

A

Fourier and Laplace Transforms

In this appendix, an interpretation of the Laplace transform is given. The explanations are given in terms of intuitive ideas and a rigorous mathematical presentation is avoided. Readers interested in the mathematical foundations are referred to previous works listed at the end of this appendix [6–11].

A.1 Fourier Series

A periodic function of time $f(t)$, with period $2p$, can be represented by [1–3]:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi}{p}t\right) + b_n \sin\left(\frac{n\pi}{p}t\right) \right], \quad (\text{A.1})$$

where a_0, a_n, b_n , are known as the *Fourier coefficients* and can be computed as:

$$\begin{aligned} a_0 &= \frac{1}{p} \int_0^{2p} f(t) dt, \\ a_n &= \frac{1}{p} \int_0^{2p} f(t) \cos\left(\frac{n\pi}{p}t\right) dt, \\ b_n &= \frac{1}{p} \int_0^{2p} f(t) \sin\left(\frac{n\pi}{p}t\right) dt. \end{aligned} \quad (\text{A.2})$$

The expressions in (A.2) are known as the *Euler formulas* and the series in (A.1) is known as the *Fourier series* of $f(t)$. Note that, according to (A.1), a periodic function can be represented as the addition of many sine and cosine functions of different *discrete frequencies*, i.e., determined by the integer n , that go from 0 to

$+\infty$. These functions are also called the frequency components of $f(t)$. Fourier coefficients represent the contribution to function $f(t)$ of the different frequency components. An important observation is that (A.1) assumes that $f(t)$ is periodic for all real t . However, not every periodic function can be represented by a Fourier series. The conditions that determine that a periodic function can be represented by a Fourier series are known as the *Dirichlet conditions*:

Theorem A.1 ([1]) *If $f(t)$ is a bounded periodic function with a finite number of maxima and minima in addition to a finite number of discontinuity points in any period, then the Fourier series in (A.1) converges to $f(t)$ at all points where $f(t)$ is continuous and it converges to the average of the left and right limits of $f(t)$ at each point where $f(t)$ is discontinuous [3].*

Definition A.1 ([3]) A function $f(t)$ is said to be an *even function* if:

$$f(-t) = f(t), \quad \forall t \in \mathcal{R},$$

and it is said to be an *odd function* if:

$$f(-t) = -f(t), \quad \forall t \in \mathcal{R}.$$

As the cosine function is even and the sine function is odd, under certain symmetry conditions in $f(t)$ the Fourier series may be composed only of sine or only of cosine functions, as stated in the following.

Theorem A.2 ([1]) *If $f(t)$ is an even function, the corresponding Fourier coefficients are given as:*

$$\begin{aligned} a_0 &= \frac{2}{p} \int_0^p f(t) dt, \\ a_n &= \frac{2}{p} \int_0^p f(t) \cos\left(\frac{n\pi}{p}t\right) dt, \quad n = 1, 2, 3, \dots \\ b_n &= 0, \quad n = 1, 2, 3, \dots \end{aligned}$$

Theorem A.3 ([1]) *If $f(t)$ is an odd function, the corresponding Fourier coefficients are given as:*

$$\begin{aligned} a_n &= 0, \quad n = 0, 1, 2, 3, \dots \\ b_n &= \frac{2}{p} \int_0^p f(t) \sin\left(\frac{n\pi}{p}t\right) dt, \quad n = 1, 2, 3, \dots \end{aligned}$$

On the other hand, any periodic function $f(t)$ can be expressed as the addition of one even function and one odd function [3]. Hence, according to Theorems A.2 and A.3, the first function contributes to a Fourier series composed only of cosine

functions and the second function contributes to a Fourier series composed only of sine functions. This explains, to some extent, why the general Fourier series contains both cosine and sine functions.

Fourier series (A.1) can be written in several equivalent ways, for instance [1]:

1.

$$f(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{p}t - \gamma_n\right),$$

$$A_0 = \frac{a_0}{2}, \quad A_n = \sqrt{a_n^2 + b_n^2}, \quad \gamma_n = \arctan\left(\frac{b_n}{a_n}\right), \quad (\text{A.3})$$

where a_0, a_n, b_n are defined in (A.2).

2.

$$f(t) = A_0 + \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{p}t + \delta_n\right),$$

$$\delta_n = \arctan\left(\frac{a_n}{b_n}\right), \quad (\text{A.4})$$

where A_0 and A_n are defined in (A.3).

Note that the phases γ_n and δ_n are different for each one of the terms of the series and, because of this, (A.3) and (A.4) can represent any periodic function $f(t)$, no matter whether it is even or odd.

3.

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{j(n\pi/p)t}, \quad (\text{A.5})$$

$$C_n = \frac{1}{2p} \int_0^{2p} f(t) e^{-j(n\pi/p)t} dt,$$

$$e^{\pm j(n\pi/p)t} = \cos\left(\frac{n\pi}{p}t\right) \pm j \sin\left(\frac{n\pi}{p}t\right).$$

According to (A.5), $f(t)$ is composed of a real part and an imaginary part. However, if $f(t)$ is a real function, this implies that its imaginary part is zero; hence, all imaginary terms in (A.5) must cancel among them. This does not mean that (A.5) only contains cosine functions because C_n is also a complex number and this implies that $f(t)$ is composed of cosine and sine functions. Note that, according to (A.5), the Fourier series of $f(t)$ is composed of functions whose frequencies take discrete values from $-\infty$ to $+\infty$ passing by zero. It is stressed that negative frequencies have no physical interpretation and, because

of that, (A.5) merely represents an abstract way of representing $f(t)$, which, however, is a convenient way of writing a *Fourier series* to define the *Fourier transform* in the following.

A.2 Fourier Transform

The *Fourier integral* of a nonperiodic function $f(t)$, is defined as [1, 3, 4]:

$$f(t) = \int_{-\infty}^{\infty} F(\omega)e^{j\omega t} d\omega, \quad (\text{A.6})$$

$$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t)e^{-j\omega t} dt. \quad (\text{A.7})$$

$F(\omega)$, a function of the *continuous frequency* ω , is known as the *Fourier transform* of $f(t)$ and both of the previous expressions are known as the *Fourier transform pair*. On the other hand, it is clear that $F(\omega)$ is a complex function. It is possible to show that, when $f(t)$ is a real function, the real part of $F(\omega)$ is an even function and its imaginary part is an odd function [3].

In [1, 3, 4], it is shown that (A.5) is the discrete version of (A.6) and (A.7). A simple way of seeing this is presented in the following. In (A.6) and (A.7), the function $f(t)$ is nonperiodic. Thus, $f(t)$ can be seen as a function with period $2p \rightarrow \infty$. Define:

$$\omega = \frac{n\pi}{p}.$$

If $f(t)$ “tends” toward a periodic function, then the period $2p$ takes finite values and ω , originally a continuous variable, “tends” toward the discrete variable $\frac{n\pi}{p}$, taking values from $-\infty$ to $+\infty$. Hence, $d\omega$ stands for the difference between two consecutive values of ω that correspond to two consecutive values of n represented by $n + 1$ and n :

$$d\omega = \frac{(n + 1)\pi}{p} - \frac{n\pi}{p} = \frac{\pi}{p}.$$

On the other hand, C_n is the discrete version of $F(\omega)$. This can be explained as follows. If $f(t)$ is a nonperiodic function, it can be seen as a function with period $2p$ that “tends” toward ∞ and, although $f(t)$ is assumed to be defined on the open interval $t \in (-\infty, +\infty)$, then the limits of the integral, which is computed on a complete period of $f(t)$, change from 0 and $2p$ to $-\infty$ and $+\infty$. Once (A.6) and (A.7) are rendered discrete in this manner, (A.5) and C_n are exactly retrieved when replacing (A.7) in (A.6).

