Appendix

A.1 Mathematical Summary

A.1.1 Some Basic Theorems of Matrix Algebra

The following scheme is called a matrix

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \tag{A.1.1}$$

where the values a_{ij} are the elements of the matrix. If its elements are real, then the matrix is called a real matrix, if they are complex, then the matrix is called a complex matrix. In general, a matrix has m rows and n columns. The dimension (the size) of the matrix is $m \times n$. A matrix of type $m \times n$ is rectangular; an $n \times n$ matrix is called square (quadratic) matrix, an $m \times 1$ matrix is a column matrix (column vector), a $1 \times n$ matrix is a row matrix (row vector), a 1×1 matrix is called a scalar.

Matrices are usually denoted by *bold* (*fat)* capital letters, the column and row vectors are denoted by *bold* lower case letters. The determinant of the square matrix $\boldsymbol{A}$ is denoted by $|\boldsymbol{A}|$ (or written as $\det(\boldsymbol{A})$).

The transpose of the matrix $\boldsymbol{A}$ is denoted by $\boldsymbol{A}^{\mathrm{T}}$, and it means the result of the mirroring of its elements for the main diagonal.

$$\boldsymbol{A}^{\mathrm{T}} = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}. \tag{A.1.2}$$

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If $\boldsymbol{A}$ is an $m \times n$ matrix, then its transpose is an $n \times m$ matrix, and it is trivial that $\left(\boldsymbol{A}^{\mathrm{T}}\right)^{\mathrm{T}} = \boldsymbol{A}$. If $\boldsymbol{A}^{\mathrm{T}} = \boldsymbol{A}$ then it is called mirror matrix.

A vector is usually considered a column matrix, and a row matrix is denoted as the transpose of a column matrix, e.g.,

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = [x_1, \ldots, x_n]^{\mathrm{T}} = [\boldsymbol{x}^{\mathrm{T}}]^{\mathrm{T}}. \tag{A.1.3}$$

The elements of the zero matrix, or zero vector, are all zeros. The diagonal matrix has elements different from zero only along the main diagonal, i.e.,

$$\boldsymbol{D} = \mathbf{diag}[a_{11}, a_{22}, \ldots, a_{nn}]. \tag{A.1.4}$$

If in a diagonal matrix all the diagonal elements are unity, then the matrix is called the unit matrix: $\boldsymbol{I} = \mathbf{diag}[1, 1, \ldots, 1]$.

Two matrices are equal if all the corresponding elements are equal. The sum of two or more matrices of the same type is obtained by summing the corresponding elements. The multiplication of a matrix by a scalar is obtained by multiplying each element of the matrix by the scalar. The most characteristic case is the multiplication of two matrices, e.g., when a matrix $\boldsymbol{A}$ of type $m \times l$ is multiplied by a matrix $\boldsymbol{B}$ of type $l \times n$,

$$\boldsymbol{C} = \boldsymbol{A}\boldsymbol{B}, \tag{A.1.5}$$

where

$$c_{ij} = \sum_{k=1}^{l} a_{ik} b_{kj}; \quad \begin{cases} i = 1, 2, \ldots, m \\ j = 1, 2, \ldots, n \end{cases} \tag{A.1.6}$$

i.e., the element in the i-th row and j-th column of the matrix $\boldsymbol{C}$ of type $m \times n$ is obtained by multiplying the i-th row of $\boldsymbol{A}$ by the j-th column of $\boldsymbol{B}$. (The number l of columns of $\boldsymbol{A}$ must be equal to the number l of the rows of $\boldsymbol{B}$.) Matrix multiplication is associative and distributive, but, in general, is not commutative: $\boldsymbol{AB} \neq \boldsymbol{BA}$. If $\boldsymbol{AB} = \boldsymbol{BA}$, then in this case the matrices are interchangeable (commutative). Note that the determinant of the square product matrix $|\boldsymbol{C}| = \det(\boldsymbol{C})$ is obtained by multiplying the determinants $|\boldsymbol{A}|$ and $|\boldsymbol{B}|$ of the factor matrices, i.e., $|\boldsymbol{C}| = |\boldsymbol{A}||\boldsymbol{B}|$.

The scalar product of two vectors having the same dimension can be expressed as the product of matrices, by

$$\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{a}^{\mathrm{T}}\boldsymbol{b} = \boldsymbol{b}^{\mathrm{T}}\boldsymbol{a} = \boldsymbol{b} \cdot \boldsymbol{a}. \tag{A.1.7}$$

If the scalar product of two different, non-zero vectors is zero, then the two vectors are called orthogonal.

The following expression represents a very important rule

$$[\boldsymbol{AB}]^{\mathrm{T}} = \boldsymbol{B}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}. \tag{A.1.8}$$

The inverse of a square, regular (nonsingular, i.e., its determinant is non-zero) matrix is a matrix, for which the following expression is valid.

$$\boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{I}. \tag{A.1.9}$$

The inverse of $\boldsymbol{A}$ is given by the rule

$$\boldsymbol{A}^{-1} = \frac{\mathbf{adj}(\boldsymbol{A})}{|\boldsymbol{A}|}. \tag{A.1.10}$$

Here $|\boldsymbol{A}|$ is the (non-zero) determinant of $\boldsymbol{A}$, and the adjunct matrix $\mathbf{adj}(\boldsymbol{A})$ of $\boldsymbol{A}$ is obtained by mirroring the matrix whose elements are sub-determinants of appropriate sign belonging to each element of $\boldsymbol{A}$. Since the rule $|\boldsymbol{AB}| = |\boldsymbol{A}||\boldsymbol{B}|$ is valid, therefore, according to $1 = |\boldsymbol{I}| = |\boldsymbol{A}^{-1}\boldsymbol{A}| = |\boldsymbol{A}^{-1}||\boldsymbol{A}|$, $\boldsymbol{A}$ has an unambiguous inverse only if $|\boldsymbol{A}| \neq 0$, i.e., the matrix $\boldsymbol{A}$ is non-singular. It is obvious that

$$[\boldsymbol{A}^{-1}]^{-1} = \boldsymbol{A} \quad \text{and} \quad [\boldsymbol{A}^{-1}]^{\mathrm{T}} = [\boldsymbol{A}^{\mathrm{T}}]^{-1}. \tag{A.1.11}$$

Furthermore if $\boldsymbol{A}$ and $\boldsymbol{B}$ are regular square matrices, then

$$[\boldsymbol{AB}]^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}. \tag{A.1.12}$$

The matrices $s\boldsymbol{I} - \boldsymbol{A}$, or $\boldsymbol{A} - s\boldsymbol{I}$, are called the characteristic matrices of the square matrix $\boldsymbol{A}$, and the equation $\mathcal{A}(s) = |s\boldsymbol{I} - \boldsymbol{A}| = 0$ is called the characteristic equation. The roots $\lambda_i (i = 1, 2, \ldots, n)$ of the characteristic equation are the eigenvalues of the matrix $\boldsymbol{A}$. Due to the main pivot theorem the eigenvectors $\boldsymbol{v}_i (i = 1, 2, \ldots n)$ of $\boldsymbol{A}$ fulfill the following vector equations:

$$\boldsymbol{A}\boldsymbol{v}_i = \lambda_i \boldsymbol{v}_i \quad (i = 1, 2, \ldots, n). \tag{A.1.13}$$

This is the definition of the eigenvectors. If the vectors $\boldsymbol{v}_j$ are linearly independent, then the matrix $\boldsymbol{A}$ has a simple structure, if the vectors are not independent, then the matrix is called deteriorated.

The CAYLEY-HAMILTON theorem has significant importance in the matrix theory: any matrix $\boldsymbol{A}$ satisfies its own characteristic equation, i.e., $\mathcal{A}(\boldsymbol{A}) = 0$. (Here in the scalar polynomial equation $\mathcal{A}(s) = 0$, s^i is replaced by $\boldsymbol{A}^i$ $(i = 1, 2, \ldots, n)$, while s^0 is by $\boldsymbol{A}^0 = \boldsymbol{I}$, so finally a matrix polynomial equation is obtained.)

In many cases it is necessary to express the inner structure of a matrix, therefore so-called block-matrices are applied, e.g.,

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}. \tag{A.1.14}$$

According to the matrix multiplication rules

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} E & F \\ G & H \end{bmatrix} = \begin{bmatrix} AE+BG & AF+BH \\ CE+DG & CF+DH \end{bmatrix}. \tag{A.1.15}$$

The determinant of a quasi-diagonal matrix is

$$\begin{vmatrix} A & B \\ \mathbf{O} & D \end{vmatrix} = \det\begin{bmatrix} A & B \\ \mathbf{O} & D \end{bmatrix} = \det(A)\det(D) = |A||D|. \tag{A.1.16}$$

The product ab^{T} is called the dyadic product. The inverse of the matrix A extended by the addition of a dyadic product can be given very simply, if the inverse of A is known:

$$\left(A + ab^{\mathrm{T}}\right)^{-1} = A^{-1} - \frac{\left(A^{-1}a\right)\left(b^{\mathrm{T}}A^{-1}\right)}{1 + b^{\mathrm{T}}A^{-1}a} \tag{A.1.17}$$

A.1.2 Some Basic Formulas of Vector Analysis

In vector analysis for EUCLIDEAN space there are scalar-scalar functions

$$f = f(x), \tag{A.1.18}$$

so-called scalar-vector functions

$$f = f(\boldsymbol{x}), \tag{A.1.19}$$

and vector-vector functions

$$\boldsymbol{f} = \boldsymbol{f}(\boldsymbol{x}). \tag{A.1.20}$$

(All these are the special cases of the most general but very rare matrix-matrix functions $F = F(X)$.) In many cases multivariable scalar-scalar, scalar-vector or vector-vector functions occur, e.g.,

$$f = f(x,u); \quad f = f(\boldsymbol{x},\boldsymbol{u}); \quad \boldsymbol{f} = \boldsymbol{f}(\boldsymbol{x},\boldsymbol{u}) \tag{A.1.21}$$

or functions containing independent variables (time or a parameter) also appear

$$f = f(x,u,t); \quad f = f(\boldsymbol{x},\boldsymbol{u},t); \quad \boldsymbol{f} = \boldsymbol{f}(\boldsymbol{x},\boldsymbol{u},t). \tag{A.1.22}$$

Certain rules for differentiation are very important. The derivative with respect to a scalar is very simple, e.g.,

$$\frac{\mathrm{d}\boldsymbol{x}(t)}{\mathrm{d}t} = \left[\frac{\mathrm{d}x_1}{\mathrm{d}t}, \frac{\mathrm{d}x_2}{\mathrm{d}t}, \ldots, \frac{\mathrm{d}x_n}{\mathrm{d}t}\right]^{\mathrm{T}} = [\dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n] = \dot{\boldsymbol{x}} \tag{A.1.23}$$

$$\frac{\mathrm{d}\boldsymbol{A}(t)}{\mathrm{d}t} = \begin{bmatrix} \dot{a}_{11} & \dot{a}_{12} & \ldots & \dot{a}_{1n} \\ \dot{a}_{21} & \dot{a}_{22} & \ldots & \dot{a}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \dot{a}_{m1} & \dot{a}_{m2} & \ldots & \dot{a}_{mn} \end{bmatrix} = \dot{\boldsymbol{A}} \tag{A.1.24}$$

The gradient of a scalar-vector function is a column vector

$$\mathbf{grad}[f(\boldsymbol{x})] = \frac{\mathrm{d}f(\boldsymbol{x})}{\mathrm{d}\boldsymbol{x}} = \left[\frac{\mathrm{d}f(\boldsymbol{x})}{\mathrm{d}x_1} \quad \frac{\mathrm{d}f(\boldsymbol{x})}{\mathrm{d}x_2} \quad \ldots \quad \frac{\mathrm{d}f(\boldsymbol{x})}{\mathrm{d}x_n}\right]^{\mathrm{T}}, \tag{A.1.25}$$

which means the application of a multivariable differential-operator

$$\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{x}} = \left[\frac{\mathrm{d}}{\mathrm{d}x_1} \quad \frac{\mathrm{d}}{\mathrm{d}x_2} \quad \ldots \quad \frac{\mathrm{d}}{\mathrm{d}x_n}\right]^{\mathrm{T}} \tag{A.1.26}$$

thus

$$\mathbf{grad}[f(\boldsymbol{x})] = \frac{\mathrm{d}}{\mathrm{d}\boldsymbol{x}} f(\boldsymbol{x}) = \frac{\mathrm{d}f(\boldsymbol{x})}{\mathrm{d}\boldsymbol{x}}. \tag{A.1.27}$$

The JACOBIAN matrix is

$$\boldsymbol{J} = \boldsymbol{J}(\boldsymbol{f},\boldsymbol{x}) = \begin{bmatrix} \frac{\mathrm{d}f_1}{\mathrm{d}x_1} & \frac{\mathrm{d}f_1}{\mathrm{d}x_2} & \ldots & \frac{\mathrm{d}f_1}{\mathrm{d}x_n} \\ \frac{\mathrm{d}f_2}{\mathrm{d}x_1} & \frac{\mathrm{d}f_2}{\mathrm{d}x_2} & \ldots & \frac{\mathrm{d}f_2}{\mathrm{d}x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\mathrm{d}f_m}{\mathrm{d}x_1} & \frac{\mathrm{d}f_m}{\mathrm{d}x_2} & \ldots & \frac{\mathrm{d}f_m}{\mathrm{d}x_n} \end{bmatrix}. \tag{A.1.28}$$

