}_i\}$, by an $n \times n$ matrix in which the element ij is

$$g_{ij} = g(\mathbf{e}_i, \mathbf{e}_j) \, . \tag{A.12}$$

If g is a non-degenerate symmetric bilinear form, then $\tilde{g}$ is the natural isomorphism of V onto V^*, while its inverse $\tilde{g}^{-1}$ is the natural isomorphism of V^* onto V: an element $\mathbf{v} \in V$ is associated to the element $\tilde{\mathbf{v}} \in V^*$ given by $\tilde{\mathbf{v}} = \tilde{g}(\mathbf{v}) = g(\mathbf{v}, \cdot)$. If $B = \{\mathbf{e}_i\}$ is a basis of V and $B^* = \{\mathbf{e}^i\}$ is its dual basis, we write $\mathbf{v} = v^i \mathbf{e}_i$ and $\tilde{\mathbf{v}} = v_i \mathbf{e}^i$. From Eq. (A.12), we find that the components of $\tilde{\mathbf{v}}$ are obtained by *lowering the indices* with g

$$v_i = g_{ij} v^j \, . \tag{A.13}$$

Denoting g^{ij} the components of the inverse matrix associated to the bilinear form g, the components of the vector $\mathbf{v}$ are obtained by *raising the indices* of the components of the vector $\tilde{\mathbf{v}}$

$$v^i = g^{ij} v_j \,. \tag{A.14}$$

Note that, by definition, $g_{ij} g^{jk} = \delta_i^k$.

Reference

1. M. Nakahara, *Geometry, Topology and Physics* (IOP, Bristol, 1990).

Appendix B
Vector Calculus

This appendix briefly reviews some basic operators and identities of vector calculus in a 3-dimensional Euclidean space with a Cartesian coordinate system (x, y, z). The metric is the Kronecker delta δ_{ij}

$$||\delta_{ij}|| = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{B.1}$$

Indices are (trivially) raised and lowered with δ^{ij} and δ_{ij}, respectively. The inner product is the algebraic operation that transforms two vectors, say $\mathbf{V}$ and $\mathbf{W}$, into the number

$$\mathbf{V} \cdot \mathbf{W} = \delta_{ij} V^i W^j , \tag{B.2}$$

where the Einstein convention of summation over repeated indices is used.

In what follows, $\phi = \phi(x, y, z)$ denotes a generic scalar function and $\mathbf{V} = (V^x, V^y, V^z)$, where $V^i = V^i(x, y, z)$, is a generic vector field.

B.1 Operators

The *del* operator ∇ is defined as

$$\nabla = \frac{\partial}{\partial x}\hat{\mathbf{x}} + \frac{\partial}{\partial y}\hat{\mathbf{y}} + \frac{\partial}{\partial z}\hat{\mathbf{z}} , \tag{B.3}$$

where $(\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}})$ is the natural basis, namely $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ are the unit vectors pointing in the direction of the axes of the Cartesian coordinate system.

The *gradient* of the scalar function ϕ is the vector field

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$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{\mathbf{x}} + \frac{\partial\phi}{\partial y}\hat{\mathbf{y}} + \frac{\partial\phi}{\partial z}\hat{\mathbf{z}}\,. \tag{B.4}$$

The components of the resulting vector field are

$$(\nabla\phi)^i = \partial^i\phi\,. \tag{B.5}$$

The *divergence* of the vector field **V** is the scalar function

$$\nabla\cdot\mathbf{V} = \frac{\partial V^x}{\partial x} + \frac{\partial V^y}{\partial y} + \frac{\partial V^z}{\partial z}\,, \tag{B.6}$$

so we can write

$$\nabla\cdot\mathbf{V} = \partial_i V^i\,. \tag{B.7}$$

The *curl* or *rotor* of the vector field **V** is the vector field

$$\nabla\times\mathbf{V} = \left(\frac{\partial V^z}{\partial y} - \frac{\partial V^y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial V^x}{\partial z} - \frac{\partial V^z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial V^y}{\partial x} - \frac{\partial V^x}{\partial y}\right)\hat{\mathbf{z}}\,, \tag{B.8}$$

The components of the resulting vector field can be written as

$$(\nabla\times\mathbf{V})^i = \varepsilon^{ijk}\partial_j V_k\,, \tag{B.9}$$

where ε^{ijk} is the Levi-Civita symbol (see next section).

The *Laplacian* of the scalar function ϕ is the scalar function

$$\Delta\phi = \nabla^2\phi = (\nabla\cdot\nabla)\,\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}\,. \tag{B.10}$$

We can also write

$$\Delta\phi = \delta^{ij}\partial_i\partial_j\phi\,. \tag{B.11}$$

The *d'Alembertian* of the scalar function ϕ is the scalar function

$$\Box\phi = \left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2\right)\phi = -\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} + \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}\,. \tag{B.12}$$

The d'Alembertian can be seen as the Laplacian in Minkowski spacetime

$$\Box\phi = \eta^{\mu\nu}\partial_\mu\partial_\nu\phi\,. \tag{B.13}$$

B.2 Levi-Civita Symbol

The *Levi-Civita symbol* ε_{ijk} is defined as

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } (i, j, k) \text{ is an } even \text{ permutation of } (1, 2, 3), \\ -1 & \text{if } (i, j, k) \text{ is an } odd \text{ permutation of } (1, 2, 3), \\ 0 & \text{if any index is repeated.} \end{cases} \tag{B.14}$$

Indices can be raised and lowered with δ^{ij} and δ_{ij}, but often the position of the indices is ignored because δ^{ij} and δ_{ij} have a trivial effect. For this reason, sometimes only lower indices are used. The following formulas hold

$$\varepsilon_{ijk}\varepsilon^{imn} = \delta^m_j \delta^n_k - \delta^n_j \delta^m_k \,, \tag{B.15}$$

$$\varepsilon_{imn}\varepsilon^{jmn} = 2\delta^j_i \,, \tag{B.16}$$

$$\varepsilon_{ijk}\varepsilon^{ijk} = 6 \,. \tag{B.17}$$

B.3 Properties

The divergence of the curl of a vector field is identically zero

$$\nabla \cdot (\nabla \times \mathbf{V}) = 0 \,. \tag{B.18}$$

This can be easily proved by rewriting this expression with the Einstein convention

$$\partial_i \left(\varepsilon^{ijk} \partial_j V_k\right) = 0 \,. \tag{B.19}$$

The expression vanishes because we sum over i and j, and ε_{ijk} and $\partial_i \partial_j$ are, respectively, antisymmetric and symmetric with respect to i and j.

The curl of the gradient of a vector field is identically zero

$$\nabla \times (\nabla \phi) = 0 \,. \tag{B.20}$$

If we rewrite this expression as

$$\varepsilon^{ijk} \partial_j \left(\partial_k \phi\right) = 0 \,, \tag{B.21}$$

we see that, again, we sum over i and j, and ε_{ijk} and $\partial_i \partial_j$ are, respectively, antisymmetric and symmetric with respect to i and j.

The following identity holds

$$\nabla \times (\nabla \times \mathbf{V}) = \nabla \left(\nabla \cdot \mathbf{V}\right) - \nabla^2 \mathbf{V} \,, \tag{B.22}$$

As in the previous cases, we write the component i

$$\varepsilon^{ijk}\partial_j\left(\nabla\times\mathbf{V}\right)_k = \varepsilon^{ijk}\partial_j\varepsilon_{kmn}\partial^m V^n\,. \tag{B.23}$$

Let us exchange i and k, so we can directly apply Eq. (B.15). We find

$$\begin{aligned}\varepsilon^{kji}\partial_j\varepsilon_{imn}\partial^m V^n &= -\varepsilon^{ijk}\partial_j\varepsilon_{imn}\partial^m V^n = \left(\delta^j_n\delta^k_m - \delta^j_m\delta^k_n\right)\partial_j\partial^m V^n\\ &= \partial^k\partial_n V^n - \partial_m\partial^m V^k\,,\end{aligned} \tag{B.24}$$

and we recover Eq. (B.22).

Appendix C
Differentiable Manifolds

Differentiable manifolds are the natural generalization of curves and surfaces in the case of spaces of arbitrary dimensions. Their key-property is that we can locally define a one-to-one correspondence between the points of a differentiable manifold of dimension n with the points of an open subset of $\mathbb{R}^n$. In this way we can "label" every point of the manifold with n numbers, representing the local coordinates of that point, and use the differential calculus developed in $\mathbb{R}^n$. If the differentiable manifold is globally different from $\mathbb{R}^n$, it is necessary to introduce more than one coordinate system to have all the points of the manifold under control. In such a case, we need that the transformations to change coordinate system are differentiable in order to have results independent of the choice of the coordinate system.

C.1 Local Coordinates

Differentiable manifold. A *differentiable manifold* of dimension n is a set M equipped with a family of bijective operators $\varphi_i : U_i \subset M \to \mathbb{R}^n$ such that:

1. $\{U_i\}$ is a family of open sets and $\bigcup_i U_i = M$;
2. $\forall i, j$ such that $U_i \cap U_j \neq \varnothing$, the operator $\lambda_{ij} = \varphi_i \varphi_j^{-1}$ from $\varphi_j(U_i \cap U_j)$ to $\varphi_i(U_i \cap U_j)$ is infinitely differentiable.

The pair (U_i, φ_i) is a local parametrization of M or a *map* of M. The family of maps $\{(U_i, \varphi_i)\}$ is called the *atlas* of M.

Figure C.1 illustrates the concepts of differentiable manifold and maps. The function φ_i can be thought of as an object with n components in which every component is a function $x^k(p)$. These n functions evaluated at a point p of the differentiable manifold are the *coordinates* of p with respect to the map (U_i, φ_i).

Let us consider an example. The spherical surface of dimension n is the subset of $\mathbb{R}^{n+1}$ defined as

C. Bambi, *Introduction to General Relativity*, Undergraduate Lecture Notes in Physics, https://doi.org/10.1007/978-981-13-1090-4

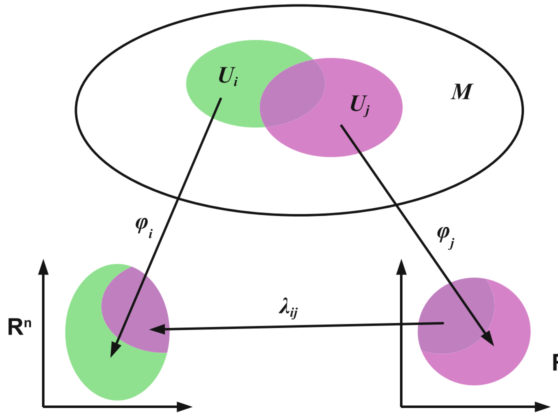

Fig. C.1 Differentiable manifold M with the maps (U_i, φ_i) and (U_j, φ_j)

$$S^n = \left\{ (x^1, x^2, \ldots, x^{n+1}) \mid \sum_{i=1}^{n+1} (x^i)^2 = 1 \right\}. \tag{C.1}$$

We define the operator

$$\varphi_1 : U_1 = S^n / \{(0, \ldots, 0, 1)\} \to \mathbb{R}^n \tag{C.2}$$

as

$$\varphi_1(x^1, x^2, \ldots, x^{n+1}) = \left(\frac{x^1}{1 - x^{n+1}}, \frac{x^2}{1 - x^{n+1}}, \ldots, \frac{x^n}{1 - x^{n+1}} \right), \tag{C.3}$$

and the operator

$$\varphi_2 : U_2 = S^n / \{(0, \ldots, 0, -1)\} \to \mathbb{R}^n \tag{C.4}$$

as

$$\varphi_2(x^1, x^2, \ldots, x^{n+1}) = \left(\frac{x^1}{1 + x^{n+1}}, \frac{x^2}{1 + x^{n+1}}, \ldots, \frac{x^n}{1 + x^{n+1}} \right). \tag{C.5}$$

The differentiable manifold S^n is parametrized by the maps (U_1, φ_1) and (U_2, φ_2). The two maps together are an atlas of S^n.

Let M and N be two differentiable manifolds, respectively of dimension m and n. The function $f : M \to N$ is said to be *differentiable* at a point $p \in M$ if, for any parametrization $\psi : V \subset N \to \mathbb{R}^n$ with $f(p) \in V$, there exists a parametrization $\varphi : U \subset M \to \mathbb{R}^m$ with $p \in U$, such that the function

$$\pi = \psi f \varphi^{-1} : \mathbb{R}^m \to \mathbb{R}^n \tag{C.6}$$

is differentiable in $\varphi(p)$. From the definition of differentiable manifold, the differentiability of f is independent of the choice of the coordinate system.

C.2 Tangent Vectors

Since a differentiable manifold of dimension n can be identified only locally with an open set of $\mathbb{R}^n$, even all objects that we can construct on the manifold are just locally defined. Tangent vectors can be defined after introducing the concept of curve.

Curve. A *curve* on a differentiable manifold M is a differentiable function $\gamma : [t_1, t_2] \subset \mathbb{R} \to M$.

With the concept of curve, we can define tangent vectors and their vector space.

Tangent vector. Let $\gamma(t)$ be a curve on a differentiable manifold M and $\mathscr{M}$ be the set of differentiable functions $f : M \to \mathbb{R}$. We call the *tangent vector* to M at $p = \gamma(t_0)$ along the direction of the curve $\gamma(t)$ the operator $X : \mathscr{M} \to \mathbb{R}$ that provides the directional derivative of the function $f \in \mathscr{M}$ along the curve $\gamma(t)$ at p; that is,

$$X[f] = \left. \frac{df[\gamma(t)]}{dt} \right|_{t=t_0} = X^\mu \frac{\partial f}{\partial x^\mu} \, . \tag{C.7}$$

The *tangent space* to the differentiable manifold M at the point p is the space generated by the tangent vectors to M at p and is denoted $T_p M$.

The tangent space at a point on a differentiable manifold is a vector space whose elements are the tangent vectors at that point.

Let us consider the map (U, φ) of the differentiable manifold M around the point $p = \varphi^{-1}(\tilde{x}^1, \tilde{x}^2, \dots, \tilde{x}^n)$. With this parametrization, the curve γ can be written as

$$\varphi[\gamma(t)] = \begin{pmatrix} x^1(t) \\ x^2(t) \\ \vdots \\ x^n(t) \end{pmatrix} , \tag{C.8}$$

and therefore

$$\gamma(t) = \varphi^{-1} \begin{pmatrix} x^1(t) \\ x^2(t) \\ \vdots \\ x^n(t) \end{pmatrix} . \tag{C.9}$$

The tangent vector X applied at the function f at $\gamma(t_0) = p$ is

$$X[f] = \frac{d}{dt} f \varphi^{-1} \left. \begin{pmatrix} x^1(t) \\ x^2(t) \\ \vdots \\ x^n(t) \end{pmatrix} \right|_{t=t_0} = \left. \frac{\partial f \varphi^{-1}}{\partial x^k} \right|_{\mathbf{x}=\tilde{\mathbf{x}}} \left. \frac{dx^k}{dt} \right|_{t=t_0} . \tag{C.10}$$

The tangent vector X can be written as

$$X = X^k \mathbf{e}_k , \tag{C.11}$$

where

$$X^k = \left. \frac{dx^k}{dt} \right|_{t=t_0} \tag{C.12}$$

are the components of the vector X, while

$$\mathbf{e}_k = \frac{\partial}{\partial x^k} \tag{C.13}$$

are the basis vectors of the tangent space at the point p of the parametrization (U, φ). The set $\{\mathbf{e}_k\}$ is called the *basis of the coordinates* and is usually denoted $\{\partial_k\}$.