From (A.5) and (A.6), the following observations are in order [1–4]. A periodic function $f(t)$ can be represented as an infinite series of sine and cosine functions with different discrete frequencies $\frac{n\pi}{p}$ and amplitudes C_n , which are functions of discrete frequency. On the other hand, a nonperiodic function $f(t)$ is composed of a large number of sinusoidal functions with a frequency that takes continuous values ω and with amplitudes $F(\omega)$ that are functions of the continuous frequency ω . C_n stands for the contribution, to a periodic function $f(t)$, of the sinusoidal functions of a specific discrete frequency $\frac{n\pi}{p}$, whereas $F(\omega)$ stands for the contribution, to a nonperiodic function $f(t)$, of the sinusoidal functions of a specific continuous frequency ω . The conditions for convergence of (A.6) to $f(t)$ and the existence of (A.7) are the following [1–3]:

1. $f(t)$ satisfies the Dirichlet conditions.
2. The integral $\int_{-\infty}^{\infty} |f(t)| dt$ exists.

Note that Dirichlet conditions are inherited from Fourier series, whereas the existence of the integral $\int_{-\infty}^{\infty} |f(t)| dt$ is a clear condition for the convergence of the integral in (A.7).

A.3 Laplace Transform

A nonperiodic function $f(t)$ that is zero for $t < 0$, can be represented as [1–4]:

$$f(t) = \frac{1}{2\pi j} \int_{a-j\infty}^{a+j\infty} F(s)e^{st} ds, \quad (\text{A.8})$$

$$F(s) = \int_0^{\infty} f(t)e^{-st} dt, \quad s = a + j\omega, \quad (\text{A.9})$$

where a and ω are real numbers. $F(s)$ is known as the *Laplace transform* of $f(t)$, whereas (A.8) and (A.9) are known as the *Laplace transform pair*. Note that (A.6) and (A.7) are similar to (A.8) and (A.9). A careful observation leads us to conclude that the former pair of expressions are obtained from the latter by assuming $a = 0$ and $f(t) = 0$ for all $t < 0$ [5], i.e., the bottom limit of the integral in (A.7) changes from $-\infty$ to 0. Conversely, (A.8) and (A.9) can be obtained from (A.6) and (A.7) [1, 3, 4] by searching for a method allowing $F(\omega)$ to be computed for some special functions $f(t)$ that are zero for negative time and such that the integral:

$$\int_0^{\infty} f(t)e^{-i\omega t} dt,$$

does not converge. This means that $f(t)$ does not become zero as $t \rightarrow \infty$; hence, the condition 2 above for the existence of $F(\omega)$ does not stand. The fundamental idea [1, 3, 4] is to compute (A.6) and (A.7) for a function:

$$g(t) = f(t)e^{-at}, \quad a > 0,$$

trying to accomplish that $g(t) \rightarrow 0$ as $t \rightarrow \infty$; hence, forcing (A.6) and (A.7) to converge. Then, the following limits are computed:

$$f(t) = \lim_{a \rightarrow 0} g(t), \quad (\text{A.10})$$

$$F(\omega) = \lim_{a \rightarrow 0} \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t)e^{-at} e^{-i\omega t} dt, \quad (\text{A.11})$$

finally obtaining the Fourier transform pair corresponding to $f(t)$. The parameter a is known as the *convergence factor* because, when introduced within a divergent integral, it leads the integrand to decrease at a rate high enough to ensure convergence of the integral.

Once this result has been found, it is natural to try to choose the convergence factor a in a more general manner, such that convergence of the involved integrals is the only thing that matters without the necessity of computing the limit of the integrals when $a \rightarrow 0$. Thus, the Laplace transform pair (A.8), (A.9), can be obtained by replacing $g(t) = f(t)e^{-at}$ instead of $f(t)$ in (A.6), (A.7), such that s is a complex variable with ω as imaginary part, which is identical to the Fourier variable, and a as a constant real part which can be positive, negative or zero. According to the above discussion, the conditions for the existence of (A.8), (A.9), can be summarized as follows [1, 2]:

1. In every finite time interval $f(t)$ is bounded and has, at most, a finite number of maxima and minima and a finite number of discontinuities.
2. There exists a real constant a such that the following integral is convergent:

$$\int_0^{\infty} |f(t)e^{-at}| dt = \int_0^{\infty} |f(t)|e^{-at} dt. \quad (\text{A.12})$$

Functions satisfying 1 are called *sectionally regular functions*. Condition 2 is commonly replaced by another more restrictive one [1, 2]:

3. There exists a constant α such that $|f(t)|e^{-\alpha t}$ remains bounded as $t \rightarrow \infty$, i.e., real finite constants α , M , T exist, such that:

$$|f(t)|e^{-\alpha t} < M, \quad \text{for all } t > T. \quad (\text{A.13})$$

Functions satisfying 3 are called *functions of exponential order*. Note that the value of α required by this condition is not unique and defines a set of values. The smallest value in this set, denoted by α_0 , is known as the *abscissa of convergence* of $f(t)$ [1]. According to condition 3, $f(t)$ can be a function that does not converge to zero when $t \rightarrow \infty$ or it can even remain growing, but its grow rate is not larger than that of an exponential function, such that there exists some α ensuring that $|f(t)e^{-\alpha t}|$ stops growing as $t \rightarrow \infty$. Hence, it is possible to select some constant

$\alpha_1 > \alpha$ such that $e^{-\alpha_1 t}$ decreases faster than the increment of $f(t)$; thus, it is ensured that:

$$\int_0^{\infty} |f(t)e^{-\alpha_1 t}| dt, \quad (\text{A.14})$$

converges [1, 2, 4]. These ideas are now formalized.

Theorem A.4 ([1]) *If $f(t)$ is sectionally regular and of exponential order with the abscissa of convergence α_0 , then for any s_0 with $\text{Re}(s_0) > \alpha_0$, the integral:*

$$F(s) = \int_0^{\infty} f(t)e^{-st} dt, \quad (\text{A.15})$$

uniformly converges for all values of s such that $\text{Re}(s) > \text{Re}(s_0)$.

According to this, (A.8), (A.9), can be written as:

$$f(t) = \frac{1}{2\pi j} \int_{a-j\infty}^{a+j\infty} F(s)e^{\text{Re}(s)t} e^{j\omega t} ds, \quad (\text{A.16})$$

$$F(s) = \int_0^{\infty} f(t)e^{-at} e^{-j\omega t} dt, \quad (\text{A.17})$$

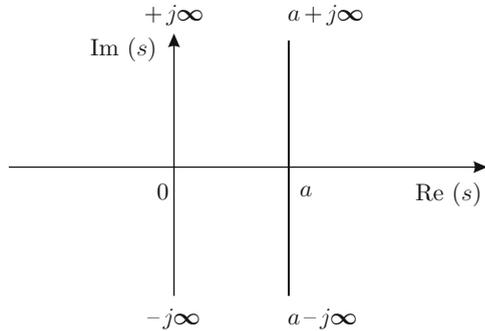
where $a > \text{Re}(s_0)$. From these expressions, a similar reasoning to that for (A.6), (A.7), can be performed to find the following interpretation for (A.16), (A.17):

The Laplace transform pair (A.8), (A.9), a nonperiodic function $f(t)$ to be represented, which is zero for $t < 0$, as the addition of many sine and cosine functions with different frequencies ω that change continuously from $-\infty$ to $+\infty$, passing by zero, and all of these functions are identically “damped” by the exponential factor e^{at} . Hence, $F(s)$ represents the contribution to the function $f(t)$ of each one of these so-called functions of the complex frequency $s = a + j\omega$. Moreover, the factor e^{at} can have any value for a such that $a > \text{Re}(s_0)$. Thus, using a different value for a , it is possible to change the “damping” rate of all the sinusoidal functions comprising $f(t)$.