Avoiding complicated notations, the JACOBIAN matrix is symbolically denoted by

$$\boldsymbol{J}(\boldsymbol{f},\boldsymbol{x}) = \frac{\mathrm{d}\boldsymbol{f}(\boldsymbol{x})}{\mathrm{d}\boldsymbol{x}^{\mathrm{T}}} \tag{A.1.29}$$

and its transpose is

$$\boldsymbol{J}^{\mathrm{T}}(\boldsymbol{f},\boldsymbol{x}) = \frac{\mathrm{d}\boldsymbol{f}^{\mathrm{T}}(\boldsymbol{x})}{\mathrm{d}\boldsymbol{x}}. \tag{A.1.30}$$

Thus the transpose of the gradient vector is

$$\mathbf{grad}^{\boldsymbol{T}}[f(\boldsymbol{x})] = \left[\frac{\mathrm{d}f(\boldsymbol{x})}{\mathrm{d}\boldsymbol{x}}\right]^{\mathrm{T}} = \frac{\mathrm{d}f(\boldsymbol{x})}{\mathrm{d}\boldsymbol{x}^{\mathrm{T}}} = \boldsymbol{J}(f,\boldsymbol{x}). \tag{A.1.31}$$

The second order derivatives of a scalar-vector function can be arranged into the HESSIAN matrix

$$\boldsymbol{H} = \boldsymbol{H}(f,\boldsymbol{x}) = \begin{bmatrix} \frac{\mathrm{d}^2 f}{\mathrm{d}x_1^2} & \frac{\mathrm{d}^2 f}{\mathrm{d}x_1 \mathrm{d}x_2} & \cdots & \frac{\mathrm{d}^2 f}{\mathrm{d}x_1 \mathrm{d}x_n} \\ \frac{\mathrm{d}^2 f}{\mathrm{d}x_2} & \frac{\mathrm{d}^2 f}{\mathrm{d}x_2^2} & \cdots & \frac{\mathrm{d}^2 f}{\mathrm{d}x_2 \mathrm{d}x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\mathrm{d}^2 f}{\mathrm{d}x_n} & \frac{\mathrm{d}^2 f}{\mathrm{d}x_n \mathrm{d}x_2} & \cdots & \frac{\mathrm{d}^2 f}{\mathrm{d}x_n^2} \end{bmatrix}. \tag{A.1.32}$$

A.2 Signals and Systems

The general topics of signals and systems directly connected to control engineering have been discussed in the main sections of this textbook. For completeness there are, however, some special fields whose effect and availability has to be known, but they cannot be connected directly to control engineering. From the subject of an excitation with special periodic signals, only the standard sine excitation was discussed for the better understanding of the frequency functions.

Dynamics of linear processes with periodic excitation

Let $u(t)$ be a function of time with a period T_p, i.e., $u\left(t+T_\mathrm{p}\right) = u(t)$. Introduce the notation $u_\mathrm{A}(t)$ for denoting the basic function (or truncated function) determining the periodic signal, which in the time domain $0<t<T_\mathrm{p}$ is equal to $u(t)$, but otherwise is zero.

$$u_\mathrm{A}(t) = \left[1(t) - 1(t-T_\mathrm{p})\right]u(t) = \begin{cases} u(t); & 0<t<T_\mathrm{p} \\ 0; & t \le 0;\ \ t > T_\mathrm{p} \end{cases}. \tag{A.2.1}$$

The periodic function $u(t)$ can be obviously constructed by repeated shifts and sums of the basic function $u_{\mathrm{A}}(t)$, according to the definition

$$u(t) = \sum_{j=0}^{\infty} u_{\mathrm{A}}\left(t - jT_{\mathrm{p}}\right) = 1_{\mathrm{A}}(t)u(t), \tag{A.2.2}$$

where $1_{\mathrm{A}}(t)$ is called the repetitive operator. Determine the LAPLACE transform of the basic function $u_{\mathrm{A}}(t)$, i.e., the function $U_{\mathrm{A}}(s)$. Due to the shift theorem

$$\mathcal{L}\left\{u_{\mathrm{A}}\left(t - jT_{\mathrm{p}}\right)\right\} = e^{-jsT_{\mathrm{p}}} U_{\mathrm{A}}(s) \tag{A.2.3}$$

and applying it to (A.2.2), the LAPLACE transform of the periodic signal $u(t)$ is

$$U(s) = \mathcal{L}\{u(t)\} = \mathcal{L}\{1_{\mathrm{A}}(t)u(t)\} = \sum_{j=0}^{\infty} e^{-jsT_{\mathrm{p}}} U_{\mathrm{A}}(s) = U_{\mathrm{A}}(s) \sum_{j=0}^{\infty} e^{-jsT_{\mathrm{p}}}. \tag{A.2.4}$$

Notice that here the summing equation for the geometric series can be applied

$$U(s) = \mathcal{L}\{1_{\mathrm{A}}(t)u(t)\} = \frac{U_{\mathrm{A}}(s)}{1 - e^{-sT_{\mathrm{p}}}}. \tag{A.2.5}$$

If the LAPLACE transform $U(s)$ of a signal can be written in the form of (A.2.5), then using the basic function $u_{\mathrm{A}}(t) = \mathcal{L}^{-1}\{U_{\mathrm{A}}(s)\}$, the time function of the periodic signal can be easily determined. The condition $u_{\mathrm{A}}(t) = 0$ for $t > T_{\mathrm{p}}$ must be fulfilled.

Next the system dynamics, i.e., the process response is investigated when a periodic signal is put to as input of an *LTI* system. The response can be gotten by the LAPLACE transform of the process output if the system is originally free of energy. The LAPLACE transform of the output by using the conventional transfer function notation $H(s) = \mathcal{B}(s)/\mathcal{A}(s)$ is

$$Y(s) = U(s)H(s) = \frac{U_{\mathrm{A}}(s)}{1 - e^{-sT_{\mathrm{p}}}} \frac{\mathcal{B}(s)}{\mathcal{A}(s)}. \tag{A.2.6}$$

In general $Y(s)$ is not the transform of a periodic signal, since the condition $\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\{U_{\mathrm{A}}(s)H(s)\} = 0$ is not fulfilled for $t > T_{\mathrm{p}}$. $H(s)$ is always (except for the case of dead-time) a rational function, but this cannot be said about $U_{\mathrm{A}}(s)$. Decompose the function $Y(s)$ into the sum of a periodic and a non-periodic function, i.e.,

$$Y(s) = \frac{U_{\mathrm{A}}(s)}{1 - e^{-sT_{\mathrm{p}}}} \frac{\mathcal{B}(s)}{\mathcal{A}(s)} = \frac{Y_{\mathrm{A}}(s)}{1 - e^{-sT_{\mathrm{p}}}} + \frac{\mathcal{C}(s)}{\mathcal{A}(s)}, \tag{A.2.7}$$

where $y_A(t) = \mathcal{L}^{-1}\{Y_A(s)\}$, $y_A(t > T_p) = 0$, and $\mathcal{C}(s)$ are unknown polynomials. From this equation the basic function of the periodic output component can be expressed as

$$Y_A(s) = \frac{\mathcal{B}(s)U_A(s) - (1 - e^{-sT_p})\boldsymbol{C}(s)}{\mathcal{A}(s)} = H(s)U_A(s) - \left(1 - e^{-sT_p}\right)\frac{\mathcal{C}(s)}{\mathcal{A}(s)}. \tag{A.2.8}$$

By transforming back $Y_A(s)$, zero has to be obtained for the time $t > T_p$. Using these conditions $\mathcal{C}(s)$ can be determined. Apply the expansion theorem and assuming single poles we get

$$y_A(t) = \sum_{i=1}^{n} \frac{\mathcal{B}(p_i)U_A(p_i) - (1 - e^{-p_i T_p})\mathcal{C}(p_i)}{\mathcal{A}'(p_i)} e^{p_i t} = 0; \quad t > T_p. \tag{A.2.9}$$

Since the factors $e^{p_i t}$ cannot be zero, therefore the function $y_A(t)$ can be zero for all time points $t > T_p$ only if the coefficients of all n factors are zero, i.e.,

$$\mathcal{C}(p_i) = \frac{\mathcal{B}(p_i)U_A(p_i)}{1 - e^{-p_i T_p}} = \alpha_i; \quad i = 1, \ldots, n. \tag{A.2.10}$$

This condition, at the same time, gives the solution for the coefficients of the unknown $\mathcal{C}(s)$, since n independent linear equations can be formulated.

$$1 + c_1 p_i + c_2 p_i^2 + \cdots + c_n p_i^n = \alpha_i; \quad i = 1, \ldots, n. \tag{A.2.11}$$

The coefficients come from the solution of these equations whose compact form is

$$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} p_1 & p_1^2 & \ldots & p_1^n \\ p_2 & p_2^2 & \ldots & p_2^n \\ \vdots & \vdots & \ddots & \vdots \\ p_n & p_n^2 & \ldots & p_n^n \end{bmatrix}^{-1} \begin{bmatrix} \alpha_1 - 1 \\ \alpha_2 - 1 \\ \vdots \\ \alpha_n - 1 \end{bmatrix}. \tag{A.2.12}$$

Based on (A.2.7) the complete time function of the process output is

$$y(t) = 1_A(t) y_A(t) + y_{tr}(t), \tag{A.2.13}$$

where $y_{tr}(t)$ is the so-called non-periodic transient factor

$$y_{tr}(t) = \mathcal{L}^{-1}\left\{\frac{\mathcal{C}(s)}{\mathcal{A}(s)}\right\} = \sum_{i=1}^{n} \frac{\mathcal{C}(p_i)}{\mathcal{A}'(p_i)} e^{p_i t} = \sum_{i=1}^{n} \frac{\mathcal{B}(p_i)}{\mathcal{A}'(p_i)} \frac{U_A(p_i)}{1 - e^{-p_i T_p}} e^{p_i t}. \tag{A.2.14}$$

Since the expansion theorem requires only the substitution values of $\mathcal{C}(p_i)$, it is not necessary to solve the system of equations (A.2.12).

Based on (A.2.8) the basic function $y_{\mathrm{A}}(t)$ of the output signal is

$$y_{\mathrm{A}}(t) = \mathcal{L}^{-1}\{H(s)U_{\mathrm{A}}(s)\} - \mathcal{L}^{-1}\left\{\frac{\mathcal{C}(s)}{\mathcal{A}(s)}\right\}; \quad 0 < t < T_{\mathrm{p}}, \tag{A.2.15}$$

which is obtained by the inverse Laplace transform. (Here the effect of $e^{-sT_{\mathrm{p}}}$ in (A.2.8) does not have to be taken into account, because the response is out of the basic period.) Applying the expansion theorem yields

$$\begin{aligned} y_{\mathrm{A}}(t) &= \mathcal{L}^{-1}\left\{\frac{\mathcal{B}(s)}{\mathcal{A}(s)}U_{\mathrm{A}}(s)\right\} - \sum_{i=1}^{n}\frac{\mathcal{C}(p_i)}{\mathcal{A}'(p_i)}e^{p_i t} \\ &= \mathcal{L}^{-1}\left\{\frac{\mathcal{B}(s)}{\mathcal{A}(s)}U_{\mathrm{A}}(s)\right\} - \sum_{i=1}^{n}\frac{\mathcal{B}(p_i)}{\mathcal{A}'(p_i)}\frac{U_{\mathrm{A}}(p_i)}{1-e^{-p_i T_{\mathrm{p}}}}e^{p_i t}; \quad 0 < t < T_{\mathrm{p}}. \end{aligned} \tag{A.2.16}$$

Note that $y_{\mathrm{A}}(t) \neq \mathcal{L}^{-1}\{H(s)U_{\mathrm{A}}(s)\}$, thus $Y_{\mathrm{A}}(s) \neq H(s)U_{\mathrm{A}}(s)$.

The process output of an *LTI* process excited by a periodic signal has two factors: a periodic signal and a transient signal. After the transient is died out only the periodic component remains. These two components appear even if the initial energy content of the process is zero (the initial state vector in the state equation is zero), i.e., the above two components must not be mistaken for the factors obtained from the solution of the homogeneous (un-excited) and inhomogeneous (excited) state equations. The above components of the response obtained for a periodic excitation appear even if the initial condition is not a zero vector. Thus, in the general case, the process response has three components.

A.3 Standard Control Engineering Signals and Notations

A.3.1 Standard Notations in Control Engineering

The design, installation, operation and maintenance of process control systems require the cooperation of the participants who are working on the solution of the task. In order to achieve this, it is required to use common notation in the documentation of each piece of equipment of the different process control functions. In the documentation, the notation of the process control equipment refers to its technical character and how it is connected to the process. Standard graphical and alphanumeric notation helps the engineers and, technicians to interpret the design documentation.

The notation systems and standard protocols may differ in different branches of the industry (chemical, energy, agriculture, etc.).

The standard DIN (Deutsches Institut für Normung) 19227 contains several graphical symbols for sensors, controllers, actuators, and control equipment. Further recommendations can be found in standards DIN 1946, 2429, 2481, 19239 and 30600.

The instrumentation and control functions are usually represented by a circle or oval curve containing letters and numbers. The letters refer to the character of the physical quantity and the control function, the numbers give the place of the equipment in the process (e.g., serial number of the valve, motor or sensor).

In the instrumentation designs [see Fig. A.3.1] the first letter of the text in the circle refers to the character of the measured or controlled quantity, e.g., the meaning of some of the first letters are: E—electrical signal, F—flowing quantity, G—movement or position, L- level, P- pressure, Q- composition or other material character (frequently it is denoted by A, too), S- speed, T- temperature, V- viscosity. The second letter means the control function, e.g., T- sensing, C- control. For example the text LC in the circle means level controller. Further letters can refer to further functions, e.g., to alarm, security operation, computer connection, transducers, etc. Figure A.3.1 illustrates the composition control of the liquid in the mixer tank and the standard notations of the valve, composition sensor and controller.

