Chapter 2

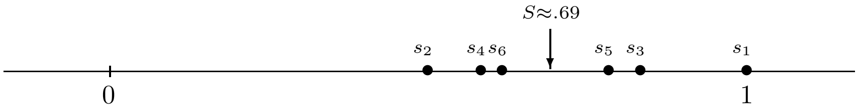
Sequences and Series

2.1 Discussion: Rearrangements of Infinite Series

Consider the infinite series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots .$$

If we naively begin adding from the left-hand side, we get a sequence of what are called *partial sums*. In other words, let s_n equal the sum of the first n terms of the series, so that $s_1 = 1$, $s_2 = 1/2$, $s_3 = 5/6$, $s_4 = 7/12$, and so on. One immediate observation is that the successive sums oscillate in a progressively narrower space. The odd sums decrease ($s_1 > s_3 > s_5 > \ldots$) while the even sums increase ($s_2 < s_4 < s_6 < \ldots$).



$$s_2 < s_4 < s_6 < \cdots S \cdots < s_5 < s_3 < s_1$$

It seems reasonable—and we will soon prove—that the sequence (s_n) eventually hones in on a value, call it S, where the odd and even partial sums "meet." At this moment, we cannot compute S precisely, but we know it falls somewhere between $7/12$ and $5/6$. Summing a few hundred terms reveals that $S \approx .69$. Whatever its value, there is now an overwhelming temptation to write

(1) $$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots$$

S. Abbott, *Understanding Analysis*, Undergraduate Texts in Mathematics, DOI 10.1007/978-1-4939-2712-8_2

meaning, perhaps, that if we could indeed add up *all* infinitely many of these numbers, then the sum would *equal* S. A more familiar example of an equation of this type might be

$$2 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \cdots,$$

the only difference being that in the second equation we have a more recognizable value for the sum.

But now for the crux of the matter. The symbols $+$, $-$, and $=$ in the preceding equations are deceptively familiar notions being used in a very unfamiliar way. The crucial question is whether or not properties of addition and equality that are well understood for finite sums remain valid when applied to infinite objects such as equation (1). The answer, as we are about to witness, is somewhat ambiguous.

Treating equation (1) in a standard algebraic way, let's multiply through by $1/2$ and add it back to equation (1):

$$
\begin{array}{lrl}
 & \frac{1}{2}S = & \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \cdots \\
 & +\ S = & 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \frac{1}{12} + \frac{1}{13} - \cdots \\
\hline
(2) & \frac{3}{2}S = & 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \frac{1}{13} \cdots
\end{array}
$$

Now, look carefully at the result. The sum in equation (2) consists *precisely* of the same terms as those in the original equation (1), only in a different order. Specifically, the series in (2) is a rearrangement of (1) where we list the first two positive terms $(1 + \frac{1}{3})$ followed by the first negative term $(-\frac{1}{2})$, followed by the next two positive terms $(\frac{1}{5} + \frac{1}{7})$ and then the next negative term $(-\frac{1}{4})$. Continuing this, it is apparent that every term in (2) appears in (1) and vice versa. The rub comes when we realize that equation (2) asserts that the sum of these rearranged, but otherwise unaltered, numbers is equal to $3/2$ its original value. Indeed, adding a few hundred terms of equation (2) produces partial sums in the neighborhood of 1.03. Addition, in this infinite setting, is not commutative!

Let's look at a similar rearrangement of the series

$$\sum_{n=0}^{\infty} (-1/2)^n.$$

This series is geometric with first term 1 and common ratio $r = -1/2$. Using the formula $1/(1-r)$ for the sum of a geometric series (Example 2.7.5), we get

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \frac{1}{64} - \frac{1}{128} + \frac{1}{256} \cdots = \frac{1}{1 - (-\frac{1}{2})} = \frac{2}{3}.$$

This time, some computational experimentation with the "two positives, one negative" rearrangement

$$1 + \frac{1}{4} - \frac{1}{2} + \frac{1}{16} + \frac{1}{64} - \frac{1}{8} + \frac{1}{256} + \frac{1}{1024} - \frac{1}{32} \cdots$$

yields partial sums quite close to $2/3$. The sum of the first 30 terms, for instance, equals .666667. Infinite addition is commutative in some instances but not in others.

Far from being a charming theoretical oddity of infinite series, this phenomenon can be the source of great consternation in many applied situations. How, for instance, should a double summation over two index variables be defined? Let's say we are given a *grid* of real numbers $\{a_{ij} : i, j \in \mathbf{N}\}$, where $a_{ij} = 1/2^{j-i}$ if $j > i$, $a_{ij} = -1$ if $j = i$, and $a_{ij} = 0$ if $j < i$.

$$\begin{bmatrix} -1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \cdots \\ 0 & -1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \cdots \\ 0 & 0 & -1 & \frac{1}{2} & \frac{1}{4} & \cdots \\ 0 & 0 & 0 & -1 & \frac{1}{2} & \cdots \\ 0 & 0 & 0 & 0 & -1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

We would like to attach a mathematical meaning to the summation

$$\sum_{i,j=1}^{\infty} a_{ij}$$

whereby we intend to include every term in the preceding array in the total. One natural idea is to temporarily fix i and sum across each row. A moment's reflection (and a fact about geometric series) shows that each row sums to 0. Summing the sums of the rows, we get

$$\sum_{i,j=1}^{\infty} a_{ij} = \sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} a_{ij} \right) = \sum_{i=1}^{\infty} (0) = 0.$$

We could just as easily have decided to fix j and sum down each column first. In this case, we have

$$\sum_{i,j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} \left(\sum_{i=1}^{\infty} a_{ij} \right) = \sum_{j=1}^{\infty} \left(\frac{-1}{2^{j-1}} \right) = -2.$$

Changing the order of the summation changes the value of the sum! One common way that double sums arise (although not this particular one) is from the multiplication of two series. There is a natural desire to write

$$\left(\sum a_i \right) \left(\sum b_j \right) = \sum_{i,j} a_i b_j,$$

except that the expression on the right-hand side makes no sense at the moment.

It is the pathologies that give rise to the need for rigor. A satisfying resolution to the questions raised will require that we be absolutely precise about what we mean as we manipulate these infinite objects. It may seem that progress is slow at first, but that is because we do not want to fall into the trap of letting the biases of our intuition corrupt our arguments. Rigorous proofs are meant to be a check on intuition, and in the end we will see that they vastly improve our mental picture of the mathematical infinite.

As a final example, consider something as intuitively fundamental as the associative property of addition applied to the series $\sum_{n=1}^{\infty}(-1)^n$. Grouping the terms one way gives

$$(-1+1)+(-1+1)+(-1+1)+(-1+1)+\cdots = 0+0+0+0+\cdots = 0,$$

whereas grouping in another yields

$$-1+(1-1)+(1-1)+(1-1)+\cdots = -1+0+0+0+\cdots = -1.$$

Manipulations that are legitimate in finite settings do not always extend to infinite settings. Deciding when they do and why they do not is one of the central themes of analysis.

2.2 The Limit of a Sequence

An understanding of infinite series depends heavily on a clear understanding of the theory of sequences. In fact, most of the concepts in analysis can be reduced to statements about the behavior of sequences. Thus, we will spend a significant amount of time investigating sequences before taking on infinite series.

Definition 2.2.1. A *sequence* is a function whose domain is $\mathbf{N}$.

This formal definition leads immediately to the familiar depiction of a sequence as an ordered list of real numbers. Given a function $f : \mathbf{N} \to \mathbf{R}$, $f(n)$ is just the nth term on the list. The notation for sequences reinforces this familiar understanding.

Example 2.2.2. Each of the following are common ways to describe a sequence.

(i) $(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots)$,

(ii) $(\frac{1+n}{n})_{n=1}^{\infty} = (\frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \cdots)$,

(iii) (a_n), where $a_n = 2^n$ for each $n \in \mathbf{N}$,

(iv) (x_n), where $x_1 = 2$ and $x_{n+1} = \frac{x_n+1}{2}$.