It is important to say that, although a was introduced at the beginning as a positive number, if $f(t)$ is a decreasing function of exponential order, then there exist some negative values for a such that $a > \text{Re}(s_0)$ and, hence, (A.8), (A.9), are still convergent. This is the reason why the word “damped”, introduced in the previous paragraph, has been written between quotation marks. Finally, it is stressed that, according to the previous discussion, (A.9) can be used to compute:

$$F(\sigma_0) = F(s)|_{s=\sigma_0} = \int_0^{\infty} f(t)e^{-\sigma_0 t} dt, \quad (\text{A.18})$$

Fig. A.1 Plane s showing the straight lines used to compute the Fourier transform, from $-j\infty$ to $+j\infty$ (imaginary axis), and the Laplace transform, from $a - j\infty$ to $a + j\infty$



which is ensured to exist for any constant complex number σ_0 such that $\text{Re}(\sigma_0) > \text{Re}(s_0)$.

The complex plane s is shown in Fig. A.1. Note that, according to (A.6), $f(t)$ can be computed as a line integral along a vertical straight line that overlaps the imaginary axis ($\text{Re}(s) = 0$), whereas, according to (A.8), $f(t)$ can be computed as a line integral along a vertical straight line that is parallel to the imaginary axis ($\text{Re}(s) = a$).

B

Bode Diagrams

In this appendix, some details are provided on how to draw Bode diagrams of some basic terms. Instrumental for these results is the use of the logarithm properties listed in (6.10), Chap. 6.

B.1 First-Order Terms

B.1.1 A Differentiator

Consider the following transfer function:

$$\frac{Y(s)}{U(s)} = G(s) = s.$$

Let $s = j\omega$ find:

$$G(j\omega) = j\omega = \omega \angle 90^\circ.$$

Hence:

$$|G(j\omega)|_{dB} = 20 \log(\omega), \quad \angle G(j\omega) = 90^\circ.$$

Consider two arbitrary frequencies with a difference of one decade between them, i.e., $\omega_2 = 10\omega_1$, and evaluate:

$$\begin{aligned} |G(j\omega_1)|_{dB} &= 20 \log(\omega_1), \\ |G(j\omega_2)|_{dB} &= 20 \log(\omega_2), \\ &= 20 \log(10\omega_1), \\ &= 20 \log(10) + 20 \log(\omega_1), \\ |G(j\omega_2)|_{dB} &= 20 + 20 \log(\omega_1). \end{aligned}$$

This means that the magnitude increases 20[dB] each time the frequency increases one decade, i.e., the magnitude Bode diagram is a straight line with a slope of +20[dB/dec]. Moreover, $|G(j\omega)|_{dB} = 0$ if $\omega = 1$. This and the constant phase $\angle G(j\omega) = 90^\circ$ explain the Bode diagrams in Fig. 6.16b.

B.1.2 An Integrator

Consider the following transfer function:

$$\frac{Y(s)}{U(s)} = G(s) = \frac{1}{s}.$$

Let $s = j\omega$ to write:

$$G(j\omega) = \frac{1}{j\omega} = \frac{1}{\omega} \angle -90^\circ.$$

Hence:

$$|G(j\omega)|_{dB} = -20 \log(\omega), \quad \angle G(j\omega) = -90^\circ. \quad (\text{B.1})$$

Consider two arbitrary frequencies with a difference of a decade between them, i.e., $\omega_2 = 10\omega_1$, and evaluate:

$$\begin{aligned} |G(j\omega_1)|_{dB} &= -20 \log(\omega_1), \\ |G(j\omega_2)|_{dB} &= -20 \log(\omega_2), \\ &= -20 \log(10\omega_1), \\ &= -20 \log(10) - 20 \log(\omega_1), \\ |G(j\omega_2)|_{dB} &= -20 - 20 \log(\omega_1). \end{aligned}$$

This means that the magnitude decreases $-20[\text{dB}]$ each time the frequency increases one decade, i.e., the magnitude Bode diagram is a straight line with a slope of $-20[\text{dB/dec}]$. Moreover, $|G(j\omega)|_{\text{dB}} = 0$ if $\omega = 1$. This and the constant phase $\angle G(j\omega) = -90^\circ$ explain the Bode diagrams in Fig. 6.16a.

Following this order of ideas, it is straightforward to verify that, in the case of the transfer function:

$$\frac{Y(s)}{U(s)} = G(s) = \frac{1}{s^n}, \quad (\text{B.2})$$

for any integer $n \geq 1$ the magnitude Bode plot is a straight line with a slope of $-20n[\text{dB/dec}]$ crossing the $0[\text{dB}]$ line at $\omega = 1$. The phase is constant at $-n90^\circ$.

B.1.3 A First-Order Pole

Consider the following transfer function:

$$\frac{Y(s)}{U(s)} = G(s) = \frac{a}{s + a}, \quad a > 0.$$

Let $s = j\omega$ to obtain:

$$G(j\omega) = \frac{a}{j\omega + a}. \quad (\text{B.3})$$

According to Sect. 6.1.1, this yields:

$$|G(j\omega)| = \frac{a}{\sqrt{\omega^2 + a^2}}, \quad \angle G(j\omega) = \arctan\left(\frac{-\omega}{a}\right). \quad (\text{B.4})$$

If $\omega \ll a$, then:

$$|G(j\omega)|_{\text{dB}} \approx 0, \quad \angle G(j\omega) \approx 0^\circ.$$

If $\omega \gg a$, then:

$$\begin{aligned} |G(j\omega)| &\approx \frac{a}{\omega}, \\ |G(j\omega)|_{\text{dB}} &= 20 \log\left(\frac{a}{\omega}\right), \\ &= 20 \log(a) - 20 \log(\omega), \\ \angle G(j\omega) &\approx -90^\circ. \end{aligned} \quad (\text{B.5})$$

According to (B.1), (B.5) and recalling that a is a positive constant, when $\omega \gg a$ the magnitude Bode diagram of (B.4) is a straight line with a slope of -20 [dB/dec], which crosses the 0[dB] line, i.e., when $20 \log(a) - 20 \log(\omega) = 0$, at $\omega = a$. Moreover, when $\omega = a$:

$$|G(j\omega)| = \frac{1}{\sqrt{2}} \Rightarrow |G(j\omega)|_{\text{dB}} = -3,$$

$$\angle G(j\omega) = -45^\circ.$$

This explains the Bode diagrams in Fig. 6.16c.

B.1.4 A First-Order Zero

Consider the following transfer function:

$$\frac{Y(s)}{U(s)} = G(s) = \frac{s + a}{a}.$$

Let $s = j\omega$ to obtain:

$$G(j\omega) = \frac{j\omega + a}{a}.$$

Notice that this function is the inverse of (B.3) and recall that $\log\left(\frac{1}{x}\right) = -\log(x)$, $\angle z = -\angle\left(\frac{1}{z}\right)$, where z is a complex number. Hence, it is straightforward to verify that the Bode diagrams in this case are the same as those obtained in Sect. B.1.3 but with an opposite sign. This explains the Bode diagrams in Fig. 6.17a.

