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Mathematical Logic

On Numbers, Sets, Structures, and Symmetry

Springer

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Preface

Logic and sermons never convince,
The damp of the night drives deeper into my soul.
(Only what proves itself to every man and woman is so,
Only what nobody denies is so.)

Walt Whitman, *Leaves of Grass*

In *Why Is There Philosophy of Mathematics at All?* [8], Ian Hacking writes:

> Yet although most members of our species have some capacity for geometrical and numerical concepts, very few human beings have much capacity for doing or even understanding mathematics. This is often held to be the consequence of bad education, but although education can surely help, there is no evidence that vast disparity of talent, or even interest in, mathematics, is a result of bad pedagogy... A paradox: we are the mathematical animal. A few of us have made astonishing mathematical discoveries, a few more of us can understand them. Mathematical applications have turned out to be a key to unlock and discipline nature to conform to some of our wishes. But the subject repels most human beings.

It all rings true to anyone who has ever taught the subject. When I teach undergraduates, I sometimes say, “Math does not make any sense, right?” to which I hear in unison “Right.” My own school experience has not been much different. Even though I was considered “good at math,” it only meant that I could follow instructions and do homework to a satisfying result. Only occasionally I’d have moments I was proud of. In my first high school year, I had a mathematics professor about whom legends were told. Everyone knew that in order to survive in his class, one had to *understand*. At any time you could be called to the blackboard to be asked a penetrating question. We witnessed humiliating moments when someone’s ignorance was ruthlessly revealed. Once, when the topic was square roots of real numbers, I was called and asked:

– Does every positive number have a square root?
– Yes.
– What is the square root of four?
– Two.
– What is the square root of two?
– The square root of two.

– Good. What is the square root of the square root of two.
– The square root of the square root of two.
– Good. Sit down.

The professor smiled. He liked my answers, and I was happy with them too. It seemed I had understood something and that this understanding was somehow complete. The truth was simple, and once you grasped it, there was nothing more to it, just plain simple truth. Perhaps such rare satisfying moments were the reason I decided to study mathematics.

In my freshmen year at the University of Warsaw, I took Analysis, Abstract Algebra, Topology, Mathematical Logic, and Introduction to Computer Science. I was completely unprepared for the level of abstraction of these courses. I thought that we would just continue with the kind of mathematics we studied in high school. I expected more of the same but more complicated. Perhaps some advanced formulas for solving algebraic and trigonometric equations and more elaborate constructions in plane and three-dimensional geometry. Instead, the course in analysis began with *defining* real numbers. Something we took for granted and we thought we knew well now needed a definition! My answers at the blackboard that I had been so proud of turned out to be rather naive. The truth was not so plain and simple after all. And to make matters worse, the course followed with a proof of existence and uniqueness of the exponential function $f(x) = e^x$, and in the process, we had to learn about complex numbers and the fundamental theorem of algebra. Instead of advanced applications of mathematics we have already learned, we were going back, asking more and more fundamental questions. It was rather unexpected. The algebra and topology courses were even harder. Instead of the already familiar planes, spheres, cones, and cylinders, now we were exposed to general algebraic systems and topological spaces. Instead of concrete objects one could try to visualize, we studied infinite spaces with surprising general properties expressed in terms of algebraic systems that were associated with them. I liked the prospect of learning all that, and in particular the promise that in the end, after all this high-level abstract stuff got sorted out, there would be a return to more down-to-earth applications. But what was even more attractive was the attempt to get to the bottom of things, to understand completely what this elaborate edifice of mathematics was founded upon. I decided to specialize in mathematical logic.

My first encounters with mathematical logic were traumatic. While still in high school, I started reading Andrzej Grzegorczyk's *Outline of Mathematical Logic*,[1] a textbook whose subtitle promised *Fundamental Results and Notions Explained with All Details*. Grzegorczyk was a prominent mathematician who made important contributions to the mathematical theory of computability and later moved to philosophy. Thanks to Google Books, we can now see a list of the words and phrases most frequently used in the book: axiom schema, computable functions, concept, empty domain, existential quantifier, false, finite number, free variable, and many

[1] An English translation is [17].

more. I found this all attractive, but I did not understand any of it, and it was not Grzegorczyk's fault. In the introduction, the author writes:

> Recent years have seen the appearance of many English-language handbooks of logic and numerous monographs on topical discoveries in the foundations of mathematics. These publications on the foundations of mathematics as a whole are rather difficult for the beginners or refer the reader to other handbooks and various piecemeal contributions and also sometimes to largely conceived 'mathematical folklore' of unpublished results. As distinct from these, the present book is as easy as possible systematic exposition of the now classical results in the foundations of mathematics. Hence the book may be useful especially for those readers who want to have all the proofs carried out in full and all the concepts explained in detail. In this sense the book is self-contained. The reader's ability to guess is not assumed, and the author's ambition was to reduce the use of such words as evident and obvious in proofs to a minimum. This is why the book, it is believed, may be helpful in teaching or learning the foundation of mathematics in those situations in which the student cannot refer to a parallel lecture on the subject.

Now that I know what Grzegorczyk is talking about, I tend to agree. When I was reading the book then, I found it almost incomprehensible. It is not badly written, it is just that the material, despite its deceptive simplicity, is hard.

In my freshmen year, Andrzej Zarach, who later became a distinguished set theorist, was finishing his doctoral dissertation and had the rather unusual idea of conducting a seminar for freshmen on set theory and Gödel's axiom of constructibility. This is an advanced topic that requires solid understanding of formal methods that cannot be expected from beginners. Zarach gave us a few lectures on axiomatic set theory, and then each of us was given an assignment for a class presentation. Mine was the Löwenheim-Skolem theorem. The Löwenheim-Skolem theorem is one of the early results in model theory. Model theory is what I list now as my research specialty. For the seminar, my job was to present the proof as given in the then recently published book *Constructible Sets with Applications* [23], by another prominent Polish logician Andrzej Mostowski. The theorem is not difficult to prove. In courses in model theory, a proof is usually given early, as it does not require much preparation. In Mostowski's book, the proof takes about one-third of the page. I was reading it and reading it, and then reading it again, and I did not understand. Not only did I not understand the idea of the proof; as far as I can recall now, I did not understand a single sentence in it. Eventually, I memorized the proof and reproduced it at the seminar in the way that clearly exposed my ignorance. It was a humiliating experience.

I brought up my early learning experiences here for just one reason: I really know what it is not to understand. I am familiar with not understanding. At the same time, I am also familiar with those extremely satisfying moments when one does finally understand. It is a very individual and private process. Sometimes moments of understanding come when small pieces eventually add up to the point when one grasps a general idea. Sometimes, it works the other way around. An understanding of a general concept can come first, and then it sheds bright light on an array of smaller related issues. There are no simple recipes for understanding. In mathematics, sometimes the only good advice is study, study, study..., but this is not what I recommend for reading this book. There are attractive areas of mathematics and its applications that cannot be fully understood without sufficient technical knowledge. It is hard, for example, to understand modern physics without a solid grasp of many areas of mathematical analysis, topology, and algebra. Here I will try to do something different. My goal is to try to explain a certain approach to the theory of mathematical structures. This material is also technical, but its nature is different. There are no prerequisites, other than some genuine curiosity about the subject. No prior mathematical experience is necessary. Somewhat paradoxically, to follow the line of thought, it may be helpful to forget some mathematics one learns in school. Everything will be built up from scratch, but this is not to say that the subject is easy.

Much of the material in this book was developed in conversations with my wife, Wanda, and friends, who are not mathematicians, but were kind and curious enough to listen to my explanations of what I do for a living. I hope it will shed some light on some areas of modern mathematics, but explaining mathematics is not the only goal. I want to present a methodological framework that potentially could be applied outside mathematics, the closest areas I can think of being architecture and visual arts. After all, everything is or has a structure.

I am very grateful to Beth Caspar, Andrew McInerney, Philip Ording, Robert Tragesser, Tony Weaver, Jim Schmerl, and Jan Zwicky who have read preliminary versions of this book and have provided invaluable advice and editorial help.

About the Content

The central topic of this book is first-order logic, the logical formalism that has brought much clarity into the study of classical mathematical number systems and is essential in the modern axiomatic approach to mathematics. There are many books that concentrate on the material leading to Gödel's famous incompleteness theorems, and on results about decidability and undecidability of formal systems. The approach in this book is different. We will see how first-order logic serves as a language in which salient features of classical mathematical structures can be described and how structures can be categorized with respect to their complexity, or lack thereof, that can be measured by the complexity of their first-order descriptions.