We can now show that, given $p \in M$ and $X \in T_p M$, it is always possible to find a curve $\gamma(t)$ such that $\gamma(t_0) = p$ and $\dot{\gamma}(t_0) = X$. It is sufficient to write the vector X in terms of some local parametrization (U, φ)

$$X = X^k \frac{\partial}{\partial x^k} \tag{C.14}$$

and solve n differential equations

$$\left. \frac{d}{dt} u^i(t) \right|_{t=t_0} = X^i . \tag{C.15}$$

The curve is

$$\gamma(t) = \varphi^{-1} \begin{pmatrix} u^1(t) \\ u^2(t) \\ \vdots \\ u^n(t) \end{pmatrix} . \tag{C.16}$$

A vector $X = X^k \mathbf{e}_k$ exists independently of the choice of the coordinate system on M and therefore it can also be written in terms of another local parametrization with coordinates $\{x'^k\}$

$$X' = X'^k \mathbf{e}'_k , \tag{C.17}$$

where

$$X'^k = \left. \frac{dx'^k}{dt'} \right|_{t'=t'_0} . \tag{C.18}$$

The components of the vector X in the two systems of coordinates are related by

$$X'^k = \frac{\partial x'^k}{\partial x^j} X^j , \tag{C.19}$$

while the relation between the two bases is

$$\mathbf{e}'_k = \frac{\partial x^j}{\partial x'^k} \mathbf{e}_j . \tag{C.20}$$

As already seen in Sect. A.2, the basis vectors of a vector space transform with the opposite rule with respect to the one for the components of vectors.

C.3 Cotangent Vectors

Cotangent vector. Let M be a differentiable manifold. The *cotangent space* at the point p is the vector space $T^*_p M$, dual of $T_p M$. A *cotangent vector* is an element of the cotangent space.

A cotangent vector is thus a linear function $\omega : T_p M \to \mathbb{R}$. If $B^* = \{\mathbf{e}^i\}$ is a basis of $T^*_p M$, we can write

$$\omega = \omega_k \mathbf{e}^k . \tag{C.21}$$

If $X = X^k e_k$ is an element of $T_p M$ written in terms of the basis B, we have

$$\omega(X) = \omega_k X^k \,. \tag{C.22}$$

It is common to use the notation $\{dx^k\}$ to indicate the basis of $T^*_p M$ with respect to the coordinates $\{x^k\}$.

Let us now consider another local parametrization in the neighborhood of the point p with local coordinates $\{x'^k\}$ and that provides the basis $\{\mathbf{e}'_k\}$ to the vector space $T_p M$ and the basis $\{\mathbf{e}'^k\}$ to $T^*_p M$. Since the quantity $\omega(X) \in \mathbb{R}$ does not depend on the choice of the basis, we have $\omega_k X^k = \omega'_k X'^k$, where ω'_k and X'^k are, respectively, the components of ω and X with respect to the new bases. From Eqs. (C.19) and (C.20), we find

$$\omega'_k = \frac{\partial x^j}{\partial x'^k}\omega_j \,, \quad \mathbf{e}'^k = \frac{\partial x'^k}{\partial x^j}\mathbf{e}^j \,. \tag{C.23}$$

C.4 Tensors

Tensors are the generalization of tangent and cotangent vectors.

Tensor. Let M be a differentiable manifold and p an element of M. A *tensor* of type (r, s) and of order $r + s$ at p is an $(r + s)$-linear function

$$\tau : \underbrace{T^*_p M \times \cdots \times T^*_p M}_{\text{r times}} \times \underbrace{T_p M \times \cdots \times T_p M}_{\text{s times}} \;\to\; \mathbb{R} \,. \tag{C.24}$$

With such a definition of tensor, tangent vectors are tensors of type $(1, 0)$, while cotangent vectors are tensors of type $(0, 1)$. The set of tensors of type (r, s) at p forms a vector space that we indicate with $\chi^r_s(M, p)$. Let $\{\mathbf{e}_k\}$ be a basis of $T_p M$ and $\{\mathbf{e}^k\}$ its dual basis. A tensor $\tau \in \chi^r_s(M, p)$ can be written as

$$\tau = \tau^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s}\, \mathbf{e}_{i_1}\, \mathbf{e}_{i_2} \, \ldots \, \mathbf{e}_{i_r}\, \mathbf{e}^{j_1}\, \mathbf{e}^{j_2} \, \ldots \, \mathbf{e}^{j_s} \,. \tag{C.25}$$

Let $\{\mathbf{e}'_k\}$ be another basis of $T_p M$ with dual basis $\{\mathbf{e}'^k\}$. Since the rules of transformation for the bases and for the dual bases are the same encountered in the previous sections for tangent and cotangent vectors, the components of the tensor τ change as follows

$$\tau'^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s} = \frac{\partial x'^{i_1}}{\partial x^{p_1}} \frac{\partial x'^{i_2}}{\partial x^{p_2}} \cdots \frac{\partial x'^{i_r}}{\partial x^{p_r}} \frac{\partial x^{q_1}}{\partial x'^{j_1}} \frac{\partial x^{q_2}}{\partial x'^{j_2}} \cdots \frac{\partial x^{q_s}}{\partial x'^{j_s}} \tau^{p_1 p_2 \ldots p_r}_{q_1 q_2 \ldots q_s} \,. \tag{C.26}$$

Tensor field. Let M be a differentiable manifold. A *tensor field* of type (r, s) on M is a function that at every point $p \in M$ associates with an element $\tau \in \chi_s^r(M, p)$ in a differentiable way.

Let τ be a tensor field of type (r, s) on a differentiable manifold M. If all the components of τ vanish in a particular coordinate system, then they vanish in any coordinate system. Such a conclusion directly follows from Eq. (C.26).

C.5 Example: Spherical Surface of Dimension 2

Let us consider the spherical surface of dimension 2. The definition in (C.1) can be rewritten as

$$S^2 = \left\{(x, y, z) \mid x^2 + y^2 + z^2 = 1\right\} . \tag{C.27}$$

The map $(\varphi, S^2/\{(0, 0, 1), (0, 0, -1)\})$ is defined as

$$\varphi \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix} = (\theta, \phi) \, , \quad \varphi^{-1}(\theta, \phi) = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix} , \tag{C.28}$$

where $\theta \in (0, \pi)$ and $\phi \in [0, 2\pi)$. A curve γ assumes the form

$$\varphi[\gamma(t)] = \begin{pmatrix} \theta(t) \\ \phi(t) \end{pmatrix} . \tag{C.29}$$

A generic $(1, 0)$ tensor field can be written as

$$V = V^\theta \mathbf{e}_\theta + V^\phi \mathbf{e}_\phi \, , \tag{C.30}$$

where $V^\theta = V^\theta(\theta, \phi)$ and $V^\phi = V^\phi(\theta, \phi)$ are the components of the tensor field with respect to the basis vectors

$$\mathbf{e}_\theta = \frac{\partial}{\partial\theta} \, , \quad \mathbf{e}_\phi = \frac{\partial}{\partial\phi} \, . \tag{C.31}$$

Figure C.2 shows the tangent space at two different points of the differentiable manifold.

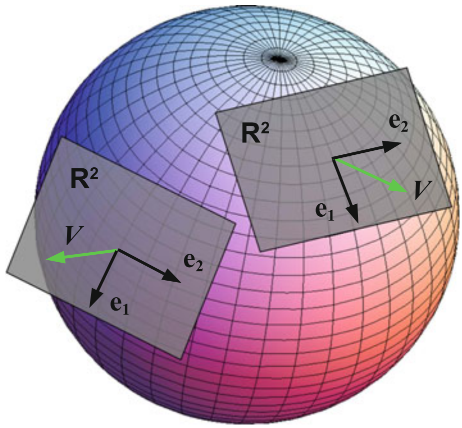

Fig. C.2 S^2 with the parametrization φ and the tangent space at two points. $\mathbf{e}_1$ and $\mathbf{e}_2$ are the two basis vectors of the parametrization φ and can be employed to write tensor fields on the manifold

Appendix D
Ellipse Equation

An ellipse is a curve in a plane with two focal points. The sum of the distances of every point on the curve from the two focal points is constant. In Fig. D.1, the two focal points are F_1 and F_2. The distance between F_1 and F_2 is $2c$:

$$\overline{F_1 F_2} = 2c\,. \tag{D.1}$$

The ellipse with focal points F_1 and F_2 and eccentricity $e = c/a$ ($0 \le e < 1$) is the set of points P such that

$$\overline{F_1 P} + \overline{F_2 P} = 2a\,. \tag{D.2}$$

We adopt a polar coordinate system (r, ϕ) centered at the point F_1. r describes the distance of the point P from F_1 and ϕ the angle $\widehat{Q F_1 P}$. Equation (D.2) can be rewritten as

$$r + \sqrt{(2c + r\cos\phi)^2 + (r\sin\phi)^2} = 2a\,. \tag{D.3}$$

We consider the square of this equations and, after some simple calculations, we find

$$r\,(a + c\cos\phi) = a^2 - c^2\,. \tag{D.4}$$

Since $e = c/a$, Eq. (D.4) can be written as

$$r\,(a + c\cos\phi) = a^2\left(1 - e^2\right)\,, \tag{D.5}$$

and in the more compact form

$$\frac{1}{r} = A + B\cos\phi\,, \tag{D.6}$$

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in Physics, https://doi.org/10.1007/978-981-13-1090-4

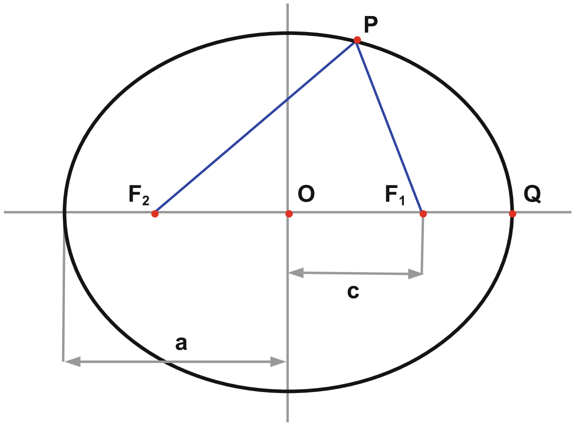

Fig. D.1 Ellipse: the focal points are F_1 and F_2

where

$$A = \frac{1}{a\left(1 - e^2\right)}, \quad B = \frac{e}{a\left(1 - e^2\right)}. \tag{D.7}$$

Appendix E
Mathematica Packages for Tensor Calculus

There are several Mathematica packages for tensor calculus available. Most of them can be downloaded from the web for free. It is also quite easy to write a Mathematica program to evaluate the main tensors and the Christoffel symbols for a given metric. In this appendix, we will introduce the package called RGTC (Riemannian Geometry and Tensor Calculus) for tensor calculus in Riemannian geometry. It is available for free at

http://www.inp.demokritos.gr/∼sbonano/RGTC/

where there is also a manual with a number of examples.

With a few simple steps, RGTC computes the explicit expressions for some tensors (Riemann, Ricci, Einstein, Weyl) and checks if the spacetime belongs to any of the following categories: flat, conformally flat, Ricci flat, Einstein space or space of constant curvature.

First, it is necessary to initialize the code with the command

```
<< EDCRGTCcode.m
```

Then, we have to define the coordinates and the metric. As an example, we can consider the Schwarzschild metric in the usual Schwarzschild coordinates. For the coordinates, we can write

```
Coord = {t,r,θ,ϕ};
```

Then we write the non-vanishing metric coefficients

```
gtt = - (1 - 2M/r);
grr = 1/(1 - 2M/r);
gpp = r^2;
gvv = r^2 Sin[θ]^2;
```

and we define the metric

```
g = {{gtt, 0, 0, 0}, {0, grr, 0, 0}, {0, 0, gpp, 0},
   { 0, 0, 0, gvv }};
```


C. Bambi, *Introduction to General Relativity*, Undergraduate Lecture Notes in Physics, https://doi.org/10.1007/978-981-13-1090-4

To launch the code, the command is

```
RGtensors[g, Coord, {1, 1, 1}]
```

and the output is something like

```
gdd = ...
LineElement = ...
gUU = ...
gUU computed in 0.007923 sec
Gamma computed in 0.003136 sec
Riemann(dddd) computed in 0.002824 sec
Riemann(Uddd) computed in 0.002247 sec
Ricci computed in 0.000167 sec
Weyl computed in 0.000015 sec
Ricci Flat
All tasks completed in 0.019439 seconds
```

where in the place of the ellipses `...` there are the expressions for the metric, the line element, and the inverse metric, respectively.

The option `{1, 1, 1}` can be manipulated to skip the evaluation of some tensors. If the first number is 0, i.e. we have `{0, 1, 1}`, the package does not compute the Riemann tensor $R^{\mu}{}_{\nu\rho\sigma}$ and instead of the line

```
Riemann(Uddd) computed in 0.002247 sec
```

the output produces the line

```
RUddd not computed
```

For `{1, 0, 1}`, the code skips the evaluation of the Weyl tensor. For `{1, 1, 0}`, it skips the evaluation of the Einstein tensor. It is just a command to save time if we do not need some tensors.

After RGTC has evaluated the tensors of the input metric, we can write their expression or work with them. For instance, if we want to visualize the Christoffel symbol Γ^t_{tr}, we write the command

```
GUdd[[1,1,2]]
```

The output will be the expression of Γ^t_{tr} of the input metric. To visualize the component R_{tt} of the Ricci tensor, we write

```
Rdd[[1,1]]
```

Note that indices run from 1 to n, where n is the number of the dimensions of the spacetime, according to the definition in `Coord`; that is, there is no index 0.

The package has some built-in functions to raise/lower indices, contract indices, evaluate the covariant derivative, etc. More details can be found in the manual of the package. We can also use standard Mathematica functions. For instance, the Kretschmann scalar can be evaluated with the command

```
Kretschmann = Simplify[ Sum[
   Rdddd[[i, j, k, l]]*RUddd[[i, m, n, o]]*gUU[[j, m]]
      *gUU[[k, n]]*gUU[[l, o]],
   {i, 1, 4}, {j, 1, 4}, {k, 1, 4}, {l, 1, 4},
      {m, 1, 4}, {n, 1, 4}, {o, 1, 4} ] ]
```

Appendix F
Interior Solution

The Schwarzschild metric found in Sect. 8.2 is the exterior vacuum solution for any spherically symmetric matter distribution in 4-dimensional Einstein's gravity; that is, it is the solution for the region $r > r_0$, where r_0 is the radius of the matter distribution. In this appendix, we want to find a simple solution for the interior region $r < r_0$.

First, we have to specify the matter energy-momentum tensor appearing on the right hand side of the Einstein equations. The simplest case is that of a perfect fluid and $T_{\mu\nu}$ reads

$$T_{\mu\nu} = (\rho + P)\,\frac{u_\mu u_\nu}{c^2} + P g_{\mu\nu}\,, \tag{F.1}$$

where ρ, P, and u^μ are, respectively, the energy density, the pressure, and the fluid 4-velocity. The most general line element for a spherically symmetric spacetime is given in (8.8). Here, we further simplify the problem, and we impose that the line element is also independent of time, namely we do not want possible radial inflows or outflows of matter. Our line element becomes

$$ds^2 = -f(r)c^2dt^2 + g(r)dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\,. \tag{F.2}$$

Since we are employing the coordinate system in which matter is at rest, the only non-vanishing component of the fluid 4-velocity is the temporal one and u^μ is

$$||u^\mu|| = \left(\frac{c}{\sqrt{f}}, \mathbf{0}\right)\,, \tag{F.3}$$

because $g_{\mu\nu}u^\mu u^\nu = -c^2$. The matter energy-momentum tensor is

$$||T_{\mu\nu}|| = \begin{pmatrix} \rho f & 0 & 0 & 0 \\ 0 & Pg & 0 & 0 \\ 0 & 0 & Pr^2 & 0 \\ 0 & 0 & 0 & Pr^2\sin^2\theta \end{pmatrix}\,, \tag{F.4}$$

C. Bambi, *Introduction to General Relativity*, Undergraduate Lecture Notes in Physics, https://doi.org/10.1007/978-981-13-1090-4

where ρ and P can be (at most) functions of the radial coordinate r.

