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Andrea Bacciotti

# Stability and Control of Linear Systems

 Springer



*To Giannina*

# Preface

This book is the natural outcome of a course I taught for many years at the Technical University of Torino, first for students enrolled in the aerospace engineering curriculum, and later for students enrolled in the applied mathematics curriculum. The aim of the course was to provide an introduction to the main notions of system theory and automatic control, with a rigorous theoretical framework and a solid mathematical background.

Throughout the book, the reference model is a finite-dimensional, time-invariant, multivariable linear system. The exposition is basically concerned with the time-domain approach, but also the frequency-domain approach is taken into consideration. In fact, the relationship between the two approaches is discussed, especially for the case of single-input–single-output systems. Of course, there are many other excellent handbooks on the same subject (just to quote a few of them, [3, 6, 8, 11, 14, 23, 25, 27, 28, 32]). The distinguishing feature of the present book is the treatment of some specific topics which are rare to find elsewhere at a graduate level. For instance, bounded-input–bounded-output stability (including a characterization in terms of canonical decompositions), static output feedback stabilization (for which a simple criterion in terms of generalized inverse matrices is proposed), controllability under constrained controls.

The mathematical theories of stability and controllability of linear systems are essentially based on linear algebra, and it has reached today a high level of advancement. During the last three decades of the past century, a great effort was done, in order to develop an analogous theory for nonlinear systems, based on differential geometry (see [7] for a historical overview). For this development, usually referred to as *geometric control theory*, we have today a rich literature ([2, 5, 13, 18–20, 26, 30]). However, I believe that the starting point for a successful approach to nonlinear systems is a wide and deep knowledge of the linear case. For this reason, while this book is limited to the linear context, in the presentation and organization of the material, as well as in the selection of topics, the final goal I had in mind is to prepare the reader for such a nonlinear extension.

Concerning the prerequisites, I assume that the reader is familiar with basic differential and integral calculus (for real functions of several real variables) and linear algebra. Some notions of complex analysis are required in the frequency-domain approach. The book can be used as a reference book for basic courses at a doctoral (or also upper undergraduate) level in mathematical control theory and in automatic control. More generally, parts of this book can be used in applied mathematics courses, where an introduction to the point of view of system theory and control philosophy is advisable. The perspective of control systems and the stability problem are indeed ubiquitous in applied sciences and witness a rapidly increasing importance in modern engineering. At a postdoctoral level, this book can be recommended for reading courses both for mathematician oriented to engineering applications and engineers with theoretical interests. To better focus on the main concepts and results, some more technical proofs are avoided or limited to special situations. However, in these cases, appropriate bibliographic references are supplied for the curious reader.

It follows a short description of the contents. The first chapter aims to introduce the reader to the “point of view” of system theory: In particular, the notions of input–output operator and external stability are given. The second chapter deals with systems without external forces which reduce, according to a more classical terminology, to homogeneous systems of linear differential equations. In view of the application, we are interested in, the representation of the general integral in terms of exponential matrix and Jordan form is crucial, and it is treated in detail. Chapter 3 is devoted to Lyapunov stability theory of the equilibrium position of a linear unforced system. The results reported in this chapter are classical but very important for the following chapters. In Chap. 4, we present some alternative approaches to the representation of solutions of a nonhomogeneous (i.e., with forcing term) system of linear differential equations: variation of constants, undetermined coefficients, Laplace transform. In Chap. 5 we finally begin the study of linear systems in a control perspective. We discuss the notions of controllability and observability, their analogies and characterizations, and the corresponding canonical forms. The final section treats shortly the controllability problem under constrained control, in view of possible applications to optimization theory. In Chap. 6, we address the bounded-input–bounded-output stability problem, and we propose a characterization using the canonical decompositions introduced in Chap. 5. Chapter 7 is devoted to various aspects of the stabilization problem: asymptotic controllability, static state feedback stabilization, static output feedback stabilization, dynamic output feedback stabilization. In particular, we re-propose in a new setting some old results about static output feedback stabilization. In author’s opinion, these results are very interesting, but neglected in the current literature. Finally, in Chap. 8, we introduce the frequency-domain approach and study the relationship with the time-domain approach. Two appendices follow. In the first one, the notions of internal stability are introduced. These notions are formulated with respect to a system of nonlinear ordinary differential equations. In fact, only in this part of the book nonlinear systems came into play. The reason of this choice is that all the aspects of the

stability notions became more evident in the nonlinear context. The second appendix is a short list of useful facts about Laplace transform.

Finally, I wish to thank students, colleagues, and coworkers who contributed in many ways to improve the content of this book. A special thanks to Luisa Mazzi and Francesca Ceragioli.