B.2 A Second-Order Transfer Function

Consider the following transfer function:

$$\frac{Y(s)}{U(s)} = G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}, \quad \omega_n > 0, \quad \zeta \geq 0. \quad (\text{B.6})$$

According to Sect. 6.1.3, we have:

$$|G(j\omega)| = \frac{\omega_n^2}{\sqrt{(\omega_n^2 - \omega^2)^2 + 4(\zeta\omega_n\omega)^2}}, \quad (\text{B.7})$$

$$\angle G(j\omega) = \arctan\left(\frac{-2\zeta\omega_n\omega}{\omega_n^2 - \omega^2}\right).$$

If $\omega \ll \omega_n$, then:

$$|G(j\omega)|_{\text{dB}} \approx 0, \quad \angle G(j\omega) = 0^\circ.$$

If $\omega \gg \omega_n$, then:

$$\begin{aligned} |G(j\omega)| &\approx \frac{\omega_n^2}{\omega^2}, & (\text{B.8}) \\ |G(j\omega)|_{\text{dB}} &= 40 \log\left(\frac{\omega_n}{\omega}\right), \\ &= 40 \log(\omega_n) - 40 \log(\omega), \\ \angle G(j\omega) &= -180^\circ. \end{aligned}$$

According to what was stated at (B.2), when $\omega \gg \omega_n$ the magnitude Bode diagram of (B.8), i.e., (B.7), is a straight line with a slope of -40 [dB/dec], which crosses the 0 [dB] line, i.e., when $40 \log(\omega_n) - 40 \log(\omega) = 0$, at $\omega = \omega_n$. Moreover, when $\omega = \omega_n$:

$$\begin{aligned} |G(j\omega)| &= \frac{1}{2\zeta} \Rightarrow |G(j\omega)|_{\text{dB}} \rightarrow \infty, \text{ as } \zeta \rightarrow 0. \\ \angle G(j\omega) &= -90^\circ. \end{aligned}$$

This explains the Bode diagrams in Fig. 6.17b.

B.3 A Second-Order Zero

Consider the following transfer function:

$$\frac{Y(s)}{U(s)} = G(s) = \frac{s^2 + 2\zeta\omega_n s + \omega_n^2}{\omega_n^2}. \quad (\text{B.9})$$

Note that this transfer function is the inverse of (B.6) and recall that $\log\left(\frac{1}{x}\right) = -\log(x)$, $\angle z = -\angle\left(\frac{1}{z}\right)$, where z is a complex number. Thus, it is straight forward to show that Bode diagrams of (B.9) are identical to Bode diagrams of (B.6) and we only have to consider that the sign changes in both the magnitude and the phase plots. This explains the Bode diagrams in Fig. 6.17c.

C

Decibels, dB

The bel, ([B]), is a unit of measure for the ratio between two quantities with unities of power. The number of bels is computed as:

$$\log\left(\frac{P_1}{P_0}\right) \text{ [B]}.$$

1 bel corresponds to a ratio 10 : 1 of power, i.e., if $P_1 = 10P_0$ then:

$$1 \text{ [B]} = \log\left(\frac{P_1}{P_0}\right) = \log(10).$$

The decibel, [dB], is defined as:

$$1 \text{ [dB]} = 0.1 \text{ [B]}.$$

Hence, if $x = \log\left(\frac{P_1}{P_0}\right)$ is in Bels and we want to convert to decibels, we have to write:

$$y = 10 \frac{\text{[dB]}}{\text{[B]}} x = 10 \log\left(\frac{P_1}{P_0}\right) \text{ [dB]},$$

where y is given in decibels. On the other hand, the power is often given as the square of signals such as voltage, electric current, etc. (also called field quantities). Thus, it is important to express the ratio of power in decibels but in terms of field quantities. In such a case:

$$y = 10 \log\left(\frac{P_1}{P_0}\right) \text{ [dB]},$$

$$\begin{aligned}
 &= 10 \log \left(\frac{F_1^2}{F_0^2} \right) [\text{dB}], \\
 &= 20 \log \left(\frac{F_1}{F_0} \right) [\text{dB}].
 \end{aligned}
 \tag{C.1}$$

The definition in (C.1) represents the expression for the number of decibels used throughout this book, i.e., for decibel units of the magnitude in frequency response of control systems.

In acoustics, decibels are used to represent an unit of pressure level:

$$20 \log \left(\frac{p}{p_0} \right) [\text{dB}]. \tag{C.2}$$

where $p_0 = 20$ [micropascals] is the threshold of perception of an average human. The pressure, p , is a field quantity and this explains (C.2).

In radiofrequency circuits, the symbol dBm is used to express the power ratio in decibels referred to as 1[milliwatt]. This means that given P in milliwatts, then:

$$10 \log \left(\frac{P}{1\text{mW}} \right) [\text{dBm}].$$

is expressed in dBm.

D

Magnetically Coupled Coils

This appendix follows the ideas presented in [12] and [13].

D.1 Invertance

When electric current through a coil induces a voltage in other coils, it is said that all these coils are magnetically coupled. In electric diagrams, an arc is depicted between coils to indicate that they are magnetically coupled, see Fig. D.1. Under these conditions, the voltage in any of these coils depends on the electric current through all the coils, i.e., according to *Faraday's Law*:

$$v_k = \frac{d\Psi_k}{dt}.$$

where v_k stands for voltage at terminals of the k -th coil and Ψ is the magnetic flux linkage in the k -th coil:

$$\Psi_k = N_k \psi_k(i_1, \dots, i_n),$$

with N_k the number of turns of the coil k and ψ_k is the flux linkage per turn of the coil k , which is a function of the electric currents through the n magnetically coupled coils with $1 \leq k \leq n$. Applying the chain rule:

$$v_k = N_k \frac{\partial \psi_k}{\partial i_1} \frac{di_1}{dt} + \dots + N_k \frac{\partial \psi_k}{\partial i_k} \frac{di_k}{dt} + \dots + N_k \frac{\partial \psi_k}{\partial i_n} \frac{di_n}{dt}. \quad (\text{D.1})$$

The *self-inductance* of coil k is defined as:

$$L_{kk} = N_k \frac{\partial \psi_k}{\partial i_k}, \quad (\text{D.2})$$

whereas:

$$L_{kj} = N_k \frac{\partial \psi_k}{\partial i_j}, \quad (\text{D.3})$$

is defined as the *mutual inductance* between the coils k and j , $1 \leq k \leq n$, $1 \leq j \leq n$, $k \neq j$. Moreover, $L_{kj} = L_{jk}$. It is not difficult to see, from (D.1), (D.2), (D.3), that it is possible to write:

$$V = M \frac{dI}{dt},$$

$$V = [v_1, v_2, \dots, v_k, \dots, v_n]^T, \quad \frac{dI}{dt} = \left[\frac{di_1}{dt}, \frac{di_2}{dt}, \dots, \frac{di_k}{dt}, \dots, \frac{di_n}{dt} \right]^T,$$

$$M = \begin{bmatrix} L_{11} & L_{12} & \cdots & L_{1k} & \cdots & L_{1n} \\ L_{21} & L_{22} & \cdots & L_{2k} & \cdots & L_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ L_{k1} & L_{k2} & \cdots & L_{kk} & \cdots & L_{kn} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ L_{n1} & L_{n2} & \cdots & L_{nk} & \cdots & L_{nn} \end{bmatrix}.$$

Hence, according to *Cramer's rule* [1], chapter 10, if M is a nonsingular matrix:

$$\frac{di_k}{dt} = \frac{\det(M_k)}{\det(M)},$$

where M_k is the matrix obtained when replacing the column k of matrix M by V , i.e.,:

$$M_k = \begin{bmatrix} L_{11} & L_{12} & \cdots & v_1 & \cdots & L_{1n} \\ L_{21} & L_{22} & \cdots & v_2 & \cdots & L_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ L_{k1} & L_{k2} & \cdots & v_k & \cdots & L_{kn} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ L_{n1} & L_{n2} & \cdots & v_n & \cdots & L_{nn} \end{bmatrix}.$$

Solving $\det(M_k)$ through the k -th column, it is possible to write:

$$\begin{aligned} \frac{di_k}{dt} &= \frac{v_1 \operatorname{cof}(M_{1k}) + v_2 \operatorname{cof}(M_{2k}) + \cdots + v_k \operatorname{cof}(M_{kk}) + \cdots + v_n \operatorname{cof}(M_{nk})}{\det(M)}, \\ &= \sum_{j=1}^{j=n} \frac{\operatorname{cof}(M_{jk})}{\det(M)} v_j, \end{aligned}$$

where $\operatorname{cof}(M_{jk})$ stands for the cofactor of the entry (j, k) of matrix M and the constant:

$$\Gamma_{kj} = \frac{\operatorname{cof}(M_{jk})}{\det(M)}, \quad (\text{D.4})$$

relating the electric current in the k -th coil to voltage in the j -th coil, is known as the *mutual inductance* between those coils. This expression indicates that inductances are functions of self-inductances and mutual inductances of the whole system for magnetically coupled coils, i.e.,:

$$\Gamma_{kj} \neq \frac{1}{L_{kj}}, \quad \text{and} \quad \Gamma_{kk} \neq \frac{1}{L_{kk}}.$$

Moreover, $\Gamma_{kj} = \Gamma_{jk}$.