On occasion, it will be more convenient to index a sequence beginning with $n = 0$ or $n = n_0$ for some natural number n_0 different from 1. These minor variations should cause no confusion. What is essential is that a sequence be an *infinite* list of real numbers. What happens at the beginning of such a list is of little importance in most cases. The business of analysis is concerned with the behavior of the infinite "tail" of a given sequence.

We now present what is arguably the most important definition in the book.

Definition 2.2.3 (Convergence of a Sequence). A sequence (a_n) *converges* to a real number a if, for every positive number ϵ, there exists an $N \in \mathbf{N}$ such that whenever $n \geq N$ it follows that $|a_n - a| < \epsilon$.

To indicate that (a_n) converges to a, we usually write either $\lim a_n = a$ or $(a_n) \to a$. The notation $\lim_{n\to\infty} a_n = a$ is also standard.

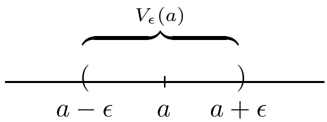
In an effort to decipher this complicated definition, it helps first to consider the ending phrase "$|a_n - a| < \epsilon$," and think about the points that satisfy an inequality of this type.

Definition 2.2.4. Given a real number $a \in \mathbf{R}$ and a positive number $\epsilon > 0$, the set

$$V_\epsilon(a) = \{x \in \mathbf{R} : |x - a| < \epsilon\}$$

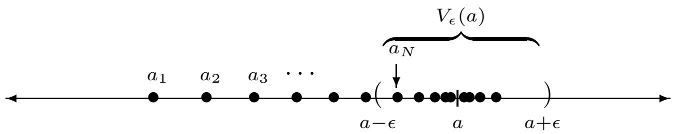
is called the *ϵ-neighborhood of a.*

Notice that $V_\epsilon(a)$ consists of all of those points whose distance from a is less than ϵ. Said another way, $V_\epsilon(a)$ is an interval, centered at a, with radius ϵ.



Recasting the definition of convergence in terms of ϵ-neighborhoods gives a more geometric impression of what is being described.

Definition 2.2.3B (Convergence of a Sequence: Topological Version). A sequence (a_n) converges to a if, given any ϵ-neighborhood $V_\epsilon(a)$ of a, there exists a point in the sequence after which all of the terms are in $V_\epsilon(a)$. In other words, every ϵ-neighborhood contains all but a finite number of the terms of (a_n).



Definition 2.2.3 and Definition 2.2.3B say precisely the same thing; the natural number N in the original version of the definition is the point where the sequence (a_n) enters $V_\epsilon(a)$, never to leave. It should be apparent that *the value of N depends on the choice of ϵ.* The smaller the ϵ-neighborhood, the larger N may have to be.

Example 2.2.5. Consider the sequence (a_n), where $a_n = 1/\sqrt{n}$.

Our intuitive understanding of limits points confidently to the conclusion that

$$\lim\left(\frac{1}{\sqrt{n}}\right) = 0.$$

Before trying to prove this not too impressive fact, let's first explore the relationship between ϵ and N in the definition of convergence. For the moment, take ϵ to be $1/10$. This defines a sort of "target zone" for the terms in the sequence. By claiming that the limit of (a_n) is 0, we are saying that the terms in this sequence eventually get arbitrarily close to 0. How close? What do we mean by "eventually"? We have set $\epsilon = 1/10$ as our standard for closeness, which leads to the ϵ-neighborhood $(-1/10, 1/10)$ centered around the limit 0. How far out into the sequence must we look before the terms fall into this interval? The 100th term $a_{100} = 1/10$ puts us right on the boundary, and a little thought reveals that

$$\text{if} \quad n > 100, \quad \text{then} \quad a_n \in \left(-\frac{1}{10}, \frac{1}{10}\right).$$

Thus, for $\epsilon = 1/10$ we choose $N = 101$ (or anything larger) as our response.

Now, our choice of $\epsilon = 1/10$ was rather whimsical, and we can do this again, letting $\epsilon = 1/50$. In this case, our target neighborhood shrinks to $(-1/50, 1/50)$, and it is apparent that we must travel farther out into the sequence before a_n falls into this interval. How far? Essentially, we require that

$$\frac{1}{\sqrt{n}} < \frac{1}{50} \quad \text{which occurs as long as} \quad n > 50^2 = 2500.$$

Thus, $N = 2501$ is a suitable response to the challenge of $\epsilon = 1/50$.

It may seem as though this duel could continue forever, with different ϵ challenges being handed to us one after another, each one requiring a suitable value of N in response. In a sense, this is correct, except that the game is effectively over the instant we recognize a *rule* for how to choose N given an *arbitrary* $\epsilon > 0$. For this problem, the desired algorithm is implicit in the algebra carried out to compute the previous response of $N = 2501$. Whatever ϵ happens to be, we want

$$\frac{1}{\sqrt{n}} < \epsilon \quad \text{which is equivalent to insisting that} \quad n > \frac{1}{\epsilon^2}.$$

With this observation, we are ready to write the formal argument.

We claim that

$$\lim\left(\frac{1}{\sqrt{n}}\right) = 0.$$

Proof. Let $\epsilon > 0$ be an arbitrary positive number. Choose a natural number N satisfying

$$N > \frac{1}{\epsilon^2}.$$

We now verify that this choice of N has the desired property. Let $n \geq N$. Then,

$$n > \frac{1}{\epsilon^2} \quad \text{implies} \quad \frac{1}{\sqrt{n}} < \epsilon, \quad \text{and hence} \quad |a_n - 0| < \epsilon. \qquad \square$$

Quantifiers

The definition of convergence given earlier is the result of hundreds of years of refining the intuitive notion of limit into a mathematically rigorous statement. The logic involved is complicated and is intimately tied to the use of the quantifiers "for all" and "there exists." Learning to write a grammatically correct convergence proof goes hand in hand with a deep understanding of why the quantifiers appear in the order that they do.

The definition begins with the phrase,

"*For all* $\epsilon > 0$, *there exists* $N \in \mathbf{N}$ such that ..."

Looking back at our first example, we see that our formal proof begins with, "Let $\epsilon > 0$ be an arbitrary positive number." This is followed by a construction of N and then a demonstration that this choice of N has the desired property. This, in fact, is a basic outline for how every convergence proof should be presented.

TEMPLATE FOR A PROOF THAT $(x_n) \to x$:

- "Let $\epsilon > 0$ be arbitrary."
- Demonstrate a choice for $N \in \mathbf{N}$. This step usually requires the most work, almost all of which is done prior to actually writing the formal proof.
- Now, show that N actually works.
- "Assume $n \geq N$."
- With N well chosen, it should be possible to derive the inequality $|x_n - x| < \epsilon$.

Example 2.2.6. Show

$$\lim \left(\frac{n+1}{n} \right) = 1.$$

As mentioned, before attempting a formal proof, we first need to do some preliminary scratch work. In the first example, we experimented by assigning specific values to ϵ (and it is not a bad idea to do this again), but let us skip straight to the algebraic punch line. The last line of our proof should be that for suitably large values of n,

$$\left| \frac{n+1}{n} - 1 \right| < \epsilon.$$

Because
$$\left|\frac{n+1}{n}-1\right|=\frac{1}{n},$$
this is equivalent to the inequality $1/n<\epsilon$ or $n>1/\epsilon$. Thus, choosing N to be an integer greater than $1/\epsilon$ will suffice.

With the work of the proof done, all that remains is the formal writeup.

Proof. Let $\epsilon>0$ be arbitrary. Choose $N\in\mathbf{N}$ with $N>1/\epsilon$. To verify that this choice of N is appropriate, let $n\in\mathbf{N}$ satisfy $n\geq N$. Then, $n\geq N$ implies $n>1/\epsilon$, which is the same as saying $1/n<\epsilon$. Finally, this means
$$\left|\frac{n+1}{n}-1\right|<\epsilon,$$
as desired. □

It is instructive to see what goes wrong in the previous example if we try to prove that our sequence converges to some limit other than 1.

Theorem 2.2.7 (Uniqueness of Limits). *The limit of a sequence, when it exists, must be unique.*

Proof. Exercise 2.2.6. □

Divergence

Significant insight into the role of the quantifiers in the definition of convergence can be gained by studying an example of a sequence that does not have a limit.