There is an other standard, KKS (Kraftwerk Kennzeichen System), which has been developed in the German electrical industry and primarily used by European firms. This notation fits the functional structure of the technology. The process control functions and notations fit with the mechanical and electrical power transmission functions and notations. The unified notations of the equipments make it possible to identify the technological units in a decomposed part of a complex technology. For example, the notation 03GCR31AA101 for a valve means that it is

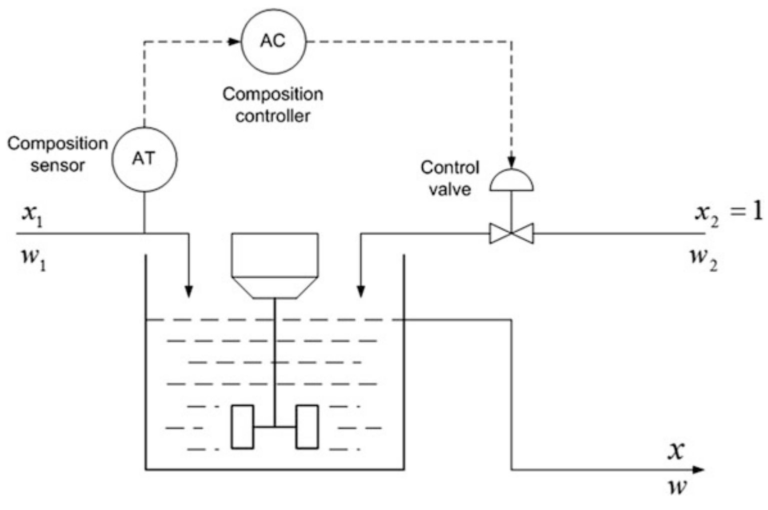

Fig. A.3.1 Typical notations applied in the instrumentation designs

Table A.3.1 Most generally used names in control engineering

Control	Disturbance, noise
Open-loop control	Manipulated variable
Closed-loop control, feedback control	Output signal, controlled variable
Process, plant	Reference signal, set-point
Sensor	Error signal
Actuator	Control signal
Controller, regulator, control algorithm	Measured output, sensor output

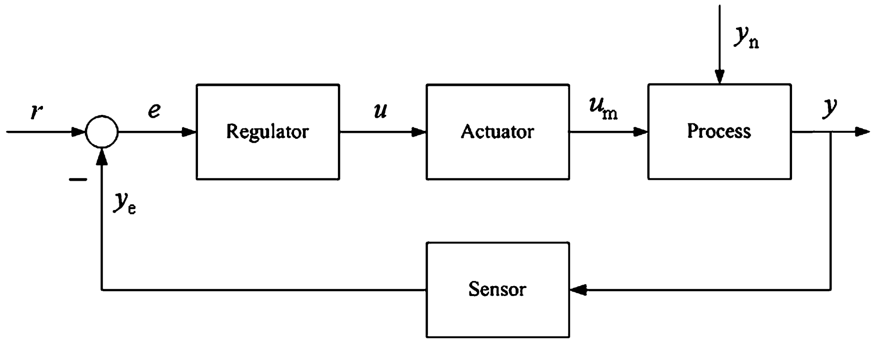

Fig. A.3.2 Operational scheme of the control loop

in system 03, GCR means the subtechnology, 31 is the serial number of the pipeline. AA means to which equipment this valve is connected, 101 is the serial number of the valve. This detailed notation makes it possible to identify unambiguously the equipments. Each technology has its own system identification notations.

A.3.2 The Names of the Most Important Signals in Control Systems

Table A.3.1 contains certain names most generally used in control engineering.

The operation scheme of a control loop is shown in Fig. A.3.2. The dynamics of the actuator and sensor are usually included in the dynamics of the controlled system. The joint scheme is shown by Fig. A.3.3.

A.4 Computer-Aided Design (*CAD*) Systems

Nowadays the design of complex systems is inconceivable without computers. The fast computers, the sophisticated developing environments and the well elaborated design algorithms make it possible to design and simulate simple, precise and

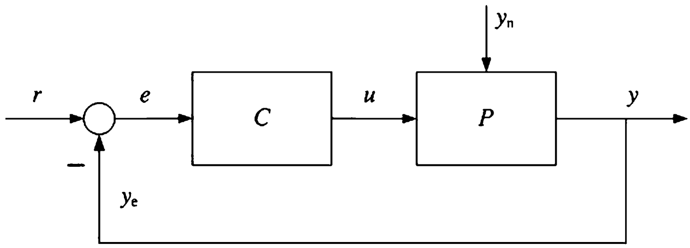

Fig. A.3.3 The joint scheme of the control loop

flexible control systems. The design consists of two phases: the design of the controllers and the simulation of the system. The control parameters are determined on the basis of the quality requirements. During the simulation the operation of the system is investigated for given parameters on the basis of a criterion or visual performance. A graphical presentation is more and more in the front, because this technique makes possible a fast, precise and information rich presentation.

There are several program packages available for the design of control systems. These can be classified in two groups. The first group contains the packages for general mathematical computations which might be extended for the design of controllers. The other group contains the industrial control systems whose main goal is to perform the control or to solve special control tasks.

A.4.1 Mathematical Program Packages

The most well-known control design packages were primarily developed for general mathematical computations, but later they were extended by special tools for helping the design procedures. There are several program packages, however, which originally were not designated for the design of controllers, but later, due to their mathematical and graphical capabilities, were applied for design, too.

MATLAB®

The program package **MATLAB®** has been elaborated for scientific and engineering computations, simulation and graphical presentation. It provides a strong background to the solution of differential equations, handling matrix algebra and the solution of other mathematical problems and to the presentation of the results in good quality and also graphically. The extended application of **MATLAB®** derives from the fact that its command set can be extended by toolboxes. A toolbox is actually a function library developed for supporting different subject areas. **MATLAB®** has very good graphical capabilities, and relatively complex design tasks can be performed within an acceptable running time range. The programming of **MATLAB®** is interactive, which means that it performs the commands row by row without translation. Its speed is based on coding the critical program parts in a lower level language, generally in C or C++, and on direct access to the system matrix structure.

The essence of the matrix programming is that the matrix operations are performed automatically by the triggered functions for all elements of the matrix instead of special embedded cycles. **MATLAB**® supports several mathematical operations, procedures (e.g., handling of complex numbers, computation of inverse and eigenvalues of a matrix, FOURIER transformation, convolution computation and determination of the roots of equations). **MATLAB**® does not support directly symbolic computations but makes that possible by the *Symbolic Math Toolbox.* The *Symbolic Toolbox* is based on **MAPLE**® but it has an interface to **MATLAB**®.

MATLAB® is primarily used in the engineering environment. If a new algorithm or theory appears then they are immediately developed in the form of toolboxes or function libraries in order to investigate and compare them with other methods.

The *Control System Toolbox* contains functions for the design and simulation of control systems. The controller can be given in transfer function or state space form. It is able to investigate continuous and discrete systems in the time and frequency domains. It can handle single and multi input-multi output, linear and nonlinear systems. The toolboxes are open, they can be easily extended with other functions and algorithms. The *Control System Toolbox* can be well used with other toolboxes, e.g., with the *Fuzzy Logic Toolbox*, *Model Predictive Control Toolbox*, *Nonlinear Control Design Blockset, System Identification Toolbox* and *Robust Control Toolbox.*

SIMULINK®is a graphical program package for modeling and simulation of dynamic systems. The simulation is interactive, therefore the effect of changing the parameters can be well presented. In **SIMULINK**® the dynamic system is given by a block-diagram, the different blocks can be copied from a library. **SIMULINK**® is able to simulate linear and nonlinear systems in the continuous, discrete and hybrid domains. **SIMULINK**® simulates the models by integrating ordinary differential equations. It can use several integrating methods. The result of the simulation can be further used by **MATLAB**® for data processing or graphical presentation. The graphical abilities of **SIMULINK**® facilitate significantly the design and simulation of the controllers.

MATHEMATICA®

MATHEMATICA® is an interactive system for mathematical computations. It supports numerical and symbolic computations and also includes a high level programming language which makes it possible for the user to develop new procedures. **MATHEMATICA**® is one of the most effective systems for general mathematical computations, which has roughly two million users all over the world.

Starting from the 60s there have been programs for special computations, but **MATHEMATICA**® with its completely new approach made it possible to handle uniformly the different fields of technical computation. Appearing in 1988 it brought significant change in the usage of the computers in several fields. The program was developed by the research group of *Wolfram Research* led by Stephen

WOLFRAM. The key development was to develop a new symbolic computer language which made first possible to handle a wide range of objects necessary for technical computations by a few basic categories (primitives). Among the developers and users a high number of mathematicians and research engineers can be found. It is very popular in education, nowadays several hundreds of textbooks are based on it and it is a very important tool among students worldwide. It is very useful in writing complex studies, reports, because it provides a uniform environment for computation, modeling, text editing and graphical presentation. One of its disadvantages is that its learning curve is quite steep, the acquirement of its basic operation is not easy. Its most important advantage is its openness, it can be easily extended to new subject areas, as, e.g., to applied mathematics, informatics, control engineering, economics, sociology, etc.

In **MATHEMATICA®** the basic arithmetic operations can be performed. It can also handle complex numbers. Its most important data structure is the list, which practically corresponds to a set. The lists can be defined as embedded, and different operations can be accomplished on them, e.g., unification, cut, adding a term and deleting a term, etc. The matrices are the special forms of the lists. The typical matrix operations can also be performed, like inversion and eigenvalue computations.

Due to its symbolic capabilities it can be well used for algebraic transformations. Several such transformations can be made very easily which are difficult to compute by hand, e.g., simplification of fractions, series expansions, decomposition into partial fractions, solving equations, minimum seeking, differentiation, and integration.

In **MATHEMATICA®** the functions are formal transformation rules. Any kind of object can appear as the input or output of a function. The function may consist of mathematical commands, program control commands (e.g., if, then, for) or it can be written even in another programming language (e.g., FORTRAN, C).

Due to its graphical capabilities the data can be presented in one, two or three dimensions.

MAPLE®

MAPLE® is a general computer algebraic system for solving mathematical problems and presenting technical figures with excellent quality. It is easy to learn and anybody can perform complex mathematical computations after a very short time. **MAPLE®** contains also high level programming languages by means of which the users can define their own procedures. Its main feature is providing symbolic computations, algebraic transformations, series expansions, integration and differential computations. It can be used in several areas of mathematics, e.g., for solving linear algebraic, statistical and group theoretical tasks. The commands can be performed interactively or in a group (*batch mode*). It can be well used in education and for development. Its capabilities can be extended by adding outer functions. It contains more than 2500 functions for different subject areas. Several of them were developed by external, independent companies, firms and research institutes. The most frequently used function libraries, toolboxes are:

- *Global Optimization Toolbox*
- *Database Integration Toolbox*
- *Fuzzy Sets*
- **MAPLE**®*Professional Math Toolbox* for **LabVIEW**®
- *Analog Filter Design Toolbox*
- *ICP* for **MAPLE**® (*Intelligent Control and Parameterization*: it makes possible the design of automatic, intelligent and robust controllers)

Its mathematical capabilities and the *ICP toolbox* provides the opportunity to solve control engineering tasks but in spite of this it is mainly used by statisticians and mathematicians and less by control engineers.

SysQuake®

SysQuake® is a very similar system to **MATLAB**® concerning its commands. It has been developed for solving design tasks interactively directly on the screen. By its help, e.g., by directly changing the place of the poles and zeros, the breakpoint frequencies, the controller or process parameters, several system attributes (Bode diagram, Nyquist diagram, root-locus, transfer functions of the closed-loop signals) can be followed simultaneously in the design procedure. The software tools for man-machine interaction can be easily realized in object-oriented structures.

A.4.2 Industrial Control Systems

Nowadays, industrial control systems have special *CAD* tools. Sometimes these do not provide a wide range of design possibilities: they are usually restricted only to those algorithms ensuring the operation of a given system. In many cases this means only a simple *PID* controller whose parameters can be set in a simulation environment. The industrial control systems are usually able to perform certain kind of automatic design, e.g., in the case adaptive systems where the parameters of the controller are automatically set based on the system's behavior. Several significant industrial companies have serious system and control design background. They can be sorted according to their functions:

- Firms producing integrated control systems, *Rockwell*, *Honeywell*. They perform the control of the whole factories, like *Rockwell Automation Ltd.*
  Rockwell Software: their program package enables the integrated control of the whole factories including automation tasks.
- Robot manufacturing firms: *Fanuk, Panasonic, ABB*. Nowadays ready made robots perform a certain part of the automated manufacturing.
- PLC producing firms: *Siemens*, *Allen Bradly* (*Rockwell*), *Toshiba*. The PLC (Programmable Logic Controller) is one of the main elements of the industrial process control systems.
- Firms producing data collecting and measurement systems, like: *National Instruments, Siemens*, etc.

Among the above firms several have also some additional activities. They generally develop program systems which can be used only for their machines and equipments. From the great number of industrial systems perhaps only the **LabVIEW®** program package developed by National Instruments is widely used and has become an accepted developing environment by other firms as well.

LabVIEW®

LabVIEW® provides a graphical developing environment for data collection, signal processing, and data presentation. It makes possible flexible, high level programming without the complexity of programming languages. It has all the programming tools (e.g., handling of data structures, cycles and events) which are given in classical programming languages, but in a simpler environment. **LabVIEW®** has also an embedded translator whose efficiency is comparable to a C translator concerning the speed and memory requirements.

