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Jakob Schwichtenberg

Physics from Symmetry

Second Edition

Springer

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NATURE ALWAYS CREATES THE BEST OF ALL OPTIONS

ARISTOTLE

AS FAR AS I SEE, ALL A PRIORI STATEMENTS IN PHYSICS HAVE THEIR ORIGIN IN SYMMETRY.

HERMANN WEYL

THE IMPORTANT THING IN SCIENCE IS NOT SO MUCH TO OBTAIN NEW FACTS AS TO DISCOVER NEW WAYS OF THINKING ABOUT THEM.

WILLIAM LAWRENCE BRAGG

Dedicated to my parents

Preface to the Second Edition

In the two years since the first edition of this book was published I've received numerous messages from readers all around the world. I was surprised by this large number of responses and how positive most of them were. Of course, I was cautiously confident that readers would like the book. Otherwise I wouldn't have spent so many months writing it. However, there is certainly no shortage of books on group theory or on the role of symmetries in physics. To quote Predrag Cvitanovic[1]

> Almost anybody whose research requires sustained use of group theory (and it is hard to think of a physical or mathematical problem that is wholly devoid of symmetry) writes a book about it.

[1] Predrag Cvitanović. *Group Theory: Birdtracks, Lie's, and Exceptional Groups*. Princeton University Press, 7 2008. ISBN 9780691118369

Moreover, I'm not a world renowned expert. Therefore, I knew no one would buy the book because my name is written on the cover. So the chances were high that "Physics from Symmetry" would simply drown in the flood of new textbooks that are published every year.

Therefore, it's reasonable to wonder: Why and how did "Physics from Symmetry" avoid this fate?

I think the main reason for the success of the first edition is what I framed as something negative above: it wasn't written by a world renowned expert. I wrote the book while I was still a student and, as I remarked in the preface to the first edition, "I wrote the book I wished had existed when I started my journey in physics". So my motivation for writing the book wasn't to create an authoritative reference or a concise text that experts would love. Instead, my only focus was to write a book that helps students understand. As a student myself I always had still fresh in memory what I found confusing and what finally helped me understand.

This point of view is nicely summarized in the following quote by C.S. Lewis[2]

> It often happens that two schoolboys can solve difficulties in their work for one another better than the master can. When you took the problem

[2] C. S. Lewis. *Reflections on the Psalms*. HarperOne, reprint edition, 2 2017. ISBN 9780062565488

> to a master, as we all remember, he was very likely to explain what you understood already, to add a great deal of information which you didn't want, and say nothing at all about the thing that was puzzling you. [...] The fellow-pupil can help more than the master because he knows less. The difficulty we want him to explain is one he has recently met. The expert met it so long ago he has forgotten. He sees the whole subject, by now, in a different light that he cannot conceive what is really troubling the pupil; he sees a dozen other difficulties which ought to be troubling him but aren't.

While most readers liked the student-friendly spirit of the first edition, it was, of course, not perfect. Several readers pointed out typos and paragraphs with confusing notation or explanations. This feedback guided me during the preparation of this second edition. I focused on correcting typos, improving the notation and I rewrote entire sections that were causing confusion.

I hope these changes make "Physics from Symmetry" even more student-friendly and useful[3].

Karlsruhe, September 2017 *Jakob Schwichtenberg*

[3] No book is ever perfect and I'm always happy to receive feedback. So if you find an error, have an idea for improvement or simply want to comment on something, always feel free to write me at `mail@jakobschwichtenberg.com`

Preface to the First Edition

> The most incomprehensible thing about the world is that it is at all comprehensible.
>
> **- Albert Einstein**[4]

[4] As quoted in Jon Fripp, Deborah Fripp, and Michael Fripp. *Speaking of Science*. Newnes, 1st edition, 4 2000. ISBN 9781878707512

In the course of studying physics I became, like any student of physics, familiar with many fundamental equations and their solutions, but I wasn't really able to see their connection.