At this point we have all the ingredients to write the Einstein equations. Unlike in Sect. 8.2, now we are not in vacuum, so the scalar curvature is non-vanishing in general and has to be calculated. Its expression is ($R_{\mu\nu}$s were calculated in Sect. 8.2)

$$\begin{aligned} R &= g^{tt} R_{tt} + g^{rr} R_{rr} + g^{\theta\theta} R_{\theta\theta} + g^{\phi\phi} R_{\phi\phi} \\ &= -\frac{f''}{2fg} + \frac{f'}{4fg}\left(\frac{f'}{f} + \frac{g'}{g}\right) - \frac{f'}{rfg} - \frac{f''}{2fg} + \frac{f'}{4fg}\left(\frac{f'}{f} + \frac{g'}{g}\right) + \frac{g'}{rg^2} \\ &\quad + \frac{2}{r^2} - \frac{2}{r^2 g} + \frac{1}{rg}\left(\frac{g'}{g} - \frac{f'}{f}\right) \\ &= -\frac{f''}{fg} + \frac{f'}{2fg}\left(\frac{f'}{f} + \frac{g'}{g}\right) + \frac{2}{r^2} - \frac{2}{r^2 g} - \frac{2}{rg}\left(\frac{f'}{f} - \frac{g'}{g}\right) . \end{aligned} \tag{F.5}$$

The non-vanishing components of the Einstein tensor $G_{\mu\nu}$ are

$$\begin{aligned} G_{tt} &= \frac{f''}{2g} - \frac{f'}{4g}\left(\frac{f'}{f} + \frac{g'}{g}\right) + \frac{f'}{rg} \\ &\quad -\frac{1}{2}\left[\frac{f''}{g} - \frac{f'}{2g}\left(\frac{f'}{f} + \frac{g'}{g}\right) - \frac{2f}{r^2} + \frac{2f}{r^2 g} + \frac{2f}{rg}\left(\frac{f'}{f} - \frac{g'}{g}\right)\right] \\ &= \frac{f}{r^2} - \frac{f}{r^2 g} + \frac{fg'}{rg^2} , \end{aligned} \tag{F.6}$$

$$\begin{aligned} G_{rr} &= -\frac{f''}{2f} + \frac{f'}{4f}\left(\frac{f'}{f} + \frac{g'}{g}\right) + \frac{g'}{rg} \\ &\quad +\frac{1}{2}\left[\frac{f''}{f} - \frac{f'}{2f}\left(\frac{f'}{f} + \frac{g'}{g}\right) - \frac{2g}{r^2} + \frac{2}{r^2} + \frac{2}{r}\left(\frac{f'}{f} - \frac{g'}{g}\right)\right] \\ &= -\frac{g}{r^2} + \frac{1}{r^2} + \frac{f'}{rf} , \end{aligned} \tag{F.7}$$

$$\begin{aligned} G_{\theta\theta} &= 1 - \frac{1}{g} + \frac{r}{2g}\left(\frac{g'}{g} - \frac{f'}{f}\right) \\ &\quad +\frac{1}{2}\left[\frac{r^2 f''}{fg} - \frac{r^2 f'}{2fg}\left(\frac{f'}{f} + \frac{g'}{g}\right) - 2 + \frac{2}{g} + \frac{2r}{g}\left(\frac{f'}{f} - \frac{g'}{g}\right)\right] \\ &= \frac{r}{2g}\left(\frac{f'}{f} - \frac{g'}{g}\right) + \frac{r^2 f''}{2fg} - \frac{r^2 f'}{4fg}\left(\frac{f'}{f} + \frac{g'}{g}\right) , \end{aligned} \tag{F.8}$$

$$G_{\phi\phi} = G_{\theta\theta} \sin^2\theta \, . \tag{F.9}$$

With the Einstein tensor $G_{\mu\nu}$ and the matter energy-momentum tensor $T_{\mu\nu}$, we can write the Einstein equations

$$G_{tt} = \frac{8\pi G_{\rm N}}{c^4} T_{tt} \rightarrow \frac{1}{r^2} - \frac{1}{r^2 g} + \frac{g'}{r g^2} = \frac{8\pi G_{\rm N}}{c^4} \rho \,, \tag{F.10}$$

$$G_{rr} = \frac{8\pi G_{\rm N}}{c^4} T_{rr} \rightarrow \frac{1}{r^2 g} - \frac{1}{r^2} + \frac{f'}{r f g} = \frac{8\pi G_{\rm N}}{c^4} P \,, \tag{F.11}$$

$$G_{\theta\theta} = \frac{8\pi G_{\rm N}}{c^4} T_{\theta\theta} \rightarrow$$
$$\frac{1}{2rg}\left(\frac{f'}{f} - \frac{g'}{g}\right) + \frac{f''}{2fg} - \frac{f'}{4fg}\left(\frac{f'}{f} + \frac{g'}{g}\right) = \frac{8\pi G_{\rm N}}{c^4} P \,. \tag{F.12}$$

The $\phi\phi$ component is as the $\theta\theta$ component with the factor $\sin^2\theta$ and therefore it is not an independent equation. The off-diagonal components are all trivial $0 = 0$. We have thus three equations and four unknown functions (f, g, ρ, and P). The system can be closed by specifying the matter equation of state. The simplest case is that in which the energy density is constant in the matter rest-frame, i.e.

$$\rho = \text{constant} \,. \tag{F.13}$$

More realistic equations of state typically require the equations to be solved numerically. Now we have three equations for three unknown functions (f, g, and P).

The covariant conservation of the matter energy-momentum tensor is a direct consequence of the Einstein equations

$$\nabla_\nu T^{\mu\nu} = \frac{\partial T^{\mu\nu}}{\partial x^\nu} + \Gamma^\mu_{\lambda\nu} T^{\lambda\nu} + \Gamma^\nu_{\lambda\nu} T^{\mu\lambda} = 0 \,. \tag{F.14}$$

Even if it does not provide an independent equation, it is sometimes more convenient to use. As it can be easily imagined from the symmetry of our system, only the equation for $\mu = r$ has a non-trivial solution. The equation reads (the Christoffel symbols were calculated in Sect. 8.2)

$$\begin{aligned}
&\frac{\partial T^{rr}}{\partial r} + \Gamma^r_{tt} T^{tt} + \Gamma^r_{rr} T^{rr} + \Gamma^r_{\theta\theta} T^{\theta\theta} + \Gamma^r_{\phi\phi} T^{\phi\phi} \\
&\qquad + \left(\Gamma^t_{rt} + \Gamma^r_{rr} + \Gamma^\theta_{r\theta} + \Gamma^\phi_{r\phi}\right) T^{rr} = 0 \,, \\
&\frac{P'}{g} - \frac{P g'}{g^2} + \frac{f'}{2g}\frac{\rho}{f} + \frac{g'}{2g}\frac{P}{g} - \frac{r}{g}\frac{P}{r^2} - \frac{r \sin^2\theta}{g}\frac{P}{r^2 \sin^2\theta} \\
&\qquad + \left(\frac{f'}{2f} + \frac{g'}{2g} + \frac{1}{r} + \frac{1}{r}\right)\frac{P}{g} = 0 \,, \\
&\frac{P'}{g} + \frac{f'}{2fg}(\rho + P) = 0 \,, \\
&P' = -\frac{f'}{2f}(\rho + P) \,.
\end{aligned} \tag{F.15}$$

From Eq. (F.10) we have

$$\begin{aligned}\frac{1}{g}-\frac{rg'}{g^2}&=1-\frac{8\pi G_\mathrm{N}}{c^4}\rho r^2\,,\\ \frac{d}{dr}\frac{r}{g}&=1-\frac{8\pi G_\mathrm{N}}{c^4}\rho r^2\,,\\ \frac{r}{g}&=r-\frac{8\pi G_\mathrm{N}}{c^4}\int_0^r\rho\,\tilde{r}^2d\tilde{r}+C\,,\end{aligned}\tag{F.16}$$

where C is an integration constant. For $r=0$, we see that $C=0$. If we employ Eq. (F.13), we find g

$$g=\frac{1}{1-\frac{8\pi G_\mathrm{N}}{3c^4}\rho r^2}\,.\tag{F.17}$$

We rewrite Eq. (F.15) as

$$\frac{d}{dr}\left(\rho+P\right)=\frac{dP}{dr}=-\frac{f'}{2f}\left(\rho+P\right)\,.\tag{F.18}$$

The solution is

$$\rho+P=\frac{C_1}{\sqrt{f}}\,,\tag{F.19}$$

where C_1 is a constant. We sum Eq. (F.10) with (F.11) and we get

$$\frac{g'}{rg^2}+\frac{f'}{rfg}=\frac{8\pi G_\mathrm{N}}{c^4}\left(\rho+P\right)=\frac{8\pi G_\mathrm{N}}{c^4}\frac{C_1}{\sqrt{f}}\,.\tag{F.20}$$

From Eq. (F.17) we can write

$$\frac{1}{g}=1-\frac{8\pi G_\mathrm{N}}{3c^4}\rho r^2\tag{F.21}$$

$$\frac{g'}{g^2}=-\frac{d}{dr}\frac{1}{g}=\frac{16\pi G_\mathrm{N}}{3c^4}\rho r\,.\tag{F.22}$$

We combine Eqs. (F.20)–(F.22) to write

$$\frac{16\pi G_\mathrm{N}}{3c^4}\rho+\frac{f'}{rf}\left(1-\frac{8\pi G_\mathrm{N}}{3c^4}\rho r^2\right)=\frac{8\pi G_\mathrm{N}}{c^4}\frac{C_1}{\sqrt{f}}\,.\tag{F.23}$$

Let us define $h=\sqrt{f}$. Equation (F.23) can be rewritten as

$$\begin{aligned}
&\frac{16\pi G_{\rm N}}{3c^4}\rho + \frac{2h'}{rh} - \frac{16\pi G_{\rm N}}{3c^4}\rho r \frac{h'}{h} = \frac{8\pi G_{\rm N}}{c^4}\frac{C_1}{h}\,,\\
&\frac{8\pi G_{\rm N}}{3c^4}\rho r h + h' - \frac{8\pi G_{\rm N}}{3c^4}\rho r^2 h' = \frac{4\pi G_{\rm N}}{c^4} r C_1\,,\\
&\frac{1 - \frac{8\pi G_{\rm N}}{3c^4}\rho r^2}{\frac{8\pi G_{\rm N}}{3c^4}\rho r} h' + h = \frac{3C_1}{2\rho}\,.
\end{aligned} \tag{F.24}$$

The homogeneous solution of this equation is

$$h = -C_2\sqrt{1 - \frac{8\pi G_{\rm N}}{3c^4}\rho r^2}\,, \tag{F.25}$$

where C_2 is a constant. An inhomogeneous solution is

$$h = \frac{3C_1}{2\rho}\,. \tag{F.26}$$

The function f is thus

$$f = \left(\frac{3C_1}{2\rho} - C_2\sqrt{1 - \frac{8\pi G_{\rm N}}{3c^4}\rho r^2}\right)^2\,, \tag{F.27}$$

and the line element of the spacetime reads

$$\begin{aligned}
ds^2 = &\left(\frac{3C_1}{2\rho} - C_2\sqrt{1 - \frac{8\pi G_{\rm N}}{3c^4}\rho r^2}\right)^2 dt^2\\
&- \frac{dr^2}{1 - \frac{8\pi G_{\rm N}}{3c^4}\rho r^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\,.
\end{aligned} \tag{F.28}$$

At this point we have two metrics. The Schwarzschild metric holds in the exterior region $r > r_0$ and is characterized by the parameter M. The matter solution holds in the interior region $r < r_0$ and has two parameters, C_1 and C_2. We can now link these constants by imposing physically reasonable conditions.

We require that at $r = r_0$, which is the surface separating the matter interior from the vacuum exterior, the metric is continuous and the pressure vanishes. The rr component of the metric tensor is continuous if

$$g_{\rm out}(r_0) = g_{\rm in}(r_0)\,, \tag{F.29}$$

where $g_{\rm out}$ and $g_{\rm in}$ are, respectively, the g_{rr} coefficients of the exterior and of the interior solutions. Plugging in their explicit expression we have

$$1 - \frac{2G_N M}{c^2 r_0} = 1 - \frac{8\pi G_N}{3c^4} \rho r_0^2 \,, \tag{F.30}$$

and we find

$$M = \frac{4\pi}{3c^2} \rho r_0^3 \,. \tag{F.31}$$

M can thus be interpreted as the *effective mass* of the body generating the gravitational field. Note that $(4/3)\pi r_0^3$ is not the volume of the massive body. Indeed r_0 is only the value of the radial coordinate of its surface. If we had not imposed the equation of state in (F.13), $\rho = \rho(r)$, and Eq. (F.31) would read

$$M = \frac{4\pi}{c^2} \int_0^{r_0} \rho \tilde{r}^2 d\tilde{r} \,. \tag{F.32}$$

The total mass of the body in this spacetime should be given by

$$M' = \frac{4\pi}{c^2} \int_0^{r_0} \rho \frac{\tilde{r}^2 d\tilde{r}}{\sqrt{1 - \frac{8\pi G_N}{3c^4} \rho \tilde{r}^2}} \,, \tag{F.33}$$

and we can thus define as the *gravitational mass defect* the quantity

$$\Delta M = M' - M \,. \tag{F.34}$$

From Eqs. (F.19) and (F.27) we can write the pressure P as

$$P = \frac{C_1}{\sqrt{f}} - \rho = \frac{C_2 \rho \sqrt{1 - \frac{8\pi G_N}{3c^4} \rho r^2} - \frac{C_1}{2}}{\frac{3C_1}{2\rho} - C_2 \sqrt{1 - \frac{8\pi G_N}{3c^4} \rho r^2}} \,. \tag{F.35}$$

The condition $P(r_0) = 0$ reads

$$\frac{C_1}{2\rho} = C_2 \sqrt{1 - \frac{8\pi G_N}{3c^4} \rho r_0^2} \tag{F.36}$$

and links together the quantities C_1, C_2, and r_0.

Lastly, we impose that even the g_{tt} coefficient is continuous at the boundary $r = r_0$

$$f_{\rm out}(r_0) = f_{\rm in}(r_0) \,. \tag{F.37}$$

We find

$$1 - \frac{2G_N M}{c^2 r_0} = \left(\frac{3C_1}{2\rho} - C_2 \sqrt{1 - \frac{8\pi G_N}{3c^4} \rho r_0^2} \right)^2 . \tag{F.38}$$

Employing Eqs. (F.31) and (F.36), we find

$$1 - \frac{8\pi G_N}{3c^4} \rho r_0^2 = 4C_2^2 \left(1 - \frac{8\pi G_N}{3c^4} \rho r_0^2 \right) ,$$
$$C_2^2 = \frac{1}{4} . \tag{F.39}$$

The solution with the negative sign is not physical because it would imply $C_1 < 0$ and then $\rho + P < 0$. Eventually, the only solution is

$$M = \frac{4\pi}{3c^2} \rho r_0^3 ,$$
$$C_1 = \rho \sqrt{1 - \frac{8\pi G_N}{3c^4} \rho r_0^2} ,$$
$$C_2 = \frac{1}{2} , \tag{F.40}$$

The three constants M, C_1, and C_2 are now completely determined by the energy density ρ and the radius of the body r_0.

The pressure P is given by

$$P = \frac{\sqrt{1 - \frac{8\pi G_N}{3c^4} \rho r^2} - \sqrt{1 - \frac{8\pi G_N}{3c^4} \rho r_0^2}}{3\sqrt{1 - \frac{8\pi G_N}{3c^4} \rho r_0^2} - \sqrt{1 - \frac{8\pi G_N}{3c^4} \rho r^2}} \rho . \tag{F.41}$$

It remains finite at $r = 0$ if the denominator in (F.41) is larger than zero

$$3\sqrt{1 - \frac{8\pi G_N}{3c^4} \rho r_0^2} - 1 > 0$$
$$1 - \frac{8\pi G_N}{3c^4} \rho r_0^2 > \frac{1}{9} ,$$
$$\frac{3\pi G_N}{c^4} \rho r_0^2 < 1 . \tag{F.42}$$

We multiply both sides by r_0 and we find the condition

$$r_0 > \frac{3\pi G_N}{c^4} \rho r_0^3 = \frac{9}{8} \left(\frac{2G_N}{c^2} \frac{4\pi}{3c^2} \rho r_0^3 \right) = \frac{9}{8} \left(\frac{2G_N M}{c^2} \right) = \frac{9}{8} r_S , \tag{F.43}$$

where r_S is the Schwarzschild radius of the body. Our solution with constant energy density is only possible if the surface of the body satisfies Eq. (F.43). There is no solution for more compact objects.