Example 2.2.8. Consider the sequence
$$\left(1,-\frac{1}{2},\frac{1}{3},-\frac{1}{4},\frac{1}{5},-\frac{1}{5},\frac{1}{5},-\frac{1}{5},\frac{1}{5},-\frac{1}{5},\frac{1}{5},-\frac{1}{5},\frac{1}{5},-\frac{1}{5},\cdots\right).$$

How can we argue that this sequence does not converge to zero? Looking at the first few terms, it seems the initial evidence actually supports such a conclusion. Given a challenge of $\epsilon=1/2$, a little reflection reveals that after $N=3$ all the terms fall into the neighborhood $(-1/2,1/2)$. We could also handle $\epsilon=1/4$. (What is the smallest possible N in this case?)

But the definition of convergence says "*For all* $\epsilon>0\ldots$," and it should be apparent that there is no response to a choice of $\epsilon=1/10$, for instance. This leads us to an important observation about the logical negation of the definition of convergence of a sequence. To prove that a particular number x is *not* the limit of a sequence (x_n), we must produce a single value of ϵ for which no $N\in\mathbf{N}$ works. More generally speaking, the negation of a statement that begins "For all P, there exists Q..." is the statement, "For at least one P, no Q is possible..." For instance, how could we disprove the spurious claim that "At every college in the United States, there is a student who is at least seven feet tall"?

We have argued that the preceding sequence does not converge to 0. Let's argue against the claim that it converges to $1/5$. Choosing $\epsilon = 1/10$ produces the neighborhood $(1/10, 3/10)$. Although the sequence continually revisits this neighborhood, there is no point at which it enters and never leaves as the definition requires. Thus, no N exists for $\epsilon = 1/10$, so the sequence does not converge to $1/5$.

Of course, this sequence does not converge to any other real number, and it would be more satisfying to simply say that this sequence does not converge.

Definition 2.2.9. A sequence that does not converge is said to *diverge*.

Although it is not too difficult, we will postpone arguing for divergence in general until we develop a more economical divergence criterion later in Section 2.5.

Exercises

Exercise 2.2.1. What happens if we reverse the order of the quantifiers in Definition 2.2.3?

Definition: A sequence (x_n) *verconges* to x if *there exists* an $\epsilon > 0$ such that *for all* $N \in \mathbf{N}$ it is true that $n \geq N$ implies $|x_n - x| < \epsilon$.

Give an example of a vercongent sequence. Is there an example of a vercongent sequence that is divergent? Can a sequence verconge to two different values? What exactly is being described in this strange definition?

Exercise 2.2.2. Verify, using the definition of convergence of a sequence, that the following sequences converge to the proposed limit.

(a) $\lim \frac{2n+1}{5n+4} = \frac{2}{5}$.

(b) $\lim \frac{2n^2}{n^3+3} = 0$.

(c) $\lim \frac{\sin(n^2)}{\sqrt[3]{n}} = 0$.

Exercise 2.2.3. Describe what we would have to demonstrate in order to disprove each of the following statements.

(a) At every college in the United States, there is a student who is at least seven feet tall.

(b) For all colleges in the United States, there exists a professor who gives every student a grade of either A or B.

(c) There exists a college in the United States where every student is at least six feet tall.

Exercise 2.2.4. Give an example of each or state that the request is impossible. For any that are impossible, give a compelling argument for why that is the case.

(a) A sequence with an infinite number of ones that does not converge to one.

(b) A sequence with an infinite number of ones that converges to a limit not equal to one.

(c) A divergent sequence such that for every $n \in \mathbf{N}$ it is possible to find n consecutive ones somewhere in the sequence.

Exercise 2.2.5. Let $[[x]]$ be the greatest integer less than or equal to x. For example, $[[\pi]] = 3$ and $[[3]] = 3$. For each sequence, find $\lim a_n$ and verify it with the definition of convergence.

(a) $a_n = [[5/n]]$,

(b) $a_n = [[(12 + 4n)/3n]]$.

Reflecting on these examples, comment on the statement following Definition 2.2.3 that "the smaller the ϵ-neighborhood, the larger N may have to be."

Exercise 2.2.6. Prove Theorem 2.2.7. To get started, assume $(a_n) \to a$ and also that $(a_n) \to b$. Now argue $a = b$.

Exercise 2.2.7. Here are two useful definitions:

(i) A sequence (a_n) is *eventually* in a set $A \subseteq \mathbf{R}$ if there exists an $N \in \mathbf{N}$ such that $a_n \in A$ for all $n \geq N$.

(ii) A sequence (a_n) is *frequently* in a set $A \subseteq \mathbf{R}$ if, for every $N \in \mathbf{N}$, there exists an $n \geq N$ such that $a_n \in A$.

(a) Is the sequence $(-1)^n$ eventually or frequently in the set $\{1\}$?

(b) Which definition is stronger? Does frequently imply eventually or does eventually imply frequently?

(c) Give an alternate rephrasing of Definition 2.2.3B using either frequently or eventually. Which is the term we want?

(d) Suppose an infinite number of terms of a sequence (x_n) are equal to 2. Is (x_n) necessarily eventually in the interval $(1.9, 2.1)$? Is it frequently in $(1.9, 2.1)$?

Exercise 2.2.8. For some additional practice with nested quantifiers, consider the following invented definition:

Let's call a sequence (x_n) *zero-heavy* if there exists $M \in \mathbf{N}$ such that for all $N \in \mathbf{N}$ there exists n satisfying $N \leq n \leq N + M$ where $x_n = 0$.

(a) Is the sequence $(0, 1, 0, 1, 0, 1, \ldots)$ zero heavy?

(b) If a sequence is zero-heavy does it necessarily contain an infinite number of zeros? If not, provide a counterexample.

(c) If a sequence contains an infinite number of zeros, is it necessarily zero-heavy? If not, provide a counterexample.

(d) Form the logical negation of the above definition. That is, complete the sentence: A sequence is *not* zero-heavy if ....

2.3 The Algebraic and Order Limit Theorems

The real purpose of creating a rigorous definition for convergence of a sequence is *not* to have a tool to verify computational statements such as $\lim 2n/(n+2) = 2$. Historically, a definition of the limit like Definition 2.2.3 came 150 years after the founders of calculus began working with intuitive notions of convergence. The point of having such a logically tight description of convergence is so that we can confidently *prove statements about convergent sequences in general.* We are ultimately trying to resolve arguments about what is and is not true regarding the behavior of limits with respect to the mathematical manipulations we intend to inflict on them.

As a first example, let us prove that convergent sequences are bounded. The term "bounded" has a rather familiar connotation but, like everything else, we need to be explicit about what it means in this context.

Definition 2.3.1. A sequence (x_n) is *bounded* if there exists a number $M > 0$ such that $|x_n| \leq M$ for all $n \in \mathbf{N}$.

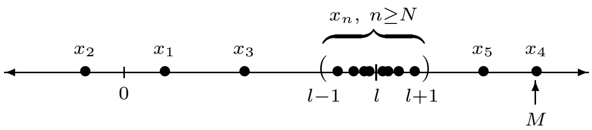
Geometrically, this means that we can find an interval $[-M, M]$ that contains every term in the sequence (x_n).

Theorem 2.3.2. *Every convergent sequence is bounded.*

Proof. Assume (x_n) converges to a limit l. This means that given a particular value of ϵ, say $\epsilon = 1$, we know there must exist an $N \in \mathbf{N}$ such that if $n \geq N$, then x_n is in the interval $(l-1, l+1)$. Not knowing whether l is positive or negative, we can certainly conclude that

$$|x_n| < |l| + 1$$

for all $n \geq N$.



We still need to worry (slightly) about the terms in the sequence that come before the Nth term. Because there are only a finite number of these, we let

$$M = \max\{|x_1|, |x_2|, |x_3|, \ldots, |x_{N-1}|, |l| + 1\}.$$

It follows that $|x_n| \leq M$ for all $n \in \mathbf{N}$, as desired. $\square$

This chapter began with a demonstration of how applying familiar algebraic properties (commutativity of addition) to infinite objects (series) can lead to paradoxical results. These examples are meant to instill in us a sense of caution and justify the extreme care we are taking in drawing our conclusions. The following theorems illustrate that sequences behave extremely well with respect to the operations of addition, multiplication, division, and order.