The effectiveness and popularity of **LabVIEW®** is due to the fact that it has several (presently about 50) program libraries, toolkits available for developers. These include different virtual tools, sample programs and documentation fitting well with the developing environments and applications. These functions are designed and optimized for such special demands which comprise a wide range of fields, from signal processing, communication to the data structure. The main toolkits are the following:

– *Application Deployment & Targeting Modules*
– *Software Engineering & Optimization Tools*
– *Data Management and Visualization*
– *Real-Time and FPGA Deployment*
– *Embedded System Deployment*
– *Signal Processing and Analysis*
– *Automated Testing*
– *Image Acquisition and Machine Vision*
– *Control Design & Simulation*
– *Industrial Control*

The *Control Design Toolkit* is able to design and analyze controllers in the **LabVIEW®** environment. The main features of the *Control Design Toolkit are:*

– The **LabVIEW®***Control Design Toolkit* can design and analyze the controllers in the **LabVIEW®** environment. It provides interactive graphical design, e.g. by the help of root-locus.
– The process and the controller can be given in transfer function and state-space forms.
– These modules are integrated with the **LabVIEW®***Simulation Module*.
– The behavior of the system can be investigated by several tools, e.g. step response function, BODE diagram, allocation of zeros and poles, etc.

LabVIEW® ensures an integrated environment for data collection, identification, controller design and simulation. The system's behavior can be graphically investigated, while its parameters can be adjusted.

A.5 Proofs and Derivations (By Chapters)

A.2.1

It is very simple to determine the Bode diagram of

$$H(s) = 1 + sT; \quad H(j\omega) = 1 + j\omega T = |H(j\omega)|e^{j\varphi(\omega)}. \tag{A.2.1}$$

The dependence of its absolute value and phase angle on the frequency is

$$|H(j\omega)| = \sqrt{1+\omega^2 T^2} = \left[10\lg\left(1+\omega^2 T^2\right)\right]\mathrm{dB}; \quad \varphi(\omega) = \mathrm{arctg}\,\omega T. \tag{A.2.2}$$

Investigating the asymptotic behavior of the functions we get

$$H(j\omega) \approx 1; \quad |H(j\omega)| \approx 0\,\mathrm{dB}; \quad \varphi(\omega) \approx 0, \quad \text{if} \quad \omega \ll \omega_1 = 1/T \tag{A.2.3}$$

and

$$\begin{aligned} &H(j\omega) \approx j\omega \mathrm{T}, \ \ |\mathrm{H}(\mathrm{j}\omega)| \approx (20\lg\omega + 20\lg\mathrm{T})\mathrm{dB}; \\ &\varphi(\omega) \approx 90^{\circ}, \quad \text{if} \quad \omega \gg \omega_1 = 1/\mathrm{T} \end{aligned} \tag{A.2.4}$$

If logarithmic scaling is applied for the frequency axis then both asymptotes of the amplitude are straight lines. On the frequency axis there are two points at a distance of a decade, for which $\omega_2 = 10\omega_1$, i.e., $\lg\omega_2 = 1 + \lg\omega_1$. Thus in logarithmic scale the decade means constant distance. So in the region $\omega \gg \omega_1$ the asymptote of the curve is a line having slope of 20 dB/decade, which cuts the 0 dB axis at ω_1 (the brake frequency). Here the actual value is

$$|H(j\omega_1)| = (20\lg 2)\,\mathrm{dB} = 3\,\mathrm{dB} \quad \text{and} \quad \varphi(\omega_1) = \mathrm{arctg}1 = 45^{\circ} \tag{A.2.5}$$

The tangents of the functions are

$$\frac{\mathrm{d}|H(j\omega)|}{\mathrm{dlg}\omega} = 10\frac{\mathrm{dlg}(1+\omega^2T^2)}{\mathrm{d}\omega}\frac{\mathrm{d}\omega}{\mathrm{dlg}\omega} = 10\frac{2\omega T^2}{1+\omega^2T^2}\omega\,\mathrm{dB/decade} \tag{A.2.6}$$

$$\frac{\mathrm{d}\varphi(\omega)}{\mathrm{dlg}\omega} = \frac{\mathrm{darctg}\omega T}{\mathrm{d}\omega}\frac{\mathrm{d}\omega}{\mathrm{dlg}\omega} = \frac{T}{1+\omega^2T^2}\frac{\omega}{\lg e}\frac{180^{\circ}}{\pi}\mathrm{degree/decade} \tag{A.2.7}$$

and their slopes at the break frequency

$$\left.\frac{\mathrm{d}|H(j\omega)|}{\mathrm{d}\lg\omega}\right|_{\omega_1} = 10\,\mathrm{dB/decade} \tag{A.2.8}$$

$$\left.\frac{\mathrm{d}\varphi(\omega)}{\mathrm{d}\lg\omega}\right|_{\omega_1} = 66\,\mathrm{degree/decade} \tag{A.2.9}$$

A.3.1

The solution of the state equation can be given by (3.18). To prove it let us differentiate the equation

$$\frac{\mathrm{d}\boldsymbol{x}(t)}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left[e^{\boldsymbol{A}t}\boldsymbol{x}(0)\right] + \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_0^t e^{\boldsymbol{A}(t-\tau)}\boldsymbol{b}u(\tau)\mathrm{d}\tau\right], \tag{A.3.1}$$

where

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[e^{\boldsymbol{A}t}\boldsymbol{x}(0)\right] = \boldsymbol{A}e^{\boldsymbol{A}t}\boldsymbol{x}(0) \tag{A.3.2}$$

and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_0^t e^{\boldsymbol{A}(t-\tau)}\boldsymbol{b}u(\tau)\mathrm{d}\tau\right] &= \int_0^t \frac{\mathrm{d}}{\mathrm{d}t}\left[e^{\boldsymbol{A}(t-\tau)}\boldsymbol{b}u(\tau)\right]\mathrm{d}\tau + \frac{\mathrm{d}t}{\mathrm{d}t}\left[e^{\boldsymbol{A}(t-\tau)}\boldsymbol{b}u(\tau)\right]_{\tau=t} \\ &\quad - \frac{\mathrm{d}0}{\mathrm{d}t}\left[e^{\boldsymbol{A}(t-\tau)}\boldsymbol{b}u(\tau)\right]_{\tau=0} = \int_0^t \boldsymbol{A}e^{\boldsymbol{A}(t-\tau)}\boldsymbol{b}u(\tau)\mathrm{d}\tau + \boldsymbol{b}u(\tau)\end{aligned} \tag{A.3.3}$$

where the expressions $\mathrm{d}t/\mathrm{d}t = 1$, $\mathrm{d}0/\mathrm{d}t = 0$ and $\left.e^{\boldsymbol{A}(t-\tau)}\right|_{\tau=t} = 1$ are taken into consideration. Thus the derivative of (3.18) is

$$\frac{\mathrm{d}\boldsymbol{x}(t)}{\mathrm{d}t} = \boldsymbol{A}e^{\boldsymbol{A}t}\boldsymbol{x}(0) + \int_0^t \boldsymbol{A}e^{\boldsymbol{A}(t-\tau)}\boldsymbol{b}u(\tau)\mathrm{d}\tau + \boldsymbol{b}u(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{b}u(t). \tag{A.3.4}$$

A.3.2

In the case of zero initial conditions (i.e. $\boldsymbol{x}(0) = \mathbf{0}$) and $d = 0$, the impulse response of a system to the excitation $u(t) = \delta(t)$ can be computed from (3.18)

$$\begin{aligned} \boldsymbol{x}(t) &= \int_0^t e^{\boldsymbol{A}(t-\tau)}\boldsymbol{b}\delta(\tau)\mathrm{d}\tau = e^{\boldsymbol{A}t}\left[\int_0^t e^{-\boldsymbol{A}\tau}\delta(\tau)\mathrm{d}\tau\right]\boldsymbol{b} = e^{\boldsymbol{A}t}\left[e^{-\boldsymbol{A}\tau}\delta(\tau)\right]_0^t\boldsymbol{b} \\ &= e^{\boldsymbol{A}t}\left[-e^{-\boldsymbol{A}t}\delta(t) + e^{-\boldsymbol{A}0}\delta(0)\right]\boldsymbol{b} = e^{\boldsymbol{A}t}\boldsymbol{b} \\ w(t) &= y(t) = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}(t) = \boldsymbol{c}^{\mathrm{T}}e^{\boldsymbol{A}t}\boldsymbol{b} \end{aligned} \tag{A.3.5}$$

which is equal to (3.25) which was obtained in the operator domain.

A.3.3

One of the most important theorems in matrix theory is the Cayley-Hamilton Theorem. A matrix fulfills its own characteristic equation, i.e., the equation $\mathcal{A}(\boldsymbol{A}) = 0 = \det(s\boldsymbol{I} - \boldsymbol{A}) = 0$ which is formally the same as

$$\mathcal{A}(\boldsymbol{A}) = \mathbf{0} \tag{A.3.6}$$

[see Appendix A.1]. Equation (A.3.7) is satisfied also by the matrix polynomial $\mathcal{P}(\boldsymbol{A})$ of matrix $\boldsymbol{A}$, but also by any such matrix function $\boldsymbol{F}(\boldsymbol{A})$ whose associated function $f(s)$ is analytical (regular) in a certain region around the origin of the s-plane. Let the basic matrix be $\boldsymbol{F}(\boldsymbol{A}) = e^{\boldsymbol{A}\tau}$, then based on the above expressions we get

$$e^{\boldsymbol{A}\tau} = \alpha_{\mathrm{o}}(\tau)\boldsymbol{I} + \alpha_1(\tau)\boldsymbol{A} + \cdots + \alpha_{\mathrm{n}-1}(\tau)\boldsymbol{A}^{n-1}. \tag{A.3.7}$$

A.5.1

The Nyquist stability criterion can be derived from the Cauchy argument principle of the theory of complex functions.

The argument principle

Let Γ be a closed curve, not cutting itself, in the complex plane, which surrounds the region D. Consider the function $f(z)$ of the complex variable z. Suppose the function $f(z)$ has P poles and Z zeros in the domain D. All poles and zeros are taken into account with their multiplicity. In all the other points of the domain the function is analytic (thus at these points it is differentiable).

Due to the argument principle, going round the curve anti-clockwise, the angle change $\Delta_\Gamma \arg f(z)$ of the function $f(z)$ is $2\pi(Z - P)$,

$$\frac{1}{2\pi}\Delta_\Gamma \arg f(z) = \frac{1}{2\pi j}\int_\Gamma \frac{f'(z)}{f(z)}\mathrm{d}z = Z - P. \tag{A.5.1}$$

Proof Assume that $f(z)$ has a zero of multiplicity m at the point $z = \alpha$. In the vicinity of the zero the function $f(z)$ can be written as: $f(z) = (z-\alpha)^m g(z)$, where $g(z)$ is an analytic function. Constitute the expression $f'(z)/f(z)$:

$$\frac{f'(z)}{f(z)} = \frac{m}{z-\alpha} + \frac{g'(z)}{g(z)}. \tag{A.5.2}$$

The second term on the right hand side of (A.5.2) is analytic at $z = \alpha$. The numerator of the first term gives the residue.

In (A.5.1) the integral around the closed curve is the sum of the residues, considering the zeros and poles it is $Z - P$. Otherwise, taking into account that

$$\frac{f'(z)}{f(z)} = \frac{\mathrm{d}}{\mathrm{d}z}\ln f(z) \tag{A.5.3}$$

the following relationship can be derived:

$$\begin{aligned}\int_\Gamma \frac{f'(z)}{f(z)}\mathrm{d}z &= \int_\Gamma \mathrm{d}(\ln f(z)) = \int_\Gamma \mathrm{d}(\ln\{|f(z)|\exp(j\arg f(z))\}) \\ &= \int_\Gamma \mathrm{d}\ln|f(z)| + j\int_\Gamma \mathrm{d}(\arg f(z)) = j\Delta_\Gamma \arg f(z) = 2\pi j(Z-P)\end{aligned} \tag{A.5.4}$$

This proves the argument principle given by (A.5.1), thus

$$\frac{1}{2\pi}\Delta_\Gamma \arg f(z) = Z - P. \tag{A.5.5}$$

The N*yquist stability criterion*

Investigate the stability of a closed control loop havingnegative feedback. The characteristic equation is

$$1 + L(s) = 0 \tag{A.5.6}$$

where $L(s)$ is the transfer function of the open loop.

Consider the closed curve on the complex plane shown in Fig. 5.17. If $L(s)$ has poles also on the imaginary axis, then pass around them at a small radius according to Fig. 5.18. The characteristic polynomial can also be written in the form of (5.31) as

$$1+L(s)=1+\frac{\mathcal{N}(s)}{\mathcal{D}(s)}=\frac{\mathcal{D}(s)+\mathcal{N}(s)}{\mathcal{D}(s)}=k\frac{(s-z_1)(s-z_2)\ldots(s-z_n)}{(s-p_1)(s-p_2)\ldots(s-p_n)}. \quad \text{(A.5.7)}$$

Let the characteristic polynomial be the function $f(z)$ to be used for a mapping. Mapping the curve of Fig. 5.17 by the characteristic polynomial, the argument principle can be applied. Since the curve of Fig. 5.17 is passed around clockwise, the number of times R the mapped curve encircles the origin is

$$\frac{1}{2\pi}\Delta_\Gamma \arg f(z)=R=P-Z. \quad \text{(A.5.8)}$$

To ensure stability, the characteristic equation must not have roots in the right half-plane, thus the condition for stability is

$$Z=0 \quad \text{(A.5.9)}$$

and from this,

$$R=P. \quad \text{(A.5.10)}$$

This means that the control system is stable if the curve mapping the curve of Fig. 5.17 by the characteristic polynomial encircles the origin anti-clockwise as many times as there are the unstable, right half-plane poles of the open-loop.