Appendix G
Metric Around a Slow-Rotating Massive Body

In this appendix, we want to derive the metric around a slow-rotating and quasi-Newtonian (i.e. the gravitational field is weak) massive body. The result can be used to see that the parameter a in the Kerr metric is the specific spin of the black hole.

For simplicity, we consider a spherically symmetric and rigidly rotating homogeneous ball of dust. The matter energy-momentum tensor reduces to $T^{\mu\nu} = \rho u^\mu u^\nu$, where ρ is the mass density (not the energy density),

$$u^\mu = (\gamma c, \gamma \mathbf{v}) \tag{G.1}$$

is the 4-velocity of every element of the ball of dust, and $\mathbf{v}$ is the 3-velocity. Let us assume that the object is rotating in the xy plane. The Lorentz factor of every element of the ball of dust is

$$\gamma = \frac{1}{\sqrt{1 - \mathbf{v}^2/c^2}} = 1 + O\left(\Omega^2\right) , \tag{G.2}$$

where $\mathbf{v}^2 = \Omega^2(x^2 + y^2)$ and $\Omega =$ constant is the angular velocity. The 3-velocity is $\mathbf{v} = (-\Omega y, \Omega x, 0)$. The matter energy-momentum tensor is

$$||T^{\mu\nu}|| = \begin{pmatrix} \rho c^2 & -\rho c \Omega y & \rho c \Omega x & 0 \\ -\rho c \Omega y & 0 & 0 & 0 \\ \rho c \Omega x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + O\left(\Omega^2\right) , \tag{G.3}$$

and, with lower indices,

$$||T_{\mu\nu}|| = \begin{pmatrix} \rho c^2 & \rho c \Omega y & -\rho c \Omega x & 0 \\ \rho c \Omega y & 0 & 0 & 0 \\ -\rho c \Omega x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + O\left(\Omega^2\right) . \tag{G.4}$$

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We can now proceed as in Sect. 12.2. We write the metric $g_{\mu\nu}$ as the Minkowski metric plus a small perturbation

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \, . \tag{G.5}$$

The Einstein equations provide the following solution for the trace-reversed perturbation $\tilde{h}_{\mu\nu}$

$$\tilde{h}_{\mu\nu} = \frac{4G_{\rm N}}{c^4} \int d^3\mathbf{x}' \frac{T_{\mu\nu}\left(\mathbf{x}'\right)}{|\mathbf{x} - \mathbf{x}'|} \, . \tag{G.6}$$

Note that here, unlike in Sect. 12.2, $T_{\mu\nu}$ is independent of time. If the ball of dust is at the origin of the coordinate system and we are interested in the metric at large radii, we can expand the term $1/|\mathbf{x} - \mathbf{x}'|$ inside the integral as

$$\frac{1}{|\mathbf{x} - \mathbf{x}'|} = \frac{1}{r} + \frac{x_i x'^i}{r^3} + \cdots \, . \tag{G.7}$$

For the tt-component we have

$$\tilde{h}_{tt} = \frac{4G_{\rm N}}{c^2 r} \int d^3\mathbf{x}' \rho + \cdots = \frac{4G_{\rm N} M}{c^2 r} + \cdots \, , \tag{G.8}$$

where M is the mass of the slow-rotating object. Since $\tilde{h} = \tilde{h}^\mu_\mu = -\tilde{h}_{tt}$, we find that the tt-component of the metric perturbation is

$$h_{tt} = \tilde{h}_{tt} - \frac{1}{2}\eta_{tt}\tilde{h} = \frac{2G_{\rm N} M}{c^2 r} \, . \tag{G.9}$$

$\tilde{h}_{ij} = 0$ because $T_{ij} = 0$ (we ignore terms of order Ω^2 or higher because the rotation is slow). The ij-components of the metric perturbation are

$$h_{ij} = -\frac{1}{2}\eta_{ij}\tilde{h} = \begin{cases} 0 & \text{if } i \neq j \, , \\ \frac{2G_{\rm N}M}{c^2 r} & \text{if } i = j \, . \end{cases} \tag{G.10}$$

Lastly, we have the terms h_{ti}s. $\tilde{h}_{ti} = h_{ti}$ because $\eta_{ti} = 0$. $h_{tz} = 0$ because $T_{tz} = 0$. For the tx-component we have

$$\begin{aligned} h_{tx} &= \frac{4G_{\rm N}}{c^3 r} \int d^3\mathbf{x}' \rho \Omega y' + \frac{4G_{\rm N}}{c^3 r^3} \int d^3\mathbf{x}' \rho \Omega y' \left(x x' + y y' + z z'\right) + \cdots \\ &= \frac{4G_{\rm N}}{c^3 r^3} \int d^3\mathbf{x}' \rho \Omega y y'^2 + \cdots \, , \end{aligned} \tag{G.11}$$

and for the ty component we have the same expression exchanging x and y and adding a minus sign

$$h_{ty} = -\frac{4G_N}{c^3 r^3} \int d^3\mathbf{x}' \rho \Omega x x'^2 + \cdots \, . \tag{G.12}$$

We introduce the spin angular momentum of the object J as

$$J = 2 \int d^3\mathbf{x}' \, \rho \Omega x'^2 = 2 \int d^3\mathbf{x}' \, \rho \Omega y'^2 = \int d^3\mathbf{x}' \, \rho \Omega \left(x'^2 + y'^2 \right) \, , \tag{G.13}$$

and the h_{ti}s terms can be written as

$$h_{tx} = \frac{2G_N J y}{c^3 r^3} + \cdots \, , \quad h_{ty} = -\frac{2G_N J x}{c^3 r^3} + \cdots \, , \quad h_{tz} = 0 \, . \tag{G.14}$$

Considering only the leading order terms in $h_{\mu\nu}$, the line element of the spacetime at large radii reads

$$\begin{aligned} ds^2 = &- \left(1 - \frac{2G_N M}{c^2 r} \right) c^2 dt^2 + \frac{4G_N J y}{c^2 r^3} dt dx - \frac{4G_N J x}{c^2 r^3} dt dy \\ &+ \left(1 + \frac{2G_N M}{c^2 r} \right) \left(dx^2 + dy^2 + dz^2 \right) \, . \end{aligned} \tag{G.15}$$

Let us now rewrite the line element in spherical coordinates (ct, r, θ, ϕ). The relation between Cartesian and spherical coordinates is

$$\begin{aligned} t &= t \, , \\ x &= r \sin\theta \cos\phi \, , \\ y &= r \sin\theta \sin\phi \, , \\ z &= r \cos\theta \, . \end{aligned} \tag{G.16}$$

The metric tensor transforms as

$$g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta} \, . \tag{G.17}$$

g_{tt} does not change, because there is no mixing between the time and the space coordinates. g_{ij}s change as in the Euclidean space (see Sect. 1.2). g_{ti}s vanish except $g_{t\phi}$

$$\begin{aligned} g_{tr} &= \frac{\partial t}{\partial t} \frac{\partial x}{\partial r} g_{tx} + \frac{\partial t}{\partial t} \frac{\partial y}{\partial r} g_{ty} \\ &= \sin\theta \cos\phi \left(\frac{2G_N J r \sin\theta \sin\phi}{c^3 r^3} \right) + \sin\theta \sin\phi \left(-\frac{2G_N J r \sin\theta \cos\phi}{c^3 r^3} \right) = 0 \, , \\ g_{t\theta} &= \frac{\partial t}{\partial t} \frac{\partial x}{\partial \theta} g_{tx} + \frac{\partial t}{\partial t} \frac{\partial y}{\partial \theta} g_{ty} \\ &= r \cos\theta \cos\phi \left(\frac{2G_N J r \sin\theta \sin\phi}{c^3 r^3} \right) + r \cos\theta \sin\phi \left(-\frac{2G_N J r \sin\theta \cos\phi}{c^3 r^3} \right) = 0 \, , \end{aligned}$$

$$
\begin{aligned}
g_{t\phi} &= \frac{\partial t}{\partial t}\frac{\partial x}{\partial \phi}g_{tx} + \frac{\partial t}{\partial t}\frac{\partial y}{\partial \phi}g_{ty} \\
&= -r\sin\theta\sin\phi\left(\frac{2G_{\rm N}Jr\sin\theta\sin\phi}{c^3r^3}\right) + r\sin\theta\cos\phi\left(-\frac{2G_{\rm N}Jr\sin\theta\cos\phi}{c^3r^3}\right) \\
&= -\frac{2G_{\rm N}J\sin^2\theta}{c^3r}\,.
\end{aligned}
\tag{G.18}
$$

The line element in spherical coordinates thus reads

$$
\begin{aligned}
ds^2 = &-\left(1-\frac{2G_{\rm N}M}{c^2r}\right)c^2dt^2 - \frac{4G_{\rm N}aM\sin^2\theta}{c^2r}dtd\phi \\
&+\left(1+\frac{2G_{\rm N}M}{c^2r}\right)\left(dr^2+r^2d\theta^2+r^2\sin^2\theta d\phi^2\right)\,,
\end{aligned}
\tag{G.19}
$$

where we have introduced the specific spin $a = J/M$. These coordinates are still isotropic. If we want Boyer–Lindquist-like coordinates, we need another coordinate transformation, as done in Sect. 9.5.

Appendix H
Friedmann–Robertson–Walker Metric

Alexander Friedmann in the early 1920s and, independently, Georges Lemaitre in the late 1920s and early 1930s were the first to employ the Friedman–Robertson–Walker metric and study the corresponding cosmological models assuming the Einstein equations. In the mid 1930s, Howard Percy Robertson and Arthur Geoffrey Walker rigorously proved that the Friedman–Robertson–Walker metric is the only geometry compatible with the Cosmological Principle. Such a statement is independent of the field equations of the gravity theory, which can only determine the scale factor $a(t)$. In this appendix, we want to outline a possible derivation of the Friedman–Robertson–Walker metric.

We want to obtain the most general metric describing a spatially homogeneous and isotropic spacetime. Isotropy means that there are no preferred directions: the spacetime should thus look spherically symmetric and we can proceed as in Sect. 8 1, finding the line element in Eq. (8.5). We can then consider a coordinate transformation to remove the off-diagonal metric coefficient and we get the metric

$$ds^2 = -f(t,r)c^2dt^2 + g(t,r)\left(dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2\right) . \tag{H.1}$$

Unlike the spherically symmetric spacetime in Chap. 8, here the spacetime is also spatially homogeneous; that is, there are no preferred points. This means, in particular, that the clock of any static observer should measure the same time and therefore $f = f(t)$ because it cannot depend on the radial coordinate r. If g_{tt} only depends on the time coordinate, we can always consider the transformation (which is a redefinition of time and is called *synchronization*)

$$dt \to dt' = \sqrt{f}dt , \tag{H.2}$$

and set $g_{tt} = -1$. Because of isotropy, we can write $g(t,r) = a^2(t)h(r)$ and the line elements becomes

$$ds^2 = -c^2dt^2 + a^2(t)dl^2 , \tag{H.3}$$

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where dl^2 is given by

$$dl^2 = h(r)\left(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right)$$
$$= h(r)\left(dx^2 + dy^2 + dz^2\right) . \tag{H.4}$$

It is straightforward to calculate the scalar curvature of the 3-metric g_{ij} with a Mathematica package for tensor calculus (see Appendix E). We find

$$R = \frac{3h'^2 - h\left(8h'/r + 4h''\right)}{2h^3} . \tag{H.5}$$

Because of homogeneity, R must be spatially constant. Imposing this condition, the solution for the function h is

$$h(r) = \frac{1}{\left(1 + kr^2/4\right)^2} , \tag{H.6}$$

where $k = 0, \pm 1$. $R = 6k$ and therefore $R > 0$, < 0, and 0 for, respectively, $k = 1, -1$, and 0. The line element of the spacetime turns out to be

$$ds^2 = -c^2 dt^2 + a^2(t)\frac{dx^2 + dy^2 + dz^2}{\left(1 + kr^2/4\right)^2}$$
$$= -c^2 dt^2 + a^2(t)\frac{dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2}{\left(1 + kr^2/4\right)^2} . \tag{H.7}$$

Note that we have obtained the most general expression for a metric describing a spatially homogeneous and isotropic spacetime. We can always rescale r to have $k = 0, \pm 1$, and therefore other values of k do not represent different metrics but just the same metric with a different radial coordinate.

With the following transformation for the radial coordinate

$$\tilde{r} = \frac{r}{1 + kr^2/4} , \tag{H.8}$$

we get the Friedman–Robertson–Walker metric in the coordinates employed in Chap. 11

$$ds^2 = -c^2 dt^2 + a^2(t)\left[\frac{d\tilde{r}^2}{1 - k\tilde{r}^2} + \tilde{r}^2 d\theta^2 + \tilde{r}^2 \sin^2\theta d\phi^2\right] . \tag{H.9}$$

Let us check that the transformation in (H.8) transforms the line element in (H.7) into the line element in (H.9). For $g_{\theta\theta}$ and $g_{\phi\phi}$, it is easy to see that

$$\frac{r^2}{\left(1+kr^2/4\right)^2}\left(d\theta^2+\sin^2\theta d\phi^2\right)=\tilde{r}^2\left(d\theta^2+\sin^2\theta d\phi^2\right) . \tag{H.10}$$

For $g_{\tilde{r}\tilde{r}}$, we can write

$$\begin{aligned} g_{rr} &= \frac{\partial\tilde{r}}{\partial r}\frac{\partial\tilde{r}}{\partial r}g_{\tilde{r}\tilde{r}} \\ &= \frac{\left(1-kr^2/4\right)}{\left(1+kr^2/4\right)^2}\frac{\left(1-kr^2/4\right)}{\left(1+kr^2/4\right)^2}\frac{1}{1-k\tilde{r}^2} \\ &= \frac{\left(1-kr^2/4\right)}{\left(1+kr^2/4\right)^2}\frac{\left(1-kr^2/4\right)}{\left(1+kr^2/4\right)^2}\frac{\left(1+kr^2/4\right)^2}{\left(1+kr^2/4\right)^2-kr^2} \\ &= \frac{1}{\left(1+kr^2/4\right)^2} , \end{aligned} \tag{H.11}$$

and we obtain the correct metric coefficient in (H.7).

Appendix I
Suggestions for Solving the Problems

I.1 Chapter 1

Problem 1.1: In Cartesian coordinates $\{x^i\} = (x, y, z)$, the metric tensor is δ_{ij}. In spherical coordinates $\{x'^i\} = (r, \theta, \phi)$, the metric tensor is given by

$$g'_{ij} = \frac{\partial x^m}{\partial x'^i} \frac{\partial x^n}{\partial x'^j} \delta_{mn} \, . \tag{I.1}$$

For $i = j = r$, we have

$$\begin{aligned} g_{rr} &= \frac{\partial x^m}{\partial r} \frac{\partial x^n}{\partial r} \delta_{mn} = \frac{\partial x}{\partial r} \frac{\partial x}{\partial r} + \frac{\partial y}{\partial r} \frac{\partial y}{\partial r} + \frac{\partial z}{\partial r} \frac{\partial z}{\partial r} \\ &= \sin^2\theta \cos^2\phi + \sin^2\theta \sin^2\phi + \cos^2\theta = 1 \, . \end{aligned} \tag{I.2}$$

For $i = r$ and $j = \theta$, we have

$$\begin{aligned} g_{r\theta} &= \frac{\partial x^m}{\partial r} \frac{\partial x^n}{\partial \theta} \delta_{mn} = \frac{\partial x}{\partial r} \frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial r} \frac{\partial y}{\partial \theta} + \frac{\partial z}{\partial r} \frac{\partial z}{\partial \theta} \\ &= r \sin\theta \cos\theta \cos^2\phi + r \sin\theta \cos\theta \sin^2\phi - r \sin\theta \cos\theta = 0 \, . \end{aligned} \tag{I.3}$$

For $i = r$ and $j = \phi$, we have

$$\begin{aligned} g_{r\phi} &= \frac{\partial x^m}{\partial r} \frac{\partial x^n}{\partial \phi} \delta_{mn} = \frac{\partial x}{\partial r} \frac{\partial x}{\partial \phi} + \frac{\partial y}{\partial r} \frac{\partial y}{\partial \phi} + \frac{\partial z}{\partial r} \frac{\partial z}{\partial \phi} \\ &= -r \sin^2\theta \cos\phi \sin\phi + r \sin^2\theta \cos\phi \sin\phi = 0 \, . \end{aligned} \tag{I.4}$$

We calculate the other components of the metric tensor in the same way and eventually we find that the only non-vanishing components are $g_{rr} = 1$, $g_{\theta\theta} = r^2$, and $g_{\phi\phi} = r^2 \sin^2\theta$. The line element in spherical coordinates is thus

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$$dl^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \,, \tag{I.5}$$

and we have verified Eq. (1.14).