D.2 Coil Polarity Marks

Consider two magnetically coupled coils as depicted in Fig. D.1a. Define the electric current sense in coil L_1 and put a polarity mark “*” at the terminal of the coil where the electric current enters the coil. A magnetic flux is produced in the core whose sense is defined according to the right-hand rule.¹ According to *Lenz Law*, a voltage is induced at the coil L_2 with a polarity such that, if the terminals of L_2 are put in short circuit, an electric current flows through L_2 producing a magnetic flux that opposes the flux produced by L_1 . Use again the right-hand rule to determine the sense of electric current through L_2 producing this effect. Put a polarity mark “*” at the terminal of L_2 where the electric current leaves the coil L_2 . This ensures that the instantaneous voltages at L_1 and L_2 , only due to effect of mutual induction, have the same polarity at the terminals with the polarity marks “*”. The mutual inductance is negative, $L_{12} = L_{21} < 0$, for coils such as those shown in Fig. D.1a.

In the case of Fig. D.1b, the senses of electric currents through L_1 and L_2 are defined such that the magnetic fluxes produced by them in the core are in the

¹*Right-hand rule*: if the right-hand fingers indicate the sense of electric current in a coil, then the thumb indicates the sense of the generated magnetic flux.

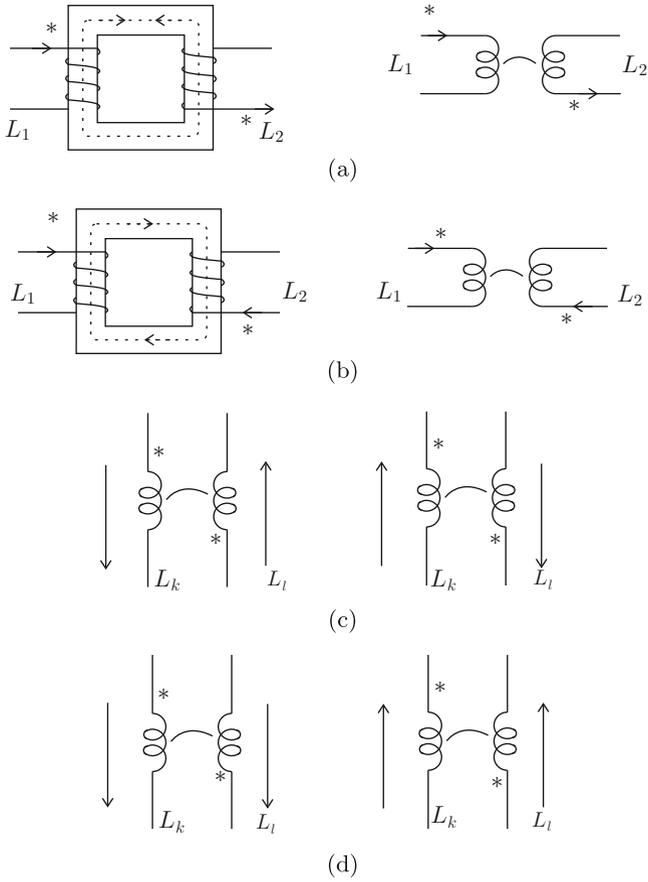


Fig. D.1 Magnetically coupled coils . (a) $L_{12} = L_{21} < 0$. (b) $L_{12} = L_{21} > 0$. (c) $L_{kl} = L_{lk} > 0$. (d) $L_{kl} = L_{lk} < 0$

same direction. In this case $L_{12} = L_{21} > 0$. Signs of the mutual inductances are determined by (D.4).

E

Euler–Lagrange Equations Subject to Constraints

Euler–Lagrange equations are used to model, among others, mechanical systems composed of several bodies that interact. Let n represent the number of degrees-of-freedom (or number of independent movements of the bodies) of the mechanical system. The Euler–Lagrange equations describing this mechanical system are written in vector form as [14]:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = \tau + \tau_Q, \tag{E.1}$$

where the Lagrangian of the system is defined as $L = K - P$ with K and P representing the total kinetic and total potential energies of the system respectively, $\tau = [\tau_1, \dots, \tau_n]^T \in \mathcal{R}^n$ is the vector of (external) generalized forces applied to the system, $\tau_Q = [\tau_{Q1}, \dots, \tau_{Qn}]^T \in \mathcal{R}^n$ is the vector of nonconservative forces, i.e., friction, applied to the system, and $q = [q_1, \dots, q_n]^T \in \mathcal{R}^n$ is the system vector of generalized coordinates, i.e., the positions defining the n degrees-of-freedom. This means that the expression in (E.1) represents n differential equations to be solved simultaneously. See Sects. 15.1 and 16.2 for some application examples of (E.1).

Some mechanical systems are subject to one linear constraint in velocities, i.e., an expression given as:

$$a_1(q)\dot{q}_1 + a_2(q)\dot{q}_2 + \dots + a_n(q)\dot{q}_n = 0,$$

which can be written in vector form as:

$$A(q)\dot{q} = 0, \quad A(q) = [a_1(q), a_2(q), \dots, a_n(q)]. \tag{E.2}$$

To take into account the constraint in (E.2) when solving (E.1), the following procedure is suggested in [15]. It is assumed that the constraint includes additional generalized forces, $\tau_c = [\tau_{c1}, \dots, \tau_{cn}]^T \in \mathcal{R}^n$, which are intended to render the

constraint always satisfied. Thus, (E.1) becomes:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = \tau + \tau_Q + \tau_c. \quad (\text{E.3})$$

The generalized forces τ_c due to the constraint are assumed not to produce any mechanical work, i.e., a basic requirement when the Euler–Lagrange equations are obtained [15]. This means that, under the effect of τ_c , the mechanical system displacement must be performed in an orthogonal direction to τ_c , i.e., $\dot{q}^T \tau_c = 0$. Then it is possible to write:

$$\dot{q}^T \tau_c = \tau_c^T \dot{q} = 0.$$

This expression and (E.2) suggest that τ_c must be parallel to $A^T(q)$, i.e., it is possible to write:

$$\tau_c = \lambda A^T(q),$$

where λ is a scalar that has to be computed. The latter expression can be used to write (E.3) as:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = \tau + \tau_Q + \lambda A^T(q). \quad (\text{E.4})$$

The unknown λ can be computed by suitable manipulation of the n equations in (E.4). See Sect. 14.1.1 for an example of application.