Mapping the curve $L(s)$ instead of the characteristic polynomial we get the so-called complete NYQUIST curve. Investigating its windings around the point $-1+0j$, the system is stable if the condition (A.5.10) is fulfilled.

A.9.1

Use the notation introduced in (3.13)

$$\boldsymbol{\Phi}(s)=(s\boldsymbol{I}-\boldsymbol{A})^{-1}=\frac{\mathbf{adj}(s\boldsymbol{I}-\boldsymbol{A})}{\det(s\boldsymbol{I}-\boldsymbol{A})}=\frac{\mathbf{adj}(s\boldsymbol{I}-\boldsymbol{A})}{\mathcal{A}(s)}=\frac{\boldsymbol{\Psi}(s)}{\mathcal{A}(s)} \quad \text{(A.9.1)}$$

to simplify the complex form $\boldsymbol{c}^{\mathrm{T}}\left(s\boldsymbol{I}-\boldsymbol{A}+\boldsymbol{b}\boldsymbol{k}^{\mathrm{T}}\right)^{-1}\boldsymbol{b}$ and use the matrix inversion lemma

$$\left(s\boldsymbol{I}-\boldsymbol{A}+\boldsymbol{b}\boldsymbol{k}^{\mathrm{T}}\right)^{-1}=\left[\boldsymbol{\Phi}^{-1}(s)+\boldsymbol{b}\boldsymbol{k}^{\mathrm{T}}\right]^{-1}=\boldsymbol{\Phi}(s)-\boldsymbol{\Phi}(s)\boldsymbol{b}\left[1+\boldsymbol{k}^{\mathrm{T}}\boldsymbol{\Phi}(s)\boldsymbol{b}\right]^{-1}\boldsymbol{k}^{\mathrm{T}}\boldsymbol{\Phi}(s) \quad \text{(A.9.2)}$$

by means of which

$$\boldsymbol{c}^{\mathrm{T}}\left(s\boldsymbol{I}-\boldsymbol{A}+\boldsymbol{b}\boldsymbol{k}^{\mathrm{T}}\right)^{-1}\boldsymbol{b}=\boldsymbol{c}^{\mathrm{T}}\boldsymbol{\Phi}(s)\boldsymbol{b}-\frac{\boldsymbol{c}^{\mathrm{T}}\boldsymbol{\Phi}(s)\boldsymbol{b}\boldsymbol{k}^{\mathrm{T}}\boldsymbol{\Phi}(s)\boldsymbol{b}}{1+\boldsymbol{k}^{\mathrm{T}}\boldsymbol{\Phi}(s)\boldsymbol{b}}=\frac{\boldsymbol{c}^{\mathrm{T}}\boldsymbol{\Phi}(s)\boldsymbol{b}}{1+\boldsymbol{k}^{\mathrm{T}}\boldsymbol{\Phi}(s)\boldsymbol{b}}. \quad \text{(A.9.3)}$$

So (9.5) can be further modified

$$T_{\mathrm{ry}}(s) = \frac{\boldsymbol{c}^{\mathrm{T}}\boldsymbol{\Phi}(s)\boldsymbol{b}k_{\mathrm{r}}}{1+\boldsymbol{k}^{\mathrm{T}}\boldsymbol{\Phi}(s)\boldsymbol{b}}. \tag{A.9.4}$$

Note that here

$$\boldsymbol{c}^{\mathrm{T}}\boldsymbol{\Phi}(s)\boldsymbol{b} = \boldsymbol{c}^{\mathrm{T}}(s\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{b} = P(s) = \frac{\mathcal{B}(s)}{\mathcal{A}(s)} \tag{A.9.5}$$

by means of which

$$\begin{aligned} T_{\mathrm{ry}}(s) &= \frac{\boldsymbol{c}^{\mathrm{T}}\boldsymbol{\Phi}(s)\boldsymbol{b}k_{\mathrm{r}}}{1+\boldsymbol{k}^{\mathrm{T}}\boldsymbol{\Phi}(s)\boldsymbol{b}} = \frac{k_{\mathrm{r}}}{1+\boldsymbol{k}^{\mathrm{T}}\frac{\boldsymbol{\Psi}(s)}{\mathcal{A}(s)}\boldsymbol{b}}P(s) = \frac{k_{\mathrm{r}}}{1+\boldsymbol{k}^{\mathrm{T}}\frac{\boldsymbol{\Psi}(s)}{\mathcal{A}(s)}\boldsymbol{b}}\frac{\mathcal{B}(s)}{\mathcal{A}(s)} \\ &= \frac{k_{\mathrm{r}}\mathcal{B}(s)}{\mathcal{A}(s)+\boldsymbol{k}^{\mathrm{T}}\boldsymbol{\Psi}(s)\boldsymbol{b}} \end{aligned} \tag{A.9.6}$$

A.9.2

The static unit gain of the transfer function$T_{\mathrm{ry}}(s)$ of the closed system can be ensured by the scaling factor k_{r}. From the condition

$$T_{\mathrm{ry}}(s)\big|_{s=0} = \boldsymbol{c}^{\mathrm{T}}\left(-\boldsymbol{A}+\boldsymbol{b}\boldsymbol{k}^{\mathrm{T}}\right)^{-1}\boldsymbol{b}k_{\mathrm{r}} = 1 \tag{A.9.7}$$

it is obtained that

$$k_{\mathrm{r}} = -1/\boldsymbol{c}^{\mathrm{T}}\left(\boldsymbol{A}-\boldsymbol{b}\boldsymbol{k}^{\mathrm{T}}\right)^{-1}\boldsymbol{b}. \tag{A.9.8}$$

Applying the matrix inversion lemma in the denominator,

$$\left(\boldsymbol{A}-\boldsymbol{b}\boldsymbol{k}^{\mathrm{T}}\right)^{-1} = \boldsymbol{A}^{-1}+\boldsymbol{A}^{-1}\boldsymbol{b}\left[1-\boldsymbol{k}^{\mathrm{T}}\boldsymbol{A}^{-1}\boldsymbol{b}\right]^{-1}\boldsymbol{k}^{\mathrm{T}}\boldsymbol{A}^{-1}, \tag{A.9.9}$$

we get that

$$\begin{aligned} \boldsymbol{c}^{\mathrm{T}}\left(\boldsymbol{A}-\boldsymbol{b}\boldsymbol{k}^{\mathrm{T}}\right)^{-1}\boldsymbol{b} &= \boldsymbol{c}^{\mathrm{T}}\boldsymbol{A}^{-1}\boldsymbol{b}+\frac{\boldsymbol{c}^{\mathrm{T}}\boldsymbol{A}^{-1}\boldsymbol{b}\boldsymbol{k}^{\mathrm{T}}\boldsymbol{A}^{-1}\boldsymbol{b}}{1-\boldsymbol{k}^{\mathrm{T}}\boldsymbol{A}^{-1}\boldsymbol{b}} = \frac{\boldsymbol{c}^{\mathrm{T}}\boldsymbol{A}^{-1}\boldsymbol{b}\left(1+\boldsymbol{k}^{\mathrm{T}}\boldsymbol{A}^{-1}\boldsymbol{b}-\boldsymbol{k}^{\mathrm{T}}\boldsymbol{A}^{-1}\boldsymbol{b}\right)}{1-\boldsymbol{k}^{\mathrm{T}}\boldsymbol{A}^{-1}\boldsymbol{b}} \\ &= \frac{\boldsymbol{c}^{\mathrm{T}}\boldsymbol{A}^{-1}\boldsymbol{b}}{1-\boldsymbol{k}^{\mathrm{T}}\boldsymbol{A}^{-1}\boldsymbol{b}} \end{aligned} \tag{A.9.10}$$

So the other form of (A.9.8) is

$$k_\mathrm{r} = \frac{-1}{\boldsymbol{c}^\mathrm{T}\left(\boldsymbol{A}-\boldsymbol{b}\boldsymbol{k}^\mathrm{T}\right)^{-1}\boldsymbol{b}} = \frac{\boldsymbol{k}^\mathrm{T}\boldsymbol{A}^{-1}\boldsymbol{b}-1}{\boldsymbol{c}^\mathrm{T}\boldsymbol{A}^{-1}\boldsymbol{b}}. \tag{A.9.11}$$

A.9.3

As was seen in the derivation of (9.10), the pole allocating state feedback vector $\boldsymbol{k}^\mathrm{T}=\boldsymbol{k}_\mathrm{c}^\mathrm{T}$ can easily be computed from the controllable canonical form. It was discussed in connection with Eq. (3.67) that all controllable systems can be rewritten into controllable canonical form by the transformation matrix $\boldsymbol{T}_\mathrm{c}=\boldsymbol{M}_\mathrm{c}^\mathrm{c}(\boldsymbol{M}_\mathrm{c})^{-1}$. From this we can get the similarity transformation (9.13) of the feedback vector

$$\boldsymbol{k}^\mathrm{T}=\boldsymbol{k}_\mathrm{c}^\mathrm{T}\boldsymbol{T}_\mathrm{c}=\boldsymbol{k}_\mathrm{c}^\mathrm{T}\boldsymbol{M}_\mathrm{c}^\mathrm{c}\boldsymbol{M}_\mathrm{c}^{-1}. \tag{A.9.12}$$

Instead of the relatively complicated transformation matrix $\boldsymbol{T}_\mathrm{c}$, another simpler method is also available. Find the matrix $\boldsymbol{T}$ of the similarity transformation by the following expressions

$$\boldsymbol{A}_\mathrm{c}=\boldsymbol{T}\boldsymbol{A}\boldsymbol{T}^{-1} \quad \text{and} \quad \boldsymbol{b}_\mathrm{c}=\boldsymbol{T}\boldsymbol{b} \tag{A.9.13}$$

The similarity transformation of the matrix $\boldsymbol{A}$ can also be expressed in the form

$$\boldsymbol{A}_\mathrm{c}\boldsymbol{T}=\boldsymbol{T}\boldsymbol{A}. \tag{A.9.14}$$

Introducing the notation $\boldsymbol{t}_i^\mathrm{T}$ for the rows of the matrix $\boldsymbol{T}$ we can write that

$$\boldsymbol{A}_\mathrm{c}\boldsymbol{T}=\begin{bmatrix} -a_1 & -a_2 & \dots & -a_{n-1} & -a_n \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} \boldsymbol{t}_1^\mathrm{T} \\ \boldsymbol{t}_2^\mathrm{T} \\ \boldsymbol{t}_3^\mathrm{T} \\ \vdots \\ \boldsymbol{t}_n^\mathrm{T} \end{bmatrix}=\boldsymbol{T}\boldsymbol{A}=\begin{bmatrix} \boldsymbol{t}_1^\mathrm{T} \\ \boldsymbol{t}_2^\mathrm{T} \\ \boldsymbol{t}_3^\mathrm{T} \\ \vdots \\ \boldsymbol{t}_n^\mathrm{T} \end{bmatrix}\boldsymbol{A}=\begin{bmatrix} \boldsymbol{t}_1^\mathrm{T}\boldsymbol{A} \\ \boldsymbol{t}_2^\mathrm{T}\boldsymbol{A} \\ \boldsymbol{t}_3^\mathrm{T}\boldsymbol{A} \\ \vdots \\ \boldsymbol{t}_n^\mathrm{T}\boldsymbol{A} \end{bmatrix}. \tag{A.9.15}$$

Executing the operations we get that

$$\boldsymbol{A}_\mathrm{c}\boldsymbol{T}=\begin{bmatrix} -a_1\boldsymbol{t}_1^\mathrm{T}-a_2\boldsymbol{t}_2^\mathrm{T} \cdots -a_{n-1}\boldsymbol{t}_{n-1}^\mathrm{T}-a_n\boldsymbol{t}_n^\mathrm{T} \\ \boldsymbol{t}_1^\mathrm{T} \\ \boldsymbol{t}_2^\mathrm{T} \\ \vdots \\ \boldsymbol{t}_{n-1}^\mathrm{T} \end{bmatrix}=\begin{bmatrix} \boldsymbol{t}_1^\mathrm{T}\boldsymbol{A} \\ \boldsymbol{t}_2^\mathrm{T}\boldsymbol{A} \\ \boldsymbol{t}_3^\mathrm{T}\boldsymbol{A} \\ \vdots \\ \boldsymbol{t}_n^\mathrm{T}\boldsymbol{A} \end{bmatrix}=\begin{bmatrix} \boldsymbol{t}_n^\mathrm{T}\boldsymbol{A}^{n-1} \\ \boldsymbol{t}_n^\mathrm{T}\boldsymbol{A}^{n-2} \\ \boldsymbol{t}_n^\mathrm{T}\boldsymbol{A}^{n-3} \\ \vdots \\ \boldsymbol{t}_n^\mathrm{T} \end{bmatrix}\boldsymbol{A}=\boldsymbol{T}\boldsymbol{A}. \tag{A.9.16}$$