Turin, Italy

Andrea Bacciotti

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# Notations and Terminology

- We denote by  $\mathbf{N}$ ,  $\mathbf{Z}$ ,  $\mathbf{R}$ , and  $\mathbf{C}$ , respectively, the set of natural, integer, real, and complex numbers. The symbol  $|a|$  denotes the modulus of  $a$  if  $a \in \mathbf{C}$ , and the absolute value of  $a$  if  $a \in \mathbf{R}$ . The symbol  $\text{sgn } a$  denotes the sign function (i.e.,  $a$  if  $a \geq 0$ ,  $-a$  if  $a < 0$ ). If  $z \in \mathbf{C}$ ,  $\bar{z}$   $\text{Re } z$ ,  $\text{im } z$  denote respectively the conjugate, the real part and the imaginary part of  $z$ .
- If  $V$  is a (real or complex) vector space of finite dimension with  $\dim V = n$ , the components of  $v \in V$  with respect to a given basis are usually denoted by  $v_1, \dots, v_n$  and we also write  $v = (v_1, \dots, v_n)$ . This notation is simple, but it may give rise to some ambiguity when we deal with several vectors distinguished by indices; in these cases, we write  $(v_i)_j$  to denote the  $j$ -th component of the  $i$ -th vector. The subspace of  $V$  generated by a subset  $U \subset V$  is denoted by  $\text{span } U$ .
- If  $v$  is an element of a finite-dimensional normed vector space  $V$ , the norm of  $v$  is generically denoted by  $\|v\|_V$  or, when the space is clear from the context, simply by  $\|v\|$ . If  $v$  is a vector of  $\mathbf{R}^n$ , and if not differently stated,  $\|v\|$  denotes the Euclidean norm, i.e.,  $\|v\| = (\sum_{i=1}^n v_i^2)^{1/2}$ .
- Let  $m$  and  $n$  be fixed. We denote  $\mathcal{M}(\mathbf{R})$  the vector space of all the matrices with  $m$  rows and  $n$  columns, with real entries. A similar notation with  $\mathbf{R}$  replaced by  $\mathbf{C}$  is adopted for matrices with complex entries. If  $M \in \mathcal{M}(\mathbf{R})$ , we may also say that  $M$  is a  $m \times n$  matrix. Of course,  $\mathcal{M}(\mathbf{R})$  can be identified with  $\mathbf{R}^{m \times n}$ : However, note that to this end,  $\mathbf{R}^{m \times n}$  should be considered different from  $\mathbf{R}^{m \times n}$ . A matrix  $M$  is said to be *square* if  $n = m$ . In this case, we may also say that  $M$  has dimension  $n$ .

To specify the entries of a matrix, we write

$$M = (m_{ij})_{i=1, \dots, n, j=1, \dots, m}$$

(the first index specifies the row, the second one the column). As for vectors, we may assign a norm to a matrix. In this book, we use the so-called Frobenius norm

$$\|M\| = \sqrt{\sum_{i,j} |m_{ij}|^2}.$$

- The identity matrix of dimension  $n$  is denoted by  $I_n$ , or simply by  $I$  when the dimension is clear from the context. If  $M$  is a square matrix, we denote by  $M^t$  the transpose of  $M$ . We also denote respectively by  $\text{tr } M$ ,  $\det M$ , and  $\text{rank } M$  the trace, the determinant, and the rank of  $M$ . The symbol  $\ker M$  represents the kernel of  $M$ , the symbol  $\text{im } M$  the image (range) of  $M$ . The characteristic polynomial of a  $n \times n$  matrix  $M$  is written  $p_M(\lambda)$ . It is defined by

$$p_M(\lambda) = \det(M - \lambda I) = (-1)^n \det(\lambda I - M).$$

Note in particular that  $\deg p_M(\lambda) = n$  (where  $\deg P(\lambda)$  denotes the degree of a polynomial  $P(\lambda)$ ) and that for each  $n$ ,  $(-1)^n p_M(\lambda)$  is a *monic* polynomial (which means that the coefficient of  $\lambda^n$  is 1).

- Recall that the eigenvalues of a square matrix  $M$  are the roots of the characteristic polynomial of  $M$ , that is, the solutions of the algebraic equation  $p_M(\lambda) = 0$ . The set of distinct eigenvalues of  $M$  constitutes the *spectrum* of  $M$ . It is denoted by  $\sigma(M)$ , and it is, in general, a subset of the complex plane  $\mathbf{C}$ . Recall also that the matrices  $A$  and  $B$  are *similar* if there exists a nonsingular matrix  $P$  such that  $B = P^{-1}AP$ .