Problem 1.2: We proceed as in Problem 1.1. In spherical coordinates $\{x^i\} = (r, \theta, \phi)$, the metric tensor g_{ij} is given in Eq. (1.15). In cylindrical coordinates $\{x'^i\} = (\rho, z, \phi')$, we calculate the metric tensor from

$$g'_{ij} = \frac{\partial x^m}{\partial x'^i} \frac{\partial x^n}{\partial x'^j} g_{mn} \,. \tag{I.6}$$

For $i = j = \rho$, we have

$$\begin{aligned} g_{\rho\rho} &= \frac{\partial x^m}{\partial \rho} \frac{\partial x^n}{\partial \rho} g_{mn} = \frac{\partial r}{\partial \rho}\frac{\partial r}{\partial \rho} + \frac{\partial \theta}{\partial \rho}\frac{\partial \theta}{\partial \rho} r^2 + \frac{\partial \phi}{\partial \rho}\frac{\partial \phi}{\partial \rho} r^2 \sin^2\theta \\ &= \frac{\rho}{\sqrt{\rho^2+z^2}} \frac{\rho}{\sqrt{\rho^2+z^2}} + \frac{z}{\rho^2+z^2} \frac{z}{\rho^2+z^2} \left(\rho^2+z^2\right) = 1 \,. \end{aligned} \tag{I.7}$$

For $i = \rho$ and $j = z$, we have

$$\begin{aligned} g_{\rho z} &= \frac{\partial x^m}{\partial \rho} \frac{\partial x^n}{\partial z} g_{mn} = \frac{\partial r}{\partial \rho}\frac{\partial r}{\partial z} + \frac{\partial \theta}{\partial \rho}\frac{\partial \theta}{\partial z} r^2 + \frac{\partial \phi}{\partial \rho}\frac{\partial \phi}{\partial z} r^2 \sin^2\theta \\ &= \frac{\rho}{\sqrt{\rho^2+z^2}} \frac{z}{\sqrt{\rho^2+z^2}} + \frac{z}{\rho^2+z^2} \frac{-\rho}{\rho^2+z^2} \left(\rho^2+z^2\right) = 0 \,. \end{aligned} \tag{I.8}$$

For $i = \rho$ and $j = \phi'$, we have

$$g_{\rho\phi'} = \frac{\partial x^m}{\partial \rho} \frac{\partial x^n}{\partial \phi'} g_{mn} = \frac{\partial r}{\partial \rho}\frac{\partial r}{\partial \phi'} + \frac{\partial \theta}{\partial \rho}\frac{\partial \theta}{\partial \phi'} r^2 + \frac{\partial \phi}{\partial \rho}\frac{\partial \phi}{\partial \phi'} r^2 \sin^2\theta = 0 \,. \tag{I.9}$$

We calculate the other components of the metric tensor in the same way and eventually we find that the only non-vanishing components are $g_{\rho\rho} = 1$, $g_{zz} = 1$, and $g_{\phi'\phi'} = \rho^2$. The line element in cylindrical coordinates is thus

$$dl^2 = d\rho^2 + dz^2 + \rho^2 d\phi'^2 \,. \tag{I.10}$$

Problem 1.3: The Jacobian of the inverse transformation of the transformation in Eq. (1.36) is

$$\left|\left| \frac{\partial x^m}{\partial x'^i} \right|\right| = ||\delta^m_i|| \,. \tag{I.11}$$

Hence the new metric is $g'_{ij} = \delta^m_i \delta^n_j \delta_{mn} = \delta_{ij}$.

Problem 1.4: The transformation from the Cartesian coordinates (x, y, z) to the Cartesian coordinates (x', y', z') is given by

$$
\begin{aligned}
x' &= x\ \cos\theta + y\ \sin\theta\,, \\
y' &= -x\ \sin\theta + y\ \cos\theta\,, \\
z' &= z\,.
\end{aligned}
\tag{I.12}
$$

The inverse transformation is

$$
\begin{aligned}
x &= x'\ \cos\theta - y'\ \sin\theta\,, \\
y &= x'\ \sin\theta + y'\ \cos\theta\,, \\
z &= z'\,.
\end{aligned}
\tag{I.13}
$$

We proceed as in the previous exercises to find the metric in the new coordinate system

$$
g'_{ij} = \frac{\partial x^m}{\partial x'^i}\frac{\partial x^n}{\partial x'^j}\delta_{mn} = \frac{\partial x}{\partial x'^i}\frac{\partial x}{\partial x'^j} + \frac{\partial y}{\partial x'^i}\frac{\partial y}{\partial x'^j} + \frac{\partial z}{\partial x'^i}\frac{\partial z}{\partial x'^j}\,. \tag{I.14}
$$

For $i = j = x'$, we have

$$
g_{x'x'} = \cos^2\theta + \sin^2\theta = 1\,. \tag{I.15}
$$

For $i = x'$ and $j = y'$, we have

$$
g_{x'y'} = -\cos\theta\sin\theta + \sin\theta\cos\theta = 0\,. \tag{I.16}
$$

For $i = x'$ and $j = z'$, we have

$$
g_{x'y'} = 0\,. \tag{I.17}
$$

We can calculate all the metric components. The result is that $g'_{ij} = \delta_{ij}$ and the expression of the Euclidean metric does not change.

Problem 1.5: The Lagrangian of a free point-like particle is

$$
L = \frac{1}{2}m g_{ij}\dot{x}^i\dot{x}^j\,. \tag{I.18}
$$

In cylindrical coordinates, the metric tensor is (see Problem 1.2)

$$
||g_{ij}|| = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \rho^2 \end{pmatrix}\,. \tag{I.19}
$$

Equation (1.18) thus becomes

$$L = \frac{1}{2} m \left(\dot{\rho}^2 + \dot{z}^2 + \rho^2 \dot{\phi}'^2 \right) . \tag{I.20}$$

The Euler–Lagrange equations are

$$\ddot{\rho} - \rho \dot{\phi}'^2 = 0 , \quad \ddot{z} = 0 , \quad \ddot{\phi}' + \frac{2}{\rho} \dot{\rho} \dot{\phi}' = 0 . \tag{I.21}$$

Problem 1.6: We have just to match the geodesic equations

$$\ddot{x}^i + \Gamma^i_{jk} \dot{x}^j \dot{x}^k = 0 , \tag{I.22}$$

with Eq. (I.21). It is straightforward to see that

$$\Gamma^{\rho}_{\phi' \phi'} = -\rho , \quad \Gamma^{\phi'}_{\rho \phi'} = \Gamma^{\phi'}_{\phi' \rho} = \frac{1}{\rho} , \tag{I.23}$$

and all other Christoffel symbols vanish.

Problem 1.7: Here the Lagrangian coordinates are (θ, ϕ). The Euler–Lagrange equations are

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0 , \quad \frac{d}{dt} \frac{\partial L}{\partial \dot{\phi}} - \frac{\partial L}{\partial \phi} = 0 . \tag{I.24}$$

We find

$$\ddot{\theta} - \sin \theta \cos \theta \dot{\phi}^2 = 0 , \quad \ddot{\phi} + 2 \cot \theta \dot{\theta} \dot{\phi} = 0 . \tag{I.25}$$

Problem 1.8: The Lagrangian does not explicitly depend on the time t, so we have the conservation of the energy E

$$E = \frac{\partial L}{\partial \dot{x}} \dot{x} + \frac{\partial L}{\partial \dot{y}} \dot{y} - L = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 \right) + \frac{1}{2} k \left(x^2 + y^2 \right) . \tag{I.26}$$

The Euler–Lagrange equations are

$$\ddot{x} + \frac{k}{m} x = 0 , \quad \ddot{y} + \frac{k}{m} y = 0 . \tag{I.27}$$

Problem 1.9: Let us consider the Galilean transformation

$$x' = x - vt , \quad y' = y , \quad z' = z , \quad t' = t , \tag{I.28}$$

We have

$$\frac{\partial}{\partial x^i} = \frac{\partial x'^m}{\partial x^i}\frac{\partial}{\partial x'^m} = \frac{\partial}{\partial x'^i} \tag{I.29}$$

and therefore $\nabla = \nabla'$. For the derivative of the temporal coordinate, we have

$$\frac{\partial}{\partial t} = -v\frac{\partial}{\partial x'} + \frac{\partial}{\partial t'} \, . \tag{I.30}$$

Maxwell's third and fourth equations would change to

$$\nabla \times \mathbf{E}' = -\frac{1}{c' + v}\left(\frac{\partial}{\partial t'} - v\frac{\partial}{\partial x'}\right)\mathbf{B}' \, , \tag{I.31}$$

$$\nabla \times \mathbf{B}' = \frac{1}{c' + v}\left(\frac{\partial}{\partial t'} - v\frac{\partial}{\partial x'}\right)\mathbf{E}' \, , \tag{I.32}$$

since $c' = c - v$ in Galilean relativity. Independently of the transformation rule of the electric and magnetic fields, in general these equations are not invariant under a Galilean transformation.

I.2 Chapter 2

Problem 2.1: The relation between Cartesian coordinates $\{x^i\} = (ct, x, y, z)$ and spherical coordinates $\{x'^i\} = (ct', r, \theta, \phi)$ is

$$t = t' \, , \quad x = r\sin\theta\cos\phi \, , \quad y = r\sin\theta\sin\phi \, , \quad z = r\cos\theta \, , \tag{I.33}$$

with inverse

$$t' = t \, , \quad r = \sqrt{x^2 + y^2 + z^2} \, , \quad \theta = \arccos\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right) \, ,$$
$$\phi = \arctan\left(\frac{y}{x}\right) \, . \tag{I.34}$$

As a tensor, $T^{\mu\nu}$ transforms according to the rule in Eq. (1.30). From the expression in Eq. (2.60), we find

$$\begin{aligned} T'^{\mu\nu} &= \frac{\partial x'^\mu}{\partial x^\alpha}\frac{\partial x'^\nu}{\partial x^\beta}T^{\alpha\beta} \\ &= \frac{1}{c^2}\frac{\partial x'^\mu}{\partial t}\frac{\partial x'^\nu}{\partial t}\varepsilon + \frac{\partial x'^\mu}{\partial x}\frac{\partial x'^\nu}{\partial x}P + \frac{\partial x'^\mu}{\partial y}\frac{\partial x'^\nu}{\partial y}P + \frac{\partial x'^\mu}{\partial z}\frac{\partial x'^\nu}{\partial z}P \end{aligned} \tag{I.35}$$

For instance, for $\mu = \nu = t'$ we have

$$T^{t't'} = \frac{\partial t'}{\partial t}\frac{\partial t'}{\partial t}\varepsilon = \varepsilon\,. \tag{I.36}$$

After calculating all the components, we find that the energy-momentum tensor of a perfect fluid in spherical coordinates and in the rest-frame of the fluid has the following form

$$||T'^{\mu\nu}|| = \begin{pmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & \frac{P}{r^2} & 0 \\ 0 & 0 & 0 & \frac{P}{r^2\sin^2\theta} \end{pmatrix}. \tag{I.37}$$

$T'^{\mu}{}_{\nu}$ is obtained by lowering the second index with $g_{\mu\nu}$

$$T'^{\mu}{}_{\nu} = g_{\nu\rho}T'^{\mu\rho}\,. \tag{I.38}$$

The result is

$$||T'^{\mu}{}_{\nu}|| = \begin{pmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}. \tag{I.39}$$

Similarly, $T'_{\mu\nu} = g_{\mu\rho}T'^{\rho}{}_{\nu}$, and we find

$$||T'_{\mu\nu}|| = \begin{pmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & Pr^2 & 0 \\ 0 & 0 & 0 & Pr^2\sin^2\theta \end{pmatrix}. \tag{I.40}$$

Problem 2.2: We have to move from the Cartesian coordinates $\{x^{\mu}\}$ to the Cartesian coordinates $\{x'^{\mu}\}$, where

$$\frac{\partial x'^{\mu}}{\partial x^{\alpha}} = \Lambda^{\mu}{}_{\alpha} \tag{I.41}$$

and $\Lambda^{\mu}{}_{\alpha}$ is the transformation in Eq. (2.8). The energy-momentum tensor in the coordinates $\{x'^{\mu}\}$ can be calculated from

$$T'^{\mu\nu} = \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}T^{\alpha\beta}\,. \tag{I.42}$$

For instance, for $\mu = \nu = t'$ we have

$$T^{t't'} = \Lambda^{t'}{}_{\alpha}\Lambda^{t'}{}_{\beta}T^{\alpha\beta} = \Lambda^{t'}{}_{t}\Lambda^{t'}{}_{t}\varepsilon + \Lambda^{t'}{}_{x}\Lambda^{t'}{}_{x}P = \gamma^2\left(\varepsilon + \beta^2 P\right). \tag{I.43}$$

The other components can be computed with the same procedure.

Problem 2.3: The Lorentz boost connecting the references frames (ct, x, y, z) and (ct', x', y', z') is

$$||\Lambda_1{}^\mu{}_\alpha|| = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \tag{I.44}$$

where $\beta = v/c$, $\gamma = 1/\sqrt{1-\beta^2}$, and we have $x'^\mu = \Lambda_1{}^\mu{}_\alpha x^\alpha$. The Lorentz boost connecting the references frames (ct', x', y', z') and (ct'', x'', y'', z'') is

$$||\Lambda_2{}^\mu_\alpha|| = \begin{pmatrix} \gamma' & -\gamma'\beta' & 0 & 0 \\ -\gamma'\beta' & \gamma' & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \tag{I.45}$$

where $\beta' = v'/c$, $\gamma' = 1/\sqrt{1-\beta'^2}$, and we have $x''^\mu = \Lambda_2{}^\mu{}_\alpha x'^\alpha$. The Lorentz boost connecting the references frames (ct, x, y, z) and (ct'', x'', y'', z'') can be found from

$$\Lambda_3{}^\mu{}_\alpha = \Lambda_2{}^\mu{}_\sigma \Lambda_1{}^\sigma{}_\alpha . \tag{I.46}$$

Problem 2.4: We can just show that the matrices in Eqs. (2.28) and (2.29) do not commute. For instance

$$\Lambda_x{}^\mu{}_\sigma \Lambda_y{}^\sigma{}_\alpha \neq \Lambda_y{}^\mu{}_\sigma \Lambda_x{}^\sigma{}_\alpha . \tag{I.47}$$

Problem 2.5: Let us indicate with Δt a time interval measured by a clock on Earth and with $\Delta\tau$ the same time interval measured by a clock on one of these satellites. Considering only the effect of the orbital motion of the satellite, we have

$$\Delta t = \frac{\Delta\tau}{\sqrt{1-\beta^2}} \approx \left(1 + \frac{\beta^2}{2}\right) \Delta\tau = \left(1 + 8.4 \cdot 10^{-11}\right) \Delta\tau , \tag{I.48}$$

where $\beta = v/c = 1.3 \cdot 10^{-5}$ is the satellite speed in units of the speed of light.