As a consequence of the equality of the two sides the following recursive relationship holds between the row vectors $\boldsymbol{t}_i^{\mathrm{T}}$, if $\boldsymbol{t}_n^{\mathrm{T}}$ is known

$$\boldsymbol{t}_{i-1}^{\mathrm{T}} = \boldsymbol{t}_i^{\mathrm{T}}\boldsymbol{A}; \quad i = n, n-1, \ldots, 2 \tag{A.9.17}$$

or in another form,

$$\boldsymbol{t}_{i-1}^{\mathrm{T}} = \boldsymbol{t}_n^{\mathrm{T}}\boldsymbol{A}^{n-i+1}; \quad i = n, n-1, \ldots, 2. \tag{A.9.18}$$

Thus the transformation matrix is

$$\boldsymbol{T}_{\mathrm{c}} = \boldsymbol{T} = \begin{bmatrix} \boldsymbol{t}_n^{\mathrm{T}}\boldsymbol{A}^{n-1} \\ \boldsymbol{t}_n^{\mathrm{T}}\boldsymbol{A}^{n-2} \\ \boldsymbol{t}_n^{\mathrm{T}}\boldsymbol{A}^{n-3} \\ \vdots \\ \boldsymbol{t}_n^{\mathrm{T}} \end{bmatrix}. \tag{A.9.19}$$

Similarly, based on (A.9.13) and (A.9.16) we get that

$$\boldsymbol{b}_{\mathrm{c}} = \boldsymbol{T}\boldsymbol{b} = \begin{bmatrix} \boldsymbol{t}_1^{\mathrm{T}} \\ \boldsymbol{t}_2^{\mathrm{T}} \\ \boldsymbol{t}_3^{\mathrm{T}} \\ \vdots \\ \boldsymbol{t}_n^{\mathrm{T}} \end{bmatrix}\boldsymbol{b} = \begin{bmatrix} \boldsymbol{t}_n^{\mathrm{T}}\boldsymbol{A}^{n-1} \\ \boldsymbol{t}_n^{\mathrm{T}}\boldsymbol{A}^{n-2} \\ \boldsymbol{t}_n^{\mathrm{T}}\boldsymbol{A}^{n-3} \\ \vdots \\ \boldsymbol{t}_n^{\mathrm{T}} \end{bmatrix}\boldsymbol{b} = \begin{bmatrix} \boldsymbol{t}_n^{\mathrm{T}}\boldsymbol{A}^{n-1}\boldsymbol{b} \\ \boldsymbol{t}_n^{\mathrm{T}}\boldsymbol{A}^{n-2}\boldsymbol{b} \\ \boldsymbol{t}_n^{\mathrm{T}}\boldsymbol{A}^{n-3}\boldsymbol{b} \\ \vdots \\ \boldsymbol{t}_n^{\mathrm{T}}\boldsymbol{b} \end{bmatrix}, \tag{A.9.20}$$

whose transposed form is

$$\boldsymbol{b}_{\mathrm{c}}^{\mathrm{T}} = \boldsymbol{t}_n^{\mathrm{T}}\begin{bmatrix} \boldsymbol{b} & \boldsymbol{A}\boldsymbol{b} & \ldots & \boldsymbol{A}^{n-2}\boldsymbol{b} & \boldsymbol{A}^{n-1}\boldsymbol{b} \end{bmatrix} = \boldsymbol{t}_n^{\mathrm{T}}\boldsymbol{M}_{\mathrm{c}}, \tag{A.9.21}$$

where $\boldsymbol{M}_{\mathrm{c}}$ is the controllability matrix. From this,

$$\boldsymbol{t}_n^{\mathrm{T}} = \boldsymbol{b}_{\mathrm{c}}^{\mathrm{T}}(\boldsymbol{M}_{\mathrm{c}})^{-1}. \tag{A.9.22}$$

Thus $\boldsymbol{t}_n^{\mathrm{T}}$ is the first row of the inverse of the controllability matrix, since

$$\boldsymbol{b}_{\mathrm{c}}^{\mathrm{T}} = [1, 0, \ldots, 0]. \tag{A.9.23}$$

Consider the transpose of the feedback vector (A.9.12)

$$\boldsymbol{k}^{\mathrm{T}} = \boldsymbol{k}_{\mathrm{c}}^{\mathrm{T}}\boldsymbol{T}_{\mathrm{c}} = [r_1 - a_1, r_2 - a_2, \ldots, r_n - a_n] \begin{bmatrix} \boldsymbol{t}_n^{\mathrm{T}}\boldsymbol{A}^{n-1} \\ \boldsymbol{t}_n^{\mathrm{T}}\boldsymbol{A}^{n-2} \\ \boldsymbol{t}_n^{\mathrm{T}}\boldsymbol{A}^{n-3} \\ \vdots \\ \boldsymbol{t}_n^{\mathrm{T}} \end{bmatrix}. \tag{A.9.24}$$

Executing the operation we get the equation

$$\boldsymbol{k}^{\mathrm{T}} = \boldsymbol{t}_n^{\mathrm{T}} \sum_{i=1}^{n} r_i \boldsymbol{A}^{n-i} - \boldsymbol{t}_n^{\mathrm{T}} \sum_{i=1}^{n} a_i \boldsymbol{A}^{n-i} \tag{A.9.25}$$

then adding $\boldsymbol{A}^n$ to both sums we get a very interesting form,

$$\boldsymbol{k}^{\mathrm{T}} = \boldsymbol{t}_n^{\mathrm{T}} \mathcal{R}(\boldsymbol{A}) - \boldsymbol{t}_n^{\mathrm{T}} \mathcal{A}(\boldsymbol{A}). \tag{A.9.26}$$

Due to the Cayley-Hamilton theorem all square matrices satisfy their characteristic polynomial, therefore $\mathcal{A}(\boldsymbol{A}) = \mathcal{R}(\boldsymbol{A}) = 0$. The final form of (A.9.26) is

$$\boldsymbol{k}^{\mathrm{T}} = \boldsymbol{t}_n^{\mathrm{T}} \mathcal{R}(\boldsymbol{A}). \tag{A.9.27}$$

This latter equation is called the Ackermann formula. The expression (A.9.12) can be evaluated much easier by computational methods than by (A.9.27).

A.9.4

Based on the diagonal canonical form, from the basic relationship (9.7) of the pole allocation we can get by equivalent rewriting that

$$\mathcal{R}(s) - \mathcal{A}(s) = \frac{\mathcal{B}(s)}{\boldsymbol{c}_{\mathrm{d}}^{\mathrm{T}}(s\boldsymbol{I} - \boldsymbol{A}_{\mathrm{d}})^{-1}\boldsymbol{b}^{\mathrm{d}}} \boldsymbol{k}_{\mathrm{d}}^{\mathrm{T}}(s\boldsymbol{I} - \boldsymbol{A}_{\mathrm{d}})^{-1}\boldsymbol{b}^{\mathrm{d}} = \mathcal{A}(s)\boldsymbol{k}_{\mathrm{d}}^{\mathrm{T}}(s\boldsymbol{I} - \boldsymbol{A}_{\mathrm{d}})^{-1}\boldsymbol{b}^{\mathrm{d}}, \tag{A.9.28}$$

which yields

$$\frac{\mathcal{R}(s)}{\mathcal{A}(s)} = 1 + \boldsymbol{k}_{\mathrm{d}}^{\mathrm{T}}(s\boldsymbol{I} - \boldsymbol{A}_{\mathrm{d}})^{-1}\boldsymbol{b}^{\mathrm{d}}. \tag{A.9.29}$$

Decomposing the left side into partial fractions, and taking the diagonal character of the system into account, it can be seen that

$$\frac{\mathcal{R}(s)}{\mathcal{A}(s)} = 1 + \sum_{i=1}^{n} \frac{k_i^{\mathrm{d}} b_i^{\mathrm{d}}}{s - \lambda_i} = 1 + \sum_{i=1}^{n} \frac{k_i^{\mathrm{d}} \beta_i}{s - \lambda_i}. \tag{A.9.30}$$

Applying the expansion theory valid for the simple poles of the partial fractions, thus multiplying both sides with $(s - \lambda_i)$ and substituting $s = \lambda_i$, we get the expression

$$k_i^d b_i^d = \prod_{j=1}^{n} \left(\lambda_i - \mu_j\right) \Big/ \prod_{\substack{j=1 \\ i \neq j}}^{n} \left(\lambda_i - \lambda_j\right) \tag{A.9.31}$$

This procedure has to be performed for all the poles.

A.9.5

Taking the matrix inversion identity (A.1.17) in Appendix A.1 into account the following steps of the rewriting can be easily followed:

$$\begin{aligned} T_{\mathrm{ry}}(s) &= \frac{\left[\boldsymbol{c}^{\mathrm{T}}(s\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{b}\right]\left[1-\boldsymbol{k}^{\mathrm{T}}\left(s\boldsymbol{I}-\boldsymbol{A}+\boldsymbol{b}\boldsymbol{k}^{\mathrm{T}}+\boldsymbol{l}\boldsymbol{c}^{\mathrm{T}}\right)^{-1}\boldsymbol{b}\right]k_{\mathrm{r}}}{1+\left[\boldsymbol{k}^{\mathrm{T}}\left(s\boldsymbol{I}-\boldsymbol{A}+\boldsymbol{b}\boldsymbol{k}^{\mathrm{T}}+\boldsymbol{l}\boldsymbol{c}^{\mathrm{T}}\right)^{-1}\boldsymbol{b}\right]\left[\boldsymbol{c}^{\mathrm{T}}(s\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{b}\right]} \\ &= \boldsymbol{c}^{\mathrm{T}}\left(s\boldsymbol{I}-\boldsymbol{A}+\boldsymbol{b}\boldsymbol{k}^{\mathrm{T}}\right)^{-1}\boldsymbol{b}k_{\mathrm{r}} = \frac{\boldsymbol{c}^{\mathrm{T}}(s\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{b}k_{\mathrm{r}}}{1+\boldsymbol{k}^{\mathrm{T}}(s\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{b}} \\ &= \frac{k_{\mathrm{r}}P(s)}{1+\boldsymbol{k}^{\mathrm{T}}(s\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{b}} = \frac{k_{\mathrm{r}}\boldsymbol{B}(s)}{\boldsymbol{R}(s)} \end{aligned} \tag{A.9.32}$$

A.9.6

The so-called *LQ* controller, discussed in 9.5, is a special case of a generally formulated optimization problem. In the general case the task is to determine the control signal $u(t)$ of the system given by the state equation

$$\dot{\boldsymbol{x}}(t) = \frac{\mathrm{d}\boldsymbol{x}(t)}{\mathrm{d}t} = \boldsymbol{f}[\boldsymbol{x}(t), u(t)], \tag{A.9.33}$$

which minimizes the general integral criterion

$$I = \frac{1}{2}\int_0^{T_{\mathrm{f}}} F[\boldsymbol{x}(t), u(t)]\mathrm{d}t = I[u(t)]. \tag{A.9.34}$$

The solution is provided by the so-called minimum principle, by means of which the so-called HAMILTON function

$$H(t) = F[\boldsymbol{x}(t), u(t)] + \boldsymbol{\lambda}(t)^{\mathrm{T}} \boldsymbol{f}[\boldsymbol{x}(t), u(t)] \tag{A.9.35}$$

has to be constructed, for which the following necessary conditions of the extremum values

$$\frac{\mathrm{d}H(t)}{\mathrm{d}u(t)} = 0; \quad \frac{\mathrm{d}H(t)}{\mathrm{d}\boldsymbol{x}(t)} = -\frac{\mathrm{d}\boldsymbol{\lambda}(t)}{\mathrm{d}t} = -\dot{\boldsymbol{\lambda}}(t) \tag{A.9.36}$$

must be fulfilled. (The sufficient condition of the minimum is that $\partial^2 H / \partial u^2 > 0$.) The HAMILTON function and the necessary condition for the minimum (A.9.36) corresponds formally to the LAGRANGE method of the conditional optimum (thus $\boldsymbol{\lambda}$ is the co-vector of the method), since the minimum of $I[u(t)]$ has to be reached under the condition (A.9.33). (Note that in the state space arbitrary motion is not allowed, only those corresponding to (A.9.33).) For the solution it is usually assumed that $\boldsymbol{\lambda}(t) = \boldsymbol{P}(t)\boldsymbol{x}(t)$, i.e., it can be derived from the state vector by a linear transformation, so

$$\dot{\boldsymbol{\lambda}}(t) = \dot{\boldsymbol{P}}(t)\boldsymbol{x}(t) + \boldsymbol{P}\dot{\boldsymbol{x}}(t). \tag{A.9.37}$$

If the upper limit of the integral is infinity $(T_{\mathrm{f}} = \infty)$ then $\boldsymbol{P}(t) = \boldsymbol{P}$ is constant, so $\dot{\boldsymbol{P}} = \boldsymbol{0}$, thus

$$\boldsymbol{\lambda}(t) = \boldsymbol{P}\boldsymbol{x}(t) \quad \text{and} \quad \dot{\boldsymbol{\lambda}}(t) = \boldsymbol{P}\dot{\boldsymbol{x}}(t). \tag{A.9.38}$$