Theorem 2.3.3 (Algebraic Limit Theorem). *Let* $\lim a_n = a$, *and* $\lim b_n = b$. *Then,*

(i) $\lim(ca_n) = ca$, *for all* $c \in \mathbf{R}$;

(ii) $\lim(a_n + b_n) = a + b$;

(iii) $\lim(a_n b_n) = ab$;

(iv) $\lim(a_n/b_n) = a/b$, *provided* $b \neq 0$.

Proof. (i) Consider the case where $c \neq 0$. We want to show that the sequence (ca_n) converges to ca, so the structure of the proof follows the template we described in Section 2.2. First, we let ϵ be some arbitrary positive number. Our goal is to find some point in the sequence (ca_n) after which we have

$$|ca_n - ca| < \epsilon.$$

Now,

$$|ca_n - ca| = |c||a_n - a|.$$

We are given that $(a_n) \to a$, so we know we can make $|a_n - a|$ as small as we like. In particular, we can choose an N such that

$$|a_n - a| < \frac{\epsilon}{|c|}$$

whenever $n \geq N$. To see that this N indeed works, observe that, for all $n \geq N$,

$$|ca_n - ca| = |c||a_n - a| < |c|\frac{\epsilon}{|c|} = \epsilon.$$

The case $c = 0$ reduces to showing that the constant sequence $(0, 0, 0, \ldots)$ converges to 0, which is easily verified.

Before continuing with parts (ii), (iii), and (iv), we should point out that the proof of (i), while somewhat short, is extremely typical for a convergence proof. Before embarking on a formal argument, it is a good idea to take an inventory of what we *want* to make less than ϵ, and what we are *given* can be made small for suitable choices of n. For the previous proof, we wanted to make $|ca_n - ca| < \epsilon$, and we were given $|a_n - a| <$ *anything we like* (for large values of n). Notice that in (i), and all of the ensuing arguments, the strategy each time is to bound the quantity we want to be less than ϵ, which in each case is

$$|(\text{terms of sequence}) - (\text{proposed limit})|,$$

with some algebraic combination of quantities over which we have control.

(ii) To prove this statement, we need to argue that the quantity

$$|(a_n + b_n) - (a + b)|$$

can be made less than an arbitrary ϵ using the assumptions that $|a_n - a|$ and $|b_n - b|$ can be made as small as we like for large n. The first step is to use the triangle inequality (Example 1.2.5) to say

$$|(a_n + b_n) - (a + b)| = |(a_n - a) + (b_n - b)| \leq |a_n - a| + |b_n - b|.$$

Again, we let $\epsilon > 0$ be arbitrary. The technique this time is to divide the ϵ between the two expressions on the right-hand side in the preceding inequality. Using the hypothesis that $(a_n) \to a$, we know there exists an N_1 such that

$$|a_n - a| < \frac{\epsilon}{2} \quad \text{whenever} \quad n \geq N_1.$$

Likewise, the assumption that $(b_n) \to b$ means that we can choose an N_2 so that

$$|b_n - b| < \frac{\epsilon}{2} \quad \text{whenever} \quad n \geq N_2.$$

The question now arises as to which of N_1 or N_2 we should take to be our choice of N. By choosing $N = \max\{N_1, N_2\}$, we ensure that if $n \geq N$, then $n \geq N_1$ and $n \geq N_2$. This allows us to conclude that

$$\begin{aligned} |(a_n + b_n) - (a + b)| &\leq |a_n - a| + |b_n - b| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

for all $n \geq N$, as desired.

(iii) To show that $(a_n b_n) \to ab$, we begin by observing that

$$\begin{aligned} |a_n b_n - ab| &= |a_n b_n - ab_n + ab_n - ab| \\ &\leq |a_n b_n - ab_n| + |ab_n - ab| \\ &= |b_n||a_n - a| + |a||b_n - b|. \end{aligned}$$

In the initial step, we subtracted and then added ab_n, which created an opportunity to use the triangle inequality. Essentially, we have broken up the distance from $a_n b_n$ to ab with a midway point and are using the sum of the two distances to overestimate the original distance. This clever trick will become a familiar technique in arguments to come.

Letting $\epsilon > 0$ be arbitrary, we again proceed with the strategy of making each piece in the preceding inequality less than $\epsilon/2$. For the piece on the right-hand side ($|a||b_n - b|$), if $a \neq 0$ we can choose N_1 so that

$$n \geq N_1 \quad \text{implies} \quad |b_n - b| < \frac{1}{|a|}\frac{\epsilon}{2}.$$

(The case when $a = 0$ is handled in Exercise 2.3.9.) Getting the term on the left-hand side ($|b_n||a_n - a|$) to be less than $\epsilon/2$ is complicated by the fact that we have a variable quantity $|b_n|$ to contend with as opposed to the constant $|a|$ we encountered in the right-hand term. The idea is to replace $|b_n|$ with a worst-case estimate. Using the fact that convergent sequences are bounded (Theorem 2.3.2), we know there exists a bound $M > 0$ satisfying $|b_n| \leq M$ for all $n \in \mathbf{N}$. Now, we can choose N_2 so that

$$|a_n - a| < \frac{1}{M}\frac{\epsilon}{2} \quad \text{whenever} \quad n \geq N_2.$$

To finish the argument, pick $N = \max\{N_1, N_2\}$, and observe that if $n \geq N$, then

$$\begin{aligned} |a_n b_n - ab| &\leq |a_n b_n - ab_n| + |ab_n - ab| \\ &= |b_n||a_n - a| + |a||b_n - b| \\ &\leq M|a_n - a| + |a||b_n - b| \\ &< M\left(\frac{\epsilon}{M2}\right) + |a|\left(\frac{\epsilon}{|a|2}\right) = \epsilon. \end{aligned}$$

(iv) This final statement will follow from (iii) if we can prove that

$$(b_n) \to b \quad \text{implies} \quad \left(\frac{1}{b_n}\right) \to \frac{1}{b}$$

whenever $b \neq 0$. We begin by observing that

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| = \frac{|b - b_n|}{|b||b_n|}.$$

Because $(b_n) \to b$, we can make the preceding numerator as small as we like by choosing n large. The problem comes in that we need a worst-case estimate on the size of $1/(|b||b_n|)$. Because the b_n terms are in the denominator, we are no longer interested in an upper bound on $|b_n|$ but rather in an inequality of the form $|b_n| \geq \delta > 0$. This will then lead to a bound on the size of $1/(|b||b_n|)$.

The trick is to look far enough out into the sequence (b_n) so that the terms are closer to b than they are to 0. Consider the particular value $\epsilon_0 = |b|/2$. Because $(b_n) \to b$, there exists an N_1 such that $|b_n - b| < |b|/2$ for all $n \geq N_1$. This implies $|b_n| > |b|/2$.

Next, choose N_2 so that $n \geq N_2$ implies

$$|b_n - b| < \frac{\epsilon|b|^2}{2}.$$

Finally, if we let $N = \max\{N_1, N_2\}$, then $n \geq N$ implies

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| = |b - b_n|\frac{1}{|b||b_n|} < \frac{\epsilon|b|^2}{2}\frac{1}{|b|\frac{|b|}{2}} = \epsilon.$$

□

Limits and Order

Although there are a few dangers to avoid (see Exercise 2.3.7), the Algebraic Limit Theorem verifies that the relationship between algebraic combinations of sequences and the limiting process is as trouble-free as we could hope for. Limits can be computed from the individual component sequences provided that each component limit exists. The limiting process is also well-behaved with respect to the order operation.

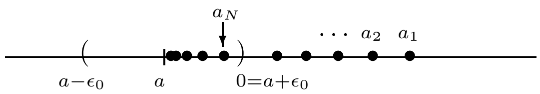
Theorem 2.3.4 (Order Limit Theorem). *Assume* $\lim a_n = a$ *and* $\lim b_n = b$.

(i) *If* $a_n \geq 0$ *for all* $n \in \mathbf{N}$, *then* $a \geq 0$.