F

Numerical Implementation of Controllers

F.1 Numerical Computation of an Integral

This mathematical operation is represented as:

$$y = \int_0^t w(s)ds,$$

and can be approximated numerically as:

$$y(k) = y(k - 1) + w(k)\Delta t, \tag{F.1}$$

where $y(k)$ is the integral computed at the present sampling time with $y(0) = 0$ and $y(k - 1)$ is the integral computed at the previous time in the past. The expression in (F.1) is obtained from the geometric interpretation of the integral of w , i.e., the area below the curve defined by w . Hence, the term $w(k)\Delta t$ stands for the area increment at each sampling time: $w(k)$ and Δt represent the height and base respectively of a rectangle that approximates such an area increment. To obtain a good approximation, Δt must be *small* compared with the control system's response time.

F.2 Numerical Differentiation

This operation allows us to compute an estimate of velocity for controllers in Sects. 11.2.1, 11.3.2 and 11.4. Consider:

$$y = \frac{dw}{dt}. \quad (\text{F.2})$$

This operation can be approximated numerically as:

$$y \approx \frac{w(k) - w(k-1)}{\Delta t}, \quad (\text{F.3})$$

where $w(k)$ stands for w at the present sampling time, $w(k-1)$ is w at the previous sampling time and Δt is the sampling period. To obtain a good approximation, Δt must be *small* compared with the control system response time.

F.3 Lead Compensator

This controller is defined by means of its transfer function:

$$\frac{I^*(s)}{E(s)} = \gamma \frac{s+d}{s+c}, \quad (\text{F.4})$$

where $E(s)$ is the Laplace transform of the system error $e = \theta_d - \theta$, whereas γ , d and c are positive constants such that $d < c$. Direct division of the fraction in (F.4) yields:

$$I^*(s) = \gamma [E(s) + V(s)], \quad (\text{F.5})$$

$$V(s) = \frac{d-c}{s+c} E(s).$$

Assuming zero initial conditions, the inverse Laplace transform is used in both expressions given in (F.5) to find:

$$i^* = \gamma[e + v], \quad (\text{F.6})$$

$$\dot{v} = -cv + (d-c)e, \quad (\text{F.7})$$

where v is the inverse Laplace transform of $V(s)$. Then, if the desired position θ_d is known and θ is measured, the signal e can be computed. Using this signal as the known variable, the differential equation in (F.7) is solved numerically, with $v(0) = 0$, to compute v and then use (F.6) to compute i^* .

Now, it is explained how to solve it numerically (F.7). According to (F.2) and (F.3), it is possible to perform a forward shift in time to approximate:

$$\dot{v} = -cv(k) + (d-c)e(k) \approx \frac{v(k+1) - v(k)}{\Delta t},$$

i.e.,

$$v(k+1) = [-cv(k) + (d-c)e(k)]\Delta t + v(k). \quad (\text{F.8})$$

Then, $v(k+1)$ can be computed iteratively from the initial value $v(0) = 0$ and the measurement of $e(k) = \theta_d(k) - \theta(k)$. The equation (F.8) is expressed in a convenient manner because it allows us to use $v(k) = v(0)$ at $k = 0$. Note that $v(k+1)$ is computed at the present sampling time but it is to be used only at the next sampling time in the future.

F.4 Controller in Fig. 14.8a

The controller to be implemented is given as:

$$i^* = \alpha[v_1 - \theta] - k_v \dot{\theta},$$

where $V_1(s) = \mathcal{L}\{v_1\}$ is defined as:

$$\begin{aligned} V_1(s) &= \gamma \frac{s+b}{s+c} E(s), \\ E(s) &= X_d(s) - X(s). \end{aligned} \quad (\text{F.9})$$

Proceeding as in Sect. F.3, it is found that the time function v_1 can be computed in discrete time as:

$$\begin{aligned} v_1(k) &= \gamma[e(k) + v(k)], \\ v(k+1) &= [-cv(k) + (b-c)e(k)]\Delta t + v(k), \end{aligned}$$

where $e(k)$ and $v_1(k)$ are the present discrete time values of the functions $\mathcal{L}\{e\} = E(s)$ and $\mathcal{L}\{v_1\} = V_1(s)$ defined in (F.9), i.e., $e(k) = x_d(k) - x(k)$, and Δt is the sampling period. Hence, $v(k+1)$ can be computed iteratively from the initial value $v(0) = 0$. Note that $v(k+1)$ is computed in the present sampling time but it is employed only at the next sampling time in the future.

Finally, the signal $\dot{\theta}$ is computed from the position measurement θ as follows:

$$\dot{\theta} \approx \frac{\theta(k) - \theta(k-1)}{\Delta t}.$$

F.5 Controllers in (12.37) and (12.40)

The controller in (12.40) is given as:

$$G_c(s) = \frac{U(s)}{E(s)} = \frac{0.4408s^3 + 5.734s^2 + 1758s + 8701}{s^3 + 94.04s^2 + 2889s + 3.41 \times 10^4},$$

where $E(s) = \theta_{2d}(s) - \theta_2(s)$. Performing the indicated division yields:

$$U(s) = \frac{-35.7188s^2 + 484.5s - 6330}{s^3 + 94.04s^2 + 2889s + 3.41 \times 10^4} E(s) + 0.4408E(s). \quad (\text{F.10})$$

Define a new variable $U_1(s)$ as:

$$U_1(s) = \frac{-35.7188s^2 + 484.5s - 6330}{s^3 + 94.04s^2 + 2889s + 3.41 \times 10^4} E(s). \quad (\text{F.11})$$

Hence, $U(s) = U_1(s) + 0.4408E(s)$. Defining:

$$\begin{aligned} \alpha_2 &= 94.04, & \alpha_1 &= 2889, & \alpha_0 &= 3.41 \times 10^4, \\ \beta_2 &= -35.7188, & \beta_1 &= 484.5, & \beta_0 &= -6330, \end{aligned}$$

the following can be written:

$$U_1(s) = \frac{\beta_2 s^2 + \beta_1 s + \beta_0}{s^3 + \alpha_2 s^2 + \alpha_1 s + \alpha_0} E(s). \quad (\text{F.12})$$

Thus, according to (F.10), (F.11) and (F.12), the diagram of one possible realization for (12.40) is depicted in Fig. F.1. On the other hand, in the computer program listed in Sect. 12.6, the controller in (12.40) is implemented using the following dynamic equation:

$$\dot{z} = Az + Be, \quad u = Cz + De, \quad (\text{F.13})$$

$$A = \begin{bmatrix} \eta_1 & \eta_2 & \eta_3 \\ \eta_4 & 0 & 0 \\ 0 & \eta_5 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ 0 \\ 0 \end{bmatrix},$$

$$C = [\sigma_1 \ \sigma_2 \ \sigma_3], \quad D = d_1,$$

$$\eta_1 = -94.04, \quad \eta_2 = -45.15, \quad \eta_3 = -16.65, \quad \eta_4 = 64, \quad \eta_5 = 32,$$

$$b_1 = 8, \quad \sigma_1 = -4.465, \quad \sigma_2 = 0.9454, \quad \sigma_3 = -0.3866, \quad d_1 = 0.4408.$$

This dynamic equation has been found using MATLAB, which converts the transfer function in (12.40) into the above dynamical equation. Hence, the most we can do is find a relationship between this system realization and that in Fig. F.1. A simulation diagram for (F.13) is depicted in Fig. F.2. Performing some block algebra in Fig. F.2, the simulation diagram shown in Fig. F.3 is found. Notice that this diagram is equivalent to that in Fig. F.1 if the following is satisfied:

$$\begin{aligned}
 -\alpha_2 &= \eta_1, & -\alpha_1 &= \eta_2\eta_4, & -\alpha_0 &= \eta_3\eta_4\eta_5, \\
 \beta_2 &= b_1\sigma_1, & \beta_1 &= b_1\eta_4\sigma_2, & \beta_0 &= b_1\eta_4\eta_5\sigma_3.
 \end{aligned}$$