I was thrilled when I understood that most of them have a common origin: **symmetry**. To me, the most beautiful thing in physics is when something incomprehensible, suddenly becomes comprehensible, because of a deep explanation. That's why I fell in love with symmetries.

For example, for quite some time I couldn't really understand spin, which is some kind of curious internal angular momentum that almost all fundamental particles carry. Then I learned that spin is a direct consequence of a symmetry, called Lorentz symmetry, and everything started to make sense.

Experiences like this were the motivation for this book and in some sense, I wrote the book I wished had existed when I started my journey in physics. Symmetries are beautiful explanations for many otherwise incomprehensible physical phenomena and this book is based on the idea that we can derive the fundamental theories of physics from symmetry.

One could say that this book's approach to physics starts at the end: Before we even talk about classical mechanics or non-relativistic quantum mechanics, we will use the (as far as we know) exact symmetries of nature to derive the fundamental equations of quantum field theory. Despite its unconventional approach, this book is about standard physics. We will not talk about speculative, experimentally unverified theories. We are going to use standard assumptions and develop standard theories.

Depending on the reader's experience in physics, the book can be used in two different ways:

- It can be used as a quick primer for those who are relatively new to physics. The starting points for classical mechanics, electrodynamics, quantum mechanics, special relativity and quantum field theory are explained and after reading, the reader can decide which topics are worth studying in more detail. There are many good books that cover every topic mentioned here in greater depth and at the end of each chapter some further reading recommendations are listed. If you feel you fit into this category, you are encouraged to start with the mathematical appendices at the end of the book[5] before going any further.
- Alternatively, this book can be used to connect loose ends for more experienced students. Many things that may seem arbitrary or a little wild when learnt for the first time using the usual historical approach, can be seen as being inevitable and straightforward when studied from the symmetry point of view.

[5] Starting with Chapter A. In addition, the corresponding appendix chapters are mentioned when a new mathematical concept is used in the text.

In any case, you are encouraged to read this book from cover to cover, because the chapters build on one another.

We start with a short chapter about special relativity, which is the foundation for everything that follows. We will see that one of the most powerful constraints is that our theories must respect special relativity. The second part develops the mathematics required to utilize symmetry ideas in a physical context. Most of these mathematical tools come from a branch of mathematics called group theory. Afterwards, the Lagrangian formalism is introduced, which makes working with symmetries in a physical context straightforward. In the fifth and sixth chapters the basic equations of modern physics are derived using the two tools introduced earlier: The Lagrangian formalism and group theory. In the final part of this book these equations are put into action. Considering a particle theory we end up with quantum mechanics, considering a field theory we end up with quantum field theory. Then we look at the non-relativistic and classical limits of these theories, which leads us to classical mechanics and electrodynamics.

Every chapter begins with a brief summary of the chapter. If you catch yourself thinking: "Why exactly are we doing this?", return to the summary at the beginning of the chapter and take a look at how this specific step fits into the bigger picture of the chapter. Every page has a big margin, so you can scribble down your own notes and ideas while reading[6].

[6] On many pages I included in the margin some further information or pictures.

I hope you enjoy reading this book as much as I have enjoyed writing it.

Karlsruhe, January 2015 *Jakob Schwichtenberg*

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I want to thank everyone who helped me create this book. I am especially grateful to Fritz Waitz, whose comments, ideas and corrections have made this book so much better. I am also very indebted to Arne Becker and Daniel Hilpert for their invaluable suggestions, comments and careful proofreading. I thank Robert Sadlier for his proofreading, Jakob Karalus for his comments and Marcel Köpke and Paul Tremper for many insightful discussions.

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Finally, my greatest debt is to my parents who always supported me and taught me to value education above all else.

If you find an error in the text I would appreciate a short email to errors@jakobschwichtenberg.com. All known errors are listed at http://physicsfromsymmetry.com/errata .

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