(ii) *If* $a_n \leq b_n$ *for all* $n \in \mathbf{N}$, *then* $a \leq b$.

(iii) *If there exists* $c \in \mathbf{R}$ *for which* $c \leq b_n$ *for all* $n \in \mathbf{N}$, *then* $c \leq b$. *Similarly, if* $a_n \leq c$ *for all* $n \in \mathbf{N}$, *then* $a \leq c$.

Proof. (i) We will prove this by contradiction; thus, let's assume $a < 0$. The idea is to produce a term in the sequence (a_n) that is also less than zero. To do this, we consider the particular value $\epsilon = |a|$. The definition of convergence guarantees that we can find an N such that $|a_n - a| < |a|$ for all $n \geq N$. In particular, this would mean that $|a_N - a| < |a|$, which implies $a_N < 0$. This contradicts our hypothesis that $a_N \geq 0$. We therefore conclude that $a \geq 0$.



(ii) The Algebraic Limit Theorem ensures that the sequence $(b_n - a_n)$ converges to $b-a$. Because $b_n - a_n \geq 0$, we can apply part (i) to get that $b-a \geq 0$.

(iii) Take $a_n = c$ (or $b_n = c$) for all $n \in \mathbf{N}$, and apply (ii). $\square$

A word about the idea of "tails" is in order. Loosely speaking, limits and their properties do not depend at all on what happens at the beginning of the sequence but are strictly determined by what happens when n gets large. Changing the value of the first ten—or ten thousand—terms in a particular sequence has no effect on the limit. Theorem 2.3.4, part (i), for instance, assumes that $a_n \geq 0$ *for all* $n \in \mathbf{N}$. However, the hypothesis could be weakened by assuming only that there exists some point N_1 where $a_n \geq 0$ for all $n \geq N_1$. The theorem remains true, and in fact the same proof is valid with the provision that when N is chosen it be at least as large as N_1.

In the language of analysis, when a property (such as non-negativity) is not necessarily possessed by some finite number of initial terms but is possessed by all terms in the sequence after some point N, we say that the sequence *eventually* has this property. (See Exercise 2.2.7.) Theorem 2.3.4, part (i), could be restated, "Convergent sequences that are eventually nonnegative converge to nonnegative limits." Parts (ii) and (iii) have similar modifications, as will many other upcoming results.

Exercises

Exercise 2.3.1. Let $x_n \geq 0$ for all $n \in \mathbf{N}$.

(a) If $(x_n) \to 0$, show that $(\sqrt{x_n}) \to 0$.

(b) If $(x_n) \to x$, show that $(\sqrt{x_n}) \to \sqrt{x}$.

Exercise 2.3.2. Using only Definition 2.2.3, prove that if $(x_n) \to 2$, then

(a) $\left(\frac{2x_n - 1}{3}\right) \to 1$;

(b) $(1/x_n) \to 1/2$.

(For this exercise the Algebraic Limit Theorem is off-limits, so to speak.)

Exercise 2.3.3 (Squeeze Theorem). Show that if $x_n \leq y_n \leq z_n$ for all $n \in \mathbf{N}$, and if $\lim x_n = \lim z_n = l$, then $\lim y_n = l$ as well.

Exercise 2.3.4. Let $(a_n) \to 0$, and use the Algebraic Limit Theorem to compute each of the following limits (assuming the fractions are always defined):

(a) $\lim \left(\frac{1+2a_n}{1+3a_n-4a_n^2}\right)$

(b) $\lim \left(\frac{(a_n+2)^2-4}{a_n}\right)$

(c) $\lim \left(\frac{\frac{2}{a_n}+3}{\frac{1}{a_n}+5}\right)$.

Exercise 2.3.5. Let (x_n) and (y_n) be given, and define (z_n) to be the "shuffled" sequence $(x_1, y_1, x_2, y_2, x_3, y_3, \ldots, x_n, y_n, \ldots)$. Prove that (z_n) is convergent if and only if (x_n) and (y_n) are both convergent with $\lim x_n = \lim y_n$.

Exercise 2.3.6. Consider the sequence given by $b_n = n - \sqrt{n^2 + 2n}$. Taking $(1/n) \to 0$ as given, and using both the Algebraic Limit Theorem and the result in Exercise 2.3.1, show $\lim b_n$ exists and find the value of the limit.

Exercise 2.3.7. Give an example of each of the following, or state that such a request is impossible by referencing the proper theorem(s):

(a) sequences (x_n) and (y_n), which both diverge, but whose sum $(x_n + y_n)$ converges;

(b) sequences (x_n) and (y_n), where (x_n) converges, (y_n) diverges, and (x_n+y_n) converges;

(c) a convergent sequence (b_n) with $b_n \neq 0$ for all n such that $(1/b_n)$ diverges;

(d) an unbounded sequence (a_n) and a convergent sequence (b_n) with $(a_n - b_n)$ bounded;

(e) two sequences (a_n) and (b_n), where $(a_n b_n)$ and (a_n) converge but (b_n) does not.

Exercise 2.3.8. Let $(x_n) \to x$ and let $p(x)$ be a polynomial.

(a) Show $p(x_n) \to p(x)$.

(b) Find an example of a function $f(x)$ and a convergent sequence $(x_n) \to x$ where the sequence $f(x_n)$ converges, but not to $f(x)$.

Exercise 2.3.9. (a) Let (a_n) be a bounded (not necessarily convergent) sequence, and assume $\lim b_n = 0$. Show that $\lim(a_n b_n) = 0$. Why are we not allowed to use the Algebraic Limit Theorem to prove this?

(b) Can we conclude anything about the convergence of $(a_n b_n)$ if we assume that (b_n) converges to some nonzero limit b?

(c) Use (a) to prove Theorem 2.3.3, part (iii), for the case when $a = 0$.

Exercise 2.3.10. Consider the following list of conjectures. Provide a short proof for those that are true and a counterexample for any that are false.

(a) If $\lim(a_n - b_n) = 0$, then $\lim a_n = \lim b_n$.

(b) If $(b_n) \to b$, then $|b_n| \to |b|$.

(c) If $(a_n) \to a$ and $(b_n - a_n) \to 0$, then $(b_n) \to a$.

(d) If $(a_n) \to 0$ and $|b_n - b| \leq a_n$ for all $n \in \mathbf{N}$, then $(b_n) \to b$.

Exercise 2.3.11 (Cesaro Means). (a) Show that if (x_n) is a convergent sequence, then the sequence given by the averages

$$y_n = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

also converges to the same limit.

(b) Give an example to show that it is possible for the sequence (y_n) of averages to converge even if (x_n) does not.

Exercise 2.3.12. A typical task in analysis is to decipher whether a property possessed by every term in a convergent sequence is necessarily inherited by the limit. Assume $(a_n) \to a$, and determine the validity of each claim. Try to produce a counterexample for any that are false.

(a) If every a_n is an upper bound for a set B, then a is also an upper bound for B.

(b) If every a_n is in the complement of the interval $(0, 1)$, then a is also in the complement of $(0, 1)$.

(c) If every a_n is rational, then a is rational.

Exercise 2.3.13 (Iterated Limits). Given a doubly indexed array a_{mn} where $m, n \in \mathbf{N}$, what should $\lim_{m,n\to\infty} a_{mn}$ represent?

(a) Let $a_{mn} = m/(m + n)$ and compute the *iterated* limits

$$\lim_{n\to\infty}\left(\lim_{m\to\infty} a_{mn}\right) \quad \text{and} \quad \lim_{m\to\infty}\left(\lim_{n\to\infty} a_{mn}\right).$$

Define $\lim_{m,n\to\infty} a_{mn} = a$ to mean that for all $\epsilon > 0$ there exists an $N \in \mathbf{N}$ such that if both $m, n \geq N$, then $|a_{mn} - a| < \epsilon$.

(b) Let $a_{mn} = 1/(m+n)$. Does $\lim_{m,n\to\infty} a_{mn}$ exist in this case? Do the two iterated limits exist? How do these three values compare? Answer these same questions for $a_{mn} = mn/(m^2 + n^2)$.

(c) Produce an example where $\lim_{m,n\to\infty} a_{mn}$ exists but where neither iterated limit can be computed.

