Martin Aigner
Günter M. Ziegler

Proofs from THE BOOK

Sixth Edition

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Including Illustrations by Karl H. Hofmann

Springer

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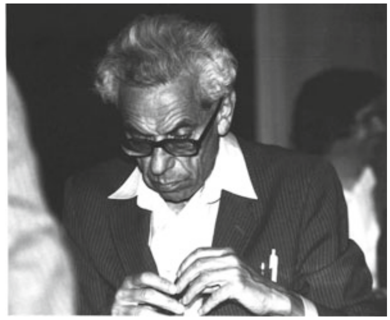
ISBN 978-3-662-57264-1 ISBN 978-3-662-57265-8 (eBook)
https://doi.org/10.1007/978-3-662-57265-8

Library of Congress Control Number: 2018940433

Printed on acid-free paper

This Springer imprint is published by the registered company Springer-Verlag GmbH, DE part of Springer Nature.
The registered company address is: Heidelberger Platz 3, 14197 Berlin, Germany

Preface



Paul Erdős

“The Book”

Paul Erdős liked to talk about The Book, in which God maintains the perfect proofs for mathematical theorems, following the dictum of G. H. Hardy that there is no permanent place for ugly mathematics. Erdős also said that you need not believe in God but, as a mathematician, you should believe in The Book. A few years ago, we suggested to him to write up a first (and very modest) approximation to The Book. He was enthusiastic about the idea and, characteristically, went to work immediately, filling page after page with his suggestions. Our book was supposed to appear in March 1998 as a present to Erdős’ 85th birthday. With Paul’s unfortunate death in the summer of 1996, he is not listed as a co-author. Instead this book is dedicated to his memory.

We have no definition or characterization of what constitutes a proof from The Book: all we offer here is the examples that we have selected, hoping that our readers will share our enthusiasm about brilliant ideas, clever insights and wonderful observations. We also hope that our readers will enjoy this despite the imperfections of our exposition. The selection is to a great extent influenced by Paul Erdős himself. A large number of the topics were suggested by him, and many of the proofs trace directly back to him, or were initiated by his supreme insight in asking the right question or in making the right conjecture. So to a large extent this book reflects the views of Paul Erdős as to what should be considered a proof from The Book.

A limiting factor for our selection of topics was that everything in this book is supposed to be accessible to readers whose backgrounds include only a modest amount of technique from undergraduate mathematics. A little linear algebra, some basic analysis and number theory, and a healthy dollop of elementary concepts and reasonings from discrete mathematics should be sufficient to understand and enjoy everything in this book.

We are extremely grateful to the many people who helped and supported us with this project — among them the students of a seminar where we discussed a preliminary version, to Benno Artmann, Stephan Brandt, Stefan Felsner, Eli Goodman, Torsten Heldmann, and Hans Mielke. We thank Margrit Barrett, Christian Bressler, Ewgenij Gawrilow, Michael Joswig, Elke Pose, and Jörg Rambau for their technical help in composing this book. We are in great debt to Tom Trotter who read the manuscript from first to last page, to Karl H. Hofmann for his wonderful drawings, and most of all to the late great Paul Erdős himself.

Berlin, March 1998 *Martin Aigner · Günter M. Ziegler*

Preface to the Sixth Edition

The idea to this project was born during some leisurely discussions at the Mathematisches Forschungsinstitut in Oberwolfach with the incomparable Paul Erdős in the mid-1990s. It is now nearly twenty years ago that we presented the first edition of our book on occasion of the International Congress of Mathematicians in Berlin 1998. At that time we could not possibly imagine the wonderful and lasting response our book about The Book would have, with all the warm letters, interesting comments and suggestions, new editions, and as of now thirteen translations. It is no exaggeration to say that it has become a part of our lives.

In addition to numerous improvements and smaller changes, many of them suggested by our readers, for the present sixth edition we wrote an entirely new chapter with Gurvits's proof of Van der Waerden's permanent conjecture, used this to derive asymptotics for the number of Latin squares, added a new, fourth proof for the Euler theorem $\sum_{n\geq 1}\frac{1}{n^2}=\pi^2/6$, and present a new geometric explanation for Heath-Brown's involution proof for the Fermat two squares theorem.

We thank everyone who helped and encouraged us over all these years. For the second edition this included Stephan Brandt, Christian Elsholtz, Jürgen Elstrodt, Daniel Grieser, Roger Heath-Brown, Lee L. Keener, Christian Lebœuf, Hanfried Lenz, Nicolas Puech, John Scholes, Bernulf Weißbach, and *many* others. The third edition benefitted especially from input by David Bevan, Anders Björner, Dietrich Braess, John Cosgrave, Hubert Kalf, Günter Pickert, Alistair Sinclair, and Herb Wilf. For the fourth edition, we were particularly indebted to Oliver Deiser, Anton Dochtermann, Michael Harbeck, Stefan Hougardy, Hendrik W. Lenstra, Günter Rote, Moritz W. Schmitt, and Carsten Schultz for their contributions. For the fifth edition, we gratefully acknowledged ideas and suggestions by Ian Agol, France Dacar, Christopher Deninger, Michael D. Hirschhorn, Franz Lemmermeyer, Raimund Seidel, Tord Sjödin, and John M. Sullivan, as well as help from Marie-Sophie Litz, Miriam Schlöter, and Jan Schneider. For the present sixth edition, very valuable hints were provided by France Dacar again, as well as by David Benko, Jan Peter Schäfermeyer, and Yuliya Semikina.

Moreover, we thank Ruth Allewelt at Springer in Heidelberg and Christoph Eyrich, Torsten Heldmann, and Elke Pose in Berlin for their continuing support throughout these years. And finally, this book would certainly not look the same without the original design suggested by Karl-Friedrich Koch, and the superb new drawings provided again and again by Karl H. Hofmann.

Berlin, March 2018 *Martin Aigner · Günter M. Ziegler*

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