The reader may proceed to use the specific numerical values given above to verify these expressions. The dynamical equation in (F.13) is solved iteratively by measuring the error e at each sample time and approximating the time derivatives as $\frac{dz_i}{dt} \approx \frac{\Delta z_i}{\Delta t}$, for $i = 1, 2, 3$, where $\Delta z_i = z_i(k + 1) - z_i(k)$, k and $k + 1$ stand for the present and the next sample time, and Δt is the sample period, to write:

$$\begin{aligned}
 \frac{\Delta z_1}{\Delta t} &= \eta_1 z_1(k) + \eta_2 z_2(k) + \eta_3 z_3(k) + b_1 e(k), \\
 \frac{\Delta z_2}{\Delta t} &= \eta_4 z_1(k), \\
 \frac{\Delta z_3}{\Delta t} &= \eta_5 z_2(k).
 \end{aligned}$$

Thus:

$$\begin{aligned}
 z_1(k + 1) &= (\eta_1 z_1(k) + \eta_2 z_2(k) + \eta_3 z_3(k) + b_1 e(k))\Delta t + z_1(k), \\
 z_2(k + 1) &= (\eta_4 z_1(k))\Delta t + z_2(k),
 \end{aligned}$$

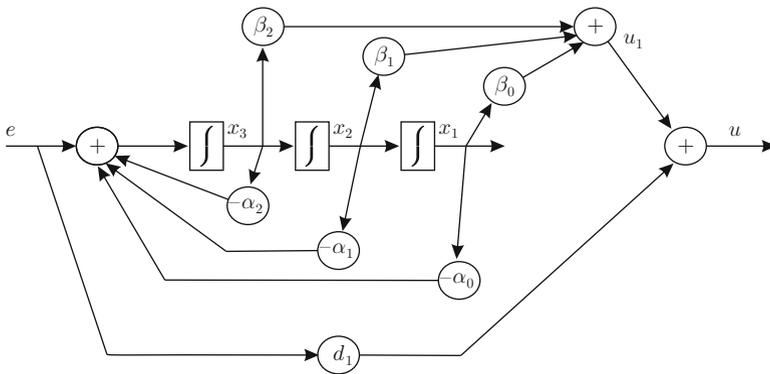


Fig. F.1 A possible realization for (12.40), $d_1 = 0.4408$

$$z_3(k+1) = (\eta_5 z_2(k)) \Delta t + z_3(k),$$

$$u(k) = \sigma_1 z_1(k) + \sigma_2 z_2(k) + \sigma_3 z_3(k) + d_1 e(k),$$

and finally assigning:

$$z_1(k+1) \Rightarrow z_1(k),$$

$$z_2(k+1) \Rightarrow z_2(k),$$

$$z_3(k+1) \Rightarrow z_3(k).$$

All initial conditions are set to zero, i.e., when $k = 0$ use $z_1(k) = 0$, $z_2(k) = 0$, $z_3(k) = 0$.

Finally, the controller in (12.37), i.e.,:

$$G_c(s) = 0.6892 \frac{(s+5)(s^2+8s+62.83^2)}{(s+36.6439)^3},$$

is implemented in the computer program listed in Sect. 12.6 using the following dynamical equation:

$$\dot{z} = Az + Be, \quad u = Cz + De,$$

$$A = \begin{bmatrix} \eta_1 & \eta_2 & \eta_3 \\ \eta_4 & 0 & 0 \\ 0 & \eta_5 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ 0 \\ 0 \end{bmatrix},$$

$$C = [\sigma_1 \ \sigma_2 \ \sigma_3], \quad D = d_1,$$

$$\eta_1 = -109.9, \quad \eta_2 = -62.94, \quad \eta_3 = -24.03, \quad \eta_4 = 64, \quad \eta_5 = 32,$$

$$b_1 = 8, \quad \sigma_1 = -8.351, \quad \sigma_2 = -0.05483, \quad \sigma_3 = -1.24, \quad d_1 = 0.6892.$$

This dynamical equation can be interpreted by proceeding exactly as in the previous case.

G

MATLAB/Simulink Code Used for Some Simulations

G.1 MATLAB/Simulink Code for Simulation in Fig. 15.9

G.1.1 Furuta Pendulum Model Block

```
function Q = modelo(theta0p,theta1,theta1p,tau)

%%-----Parameters-----%%
g = 9.81;
l1 = 129e-3;
L0 = 155e-3;
m1 = 22.18e-3;
J1 = 184.5e-6;
I0 = 238.49e-6;
%%----- M(q)qpp + C(q, qp)qp + G(q) = T-----%%
%%----- q=[theta0;theta1]-----%%
M11= I0 + m1*(L0^2) + (l1^2)*m1*(sin(theta1)^2);
M12= m1*L0*l1*cos(theta1);
M21= m1*L0*l1*cos(theta1);
M22= J1 + m1*(l1^2);
M= [M11 M12;
M21 M22];
C11= 2*(l1^2)*m1*sin(theta1)*cos(theta1)*theta1p;
C12= -m1*L0*l1*sin(theta1)*theta1p;
C21= -m1*(l1^2)*sin(theta1)*cos(theta1)*theta0p;
C22= 0;
C= [C11 C12;
C21 C22];
G11= 0;
G21= -m1*g*l1*sin(theta1);
G= [G11;
G21];
T= [tau;
0];
```

```

Qp= [theta0p;
theta1p];
Qpp= M \ ( T - C*Qp - G );
theta0pp=Qpp(1,1);
theta1pp=Qpp(2,1);
Q=[theta0pp;theta1pp];

```

G.1.2 Swing-Up Control Block

```

function [tauS,E,V] = fcn(theta0,theta0p,theta1,theta1p)
% Parameters
g = 9.81;
lpc = 129e-3;
lb = 155e-3;
mp = 22.18e-3;
J1 = 184.5e-6;
I0 = 238.49e-6;
ke = 1900;
kw =1;
kdelta = 3;
ktheta = 3;
qp=[theta0p;theta1p];
M11= I0 + mp*(lb^2) + (lpc^2)*mp*(sin(theta1)^2);
M12= mp*lb*lpc*cos(theta1);
M21= mp*lb*lpc*cos(theta1);
M22= J1 + mp*(lpc^2);
Dq=[M11 M12;
M21 M22];
detD=det(Dq);
E= (0.5)*qp'*Dq*qp + mp*g*lpc*(cos(theta1)-1);
F= -(J1 + mp*(lpc^2))*mp*(lpc^2)*sin(2*theta1)*theta0p*
theta1p-... (0.5)*(mp^2)*(lpc^3)*lb*cos(theta1)*sin(2*theta1)*
(theta0p^2)+... (J1 + mp*(lpc^2))*mp*lpc*lb*sin(theta1)*
(theta1p^2)-... (mp^2)*(lpc^2)*lb*g*cos(theta1)*sin(theta1);
tauS= (-kw*F-detD*(kdelta*theta0p + ktheta*theta0))...
/(detD*ke*E + kw*(J1 + mp*(lpc^2)));
V = 0.5*ke*E^2 + 0.5*kw*theta0p^2 + 0.5*ktheta*theta0^2;

```

G.1.3 Stabilizing Control Block

```

function tauE = fcn(theta0,theta0p,theta1,theta1p)
if(theta1>pi)
theta1d=2*pi;
else
theta1d=0;
end
x=[theta0 theta0p (theta1-theta1d) theta1p];
K=[-0.0118 -0.0118 -0.2742 -0.0298];

```

```
tauE = -K*x';
```

G.1.4 Switching Condition Block

```
function tau = fcn(theta1,theta1p,tauS,tauE)
if (theta1>pi)
theta1d=2*pi;
else
theta1d=0;
end
if sqrt(3*(theta1-theta1d)^2+0.1*(theta1p*theta1p)) < 0.2
tau = tauE;
else
tau= tauS;
end
```