(d) Assume $\lim_{m,n\to\infty} a_{mn} = a$, and assume that for each fixed $m \in \mathbf{N}$, $\lim_{n\to\infty}(a_{mn}) \to b_m$. Show $\lim_{m\to\infty} b_m = a$.

(e) Prove that if $\lim_{m,n\to\infty} a_{mn}$ exists and the iterated limits both exist, then all three limits must be equal.

2.4 The Monotone Convergence Theorem and a First Look at Infinite Series

We showed in Theorem 2.3.2 that convergent sequences are bounded. The converse statement is certainly not true. It is not too difficult to produce an example of a bounded sequence that does not converge. On the other hand, if a bounded sequence is *monotone*, then in fact it does converge.

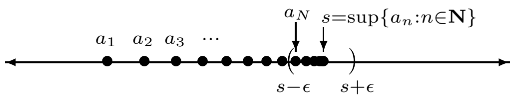
Definition 2.4.1. A sequence (a_n) is *increasing* if $a_n \leq a_{n+1}$ for all $n \in \mathbf{N}$ and *decreasing* if $a_n \geq a_{n+1}$ for all $n \in \mathbf{N}$. A sequence is *monotone* if it is either increasing or decreasing.

Theorem 2.4.2 (Monotone Convergence Theorem). *If a sequence is monotone and bounded, then it converges.*

Proof. Let (a_n) be monotone and bounded. To prove (a_n) converges using the definition of convergence, we are going to need a candidate for the limit. Let's assume the sequence is increasing (the decreasing case is handled similarly), and consider the *set* of points $\{a_n : n \in \mathbf{N}\}$. By assumption, this set is bounded, so we can let

$$s = \sup\{a_n : n \in \mathbf{N}\}.$$

It seems reasonable to claim that $\lim a_n = s$.



To prove this, let $\epsilon > 0$. Because s is the least upper bound for $\{a_n : n \in \mathbf{N}\}$, $s - \epsilon$ is not an upper bound, so there exists a point in the sequence a_N such that $s - \epsilon < a_N$. Now, the fact that (a_n) is increasing implies that if $n \geq N$, then $a_N \leq a_n$. Hence,

$$s - \epsilon < a_N \leq a_n \leq s < s + \epsilon,$$

which implies $|a_n - s| < \epsilon$, as desired. □

The Monotone Convergence Theorem is extremely useful for the study of infinite series, largely because it asserts the convergence of a sequence without explicit mention of the actual limit. This is a good moment to do some preliminary investigations, so it is time to formalize the relationship between sequences and series.

Definition 2.4.3 (Convergence of a Series). Let (b_n) be a sequence. An *infinite series* is a formal expression of the form

$$\sum_{n=1}^{\infty} b_n = b_1 + b_2 + b_3 + b_4 + b_5 + \cdots .$$

We define the corresponding *sequence of partial sums* (s_m) by

$$s_m = b_1 + b_2 + b_3 + \cdots + b_m,$$

and say that the series $\sum_{n=1}^{\infty} b_n$ *converges to* B if the sequence (s_m) converges to B. In this case, we write $\sum_{n=1}^{\infty} b_n = B$.

Example 2.4.4. Consider

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Because the terms in the sum are all positive, the sequence of partial sums given by

$$s_m = 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{m^2}$$

is increasing. The question is whether or not we can find some upper bound on (s_m). To this end, observe

$$\begin{aligned} s_m &= 1+\frac{1}{2\cdot 2}+\frac{1}{3\cdot 3}+\frac{1}{4\cdot 4}+\cdots+\frac{1}{m^2} \\ &< 1+\frac{1}{2\cdot 1}+\frac{1}{3\cdot 2}+\frac{1}{4\cdot 3}+\cdots+\frac{1}{m(m-1)} \\ &= 1+\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{(m-1)}-\frac{1}{m}\right) \\ &= 1+1-\frac{1}{m} \\ &< 2. \end{aligned}$$

Thus, 2 is an upper bound for the sequence of partial sums, so by the Monotone Convergence Theorem, $\sum_{n=1}^{\infty} 1/n^2$ converges to some (for the moment) unknown limit less than 2. (Finding the value of this limit is the subject of Sections 6.1 and 8.3.)

Example 2.4.5 (Harmonic Series). This time, consider the so-called *harmonic series*

$$\sum_{n=1}^{\infty}\frac{1}{n}.$$

Again, we have an increasing sequence of partial sums,

$$s_m = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{m},$$

that upon naive inspection appears as though it may be bounded. However, 2 is no longer an upper bound because

$$s_4 = 1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right) > 1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right) = 2.$$

A similar calculation shows that $s_8 > 2\frac{1}{2}$, and we can see that in general

$$\begin{aligned} s_{2^k} &= 1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\cdots+\frac{1}{8}\right)+\cdots+\left(\frac{1}{2^{k-1}+1}+\cdots+\frac{1}{2^k}\right) \\ &> 1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\cdots+\frac{1}{8}\right)+\cdots+\left(\frac{1}{2^k}+\cdots+\frac{1}{2^k}\right) \\ &= 1+\frac{1}{2}+2\left(\frac{1}{4}\right)+4\left(\frac{1}{8}\right)+\cdots+2^{k-1}\left(\frac{1}{2^k}\right) \\ &= 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots+\frac{1}{2} \\ &= 1+k\left(\frac{1}{2}\right), \end{aligned}$$

which is unbounded. Thus, despite the incredibly slow pace, the sequence of partial sums of $\sum_{n=1}^{\infty} 1/n$ eventually surpasses every number on the positive real line. Because convergent sequences are bounded, the harmonic series diverges.

The previous example is a special case of a general argument that can be used to determine the convergence or divergence of a large class of infinite series.

Theorem 2.4.6 (Cauchy Condensation Test). *Suppose (b_n) is decreasing and satisfies $b_n \geq 0$ for all $n \in \mathbf{N}$. Then, the series $\sum_{n=1}^{\infty} b_n$ converges if and only if the series*

$$\sum_{n=0}^{\infty} 2^n b_{2^n} = b_1 + 2b_2 + 4b_4 + 8b_8 + 16b_{16} + \cdots$$

converges.

Proof. First, assume that $\sum_{n=0}^{\infty} 2^n b_{2^n}$ converges. Theorem 2.3.2 guarantees that the partial sums

$$t_k = b_1 + 2b_2 + 4b_4 + \cdots + 2^k b_{2^k}$$

are bounded; that is, there exists an $M > 0$ such that $t_k \leq M$ for all $k \in \mathbf{N}$. We want to prove that $\sum_{n=1}^{\infty} b_n$ converges. Because $b_n \geq 0$, we know that the partial sums are increasing, so we only need to show that

$$s_m = b_1 + b_2 + b_3 + \cdots + b_m$$

is bounded.

Fix m and let k be large enough to ensure $m \leq 2^{k+1} - 1$. Then, $s_m \leq s_{2^{k+1}-1}$ and

$$\begin{aligned} s_{2^{k+1}-1} &= b_1 + (b_2 + b_3) + (b_4 + b_5 + b_6 + b_7) + \cdots + (b_{2^k} + \cdots + b_{2^{k+1}-1}) \\ &\leq b_1 + (b_2 + b_2) + (b_4 + b_4 + b_4 + b_4) + \cdots + (b_{2^k} + \cdots + b_{2^k}) \\ &= b_1 + 2b_2 + 4b_4 + \cdots + 2^k b_{2^k} = t_k. \end{aligned}$$

Thus, $s_m \leq t_k \leq M$, and the sequence (s_m) is bounded. By the Monotone Convergence Theorem, we can conclude that $\sum_{n=1}^{\infty} b_n$ converges.

The proof that $\sum_{n=0}^{\infty} 2^n b_{2^n}$ diverges implies $\sum_{n=1}^{\infty} b_n$ diverges is similar to Example 2.4.5. The details are requested in Exercise 2.4.9. ☐

Corollary 2.4.7. *The series $\sum_{n=1}^{\infty} 1/n^p$ converges if and only if $p > 1$.*

A rigorous argument for this corollary requires a few basic facts about geometric series. The proof is requested in Exercise 2.7.5 at the end of Section 2.7 where geometric series are discussed.