The *LQ* regulator of the *LTI* process has to solve the task

$$I = \frac{1}{2} \int_0^{\infty} \left[\boldsymbol{x}^{\mathrm{T}}(t) \boldsymbol{W}_{\mathrm{x}} \boldsymbol{x}(t) + W_{\mathrm{u}} u^2(t)\right] \mathrm{d}t = \min_{u(t)} \tag{A.9.39}$$

under the condition of linear system dynamics

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{b}u(t). \tag{A.9.40}$$

The HAMILTON function now is

$$H(t) = \frac{1}{2} \left[\boldsymbol{x}^{\mathrm{T}}(t) \boldsymbol{W}_{\mathrm{x}} \boldsymbol{x}(t) + W_{\mathrm{u}} u^2(t)\right] + \boldsymbol{\lambda}^{\mathrm{T}} [\boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{b}u(t)], \tag{A.9.41}$$

whose second order derivate is $\partial^2 H / \partial u^2 = W_{\mathrm{u}} > 0$, so the necessary condition is, at the same time sufficient, too. The necessary condition, on the one hand, is

$$\frac{\mathrm{d}H(t)}{\mathrm{d}u(t)} = W_{\mathrm{u}}u(t) + \boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{b} = W_{\mathrm{u}}u(t) + \boldsymbol{b}^{\mathrm{T}}\boldsymbol{\lambda} = 0 \tag{A.9.42}$$

from which the optimal control is

$$u(t) = -\frac{1}{W_{\mathrm{u}}}\boldsymbol{b}^{\mathrm{T}}\boldsymbol{\lambda}(t) = -\frac{1}{W_{\mathrm{u}}}\boldsymbol{b}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}(t) = -\boldsymbol{k}_{\mathrm{LQ}}^{\mathrm{T}}\boldsymbol{x}(t) \tag{A.9.43}$$

On the other, the matrix $\boldsymbol{P}$ in Eq. (A.9.43) has to be determined. For this, consider the complete state equation of the closed system

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}\boldsymbol{k}_{\mathrm{LQ}}^{\mathrm{T}}\boldsymbol{x} = \left(\boldsymbol{A} - \boldsymbol{b}\boldsymbol{k}_{\mathrm{LQ}}^{\mathrm{T}}\right)\boldsymbol{x} = \left(\boldsymbol{A} - \frac{1}{W_{\mathrm{u}}}\boldsymbol{b}\boldsymbol{b}^{\mathrm{T}}\boldsymbol{P}\right)\boldsymbol{x} = \bar{\boldsymbol{A}}\boldsymbol{x}, \tag{A.9.44}$$

which has the same form as for the state feedback. Thus the *LQ* regulator is a state feedback controller. Based on Eqs. (A.9.36) and (A.9.44) the co-vector is

$$\dot{\boldsymbol{\lambda}} = \boldsymbol{P}\dot{\boldsymbol{x}} = \left(\boldsymbol{P}\boldsymbol{A} - \frac{1}{W_{\mathrm{u}}}\boldsymbol{P}\boldsymbol{b}\boldsymbol{b}^{\mathrm{T}}\boldsymbol{P}\right)\boldsymbol{x} = \boldsymbol{P}\bar{\boldsymbol{A}}x, \tag{A.9.45}$$

which has to satisfy the equation

$$\begin{aligned}\dot{\boldsymbol{\lambda}}(t) &= -\frac{\mathrm{d}H(t)}{\mathrm{d}\boldsymbol{x}(t)} = -\boldsymbol{W}_{\mathrm{x}}\boldsymbol{x}(t) - \boldsymbol{A}^{\mathrm{T}}\boldsymbol{\lambda}(t) = -\boldsymbol{W}_{\mathrm{x}}\boldsymbol{x}(t) - \boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}(t) \\ &= -\left(\boldsymbol{W}_{\mathrm{x}} + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}\right)\boldsymbol{x}(t)\end{aligned} \tag{A.9.46}$$

coming from the necessary condition (A.9.36). Comparing the last two equations, the following equality

$$\boldsymbol{P}\boldsymbol{A} - \frac{1}{W_{\mathrm{u}}}\boldsymbol{P}\boldsymbol{b}\boldsymbol{b}^{\mathrm{T}}\boldsymbol{P} = -\left(\boldsymbol{W}_{\mathrm{x}} + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}\right) \tag{A.9.47}$$

is obtained for symmetric $\boldsymbol{P}$. By rewriting we get the so-called nonlinear algebraic Riccati matrix equation

$$\boldsymbol{P}\boldsymbol{A} + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{P} - \frac{1}{W_{\mathrm{u}}}\boldsymbol{P}\boldsymbol{b}\boldsymbol{b}^{\mathrm{T}}\boldsymbol{P} = \boldsymbol{W}_{\mathrm{x}}, \tag{A.9.48}$$

which has no explicit algebraic solution, but there are several fast numerical methods available for its computation.

The joint state equation of the state vector and co-vector of the closed system can be easily written using Eqs. (A.9.44) and (A.9.46)

$$\begin{bmatrix} \dot{\boldsymbol{x}} \\ \dot{\boldsymbol{\lambda}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} & \frac{1}{W_\mathrm{u}}\boldsymbol{b}\boldsymbol{b}^\mathrm{T}\boldsymbol{P} \\ \boldsymbol{W}_\mathrm{x} & -\boldsymbol{A}^\mathrm{T} \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{\lambda} \end{bmatrix}. \tag{A.9.49}$$

Note that if the upper limit of the integral is finite $(T_\mathrm{f} < \infty)$ then $\boldsymbol{P} = \boldsymbol{P}(t)$ depends on the time and the RICCATI matrix equation has to be solved in advance for the domain $0 \le t \le T_\mathrm{f}$.

Next it will be shown that the solution $\boldsymbol{P}$ of the RICCATI matrix equation has exceptional meaning. Substitute Eq. (A.9.41) of the optimal control into the criterion (A.9.37)

$$\begin{aligned} I &= \frac{1}{2}\int_0^\infty \left[\boldsymbol{x}^\mathrm{T}(t)\boldsymbol{W}_\mathrm{x}\boldsymbol{x}(t) + W_\mathrm{u}u(t)\right]\mathrm{d}t = \frac{1}{2}\int_0^\infty \left\{\boldsymbol{x}^\mathrm{T}(t)\boldsymbol{W}_\mathrm{x}\boldsymbol{x}(t) + W_\mathrm{u}\left[-\boldsymbol{k}_\mathrm{LQ}^\mathrm{T}\boldsymbol{x}(t)\right]^2\right\}\mathrm{d}t \\ &= \frac{1}{2}\int_0^\infty \left[\boldsymbol{x}^\mathrm{T}(t)\boldsymbol{W}_\mathrm{x}\boldsymbol{x}(t) + \frac{1}{W_\mathrm{u}}\boldsymbol{x}^\mathrm{T}(t)\boldsymbol{P}^\mathrm{T}\boldsymbol{b}\boldsymbol{b}^\mathrm{T}\boldsymbol{P}\boldsymbol{x}(t)\right]\mathrm{d}t = \frac{1}{2}\int_0^\infty \left[\boldsymbol{x}^\mathrm{T}(t)\bar{\boldsymbol{W}}_\mathrm{x}\boldsymbol{x}(t)\right]\mathrm{d}t \end{aligned} \tag{A.9.50}$$

where

$$\bar{\boldsymbol{W}}_\mathrm{x} = \boldsymbol{W}_\mathrm{x} + \frac{1}{W_\mathrm{u}}\boldsymbol{P}^\mathrm{T}\boldsymbol{b}\boldsymbol{b}^\mathrm{T}\boldsymbol{P}. \tag{A.9.51}$$

The solution for the closed system of (A.9.44) without excitation is

$$\boldsymbol{x}(t) = e^{\bar{\boldsymbol{A}}t}\boldsymbol{x}(0) \tag{A.9.52}$$

so the criterion (A.9.50) for the case without excitation is

$$I = \frac{1}{2}\int_0^\infty \left[\boldsymbol{x}^\mathrm{T}(0)e^{\bar{\boldsymbol{A}}^\mathrm{T}t}\bar{\boldsymbol{W}}_\mathrm{x}e^{\bar{\boldsymbol{A}}t}\boldsymbol{x}(0)\right]\mathrm{d}t = \frac{1}{2}\boldsymbol{x}^\mathrm{T}(0)\boldsymbol{P}\boldsymbol{x}(0), \tag{A.9.53}$$

where it is assumed that

$$\boldsymbol{P} = \int_0^\infty e^{\bar{\boldsymbol{A}}^\mathrm{T}t}\bar{\boldsymbol{W}}_\mathrm{x}e^{\bar{\boldsymbol{A}}t}\mathrm{d}t. \tag{A.9.54}$$

To prove this, carry out the integration

$$\boldsymbol{P} = \int_0^{\infty} e^{\bar{\boldsymbol{A}}^{\mathrm{T}}t}\bar{\boldsymbol{W}}_{\mathrm{x}}e^{\bar{\boldsymbol{A}}t}\mathrm{d}t = \left[e^{\bar{\boldsymbol{A}}^{\mathrm{T}}t}\bar{\boldsymbol{W}}_{\mathrm{x}}\bar{\boldsymbol{A}}^{-1}e^{\bar{\boldsymbol{A}}t}\right]_0^{\infty} - \int_0^{\infty} \bar{\boldsymbol{A}}^{\mathrm{T}}e^{\bar{\boldsymbol{A}}^{\mathrm{T}}t}\bar{\boldsymbol{W}}_{\mathrm{x}}\bar{\boldsymbol{A}}^{-1}e^{\bar{\boldsymbol{A}}t}\mathrm{d}t. \quad \text{(A.9.55)}$$

Furthermore, if $\bar{\boldsymbol{A}}$ is stable, then

$$\boldsymbol{P} = -\bar{\boldsymbol{W}}_{\mathrm{x}}\bar{\boldsymbol{A}}^{-1} - \bar{\boldsymbol{A}}^{\mathrm{T}}\left(\int_0^{\infty} e^{\bar{\boldsymbol{A}}^{\mathrm{T}}t}\bar{\boldsymbol{W}}_{\mathrm{x}}e^{\bar{\boldsymbol{A}}t}\mathrm{d}t\right)\bar{\boldsymbol{A}}^{-1} = -\bar{\boldsymbol{W}}_{\mathrm{x}}\bar{\boldsymbol{A}}^{-1} - \bar{\boldsymbol{A}}^{\mathrm{T}}\boldsymbol{P}\bar{\boldsymbol{A}}^{-1}. \quad \text{(A.9.56)}$$

Here, it has been used that $\bar{\boldsymbol{A}}^{-1}e^{\bar{\boldsymbol{A}}t} = e^{\bar{\boldsymbol{A}}t}\bar{\boldsymbol{A}}^{-1}$. Finally the equation

$$\boldsymbol{P}\bar{\boldsymbol{A}} + \bar{\boldsymbol{A}}^{\mathrm{T}}\boldsymbol{P} = -\bar{\boldsymbol{W}}_{\mathrm{x}} \quad \text{(A.9.57)}$$

is obtained, which is called the LYAPUNOV equation. The equation is only virtually linear in $\boldsymbol{P}$, since $\bar{\boldsymbol{W}}_{\mathrm{x}}$ and $\bar{\boldsymbol{A}}$ also depend on $\boldsymbol{P}$. Rewriting the equation we get again the algebraic RICCATI matrix equation (A.9.48). By this, on the one hand, the relationship (A.9.54) is proved for $\boldsymbol{P}$, on the other hand the meaning of $\boldsymbol{P}$ is also shown: namely that it is the quadratic cost function matrix associated with the control ensuring the minimum of the criterion (A.9.37) for the case without excitation.

It is also interesting to investigate how the HAMILTON function itself changes in time. Determine the time derivates

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \left[\frac{\mathrm{d}H}{\mathrm{d}\boldsymbol{x}}\right]^{\mathrm{T}}\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} + \left[\frac{\mathrm{d}H}{\mathrm{d}u}\right]^{\mathrm{T}}\frac{\mathrm{d}u}{\mathrm{d}t} + \left[\frac{\mathrm{d}H}{\mathrm{d}\boldsymbol{\lambda}}\right]^{\mathrm{T}}\frac{\mathrm{d}\boldsymbol{\lambda}}{\mathrm{d}t} \quad \text{(A.9.58)}$$

Since based on (A.9.33) and (A.9.35) it can be stated that

$$\frac{\mathrm{d}H}{\mathrm{d}\boldsymbol{\lambda}} = \frac{\mathrm{d}H}{\mathrm{d}\boldsymbol{x}} = \boldsymbol{f} \quad \text{(A.9.59)}$$

and taking Eq. (A.9.36) into account, we get

$$\left[\frac{\mathrm{d}H}{\mathrm{d}\boldsymbol{x}}\right]^{\mathrm{T}}\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} + \left[\frac{\mathrm{d}H}{\mathrm{d}\boldsymbol{\lambda}}\right]^{\mathrm{T}}\frac{\mathrm{d}\boldsymbol{\lambda}}{\mathrm{d}t} = 0. \quad \text{(A.9.60)}$$

Thus finally

$$\frac{\mathrm{d}H}{\mathrm{d}t} = 0, \quad \text{(A.9.61)}$$

i.e., the HAMILTON function is constant (assuming that neither the control, nor the state vector have restrictions). Thus in the case without limitation, the HAMILTON function is time invariant and it is invariant also for the input (see the necessary condition (A.9.36) for the extremum).