An eigenvector of  $M$  corresponding to an eigenvalue  $\lambda$  is a nontrivial solution of the linear algebraic system  $(M - \lambda I)v_0 = 0$ . The dimension of the subspace generated by all the eigenvalues of an eigenvalue  $\lambda$  of  $A$  is called the *geometric multiplicity* of  $\lambda$ . The geometric multiplicity is less than or equal to the algebraic multiplicity of  $\lambda$ .

Let  $v_0$  be an eigenvector of  $M$ ; the finite sequence of vectors  $v_1, \dots, v_k$  forms a chain of generalized eigenvectors generated by  $v_0$  if  $(M - \lambda I)v_1 = v_0$ ,  $(M - \lambda I)v_2 = v_1, \dots, (M - \lambda I)v_k = v_{k-1}$ .

- If  $A$  is a subset of  $\mathbf{R}^n$ , we denote respectively by  $\overset{\circ}{A}$ ,  $\bar{A}$ ,  $\partial A$  the set of the interior points of  $A$ , the closure of  $A$ , the boundary of  $A$  (in the topology of  $\mathbf{R}^n$ ).
- If  $A$  and  $B$  are two arbitrary sets,  $\mathcal{F}(A, B)$  denotes the set of all the functions from  $A$  to  $B$ . In particular:
  - $\mathcal{C}(I, U)$  denotes the set of all the continuous functions defined in  $I$  with values in  $U$ , where  $I$  is an interval (open or closed, bounded or unbounded) of real numbers and  $U \subseteq \mathbf{R}^n$ ;
  - $\mathcal{PC}([a, b], U)$  denotes the set of all the piecewise continuous,<sup>1</sup> right-continuous functions defined on  $[a, b]$  with values in  $U$ , where  $a$  and  $b$  are real numbers ( $a < b$ ) and  $U \subseteq \mathbf{R}^n$ ;

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<sup>1</sup>Recall that a function is piecewise continuous on a compact interval  $[a, b]$  if in this interval it has at most finitely many discontinuity points, and each possible discontinuity point is a jump.

- $\mathcal{PC}([a, +\infty), U)$ , where  $a \in \mathbf{R}$  and  $U \subseteq \mathbf{R}^n$ , denotes the set of all the piecewise continuous<sup>2</sup> right-continuous functions defined on the interval  $[a, +\infty)$  with values in  $U$  (the sets  $\mathcal{PC}((-\infty, +\infty), U)$  and  $\mathcal{PC}((-\infty, b], U)$  are defined in analogous way);
- $\mathcal{B}(I, \mathbf{R}^n)$  denotes the set of all the bounded functions defined in the interval  $I$  with values in  $\mathbf{R}^n$ ;
- we may use also the notations

$$\mathcal{CB}([a, +\infty), \mathbf{R}^n) \quad \text{and} \quad \mathcal{PCB}([a, +\infty), \mathbf{R}^n)$$

to denote the sets of all the bounded functions which belong respectively to the sets

$$\mathcal{C}([a, +\infty), \mathbf{R}^n) \quad \text{and} \quad \mathcal{PC}([a, +\infty), \mathbf{R}^n).$$

- If the function  $f(\cdot)$  is an element of a functional normed vector space  $\mathcal{V}$ , its norm is denoted by  $\|f(\cdot)\|_{\mathcal{V}}$ . In particular, if  $f(\cdot) \in \mathcal{B}(I, \mathbf{R}^n)$ , we will write  $\|f(\cdot)\|_{\infty} = \sup_{t \in I} \|f(t)\|$  (norm of the uniform convergence).
- Depending on the circumstances, for the derivative of a function  $f(t) : \mathbf{R} \rightarrow \mathbf{R}^n$ , the following symbols can be used:  $\frac{df}{dt}$ ,  $f'(t)$ ,  $\dot{f}(t)$ ,  $(Df)(t)$ . For higher-order derivatives, we write  $f^{(k)}(t)$ .
- A rational function has the form  $R(s) = N(s)/D(s)$  where  $N(s)$  and  $D(s)$  are polynomials. It is usually thought of as a function from  $\mathbf{C}$  to  $\mathbf{C}$ . A rational function is said to be *proper* if  $\deg N(s) < \deg D(s)$ . Other agreements about rational functions will be specified later in Chap. 8 (see in particular Remark 8.2)

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<sup>2</sup> Recall that a function is piecewise continuous on an unbounded interval  $I$  if it is piecewise continuous on every compact interval  $[c, b] \subset I$ .