Exercises

Exercise 2.4.1. (a) Prove that the sequence defined by $x_1 = 3$ and

$$x_{n+1} = \frac{1}{4 - x_n}$$

converges.

(b) Now that we know $\lim x_n$ exists, explain why $\lim x_{n+1}$ must also exist and equal the same value.

(c) Take the limit of each side of the recursive equation in part (a) to explicitly compute $\lim x_n$.

Exercise 2.4.2. (a) Consider the recursively defined sequence $y_1 = 1$,

$$y_{n+1} = 3 - y_n,$$

and set $y = \lim y_n$. Because (y_n) and (y_{n+1}) have the same limit, taking the limit across the recursive equation gives $y = 3 - y$. Solving for y, we conclude $\lim y_n = 3/2$.

What is wrong with this argument?

(b) This time set $y_1 = 1$ and $y_{n+1} = 3 - \frac{1}{y_n}$. Can the strategy in (a) be applied to compute the limit of this sequence?

Exercise 2.4.3. (a) Show that

$$\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, \ldots$$

converges and find the limit.

(b) Does the sequence

$$\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots$$

converge? If so, find the limit.

Exercise 2.4.4. (a) In Section 1.4 we used the Axiom of Completeness (AoC) to prove the Archimedean Property of $\mathbf{R}$ (Theorem 1.4.2). Show that the Monotone Convergence Theorem can also be used to prove the Archimedean Property without making any use of AoC.

(b) Use the Monotone Convergence Theorem to supply a proof for the Nested Interval Property (Theorem 1.4.1) that doesn't make use of AoC.

These two results suggest that we could have used the Monotone Convergence Theorem in place of AoC as our starting axiom for building a proper theory of the real numbers.

Exercise 2.4.5 (Calculating Square Roots). Let $x_1 = 2$, and define

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right).$$

(a) Show that x_n^2 is always greater than or equal to 2, and then use this to prove that $x_n - x_{n+1} \geq 0$. Conclude that $\lim x_n = \sqrt{2}$.

(b) Modify the sequence (x_n) so that it converges to $\sqrt{c}$.

Exercise 2.4.6 (Arithmetic–Geometric Mean). (a) Explain why $\sqrt{xy} \leq (x+y)/2$ for any two positive real numbers x and y. (The geometric mean is always less than the arithmetic mean.)

(b) Now let $0 \leq x_1 \leq y_1$ and define

$$x_{n+1} = \sqrt{x_n y_n} \quad \text{and} \quad y_{n+1} = \frac{x_n + y_n}{2}.$$

Show $\lim x_n$ and $\lim y_n$ both exist and are equal.

Exercise 2.4.7 (Limit Superior). Let (a_n) be a bounded sequence.

(a) Prove that the sequence defined by $y_n = \sup\{a_k : k \geq n\}$ converges.

(b) The *limit superior* of (a_n), or $\limsup a_n$, is defined by

$$\limsup a_n = \lim y_n,$$

where y_n is the sequence from part (a) of this exercise. Provide a reasonable definition for $\liminf a_n$ and briefly explain why it always exists for any bounded sequence.

(c) Prove that $\liminf a_n \leq \limsup a_n$ for every bounded sequence, and give an example of a sequence for which the inequality is strict.

(d) Show that $\liminf a_n = \limsup a_n$ if and only if $\lim a_n$ exists. In this case, all three share the same value.

Exercise 2.4.8. For each series, find an explicit formula for the sequence of partial sums and determine if the series converges.

$$\text{(a)} \sum_{n=1}^{\infty} \frac{1}{2^n} \qquad \text{(b)} \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \qquad \text{(c)} \sum_{n=1}^{\infty} \log\left(\frac{n+1}{n}\right)$$

(In (c), $\log(x)$ refers to the natural logarithm function from calculus.)

Exercise 2.4.9. Complete the proof of Theorem 2.4.6 by showing that if the series $\sum_{n=0}^{\infty} 2^n b_{2^n}$ diverges, then so does $\sum_{n=1}^{\infty} b_n$. Example 2.4.5 may be a useful reference.

Exercise 2.4.10 (Infinite Products). A close relative of infinite series is the *infinite product*

$$\prod_{n=1}^{\infty} b_n = b_1 b_2 b_3 \cdots$$

which is understood in terms of its sequence of *partial products*

$$p_m = \prod_{n=1}^{m} b_n = b_1 b_2 b_3 \cdots b_m.$$

Consider the special class of infinite products of the form

$$\prod_{n=1}^{\infty}(1+a_n) = (1+a_1)(1+a_2)(1+a_3)\cdots, \qquad \text{where } a_n \geq 0.$$

(a) Find an explicit formula for the sequence of partial products in the case where $a_n = 1/n$ and decide whether the sequence converges. Write out the first few terms in the sequence of partial products in the case where $a_n = 1/n^2$ and make a conjecture about the convergence of this sequence.

(b) Show, in general, that the sequence of partial products converges if and only if $\sum_{n=1}^{\infty} a_n$ converges. (The inequality $1 + x \leq 3^x$ for positive x will be useful in one direction.)

2.5 Subsequences and the Bolzano–Weierstrass Theorem

In Example 2.4.5, we showed that the sequence of partial sums (s_m) of the harmonic series does not converge by focusing our attention on a particular *subsequence* (s_{2^k}) of the original sequence. For the moment, we will put the topic of infinite series aside and more fully develop the important concept of subsequences.

Definition 2.5.1. Let (a_n) be a sequence of real numbers, and let $n_1 < n_2 < n_3 < n_4 < n_5 < \ldots$ be an increasing sequence of natural numbers. Then the sequence

$$(a_{n_1}, a_{n_2}, a_{n_3}, a_{n_4}, a_{n_5}, \ldots)$$

is called a *subsequence* of (a_n) and is denoted by (a_{n_k}), where $k \in \mathbf{N}$ indexes the subsequence.

Notice that the order of the terms in a subsequence is the same as in the original sequence, and repetitions are not allowed. Thus if

$$(a_n) = \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \cdots\right),$$

then

$$\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \cdots\right) \quad \text{and} \quad \left(\frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, \frac{1}{10000}, \cdots\right)$$

are examples of legitimate subsequences, whereas

$$\left(\frac{1}{10}, \frac{1}{5}, \frac{1}{100}, \frac{1}{50}, \frac{1}{1000}, \frac{1}{500}, \cdots\right) \quad \text{and} \quad \left(1, 1, \frac{1}{3}, \frac{1}{3}, \frac{1}{5}, \frac{1}{5}, \cdots\right)$$

are not.

Theorem 2.5.2. *Subsequences of a convergent sequence converge to the same limit as the original sequence.*

Proof. Assume $(a_n) \to a$, and let (a_{n_k}) be a subsequence. Given $\epsilon > 0$, there exists N such that $|a_n - a| < \epsilon$ whenever $n \geq N$. Because $n_k \geq k$ for all k, the same N will suffice for the subsequence; that is, $|a_{n_k} - a| < \epsilon$ whenever $k \geq N$. $\square$

This not too surprising result has several somewhat surprising applications. It is the key ingredient for understanding when infinite sums are associative (Exercise 2.5.3). We can also use it in the following clever way to compute values of some familiar limits.

Example 2.5.3. Let $0 < b < 1$. Because

$$b > b^2 > b^3 > b^4 > \cdots > 0,$$

the sequence (b^n) is decreasing and bounded below. The Monotone Convergence Theorem allows us to conclude that (b^n) converges to some l satisfying $b > l \geq 0$. To compute l, notice that (b^{2n}) is a subsequence, so $(b^{2n}) \to l$ by Theorem 2.5.2. But $b^{2n} = b^n \cdot b^n$, so by the Algebraic Limit Theorem, $(b^{2n}) \to l \cdot l = l^2$. Because limits are unique (Theorem 2.2.7), $l^2 = l$, and thus $l = 0$.

Without much trouble (Exercise 2.5.7), we can generalize this example to conclude $(b^n) \to 0$ if and only if $-1 < b < 1$.