A.11.1

Based on the transfer function of the zero order hold

$$W_{\mathrm{ZOH}}(s) = \frac{1 - e^{-sT_s}}{s} \tag{A.11.1}$$

the frequency function of the holding element is

$$\begin{aligned} W_{\mathrm{ZOH}}(s)\big|_{s=j\omega} &= \frac{1 - e^{-j\omega \mathrm{T_s}}}{j\omega} = \frac{2e^{-j\omega \mathrm{T_s}/2}\left(e^{j\omega \mathrm{T_s}/2} - e^{-j\omega \mathrm{T_s}/2}\right)}{2j\omega} \\ &= T_s \frac{\sin(\omega \mathrm{T_s}/2)}{\omega \mathrm{T_s}/2} e^{-j\omega \mathrm{T_s}/2} \end{aligned} \tag{A.11.2}$$

Based on the above the absolute value function of the zero order hold can be written as

$$|W_{\mathrm{ZOH}}(j\omega)| = T_s \left| \frac{\sin(\omega T_s/2)}{\omega T_s/2} \right| \tag{A.11.3}$$

and its phase function is

$$\angle\{W_{\mathrm{ZOH}}(j\omega)\} = \angle\{\sin(\omega T_s/2)\} - \omega T_s/2 \tag{A.11.4}$$

Both components of the frequency function are drawn for the choice $T_s = 1$ s in Figs. A.11.1 and A.11.2. It can be seen that the absolute value function becomes

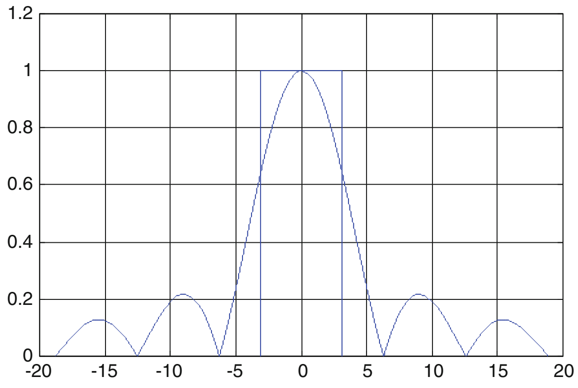

Fig. A.11.1 The absolute value of the frequency function of the zero order hold

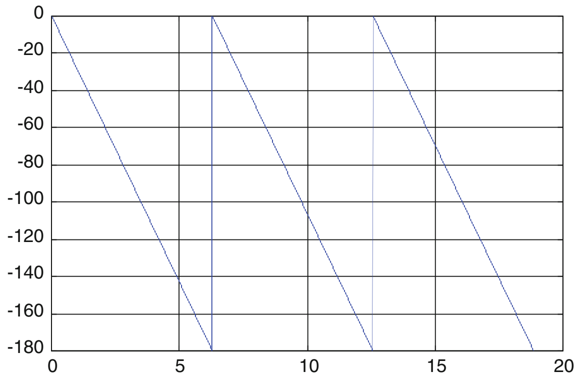

Fig. A.11.2 The phase function of the frequency function of the zero order hold

zero at the sampling frequency $\omega_s = 2\pi/T_s = 6.28\,\text{rad/s}$ and at its integer multiples, and the phase function has a linear character, its value corresponds to a delay term of $T_h = T_s/2$, at the singular points the phase changes by $\pm 180°$.

In Fig. A.11.1 the characteristic of the active linear filter in the region $\omega \leq \omega_s/2 = \omega_{max}$ is also shown. It can be seen, that the amplitude distortion can be neglected only in the lowest frequency region.

A.11.2

Let us start from the expression

$$\mathbf{Z}\{f[k]\} = F(z) = \sum_{k=0}^{\infty} f[k]z^{-k} = f[0] + f[1]z^{-1} + f[2]z^{-2} + \cdots + f[k]z^{-k} + \cdots \tag{A.11.1}$$

defining the z-transform of a discrete signal $f[k](k = 0, 1, 2, \ldots)$ as an infinite geometric progression. Multiplying both sides by the factor z^{k-1}

$$F(z)z^{k-1} = f[0]z^{k-1} + f[1]z^{k-2} + f[2]z^{k-3} + \cdots + f[k]z^{-1} + \cdots \tag{A.11.2}$$

is obtained, which is actually the Laurent series of the expression $z^{k-1}F(z)$ at $z = 0$. Consider now a circle C around the origin of the complex plane, which includes all the poles of $z^{k-1}F(z)$. Since in the above expression the coefficient of z^{-1} is $f[k]$, and at the same time, this coefficient is the residue of $z^{k-1}F(z)$, we obtain

$$f[k] = \mathcal{Z}^{-1}\{F(z)\} = \frac{1}{2\pi j}\oint_C F(z)z^{k-1}\mathrm{d}z. \tag{A.11.3}$$

A.16.1

Due to the definition of $\mathcal{H}_2$ and the PARSEVAL theorem

$$\|H(j\omega)\|_2 = \sqrt{\int_0^\infty \|w(t)\|^2 \mathrm{d}t} = \sqrt{\int_0^\infty \boldsymbol{c}^{\mathrm{T}} e^{\boldsymbol{A}t} \boldsymbol{b}\boldsymbol{b}^{\mathrm{T}} e^{\boldsymbol{A}^{\mathrm{T}}t} \boldsymbol{c} \mathrm{d}t}. \tag{A.16.1}$$

Thus

$$\|H(j\omega)\|_2^2 = \boldsymbol{c}^{\mathrm{T}} \left[\int_0^\infty e^{\boldsymbol{A}t} \boldsymbol{b}\boldsymbol{b}^{\mathrm{T}} e^{\boldsymbol{A}^{\mathrm{T}}t} \mathrm{d}t \right] \boldsymbol{c}. \tag{A.16.2}$$

Introduce the notation

$$\boldsymbol{L} = \int_0^\infty e^{\boldsymbol{A}t} \boldsymbol{b}\boldsymbol{b}^{\mathrm{T}} e^{\boldsymbol{A}^{\mathrm{T}}t} \mathrm{d}t \tag{A.16.3}$$

so finally the $\mathcal{H}_2$ norm can be computed as

$$\|H(j\omega)\|_2 = \sqrt{\boldsymbol{c}^{\mathrm{T}} \boldsymbol{L} \boldsymbol{c}}. \tag{A.16.4}$$

Differentiate the integral in the A.16.3

$$\frac{\mathrm{d}}{\mathrm{d}t} e^{\boldsymbol{A}t} \boldsymbol{b}\boldsymbol{b}^{\mathrm{T}} e^{\boldsymbol{A}^{\mathrm{T}}t} = \boldsymbol{A} e^{\boldsymbol{A}t} \boldsymbol{b}\boldsymbol{b}^{\mathrm{T}} e^{\boldsymbol{A}^{\mathrm{T}}t} + e^{\boldsymbol{A}t} \boldsymbol{b}\boldsymbol{b}^{\mathrm{T}} e^{\boldsymbol{A}^{\mathrm{T}}t} \boldsymbol{A}^{\mathrm{T}} \tag{A.16.5}$$

then integrate both sides of the equation over the domain $[0, \infty]$:

$$\left[e^{\boldsymbol{A}t} \boldsymbol{b}\boldsymbol{b}^{\mathrm{T}} e^{\boldsymbol{A}^{\mathrm{T}}t} \right]_0^\infty = \boldsymbol{A} \left[\int_0^\infty e^{\boldsymbol{A}t} \boldsymbol{b}\boldsymbol{b}^{\mathrm{T}} e^{\boldsymbol{A}^{\mathrm{T}}t} \mathrm{d}t \right] + \left[\int_0^\infty e^{\boldsymbol{A}t} \boldsymbol{b}\boldsymbol{b}^{\mathrm{T}} e^{\boldsymbol{A}^{\mathrm{T}}t} \mathrm{d}t \right] \boldsymbol{A}^{\mathrm{T}}, \tag{A.16.6}$$

from which by simple computation and considering (A.16.3) we get the system of linear equations

$$-\boldsymbol{b}\boldsymbol{b}^{\mathrm{T}} = \boldsymbol{A}\boldsymbol{L} + \boldsymbol{L}\boldsymbol{A}^{\mathrm{T}} \tag{A.16.7}$$

for $\boldsymbol{L}$.

A.16.2

Rewriting the criterion (16.27) in detail the form

$$V(\hat{\boldsymbol{p}}) = \frac{1}{2}\left[\boldsymbol{y}^{\mathrm{T}}\boldsymbol{y} - 2\boldsymbol{y}^{\mathrm{T}}\boldsymbol{F}(\boldsymbol{u})\hat{\boldsymbol{p}} + \hat{\boldsymbol{p}}^{\mathrm{T}}\boldsymbol{F}^{\mathrm{T}}(\boldsymbol{u})\boldsymbol{F}(\boldsymbol{u})\hat{\boldsymbol{p}}\right] \tag{A.16.8}$$

is obtained, and making its gradient equal to zero yields

$$\frac{\mathrm{d}V(\hat{\boldsymbol{p}})}{\mathrm{d}\hat{\boldsymbol{p}}} = -\boldsymbol{F}^{\mathrm{T}}(\boldsymbol{u})\boldsymbol{y} + \boldsymbol{F}^{\mathrm{T}}(\boldsymbol{u})\boldsymbol{F}(\boldsymbol{u})\hat{\boldsymbol{p}} = \boldsymbol{0}. \tag{A.16.9}$$

Solving the equation for $\hat{\boldsymbol{p}}$, the best parameter estimator is obtained in the form

$$\hat{\boldsymbol{p}} = \left[\boldsymbol{F}^{\mathrm{T}}(\boldsymbol{u})\boldsymbol{F}(\boldsymbol{u})\right]^{-1}\boldsymbol{F}^{\mathrm{T}}(\boldsymbol{u})\boldsymbol{y}. \tag{A.16.10}$$

A.16.3

Assume that processing N data pairs the off-line LS estimation of the parameters is available as

$$\hat{\boldsymbol{p}}[N] = \left[\boldsymbol{F}^{\mathrm{T}}[N]\boldsymbol{F}[N]\right]^{-1}\boldsymbol{F}^{\mathrm{T}}[N]\boldsymbol{y}_N, \tag{A.16.11}$$

then compute the LS estimation for the $(N+1)$-th point

$$\begin{aligned}\hat{\boldsymbol{p}}[N+1] &= \left[\boldsymbol{F}^{\mathrm{T}}[N+1]\boldsymbol{F}[N+1]\right]^{-1}\boldsymbol{F}^{\mathrm{T}}[N+1]\boldsymbol{y}_{N+1}\\ &= \left\{\begin{bmatrix}\boldsymbol{F}[N]\\ \boldsymbol{f}^{\mathrm{T}}[N+1]\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\boldsymbol{F}[N]\\ \boldsymbol{f}^{\mathrm{T}}[N+1]\end{bmatrix}\right\}^{-1}\begin{bmatrix}\boldsymbol{F}[N]\\ \boldsymbol{f}^{\mathrm{T}}[N+1]\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\boldsymbol{y}_N\\ y[N+1]\end{bmatrix}\\ &= \left\{\boldsymbol{F}^{\mathrm{T}}[N]\boldsymbol{F}^{\mathrm{T}}[N] + \boldsymbol{f}[N+1]\boldsymbol{f}^{\mathrm{T}}[N+1]\right\}^{-1}\left\{\boldsymbol{F}^{\mathrm{T}}[N]\boldsymbol{y}_N + \boldsymbol{f}[N+1]y[N+1]\right\}\end{aligned} \tag{A.16.12}$$

For the solution it is required to compute the inverse of the matrix extended by the dyadic product

$$\boldsymbol{R}[N+1] = \left\{\boldsymbol{F}^{\mathrm{T}}[N+1]\boldsymbol{F}^{\mathrm{T}}[N+1]\right\}^{-1} = \left\{\boldsymbol{F}^{\mathrm{T}}[N]\boldsymbol{F}^{\mathrm{T}}[N] + \boldsymbol{f}[N+1]\boldsymbol{f}^{\mathrm{T}}[N+1]\right\}^{-1}. \tag{A.16.13}$$

According to A.1.17 in Appendix A.1,

$$\left\{\boldsymbol{F}^{\mathrm{T}}[N+1]\boldsymbol{F}^{\mathrm{T}}[N+1]\right\}^{-1} = \left\{\boldsymbol{F}^{\mathrm{T}}[N]\boldsymbol{F}^{\mathrm{T}}[N]\right\}^{-1} - \frac{\left\{\boldsymbol{F}^{\mathrm{T}}[N]\boldsymbol{F}^{\mathrm{T}}[N]\right\}^{-1}\boldsymbol{f}[N+1]\boldsymbol{f}^{\mathrm{T}}[N+1]\left\{\boldsymbol{F}^{\mathrm{T}}[N]\boldsymbol{F}^{\mathrm{T}}[N]\right\}^{-1}}{1+\boldsymbol{f}^{\mathrm{T}}[N+1]\left\{\boldsymbol{F}^{\mathrm{T}}[N]\boldsymbol{F}^{\mathrm{T}}[N]\right\}^{-1}\boldsymbol{f}[N+1]}. \tag{A.16.14}$$

Using the definition of $\boldsymbol{R}[N]$ by (16.54) we get

$$\boldsymbol{R}[N+1] = \boldsymbol{R}[N] - \frac{\boldsymbol{R}[N]\boldsymbol{f}(N+1)\boldsymbol{f}^{\mathrm{T}}(N+1)\boldsymbol{R}[N]}{1+\boldsymbol{f}^{\mathrm{T}}(N+1)\boldsymbol{R}[N]\boldsymbol{f}(N+1)} \tag{A.16.13}$$

Substituting the above recursive equation of the convergence matrix into (A.16.12), the recursive equation of the parameter estimation is obtained as

$$\hat{\boldsymbol{p}}[N+1] = \hat{\boldsymbol{p}}[N] + \boldsymbol{R}[N+1]\boldsymbol{f}(N+1)\left\{y[N+1] - \boldsymbol{f}^{\mathrm{T}}(N+1)\hat{\boldsymbol{p}}[N]\right\}. \tag{A.16.14}$$

The term "recursive" comes from the fact that the renewal equations of both $\boldsymbol{R}[N]$ and $\hat{\boldsymbol{p}}[N]$ can be computed from the previous values by adding a new term.

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