Appendix A
Derivation of the Acceleration in Circular Motion

Let us first recall the definitions of position vector, velocity and acceleration of a particle. In a given reference frame (i.e., coordinate system) the position of a particle is given by a vector extending from the coordinate frame origin to the instantaneous position of the particle. This vector could be a function of time, thus specifying the particle trajectory

$$\mathbf{r}(t) = (x(t), y(t), z(t)). \tag{A.1}$$

The velocity of the particle is defined as the change of position per unit time

$$\mathbf{v} = \lim_{\triangle t \to 0} \frac{\triangle \mathbf{r}}{\triangle t} = \frac{d\mathbf{r}}{dt}. \tag{A.2}$$

The acceleration is defined as the change of velocity per unit time,

$$\mathbf{a} = \lim_{\triangle t \to 0} \frac{\triangle \mathbf{v}}{\triangle t} = \frac{d\mathbf{v}}{dt}. \tag{A.3}$$

We note that this is a vector equation, so the change in velocity refers to both a change in the magnitude of velocity, as well as to a change of velocity direction.

Circular motion is described by a rotating vector $\mathbf{r}(t)$ of fixed length, $|r| = constant$. The position vector is thus determined by the radius of the circle r, and by the angle $\theta(t)$ of $\mathbf{r}$ with respect to the x-axis (Fig. A.1).

Let us now introduce a set of three orthogonal unit vectors: $\mathbf{e}_r$ in the direction of $\mathbf{r}$, $\mathbf{e}_t$ in the direction of the tangent to the circle and $\mathbf{e}_z$ in the direction of the z axis. The relation between the three unit vectors is given by $\mathbf{e}_t = \mathbf{e}_z \times \mathbf{e}_r$.

We define the angular velocity vector as being perpendicular to the plane of the circular trajectory with magnitude equal to the time derivative of the angle θ

$$\boldsymbol{\omega} = \dot{\theta}\mathbf{e}_z. \tag{A.4}$$

Let us proceed to calculate the velocity

Fig. A.1 Parameters and
variables in circular motion
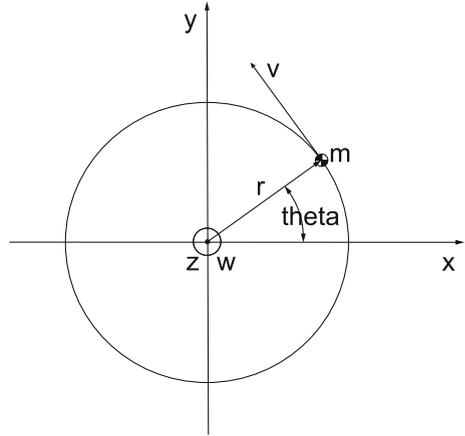
$$\mathbf{v} = \frac{d\mathbf{r}}{dt}. \tag{A.5}$$

The direction of velocity is given by the tangent to the circle: $\mathbf{e}_t = \mathbf{e}_z \times \mathbf{e}_r$. The magnitude of velocity is given by the length of the infinitesimal circular arc $ds = r d\theta$ divided by the infinitesimal time dt, that the particle requires to traverse this path

$$\frac{ds}{dt} = r\frac{d\theta}{dt} = r\dot{\theta}. \tag{A.6}$$

Including the tangential direction of velocity gives

$$\mathbf{v} = r\dot{\theta}\mathbf{e}_t = \dot{\theta}\mathbf{e}_z \times r\mathbf{e}_r = \omega \times \mathbf{r}. \tag{A.7}$$

In order to obtain the acceleration, we calculate the time derivative of velocity

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}(\omega \times \mathbf{r}). \tag{A.8}$$

We differentiate the vector product as one would differentiate a normal product of two functions

$$\mathbf{a} = \frac{d\omega}{dt} \times \mathbf{r} + \omega \times \frac{d\mathbf{r}}{dt}. \tag{A.9}$$

Defining the angular acceleration α as the time derivative of angular velocity $\alpha = \frac{d\omega}{dt}$, we see that the first term corresponds to the tangential acceleration

$$\mathbf{a}_t = \alpha \times \mathbf{r}. \tag{A.10}$$

In the second term we insert the expression for velocity

$$\frac{d\mathbf{r}}{dt} = \mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}, \tag{A.11}$$

and we get a double vector product $\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$. Using the identity $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$ from vector algebra and noting that $\boldsymbol{\omega}$ and $\mathbf{r}$ are orthogonal, we obtain for the second term in the equation for acceleration

$$\boldsymbol{\omega} \times \frac{d\mathbf{r}}{dt} = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) = \boldsymbol{\omega}(\boldsymbol{\omega} \cdot \mathbf{r}) - \mathbf{r}(\boldsymbol{\omega} \cdot \boldsymbol{\omega}) = -\omega^2\mathbf{r}, \tag{A.12}$$

which is the radial (or centripetal) component of acceleration. So we finally have

$$\mathbf{a} = \mathbf{a}_t + \mathbf{a}_r = \boldsymbol{\alpha} \times \mathbf{r} - \omega^2\mathbf{r}. \tag{A.13}$$

Further Reading

1. Bajd T, Mihelj M, Munih M (2013) Introduction to Robotics, Springer
2. Craig JJ (2005) Introduction to Robotics—Mechanics and Control, Pearson Prentice Hall
3. Kajita S, Hirukawa H, Harada K, Yokoi K (2014) Introduction to Humanoid Robotics, Springer
4. Klančar G, Zdešar A, Blažič S, Škrjanc I (2017) Wheeled Mobile Robotics - From Fundamentals Towards Autonomous Systems, Elsevier
5. Lenarčič J, Bajd T, Stanišić MM (2013) Robot Mechanisms, Springer
6. Merlet J-P (2006) Parallel Robots (Second Edition), Springer
7. Mihelj M, Podobnik J (2012) Haptics for Virtual Reality and Teleoperation, Springer
8. Mihelj M, Novak D, Beguš S (2014) Virtual Reality Technology and Applications, Springer
9. Natale C (2003) Interaction Control of Robot Manipulators, Springer
10. Nof SY (1999) Handbook of Industrial Robotics, John Wiley & Sons
11. Paul RP (1981) Robot Manipulators: Mathematics, Programming, and Control, The MIT Press
12. Sciavico L, Siciliano B (2002) Modeling and Control of Robot Manipulators, Springer
13. Spong MW, Hutchinson S, Vidyasagar M (2006) Robot Modeling and Control, John Wiley & Sons
14. Tsai LW (1999) Robot Analysis: The Mechanics of Serial and Parallel Manipulators, John Wiley & Sons
15. Xie M (2003) Fundamentals of Robotics—Linking Perception to Action, World Scientific

Index