I.3 Chapter 3

Problem 3.1: We follow the 4-dimensional formalism. From the action in Eq. (3.22), in spherical coordinates the Lagrangian is

$$L = -\frac{1}{2} m \left(c^2 \dot{t}^2 - \dot{r}^2 - r^2 \dot{\theta}^2 - r^2 \sin^2 \theta \dot{\phi}^2 \right) . \tag{I.49}$$

We employ Eq. (3.23) to calculate the components of the conjugate momentum. For $\mu = t$, we have

$$p_t = \frac{\partial L}{\partial \dot{x}^0} = \frac{1}{c} \frac{\partial L}{\partial \dot{t}} = -mc\dot{t} . \tag{I.50}$$

For $\mu = r, \theta$, and ϕ, we find

$$p_r = m\dot{r} , \quad p_\theta = mr^2 \dot{\theta} , \quad p_\phi = mr^2 \sin^2 \theta \dot{\phi} . \tag{I.51}$$

The components of the 4-momentum can be obtained raising the index in p_μ with the inverse of the metric tensor. For $\mu = t$, we have

$$p^t = g^{t\mu} p_\mu = mc\dot{t} . \tag{I.52}$$

Similarly, for the spatial components we get

$$p^r = m\dot{r} , \quad p^\theta = m\dot{\theta} , \quad p^\phi = m\dot{\phi} . \tag{I.53}$$

Problem 3.2: The Lagrangian in Eq. (I.49) does not depend on the coordinates t and ϕ, and therefore we have the conservation of the energy E and of the axial component of the angular momentum L_z

$$E = -p_t = mc\dot{t} , \quad L_z = p_\phi = mr^2 \sin^2 \theta \dot{\phi} . \tag{I.54}$$

Note that the energy E is defined as $-p_t$. If we adopt a metric with signature $(+---)$, as it is common in particle physics, we would define the energy $E = p_t$ and the axial component of the angular momentum $L_z = -p_\phi$. The system has also a third constant of motion, which is associated to the conservation of the norm of the 4-velocity and follows from Eq. (3.24).

Problem 3.3: In cylindrical coordinates, the Lagrangian in (I.49) becomes

$$L = -\frac{1}{2} m \left(c^2 \dot{t}^2 - \dot{\rho}^2 - \dot{z}^2 - \rho^2 \dot{\phi}^2 \right) . \tag{I.55}$$

The components of the conjugate momentum are

$$p_t = -mc\dot{t} , \quad p_\rho = m\dot{\rho} , \quad p_z = m\dot{z} , \quad p_\phi = m\rho^2 \dot{\phi} . \tag{I.56}$$

The components of the 4-momentum are obtained raising the index of p_μ with the inverse of the metric tensor

$$p^t = mc\dot{t}\,, \quad p^\rho = m\dot{\rho}\,, \quad p^z = m\dot{z}\,, \quad p^\phi = m\dot{\phi}\,. \tag{I.57}$$

The Lagrangian does not depend on the coordinates t, z, and ϕ, so we have the conservation of the energy $-p_t$, of the momentum along the z axis p_z $(=p^z)$, and of the axial component of the angular momentum p_ϕ.

Problem 3.4: We assume that the high-energy photon moves in the xy plane and the CMB photon moves along the x axis. Their 4-momenta are, respectively,

$$||p^\mu_\gamma|| = (p, p\cos\theta, p\sin\theta, 0)\,, \quad ||p^\mu_{CMB}|| = (q, q, 0, 0)\,, \tag{I.58}$$

The reaction is energetically allowed when

$$-\,p^\mu_i p^i_\mu \ge 4m_e^2c^2\,, \tag{I.59}$$

where $p^\mu_i = p^\mu_\gamma + p^\mu_{CMB}$. We find

$$\begin{aligned}&p^2 + q^2 + 2pq - p^2\cos^2\theta - q^2 - 2pq\cos\theta - p^2\sin^2\theta \ge 4m_e^2c^2\,,\\ &2pq\,(1-\cos\theta) \ge 4m_e^2c^2\,.\end{aligned} \tag{I.60}$$

The average energy of CMB photons is $\langle qc\rangle = 2\cdot 10^{-4}$ eV. Ignoring the term $pq\cos\theta$ in Eq. (I.60), we find that the threshold energy for the high-energy photon is $E_\gamma \sim 10^{15}$ eV.

Problem 3.5: The binding energy of iron-56 is

$$\begin{aligned}E_{\rm B} &= (26\cdot m_pc^2 + 30\cdot m_nc^2 - Mc^2)\\ &= (26\cdot 0.938 + 30\cdot 0.940 - 52.103)\,\text{GeV} = 485\,\text{MeV}\,.\end{aligned} \tag{I.61}$$

The binding energy per nucleon is $\varepsilon_{\rm B} = E_{\rm B}/56 = 8.7$ MeV.

Problem 3.6: The first part in the Euler–Lagrange equation is

$$\begin{aligned}&\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\sigma}\left[\sqrt{-g}\,\frac{\partial\mathcal{L}}{\partial\left(\partial_\sigma\phi\right)}\right]\\ &\quad= -\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\sigma}\left[\sqrt{-g}\frac{\hbar}{2}g^{\mu\nu}\frac{\partial\left(\partial_\mu\phi\right)}{\partial\left(\partial_\sigma\phi\right)}\left(\partial_\nu\phi\right) + \sqrt{-g}\frac{\hbar}{2}g^{\mu\nu}\left(\partial_\mu\phi\right)\frac{\partial\left(\partial_\nu\phi\right)}{\partial\left(\partial_\sigma\phi\right)}\right]\\ &\quad= -\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\sigma}\left[\sqrt{-g}\frac{\hbar}{2}g^{\mu\nu}\delta^\sigma_\mu\left(\partial_\nu\phi\right) + \sqrt{-g}\frac{\hbar}{2}g^{\mu\nu}\left(\partial_\mu\phi\right)\delta^\sigma_\nu\right]\\ &= -\frac{\hbar}{\sqrt{-g}}\frac{\partial}{\partial x^\sigma}\left[\sqrt{-g}\,g^{\nu\sigma}\left(\partial_\nu\phi\right)\right] = -\hbar\,\Box\phi\,,\end{aligned} \tag{I.62}$$

where we have introduced the operator $\Box$

$$\Box = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\mu} \sqrt{-g}\, g^{\mu\nu} \frac{\partial}{\partial x^\nu} \tag{I.63}$$

The second part in the Euler–Lagrange equation reads

$$\frac{\partial \mathscr{L}}{\partial \phi} = -\frac{m^2 c^2}{\hbar} \phi \,. \tag{I.64}$$

In the end, the Euler–Lagrange equation can be written in the following form

$$\left(\Box - \frac{m^2 c^2}{\hbar^2} \right) \phi = 0 \,. \tag{I.65}$$

This is the Klein–Gordon equation.

Problem 3.7: The problem reduces to writing the operator $\Box$ in Cartesian and spherical coordinates. In Cartesian coordinates, this is trivial, as $g^{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ and therefore

$$\Box = \eta^{\mu\nu} \frac{\partial^2}{\partial x^\mu \partial x^\nu} = -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \,. \tag{I.66}$$

In spherical coordinates, we have $\sqrt{-g} = r^2 \sin\theta$ and therefore

$$\Box = -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\cot\theta}{r^2} \frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2}{\partial \phi^2} \,. \tag{I.67}$$

Problem 3.8: From Eq. (I.62) we know that

$$\frac{\partial \mathscr{L}}{\partial \left(\partial_\mu \phi \right)} = -\hbar\, g^{\mu\sigma} \left(\partial_\sigma \phi \right) = -\hbar \left(\partial^\mu \phi \right) \,. \tag{I.68}$$

The energy-momentum tensor is thus

$$T^\mu_\nu = \hbar \left(\partial^\mu \phi \right) \left(\partial_\nu \phi \right) - \delta^\mu_\nu \left[\frac{\hbar}{2} \eta^{\sigma\rho} \left(\partial_\sigma \phi \right) \left(\partial_\rho \phi \right) + \frac{1}{2} \frac{m^2 c^2}{\hbar} \phi^2 \right] , \tag{I.69}$$

or

$$T^{\mu\nu} = \hbar \left(\partial^\mu \phi \right) \left(\partial^\nu \phi \right) - \frac{\hbar}{2} \eta^{\mu\nu} \left[\eta^{\sigma\rho} \left(\partial_\sigma \phi \right) \left(\partial_\rho \phi \right) + \frac{m^2 c^2}{\hbar^2} \phi^2 \right] . \tag{I.70}$$

Problem 3.9: In Eq. (3.104) we see $\eta^{\mu\nu}$, which is the metric tensor in Cartesian coordinates. If we replace $\eta^{\mu\nu}$ with $g^{\mu\nu}$, we have

$$T^{\mu\nu} = (\varepsilon + P) \frac{U^\mu U^\nu}{c^2} + P g^{\mu\nu} \,. \tag{I.71}$$

This is the energy-momentum tensor of a perfect fluid in a general coordinate system. For spherical coordinates $g_{\mu\nu} = \mathrm{diag}\left(-1, 1, r^2, r^2 \sin^2\theta\right)$, and in the fluid rest-frame we have

$$||T^{\mu\nu}|| = \begin{pmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & \frac{P}{r^2} & 0 \\ 0 & 0 & 0 & \frac{P}{r^2 \sin^2\theta} \end{pmatrix} , \quad ||T_{\mu\nu}|| = \begin{pmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & Pr^2 & 0 \\ 0 & 0 & 0 & Pr^2 \sin^2\theta \end{pmatrix} . \tag{I.72}$$

I.4 Chapter 4

Problem 4.1: We can write

$$F^{\mu\nu} F_{\mu\nu} = F^{\mu t} F_{\mu t} + F^{\mu x} F_{\mu x} + F^{\mu y} F_{\mu y} + F^{\mu z} F_{\mu z} . \tag{I.73}$$

The first term on the right hand side is

$$F^{\mu t} F_{\mu t} = -E_x^2 - E_y^2 - E_z^2 . \tag{I.74}$$

Similarly, we calculate the other terms

$$\begin{aligned} F^{\mu x} F_{\mu x} &= -E_x^2 + B_z^2 + B_y^2 , \\ F^{\mu y} F_{\mu y} &= -E_y^2 + B_z^2 + B_x^2 , \\ F^{\mu z} F_{\mu z} &= -E_z^2 + B_y^2 + B_x^2 . \end{aligned} \tag{I.75}$$

So we find

$$F^{\mu\nu} F_{\mu\nu} = 2\left(\mathbf{B}^2 - \mathbf{E}^2\right) , \tag{I.76}$$

where we have defined $\mathbf{B}^2 = B_x^2 + B_y^2 + B_z^2$ and $\mathbf{E}^2 = E_x^2 + E_y^2 + E_z^2$.
$\varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}$ can be calculated with the same approach and the result is

$$\varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} = 4\mathbf{E} \cdot \mathbf{B} , \tag{I.77}$$

where $\mathbf{E} \cdot \mathbf{B} = E_x B_x + E_y B_y + E_z B_z$.

Problem 4.2: Let us write the component i of the left hand side. We have

$$[\nabla (\mathbf{V} \cdot \mathbf{W})]^i = \partial^i \left(V^j W_j\right) = \left(\partial^i V^j\right) W_j + V^j \left(\partial^i W_j\right) . \tag{I.78}$$

Let us now consider the right hand side. The component i of the first term is

$$[(\mathbf{W} \cdot \nabla) \mathbf{V}]^i = W^j \partial_j V^i . \tag{I.79}$$

The second term is similar to this with **V** and **W** exchanged. The component i of the third term is

$$\begin{aligned}[\mathbf{W} \times (\nabla \times \mathbf{V})]^i &= \varepsilon^{ijk} W_j \, (\nabla \times \mathbf{V})_k = \varepsilon^{ijk} W_j \varepsilon_{klm} \partial^l V^m = \varepsilon^{kij} \varepsilon_{klm} W_j \partial^l V^m \\ &= \left(\delta^i_l \delta^j_m - \delta^i_m \delta^j_l\right) W_j \partial^l V^m = W_j \partial^i V^j - W_j \partial^j V^i \, . \end{aligned} \tag{I.80}$$

The fourth term is similar to the third term with **V** and **W** exchanged. If we combine Eqs. (I.79) and (I.80), we have

$$[(\mathbf{W} \cdot \nabla) \, \mathbf{V}]^i + [\mathbf{W} \times (\nabla \times \mathbf{V})]^i = W_j \left(\partial^i V^j\right) \, . \tag{I.81}$$

Similarly, we have

$$[(\mathbf{V} \cdot \nabla) \, \mathbf{W}]^i + [\mathbf{V} \times (\nabla \times \mathbf{W})]^i = V_j \left(\partial^i W^j\right) \, , \tag{I.82}$$

and the sum of Eqs. (I.81) and (I.82) gives the expression in Eq. (I.78).

Problem 4.3: The Faraday tensor in the first reference frame is

$$||F_{\mu\nu}|| = \begin{pmatrix} 0 & -E & 0 & 0 \\ E & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \tag{I.83}$$

The Faraday tensor in the second reference frame can be calculated from

$$F'_{\mu\nu} = \Lambda^\alpha_\mu \Lambda^\beta_\nu F_{\alpha\beta} = \Lambda^t_\mu \Lambda^x_\nu F_{tx} + \Lambda^x_\mu \Lambda^t_\nu F_{xt} \, , \tag{I.84}$$

where

$$||\Lambda^\mu_\nu|| = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \tag{I.85}$$

$\beta = v/c$ and $\gamma = 1/\sqrt{1-\beta^2}$. The result is that $F'_{\mu\nu} = F_{\mu\nu}$.

Problem 4.4: The faraday tensor is given in Eq. (I.83), so the only non-vanishing components are $F_{xt} = -F_{tx} = E$ and $F^{tx} = -F^{xt} = E$. We have $F^{\rho\sigma} F_{\rho\sigma} = -2E^2$. The energy momentum tensor is

$$||T^{\mu\nu}|| = \frac{1}{4\pi} ||F^{\mu\rho} F^\nu{}_\rho|| - \frac{1}{16\pi} ||\eta^{\mu\nu}|| F^{\rho\sigma} F_{\rho\sigma} = \frac{E^2}{8\pi} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \tag{I.86}$$

and its trace vanishes, $T^\mu_\mu = E^2/8\pi\,(-1-1+1+1) = 0$.

I.5 Chapter 5

Problem 5.1: From Eq. (5.50) we have

$$\begin{aligned}
\nabla_\mu A_{\alpha\beta} &= \frac{\partial A_{\alpha\beta}}{\partial x^\mu} - \Gamma^\sigma_{\mu\alpha} A_{\sigma\beta} - \Gamma^\sigma_{\mu\beta} A_{\alpha\sigma} \, , \\
\nabla_\mu A^{\alpha\beta} &= \frac{\partial A^{\alpha\beta}}{\partial x^\mu} + \Gamma^\alpha_{\mu\sigma} A^{\sigma\beta} + \Gamma^\beta_{\mu\sigma} A^{\alpha\sigma} \, , \\
\nabla_\mu A^\alpha{}_\beta &= \frac{\partial A^\alpha{}_\beta}{\partial x^\mu} + \Gamma^\alpha_{\mu\sigma} A^\sigma{}_\beta - \Gamma^\sigma_{\mu\beta} A^\alpha{}_\sigma \, , \\
\nabla_\mu A_\alpha{}^\beta &= \frac{\partial A_\alpha{}^\beta}{\partial x^\mu} - \Gamma^\sigma_{\mu\alpha} A_\sigma{}^\beta + \Gamma^\beta_{\mu\sigma} A_\alpha{}^\sigma \, .
\end{aligned} \tag{I.87}$$

Problem 5.2: It is straightforward to compute these quantities with Cartesian coordinates, where the metric tensor is $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$. In Cartesian coordinates, all the components of the Riemann tensor vanish, and therefore all the components of the Ricci tensor and the scalar curvature vanish as well. From the transformation rules for tensors and scalars, we see that all the components of the Riemann tensor, the Ricci tensor, and the scalar curvature are identically zero in the Minkowski spacetime.