G.1.5 Code to Plot Variables

```
clc,close all;

figure(1)
plot(t,theta0,'b','Linewidth',1.5);
grid on
axis([0 35 -0.25 0.25]);
set(gca,'XTick',[0 10 20 30],'YTick',[-0.2 0 0.2],'FontName','Times New Roman','FontSize',24)
text(16,0.195,'$\theta_0$', 'Interpreter','latex','FontSize',24);
text(30.75,-0.22,'$\tau_S$ [s]', 'Interpreter','latex','FontSize',24);
text(1.6,0.155,' [rad]', 'Interpreter','latex','FontSize',24,'rotation',90);

figure(2)
plot(t,theta1,'b','Linewidth',1.5);
grid on
axis([0 35 -0 9]);
set(gca,'XTick',[0 10 20 30],'YTick',[0 3 6 9],'FontName','Times New Roman','FontSize',24)
text(16,8,'$\theta_1$', 'Interpreter','latex','FontSize',24);
text(30.75,0.55,'$\tau_S$ [s]', 'Interpreter','latex','FontSize',24);
text(1.6,7.3,' [rad]', 'Interpreter','latex','FontSize',24,'rotation',90);

figure(3)
plot(t,theta1,theta1p,'b','Linewidth',1.5);
grid on
axis([-2 9 -20 20]);
set(gca,'XTick',[0 3 6 9],'YTick',[-20 -10 0 10 20],'FontName','Times New Roman','FontSize',24)
text(6.35,-17.5,'$\theta_1$ [rad]', 'Interpreter','latex','FontSize',24);
text(-1.37,4.5,'$\dot{\theta}_1$ [rad/s]', 'Interpreter','latex','FontSize',24,'rotation',90);

figure(4)
plot(t,tauF,'b','Linewidth',1.5);
grid on
axis([0 35 -0.1 0.1]);
set(gca,'XTick',[0 10 20 30],'YTick',[-0.1 -0.05 0 0.05 0.1],'FontName','Times New Roman','FontSize',24)
text(16,0.08,'$\tau_A$', 'Interpreter','latex','FontSize',24);
text(30.7,-0.088,'$\tau_S$ [s]', 'Interpreter','latex','FontSize',24);
text(1.6,0.06,' [Nm]', 'Interpreter','latex','FontSize',24,'rotation',90);

figure(5)
plot(t,E,'b','Linewidth',1.5);
grid on
axis([0 35 -0.075 0.075]);
set(gca,'XTick',[0 10 20 30],'YTick',[-0.05 0 0.05],'FontName','Times New Roman','FontSize',24)
text(13,0.057,'$E\left(q,\dot{q}\right)$', 'Interpreter','latex','FontSize',24);
text(30.75,-0.0665,'$\tau_S$ [s]', 'Interpreter','latex','FontSize',24);
text(1.6,0.06,' [J]', 'Interpreter','latex','FontSize',24,'rotation',90);

figure(6)
plot(t,V,'b','Linewidth',1.5);
grid on
axis([0 35 -1 4]);
```

```

set(gca,'XTick',[0 10 20 30],'YTick',[0 2 4],'FontName','Times New Roman','FontSize',24)
text(13,3.5,'$\left(q,\dot{q}\right)$','Interpreter','latex','FontSize',24);
text(30.75,-0.71,'$s$ $\mathrm{s}$!','Interpreter','latex','FontSize',24);

```

G.2 MATLAB/Simulink Code for Simulation in Fig. 16.6

G.2.1 Inertia Wheel Pendulum Block

```

function [Sys,x0]=InertialWheelPendulum(t,x,u,flag)
N=3;
switch flag,
case 0,
[Sys,x0] = Initialization(N);
case 1,
Sys = State_Equations(t,x,u,N);
case 3,
Sys = Output(t,x,u);
case 9,
Sys = [];
otherwise
Sys = [];
end

```

```

function [Sys,x0]=Initialization(N)
NumContStates = N;
NumDiscStates = 0;
NumOutputs = N;
NumInputs = 1;
DirFeedthrough = 0;
NumSampleTimes = 1;
Sys = [NumContStates NumDiscStates
NumOutputs NumInputs DirFeedthrough NumSampleTimes];
x0 = [0 0.00001 0];

```

```

function Sys = State_Equations(t,x,u,N)
mbg=0.12597;
R=4.172;
km=0.00775;
d11=0.0014636;
d12=0.0000076;
d21=d12;
d22=d21;
D=[d11 d12;d21 d22];

```

```

Di=inv(D);
di11=Di(1,1);
di12=Di(1,2);
di21=Di(2,1);
di22=Di(2,2);
xp(1)=x(2);
xp(2)=-di11*mbg*sin(x(1))+(km/R)*di12*u;
xp(3)=-di21*mbg*sin(x(1))+(km/R)*di22*u;
Sys=[xp'];

function Sys = Output(t,x,u)
Sys = [x];

```

G.2.2 Nonlinear Controller Block

```

function [nlc,V,Verror] = fcn(x,d,V0)
%#codegen
mgl=0.12597;
mbg=mgl;
R=4.172;
km=0.00775;
Kd=20;
d11=0.0014636;
d12=0.0000076;
d21=d12;
d22=d21;
J=(d11*d22-d12*d21)/d12;
if x(2)>d
satx2=d;
elseif x(2)<-d
satx2=-d;
else satx2=x(2);
end
%-----energy-----
V=(J/2)*x(2)^2+mbg*(1-cos(x(1)));

%-----nonlinear control-----
nlc=(R/km)*Kd*satx2*(V-V0);

%-----energy error-----
Verror=V-V0;

```

G.2.3 Linear Controller Block

```
function lc = fcn(x,K)
%#codegen}
n=1;
c=-570;
xd=[n*pi; 0; c];
z=x-xd;
%-----linear control-----
lc =-K'*z;
```

G.2.4 Controller Commutation Block

```
function controller = fcn(nlc, lc, x)
%#codegen
n=1;
c=-570;
xd=[n*pi; 0; c];
z=x-xd;
if z(1)^2+z(2)^2<=0.01
%-----linear control action-----
controller=lc;
%-----nonlinear control action-----
else controller=nlc;
end
```

G.2.5 Numerical Parameters

This code must be run before running the simulation in Fig. 16.6 to fix the numerical values of the parameters.

```
clc,clear all;
% Motor parameters
R=4.172;
km=0.00775;
Umax=13;
% IWP Model
g=9.81;
mg1=0.12597;
mbg=mg1
d11=0.0014636;
d12=0.0000076;
d21=d12;
```

```

d22=d21;
J= (d11*d22-d12*d21)/d12;
D=[d11 d12;d21 d22];
Di=inv(D);
di11=Di(1,1)
di12=Di(1,2)
di21=Di(2,1)
di22=Di(2,2)
% Linear approximate model of IWP
A=[0 1 0;di11*mbg 0 0;di21*mbg 0 0]
B=[0;di12*km/R;di22*km/R]
% Controllability determination
disp('Is system controllable?');
Pc=ctrb(A,B);
if rank(Pc) == size(Pc)
disp('Yes. ');
else
disp('No. ');
end
% Nonlinear controller
Kd=20;
V0=2*mbg
d=(Umax*km)/(R*Kd*V0)
%Linear state feedback controller at the operation point
n=1
c=-570
xd=[n*pi 0 c]
% Desired closed-loop eigenvalues
lambda1= -9.27 + 20.6i;
lambda2= -9.27 - 20.6i;
lambda3= -0.719;
Vp=[lambda1 lambda2 lambda3]
K = place(A,B,Vp)
% Verifying closed-loop eigenvalues
Vp_=eig(A-B*K)

```

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