All kinds of geometric, combinatorial, and algebraic objects are called structures, but for us the word "structure" will have a strict meaning determined by a formal definition. Part I of the book presents a framework in which such formal definitions can be given. The exposition in this part is written for the reader for whom this material is entirely new. All necessary background is provided, sometimes in a repetitive fashion.

The role of exercises is to give the reader a chance to revisit the main ideas presented in each chapter. Newly learned concepts become meaningful only after we "internalize" them. Only then, can one question their soundness, look for alternatives, and think of examples and situations when they can be applied, and, sometimes more importantly, when they cannot. Internalizing takes time, so one has to be patient. Exercises should help. To the reader who has no prior preparation in abstract mathematics or mathematical logic, the exercises may look intimidating, but they are different, and much easier, than in a mathematics textbook. Most of them only require checking appropriate definitions and facts, and most of them have pointers and hints. The exercises that are marked by asterisks are for more advanced readers.

All instructors will have their own way of introducing the material covered in Part I. A selection from Chapters 1 through 6 can be chosen for individual reading, and exercises can be assigned based on how advanced the students in the class are. My suggestion is to not skip Chapter 2, where the idea of *logical seeing* is first introduced. That term is often used in the second part of the book. I would also recommend not to skip the development of axiomatic set theory, which is discussed in Chapter 6. It is done there rigorously but in a less technical fashion than one usually sees in textbooks on mathematical logic.

Here is a brief overview of the chapters in Part I. All chapters in both parts have more extensive introductions:

- Chapter 1 begins with a detailed discussion of a formalization of the statement "there are infinitely many prime numbers," followed by an introduction of the full syntax of first-order logic and Alfred Tarski's definition of truth.
- Chapter 2 introduces the model-theoretic concept of symmetry (automorphism) using simple finite graphs as examples. The idea of "logical seeing" is discussed.
- Short Chapter 3 is devoted to the elusive concept of natural number.
- In Chapter 4, building upon the structure of the natural numbers, a detailed formal reconstruction of the arithmetic structures of the integers (whole numbers) and the rational numbers (fractions) in terms of first-order logic is given. This chapter is important for further developments.
- Chapter 5 provides motivation for grounding the rest of the discussion in axiomatic set theory. It addresses important questions: What is a real number, and how can a continuous real line be made of points?
- Chapter 6 is a short introduction to the axioms of Zermelo-Fraenkel set theory.

Part II is more advanced. Its aim is to give a gentle introduction to model theory and to explain some classical and some recent results on the classification of first-order structures. A few detailed proofs are included. Undoubtedly, this part will be more challenging for the reader who has no prior knowledge of mathematical logic; nevertheless, it is written with such a reader in mind.

- Chapter 7 formally introduces ordered pairs, Cartesian products, relations, and first-order definability. It concludes with an example of a variety of structures on a one-element domain and an important structure with a two-element domain.
- Chapter 8 is devoted to a detailed discussion of definable elements and, in particular, definability of numbers in the field of real numbers.
- In Chapter 9, types and symmetries are defined for arbitrary structures. The concepts of minimality and order-minimality are illustrated by examples of ordering relations on sets of numbers.

- Chapter 10 introduces the concept of geometry of definable sets motivated by the example of geometry of conic sections in the ordered field of real numbers. The chapter ends with a discussion of Diophantine equations and Hilbert's 10th problem.
- In Chapter 11, it is shown how the fundamental compactness theorem is used to construct elementary extensions of structures.
- Chapter 12 is devoted to elementary extensions admitting symmetries. A proof of minimality of the ordered set of natural numbers is given.
- In Chapter 13, formal arguments are given to show why the fields of real and complex numbers are considered tame and why the field of rational numbers is wild.
- Chapter 14 includes a further discussion of first-order definability and a brief sketch of definability in higher-order logics. Well-orderings and the Mandelbrot set are used as examples.
- Chapter 15 is an extended summary of Part II. The reader who is familiar with first-order logic may want to read this chapter first. The chapter is followed by suggestions for further reading.

In Appendix A, the reader will find complete proofs of irrationality of the square root of two (Tennenbaum's proof), Cantor's theorem on non-denumerability of the set of all real numbers, first-order categoricity of structures with finite domains, existence of proper elementary extensions of structures with infinite domains, and a "nonstandard" proof of the Infinite Ramsey's theorem for partitions of sets of pairs.

Appendix B contains a brief discussion of Hilbert's program for foundations of mathematics.

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