Problem 5.3: We write $R_{\alpha\mu\beta\nu}$ with the help of Eq. (5.76)

$$\begin{aligned}
R_{\alpha\mu\beta\nu} = \frac{1}{2} &\left(\frac{\partial^2 g_{\alpha\nu}}{\partial x^\mu \partial x^\beta} + \frac{\partial^2 g_{\mu\beta}}{\partial x^\alpha \partial x^\nu} - \frac{\partial^2 g_{\alpha\beta}}{\partial x^\mu \partial x^\nu} - \frac{\partial^2 g_{\mu\nu}}{\partial x^\alpha \partial x^\beta} \right) \\
&+ g_{\kappa\lambda} \left(\Gamma^\lambda_{\mu\beta} \Gamma^\kappa_{\alpha\nu} - \Gamma^\lambda_{\mu\nu} \Gamma^\kappa_{\alpha\beta} \right) \, .
\end{aligned} \tag{I.88}$$

The Ricci tensor is

$$\begin{aligned}
R_{\mu\nu} &= g^{\alpha\beta} R_{\alpha\mu\beta\nu} \\
&= \frac{1}{2} g^{\alpha\beta} \left(\frac{\partial^2 g_{\alpha\nu}}{\partial x^\mu \partial x^\beta} + \frac{\partial^2 g_{\mu\beta}}{\partial x^\alpha \partial x^\nu} \right) - \frac{1}{2} g^{\alpha\beta} \frac{\partial^2 g_{\alpha\beta}}{\partial x^\mu \partial x^\nu} - \frac{1}{2} g^{\alpha\beta} \frac{\partial^2 g_{\mu\nu}}{\partial x^\alpha \partial x^\beta} \\
&\quad + \frac{1}{2} g^{\alpha\beta} g_{\kappa\lambda} \Gamma^\lambda_{\mu\beta} \Gamma^\kappa_{\alpha\nu} - \frac{1}{2} g^{\alpha\beta} \Gamma^\lambda_{\mu\nu} \Gamma^\kappa_{\alpha\beta} \, ,
\end{aligned} \tag{I.89}$$

and is explicitly symmetric in the indices μ and ν.

I.6 Chapter 6

Problem 6.1: Since the metric matrix is diagonal, it is straightforward to find the vierbeins. $E^{\mu}_{(\alpha)}$ will be zero for $\alpha \neq \mu$ and $1/\sqrt{|g_{\mu\mu}|}$ for $\alpha = \mu$. We thus have

$$E_{(t)} = \left(\frac{1}{\sqrt{f}}, 0, 0, 0 \right) , \quad E_{(r)} = \left(0, \sqrt{f}, 0, 0 \right) ,$$
$$E_{(\theta)} = \left(0, 0, \frac{1}{r}, 0 \right) , \quad E_{(\phi)} = \left(0, 0, 0, \frac{1}{r \sin\theta} \right) . \tag{I.90}$$

Problem 6.2: If the observer has constant spatial coordinates, in the line element $dx^i = 0$ and therefore

$$d\tau^2 = -g_{tt} dt^2 = \left(1 - \frac{r_{\rm Sch} r}{r^2 + a^2 \cos^2\theta} \right) dt^2 . \tag{I.91}$$

Note that this requires $r > r_{\rm sl} = r_{\rm Sch}/2 + \sqrt{r^2_{\rm Sch}/4 - a^2 \cos^2\theta}$. For $r < r_{\rm sl}$ we have $g_{tt} > 0$ and therefore an observer with constant spatial coordinates would follow a space-like trajectory, which is not allowed. Observers with $r < r_{\rm sl}$ are allowed but they have to move. We will discuss this point in Sect. 10.3.3.

Problem 6.3: For a general reference frame, we simply have to replace the partial derivative ∂_μ with the covariant derivative ∇_μ. The result is that the equation now reads $\nabla_\mu J^\mu = 0$. Note that $\partial_\mu J^\mu = 0$ holds in an inertial reference frame in flat spacetime in Cartesian coordinates. $\nabla_\mu J^\mu = 0$ holds for any other case, including when we do not have a Cartesian coordinate system, or when the spacetime is flat but the reference frame is not inertial, or in a curved spacetime.

In a general reference frame, we have

$$\nabla_\mu J^\mu = \frac{\partial J^\mu}{\partial x^\mu} + \Gamma^\mu_{\mu\nu} J^\nu = 0 . \tag{I.92}$$

To write this expression in an inertial reference frame in flat spacetime in spherical coordinates, it is necessary to evaluate the Christoffel symbols and then plug them into Eq. (I.92). Alternatively, it is possible to employ the formula (5.64) and write

$$\begin{aligned} \nabla_\mu J^\mu &= \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial t} \left(J^t r^2 \sin\theta \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial r} \left(J^r r^2 \sin\theta \right) \\ &\quad + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(J^\theta r^2 \sin\theta \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \phi} \left(J^\phi r^2 \sin\theta \right) \\ &= \frac{\partial J^t}{\partial t} + \frac{2 J^r}{r} + \frac{\partial J^r}{\partial r} + \frac{\partial J^\theta}{\partial \theta} + \cot\theta J^\theta + \frac{\partial J^\phi}{\partial \phi} . \end{aligned} \tag{I.93}$$

Problem 6.4: In the case of minimal coupling, we can just apply our standard recipe and replace partial derivatives with covariant derivatives. The result is

$$\left(\nabla_\mu \partial^\mu - \frac{m^2 c^2}{\hbar^2}\right)\phi = 0\,. \tag{I.94}$$

Note that we have $\nabla_\mu \partial^\mu$ instead of $\nabla_\mu \nabla^\mu$ because ϕ is a scalar and therefore $\nabla^\mu \phi = \partial^\mu \phi$. Since $\partial^\mu \phi$s are the components of a vector field, we need the covariant derivative and therefore we write $\nabla_\mu \partial^\mu \phi$. If in flat spacetime in Cartesian coordinates we had the expression

$$\partial_\mu \partial^\mu A^\nu\,, \tag{I.95}$$

where A^ν is some vector field, then the generalization to curved spacetime would be

$$\nabla_\mu \nabla^\mu A^\nu\,. \tag{I.96}$$

In the case of non-minimal coupling, we have an extra term coming from $\partial \mathscr{L}/\partial\phi$ and the field equation becomes

$$\left(\nabla_\mu \partial^\mu - \frac{m^2 c^2}{\hbar^2} + \frac{2\xi R}{\hbar}\right)\phi = 0\,. \tag{I.97}$$

Problem 6.5: We have to change sign in front of the term with $g^{\mu\nu}$. The Lagrangian density reads

$$\mathscr{L} = \frac{\hbar}{2} g^{\mu\nu}\left(\partial_\mu \phi\right)\left(\partial_\nu \phi\right) - \frac{1}{2}\frac{m^2 c^2}{\hbar}\phi^2 + \xi R \phi^2\,. \tag{I.98}$$

The Klein–Gordon equation becomes

$$\left(\nabla_\mu \partial^\mu + \frac{m^2 c^2}{\hbar^2} - \frac{2\xi R}{\hbar}\right)\phi = 0\,. \tag{I.99}$$

Problem 6.6: The Minkowski metric $\eta^{\mu\nu}$ has to be replaced by the general expression for the metric tensor $g^{\mu\nu}$. The energy-momentum tensor of a perfect fluid reads

$$T^{\mu\nu} = (\varepsilon + P)\,\frac{U^\mu U^\nu}{c^2} + P g^{\mu\nu}\,. \tag{I.100}$$

I.7 Chapter 7

Problem 7.1: We already know from Sect. 7.4 that the action

$$S = -\frac{\hbar}{2c}\int\left[g^{\mu\nu}\left(\partial_\mu\phi\right)\left(\partial_\nu\phi\right) + \frac{m^2c^2}{\hbar^2}\phi^2\right]\sqrt{-g}\,d^4x\,, \tag{I.101}$$

leads to the energy-momentum tensor

$$T^{\mu\nu} = \hbar\left(\partial^\mu\phi\right)\left(\partial^\nu\phi\right) - \frac{\hbar}{2}g^{\mu\nu}\left[g^{\rho\sigma}\left(\partial_\rho\phi\right)\left(\partial_\sigma\phi\right) + \frac{m^2c^2}{\hbar^2}\phi^2\right]. \tag{I.102}$$

Now we have to evaluate the contribution from

$$S_{R\phi^2} = \frac{1}{c}\int \xi R\phi^2\sqrt{-g}\,d^4x\,. \tag{I.103}$$

Instead of Eq. (7.38), now we have

$$\begin{aligned}\delta S_{R\phi^2} &= \frac{1}{c}\int \xi\phi^2\left(\frac{1}{2}g^{\mu\nu}R - R^{\mu\nu}\right)\sqrt{-g}\left(\delta g_{\mu\nu}\right)\,d^4x \\ &\quad + \frac{1}{c}\int \xi\phi^2\nabla_\rho H^\rho\sqrt{-g}\,d^4x\,. \end{aligned} \tag{I.104}$$

The first term on the right hand side leads to the left hand side of the Einstein equations with an effective Einstein constant $\kappa_{\rm eff} = 1/(2\xi\phi^2)$. The second term on the right hand side cannot be ignored now, as we did in Sect. 7.3, and contributes to the energy-momentum tensor of the scalar field. In Eq. (7.37), H^ρ is written in terms of $\delta\Gamma^\kappa_{\mu\nu}$ and now we have to write it extracting $\delta g_{\mu\nu}$. After some tedious but straightforward calculation, we can recast the second term on the right hand side in Eq. (I.104) in the form (7.42) with the energy-momentum tensor

$$T^{\mu\nu}_{R\phi^2} = -2\xi\left(g^{\mu\nu}\Box - \nabla^\mu\partial^\nu\right)\phi^2\,, \tag{I.105}$$

where $\Box = \nabla_\sigma\partial^\sigma$. The final energy-momentum tensor of the scalar field ϕ is $T^{\mu\nu}_\phi = T^{\mu\nu} + T^{\mu\nu}_{R\phi^2}$, where $T^{\mu\nu}$ is given in Eq. (I.102).

Problem 7.2: The equations of motion for the gravity sector are the Einstein equations and can be obtained by considering the variation $g_{\mu\nu} \rightarrow g'_{\mu\nu} = g_{\mu\nu} + \delta g_{\mu\nu}$. The result is

$$2\xi\phi^2\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) = T^\phi_{\mu\nu}\,, \tag{I.106}$$

where $T^\phi_{\mu\nu}$ is the energy-momentum tensor of the scalar field found in Problem 7.1.

The equations of motion for the matter sector can be obtained by considering the variations in ϕ and $\partial_\mu\phi$. The result is

$$\left(\Box - \frac{m^2c^2}{\hbar^2} + \frac{2\xi R}{\hbar}\right)\phi = 0\,, \tag{I.107}$$

where $\Box = \nabla_\mu \partial^\mu$.

Problem 7.3: Without a cosmological constant, the action is

$$S = \frac{1}{2\kappa c} \int R\sqrt{-g}\, d^4x + S_{\rm m} \,. \tag{I.108}$$

When we consider the variation $g_{\mu\nu} \to g'_{\mu\nu} = g_{\mu\nu} + \delta g_{\mu\nu}$, we find

$$\delta S = \frac{1}{2\kappa c} \int \left(\frac{1}{2} g^{\mu\nu} R - R^{\mu\nu} + \kappa T^{\mu\nu} \right) \sqrt{-g} \left(\delta g_{\mu\nu} \right) \, d^4x \,, \tag{I.109}$$

which leads the the Einstein equations without a cosmological constant.

Now we want to include the cosmological constant Λ. The variation $g_{\mu\nu} \to g'_{\mu\nu} = g_{\mu\nu} + \delta g_{\mu\nu}$ should lead to

$$\delta S = \frac{1}{2\kappa c} \int \left(\frac{1}{2} g^{\mu\nu} R - R^{\mu\nu} - \Lambda g^{\mu\nu} + \kappa T^{\mu\nu} \right) \sqrt{-g} \left(\delta g_{\mu\nu} \right) \, d^4x \,. \tag{I.110}$$

From Eq. (7.29), we see that the Einstein–Hilbert action should be

$$S'_{\rm EH} = \frac{1}{2\kappa c} \int (R - 2\Lambda) \sqrt{-g}\, d^4x \,. \tag{I.111}$$

I.8 Chapter 8

Problem 8.1: We write the geodesic equations as

$$\frac{d}{d\tau} \left(g_{\mu\nu} \dot{x}^\nu \right) = \frac{1}{2} \frac{\partial g_{\nu\rho}}{\partial x^\mu} \dot{x}^\nu \dot{x}^\rho \,, \tag{I.112}$$

where the dot $\dot{}$ stands for the derivative with respect to the proper time of the particle, τ. For simplicity and without loss of generality, we consider orbits in the equatorial plane, so $\theta = \pi/2$ and $\dot{\theta} = 0$. In the case of circular orbits, we have $\dot{r} = \ddot{r} = 0$ and for $\mu = r$ Eq. (I.112) reduces to

$$\frac{\partial g_{tt}}{\partial r} \dot{t}^2 + \frac{\partial g_{\phi\phi}}{\partial r} \dot{\phi}^2 = 0 \tag{I.113}$$

when we consider the Schwarzschild metric, because only the diagonal metric coefficients are non-vanishing. The angular velocity of the particle is

$$\Omega(r) = \frac{\dot{\phi}}{\dot{t}} = \sqrt{-\frac{\partial_r g_{tt}}{\partial_r g_{\phi\phi}}} = \sqrt{\frac{r_{\rm S}}{2r^3}} \,. \tag{I.114}$$

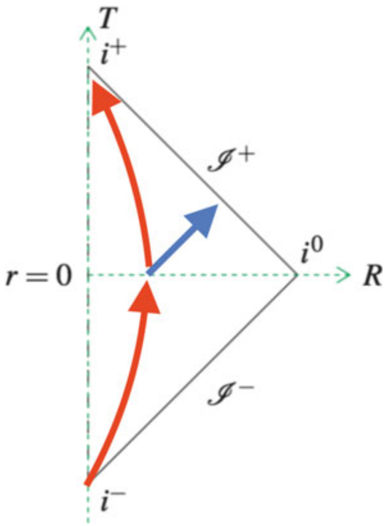

Fig. I.1 Penrose diagram for the Minkowski spacetime, trajectory of the massive particle (red arrows), and trajectory of the electromagnetic pulse (blue arrow)

From $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = -1$ with $\dot{r} = \dot{\theta} = 0$, we can write

$$g_{tt}\dot{t}^2 + g_{\phi\phi}\dot{\phi}^2 = \dot{t}^2\left(g_{tt} + \Omega^2 g_{\phi\phi}\right) = -1\,,$$
$$\Rightarrow \dot{t} = \frac{1}{\sqrt{-g_{tt} - \Omega^2 g_{\phi\phi}}} = \sqrt{\frac{2r}{2r - 3r_S}}\,. \tag{I.115}$$

Since $\dot{t} = dt/d\tau$, the relation between the particle proper time τ and the coordinate time t is

$$\Delta t = \Delta\tau\sqrt{\frac{2r}{2r - 3r_S}}\,. \tag{I.116}$$

Note that for $r \to 3r_S/2$ we have $\Delta\tau \to 0$. As we will see in Sect. 10.3.1, $r = 3r_S/2$ is the photon orbit of the Schwarzschild spacetime. Massive particles can orbit at the photon orbit in the limit $v \to c$ and therefore have Lorentz factor $\gamma \to \infty$.

Problem 8.2: The possible trajectory of the massive particle is shown in red in Fig. I.1, while the trajectory of the electromagnetic pulse is in blue. Note that the trajectory of the massive particle starts from past time-like infinity, ends at future time-like infinity, and it is always inside the light-cone of the particle (the particle velocity is lower than the speed of light). The trajectory of the electromagnetic pulse is a straight line at 45°, starts at $t = 0$, and reaches future null infinity.

Problem 8.3: The future-light cones of an event in region I, of an event inside the black hole (region II), and of an event inside the white hole (region IV) are shown in, respectively, the top, central, and bottom panels in Fig. I.2. Time-like and null trajectories in region I can either fall to the singularity of the black hole at $r = 0$ or reach future time-like infinity (time-like trajectories) and future null infinity (null trajectories). All time-like and null trajectories in region II must end at the singularity of the black hole at $r = 0$. Time-like and null trajectories in region IV may go to region I, region II, or region III.

I.9 Chapter 10

Problem 10.1: From the line element in Eq. (10.4) we see that the metric tensor of the Reissner–Nordström spacetime is

$$||g_{\mu\nu}|| = \begin{pmatrix} -f & 0 & 0 & 0 \\ 0 & \frac{1}{f} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix} , \tag{I.117}$$

where (in units in which $G_{\rm N} = c = 4\pi\varepsilon_0 = 1$)

$$f = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} . \tag{I.118}$$

By definition of inverse metric $g_{\mu\nu}g^{\nu\rho} = \delta^\rho_\mu$. Since the metric matrix in Eq. (I.117) is diagonal, the inverse is

$$||g^{\mu\nu}|| = \begin{pmatrix} -\frac{1}{f} & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2 \sin^2\theta} \end{pmatrix} . \tag{I.119}$$

Problem 10.2: From the line element in Eq. (10.6) we see that the metric tensor of the Kerr spacetime is

$$||g_{\mu\nu}|| = \begin{pmatrix} -\left(1 - \frac{2Mr}{\Sigma}\right) & 0 & 0 & -\frac{2aMr\sin^2\theta}{\Sigma} \\ 0 & \frac{\Sigma}{\Delta} & 0 & 0 \\ 0 & 0 & \Sigma & 0 \\ -\frac{2aMr\sin^2\theta}{\Sigma} & 0 & 0 & \left(r^2 + a^2 + \frac{2a^2Mr\sin^2\theta}{\Sigma}\right)\sin^2\theta \end{pmatrix} . \tag{I.120}$$

We have to find the inverse matrix. For the metric coefficients involving at least one index r or θ, it is straightforward because we can still treat the matrix as diagonal. For the metric coefficients g_{tt}, $g_{t\phi}$, $g_{\phi t}$, and $g_{\phi\phi}$ the problem reduces to find the inverse matrix of a symmetric matrix 2×2. For a general symmetric matrix 2×2

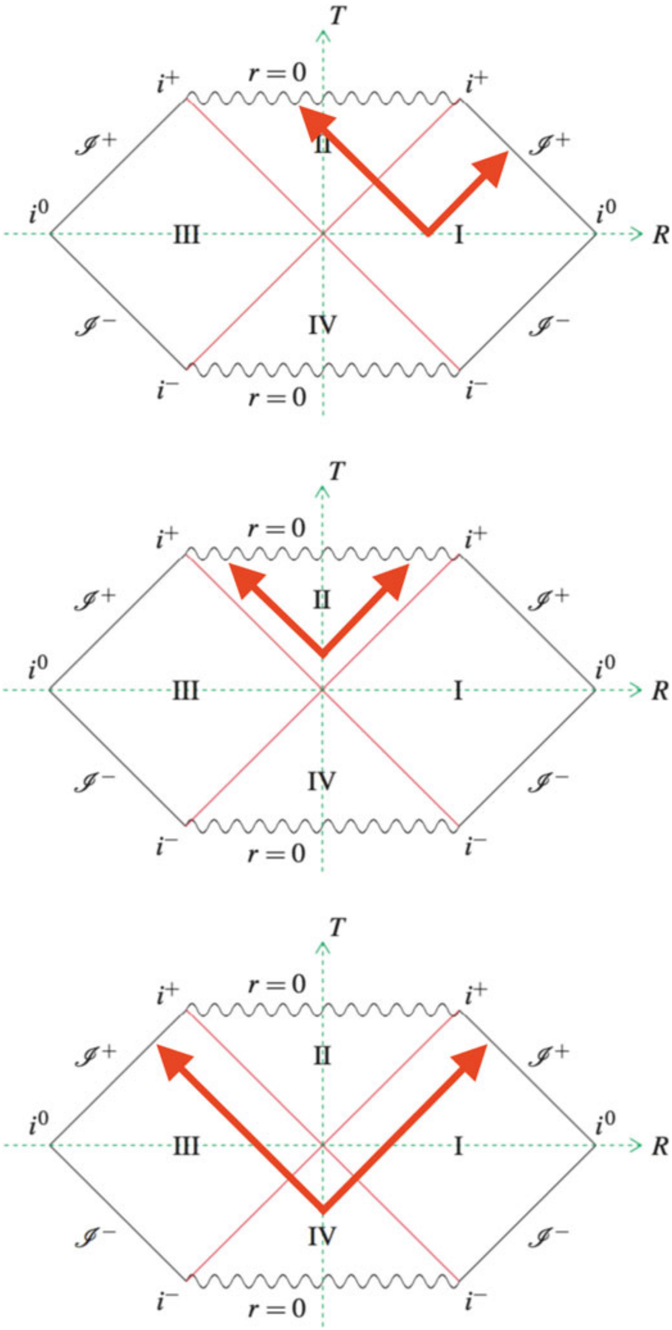

Fig. I.2 Schwarzschild spacetime. The top diagram shows the future light-cone of an event in region I, the central diagram shows the future light-cone of an event inside the black hole (region II), and the bottom diagram shows the future light-cone of an event inside the white hole (region IV)

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix} , \tag{I.121}$$

the inverse matrix is

$$A^{-1} = \frac{1}{ac - b^2} \begin{pmatrix} c & -b \\ -b & a \end{pmatrix} = \frac{1}{\det|A|} \begin{pmatrix} c & -b \\ -b & a \end{pmatrix} . \tag{I.122}$$

For the metric of the Kerr spacetime in Boyer–Lindquist coordinates, we find

$$||g^{\mu\nu}|| = \begin{pmatrix} -\frac{(r^2+a^2)^2 - a^2 \Delta \sin^2\theta}{\Sigma\Delta} & 0 & 0 & -\frac{2aMr}{\Sigma\Delta} \\ 0 & \frac{\Delta}{\Sigma} & 0 & 0 \\ 0 & 0 & \frac{1}{\Sigma} & 0 \\ -\frac{2aMr}{\Sigma\Delta} & 0 & 0 & \frac{\Delta - a^2\sin^2\theta}{\Sigma\Delta\sin^2\theta} \end{pmatrix} . \tag{I.123}$$

Problem 10.3: The left hand side of Eq. (10.21) can be written as

$$\begin{aligned} \frac{d}{d\tau}\left(g_{\mu\nu}\dot{x}^\nu\right) &= \frac{\partial g_{\mu\nu}}{\partial x^\rho}\dot{x}^\nu\dot{x}^\rho + g_{\mu\nu}\ddot{x}^\nu \\ &= \frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^\rho}\dot{x}^\nu\dot{x}^\rho + \frac{1}{2}\frac{\partial g_{\mu\rho}}{\partial x^\nu}\dot{x}^\nu\dot{x}^\rho + g_{\mu\nu}\ddot{x}^\nu . \end{aligned} \tag{I.124}$$

Equation (10.21) thus becomes

$$g_{\mu\nu}\ddot{x}^\nu + \frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^\rho}\dot{x}^\nu\dot{x}^\rho + \frac{1}{2}\frac{\partial g_{\mu\rho}}{\partial x^\nu}\dot{x}^\nu\dot{x}^\rho = \frac{1}{2}\frac{\partial g_{\nu\rho}}{\partial x^\mu}\dot{x}^\nu\dot{x}^\rho \tag{I.125}$$

We move the term on the right hand side to the left hand side and we multiply everything by $g^{\sigma\mu}$ summing over the repeated index μ. We find

$$\ddot{x}^\sigma + \frac{1}{2}g^{\sigma\mu}\left(\frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^\rho} + \frac{1}{2}\frac{\partial g_{\rho\mu}}{\partial x^\nu} - \frac{\partial g_{\rho\nu}}{\partial x^\mu}\right)\dot{x}^\nu\dot{x}^\rho = \ddot{x}^\sigma + \Gamma^\sigma_{\nu\rho}\dot{x}^\nu\dot{x}^\rho = 0 , \tag{I.126}$$

which are the geodesic equations in their standard form.

Problem 10.4: Let us model Earth as a uniform sphere. Its moment of inertia is

$$I = \frac{2}{5}MR^2 , \tag{I.127}$$

where $M = 6.0 \cdot 10^{24}$ kg is Earth's mass and $R = 6.4 \cdot 10^6$ m is Earth's radius. Earth's spin angular momentum is $J = I\omega$, where $\omega = 7.3 \cdot 10^{-5}$ rad/s is Earth's angular velocity. The spin parameter of Earth is

$$a_* = \frac{cJ}{G_\mathrm{N}M^2} = 897 \gg 1 . \tag{I.128}$$

Note that $a_* \gg 1$ does not imply that the effect of frame dragging is strong, because Earth's physical radius is much larger than Earth's gravitational radius.

I.10 Chapter 11

Problem 11.1: If we want to do the calculations by hand, we can start deriving the geodesic equations in the Friedmann–Robertson–Walker metric in order to get the Christoffel symbols, then we calculate the Ricci tensor and the scalar curvature, and eventually we write the tt component of the Einstein equations in which matter is described by a perfect fluid in its rest-frame. We find the first Friedmann equation.

Alternatively, we can use the RGTC package presented in Appendix E. We initialize the code

```
<< EDCRGTCcode.m
```

We define the coordinates

```
Coord = {t,r,θ,φ};
```

We define the non-vanishing metric coefficients

```
gtt = - 1;
grr = a[t]^2/(1 - k r^2);
gpp = a[t]^2 r^2;
gvv = a[t]^2  r^2 Sin[θ]^2;
```

and then the metric

```
g = {{gtt, 0, 0, 0}, {0, grr, 0, 0}, {0, 0, gpp, 0},
   { 0, 0, 0, gvv }};
```

We launch the code with the command

```
RGtensors[g, Coord, {1, 1, 1}]
```

At this point, we ask the code to provide us the tt component of the Einstein tensor G^t_t (note: an upper index and a lower index because this is what the code calculates as default)

```
EUd[[1,1]]
```

The output is

$$-3\frac{\dot{a}^2+k}{a^2}\,. \tag{I.129}$$

Since we have used units in which $c = 1$, we reintroduce the speed of light

$$- 3 \frac{\dot{a}^2 + kc^2}{a^2 c^2} \, . \tag{I.130}$$

The tt component of the Einstein equations (with an upper index and a lower index) is

$$- 3 \frac{\dot{a}^2 + kc^2}{a^2 c^2} = G^t_t = \frac{8\pi G_\mathrm{N}}{c^4} T^t_t = - \frac{8\pi G_\mathrm{N}}{c^4} \rho \, , \tag{I.131}$$

and we find the first Friedmann equation.

Problem 11.2: As in the previous exercise, we can calculate these quantities either by hand or with a program like the RGTC package. In the latter case, we proceed with the same commands as in the previous exercise and then we ask the code to provide us the Kretschmann scalar

```
Kretschmann = Simplify[ Sum[
   Rdddd[[i, j, k, l]]*RUddd[[i, m, n, o]]*gUU[[j, m]]
      *gUU[[k, n]]*gUU[[l, o]],
   {i, 1, 4}, {j, 1, 4}, {k, 1, 4}, {l, 1, 4},
      {m, 1, 4}, {n, 1, 4}, {o, 1, 4} ] ]
```

and the scalar curvature

```
ScalarCurvature = Simplify[ Sum[ Rdd[[i, j]]*gUU[[i, j]],
                            {i, 1, 4}, {j, 1, 4} ] ]
```

Reintroducing the speed of light c, we obtain the expressions in Eqs. (11.3) and (11.4).

Problem 11.3: Let us consider a small change in either the value of the matter energy density ρ, the cosmological constant Λ, or the scale factor a. The result is that the universe either starts expanding forever ($a \to \infty$) or recollapses to a singular solution ($a = 0$).

Problem 11.4: If we include a radiation component, we have to add the following energy density to the list in Eq. (11.62)

$$\rho_\mathrm{r} = \rho^0_\mathrm{r} \, (1 + z)^4 \ , \tag{I.132}$$

and Eq. (11.63) becomes

$$H^2 = H_0^2 \left[\Omega^0_\mathrm{r} \, (1 + z)^4 + \Omega^0_\mathrm{m} \, (1 + z)^3 + \Omega^0_\Lambda + \Omega^0_k \, (1 + z)^2 \right] . \tag{I.133}$$

Now $\Omega^0_k = 1 - \Omega^0_\mathrm{r} - \Omega^0_\mathrm{m} - \Omega^0_\Lambda$ and Eq. (11.67) becomes

$$\tau = \frac{1}{H_0} \int_0^\infty \frac{d\tilde{z}}{1 + \tilde{z}} \frac{1}{\sqrt{\tilde{z} \, (2 + \tilde{z}) \, (1 + \tilde{z})^2 \, \Omega^0_\mathrm{r} + \left(1 + \Omega^0_\mathrm{m} \tilde{z}\right) (1 + \tilde{z})^2 - \tilde{z} \, (2 + \tilde{z}) \, \Omega^0_\Lambda}} \, . \tag{I.134}$$

Let us consider the situation in our Universe. If we ignore the contribution from radiation, we have $\Omega^0_{\rm m} = 0.31$ and $\Omega^0_\Lambda = 0.69$, and the integral gives the numerical factor 0.9553. The contribution of radiation today is $\Omega^0_{\rm r} = 5 \cdot 10^{-5}$. If we take this contribution into account, the integral in Eq. (I.134) gives the numerical factor 0.9551. Note that here we are ignoring the possibility that some matter is relativistic at some early time and becomes non-relativistic at a later time.

I.11 Chapter 12

Problem 12.1: For $M = 10^6\ M_\odot$, the maximum frequency is $\nu_{\rm max} \sim 10$ mHz. For $M = 10^9\ M_\odot$, we have $\nu_{\rm max} \sim 10$ nHz. This is consistent with the expected signal from these objects in Fig. 12.4.

I.12 Chapter 13

Problem 13.1: Let us use units in which $c = \hbar = 1$ for simplicity. The area of the event horizon is $A_{\rm H} \sim r_{\rm g}^2$, where $r_{\rm g} = G_{\rm N} M$ is the gravitational radius of the black hole. Since the black hole temperature is $T_{\rm BH} \sim 1/r_{\rm g}$, the black hole luminosity is

$$L_{\rm BH} \sim A_{\rm H} T^4_{\rm BH} \sim \frac{1}{r^2_{\rm g}} = \frac{1}{G^2_{\rm N} M^2} \, . \tag{I.135}$$

We write $L_{\rm BH} = dM/dt$ and Eq. (I.135) gives

$$G^2_{\rm N} M^2\, dM \sim dt \, . \tag{I.136}$$

We integrate both sides and we get a rough estimate of the evaporation time

$$\tau_{\rm evap} = \int dt \sim \int G^2_{\rm N} M^2\, dM = \frac{1}{3} G^2_{\rm N} M^3_0 \, , \tag{I.137}$$

where M_0 is the initial mass of the black hole. Since $G^2_{\rm N} = T_{\rm Pl}/M^3_{\rm Pl}$, we have

$$\tau_{\rm evap} \sim \left(\frac{M_0}{M_{\rm Pl}} \right)^3 T_{\rm Pl} \sim 10^{-44} \left(\frac{M_0}{10^{-5}\ {\rm g}} \right)^3 {\rm s} \, , \tag{I.138}$$

For $M_0 = M_\odot \sim 10^{33}$ g, we find $\tau_{\rm evap} \sim 10^{70}$ s, which is about 10^{63} years and is much longer than the age of the Universe (about 10^{10} years). A more accurate calculation would lead to $\tau_{\rm evap} \sim 10^{74}$ s.

Subject Index

C. Bambi, *Introduction to General Relativity*, Undergraduate Lecture Notes in Physics, https://doi.org/10.1007/978-981-13-1090-4