Example 2.5.4 (Divergence Criterion). Theorem 2.5.2 is also useful for providing economical proofs for divergence. In Example 2.2.8, we were quite sure that

$$\left(1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, -\frac{1}{5}, \frac{1}{5}, -\frac{1}{5}, \frac{1}{5}, -\frac{1}{5}, \frac{1}{5}, -\frac{1}{5}, \frac{1}{5}, -\frac{1}{5}, \cdots\right)$$

did not converge to any proposed limit. Notice that

$$\left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \cdots\right)$$

is a subsequence that converges to $1/5$. Also,

$$\left(-\frac{1}{5}, -\frac{1}{5}, -\frac{1}{5}, -\frac{1}{5}, -\frac{1}{5}, \cdots\right)$$

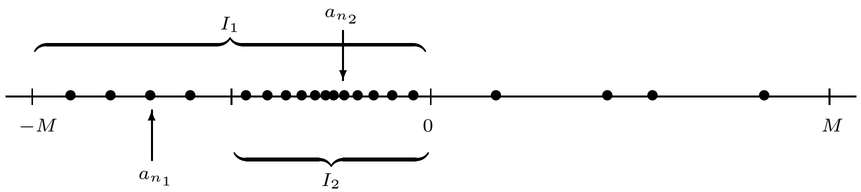
is a different subsequence of the original sequence that converges to $-1/5$. Because we have two subsequences converging to two different limits, we can rigorously conclude that the original sequence diverges.

The Bolzano–Weierstrass Theorem

In the previous example, it was rather easy to spot a convergent subsequence (or two) hiding in the original sequence. For *bounded* sequences, it turns out that it is always possible to find at least one such convergent subsequence.

Theorem 2.5.5 (Bolzano–Weierstrass Theorem). *Every bounded sequence contains a convergent subsequence.*

Proof. Let (a_n) be a bounded sequence so that there exists $M > 0$ satisfying $|a_n| \leq M$ for all $n \in \mathbf{N}$. Bisect the closed interval $[-M, M]$ into the two closed intervals $[-M, 0]$ and $[0, M]$. (The midpoint is included in both halves.) Now, it must be that at least one of these closed intervals contains an infinite number of the terms in the sequence (a_n). Select a half for which this is the case and label that interval as I_1. Then, let a_{n_1} be some term in the sequence (a_n) satisfying $a_{n_1} \in I_1$.



Next, we bisect I_1 into closed intervals of equal length, and let I_2 be a half that again contains an infinite number of terms of the original sequence. Because there are an infinite number of terms from (a_n) to choose from, we can select an a_{n_2} from the original sequence with $n_2 > n_1$ and $a_{n_2} \in I_2$. In general, we construct the closed interval I_k by taking a half of I_{k-1} containing an infinite number of terms of (a_n) and then select $n_k > n_{k-1} > \cdots > n_2 > n_1$ so that $a_{n_k} \in I_k$.

We want to argue that (a_{n_k}) is a convergent subsequence, but we need a candidate for the limit. The sets

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$$

form a nested sequence of closed intervals, and by the Nested Interval Property there exists at least one point $x \in \mathbf{R}$ contained in every I_k. This provides us with the candidate we were looking for. It just remains to show that $(a_{n_k}) \to x$.

Let $\epsilon > 0$. By construction, the length of I_k is $M(1/2)^{k-1}$ which converges to zero. (This follows from Example 2.5.3 and the Algebraic Limit Theorem.) Choose N so that $k \geq N$ implies that the length of I_k is less than ϵ. Because x and a_{n_k} are both in I_k, it follows that $|a_{n_k} - x| < \epsilon$. $\square$

Exercises

Exercise 2.5.1. Give an example of each of the following, or argue that such a request is impossible.

(a) A sequence that has a subsequence that is bounded but contains no subsequence that converges.

(b) A sequence that does not contain 0 or 1 as a term but contains subsequences converging to each of these values.

(c) A sequence that contains subsequences converging to every point in the infinite set $\{1, 1/2, 1/3, 1/4, 1/5, \ldots\}$.

(d) A sequence that contains subsequences converging to every point in the infinite set $\{1, 1/2, 1/3, 1/4, 1/5, \ldots\}$, and no subsequences converging to points outside of this set.

Exercise 2.5.2. Decide whether the following propositions are true or false, providing a short justification for each conclusion.

(a) If every proper subsequence of (x_n) converges, then (x_n) converges as well.

(b) If (x_n) contains a divergent subsequence, then (x_n) diverges.

(c) If (x_n) is bounded and diverges, then there exist two subsequences of (x_n) that converge to different limits.

(d) If (x_n) is monotone and contains a convergent subsequence, then (x_n) converges.

Exercise 2.5.3. (a) Prove that if an infinite series converges, then the associative property holds. Assume $a_1 + a_2 + a_3 + a_4 + a_5 + \cdots$ converges to a limit L (i.e., the sequence of partial sums $(s_n) \to L$). Show that any regrouping of the terms

$$(a_1 + a_2 + \cdots + a_{n_1}) + (a_{n_1+1} + \cdots + a_{n_2}) + (a_{n_2+1} + \cdots + a_{n_3}) + \cdots$$

leads to a series that also converges to L.

(b) Compare this result to the example discussed at the end of Section 2.1 where infinite addition was shown not to be associative. Why doesn't our proof in (a) apply to this example?

Exercise 2.5.4. The Bolzano–Weierstrass Theorem is extremely important, and so is the strategy employed in the proof. To gain some more experience with this technique, assume the Nested Interval Property is true and use it to provide a proof of the Axiom of Completeness. To prevent the argument from being circular, assume also that $(1/2^n) \to 0$. (Why precisely is this last assumption needed to avoid circularity?)

Exercise 2.5.5. Assume (a_n) is a bounded sequence with the property that every convergent subsequence of (a_n) converges to the same limit $a \in \mathbf{R}$. Show that (a_n) must converge to a.

Exercise 2.5.6. Use a similar strategy to the one in Example 2.5.3 to show $\lim b^{1/n}$ exists for all $b \geq 0$ and find the value of the limit. (The results in Exercise 2.3.1 may be assumed.)

Exercise 2.5.7. Extend the result proved in Example 2.5.3 to the case $|b| < 1$; that is, show $\lim(b^n) = 0$ if and only if $-1 < b < 1$.

Exercise 2.5.8. Another way to prove the Bolzano–Weierstrass Theorem is to show that every sequence contains a monotone subsequence. A useful device in this endeavor is the notion of a *peak term*. Given a sequence (x_n), a particular term x_m is a peak term if no later term in the sequence exceeds it; i.e., if $x_m \geq x_n$ for all $n \geq m$.

(a) Find examples of sequences with zero, one, and two peak terms. Find an example of a sequence with infinitely many peak terms that is not monotone.

(b) Show that every sequence contains a monotone subsequence and explain how this furnishes a new proof of the Bolzano–Weierstrass Theorem.

Exercise 2.5.9. Let (a_n) be a bounded sequence, and define the set

$$S = \{x \in \mathbf{R} : x < a_n \text{ for infinitely many terms } a_n\}.$$

Show that there exists a subsequence (a_{n_k}) converging to $s = \sup S$. (This is a direct proof of the Bolzano–Weierstrass Theorem using the Axiom of Completeness.)

2.6 The Cauchy Criterion

The following definition bears a striking resemblance to the definition of convergence for a sequence.

Definition 2.6.1. A sequence (a_n) is called a *Cauchy sequence* if, for every $\epsilon > 0$, there exists an $N \in \mathbf{N}$ such that whenever $m, n \geq N$ it follows that $|a_n - a_m| < \epsilon$.

To make the comparison easier, let's restate the definition of convergence.

Definition 2.2.3. A sequence (a_n) *converges* to a real number a if, for every $\epsilon > 0$, there exists an $N \in \mathbf{N}$ such that whenever $n \geq N$ it follows that $|a_n - a| < \epsilon$.

As we have discussed, the definition of convergence asserts that, given an arbitrary positive ϵ, it is possible to find a point in the sequence after which the terms of the sequence are all closer to the limit a than the given ϵ. On the other hand, a sequence is a Cauchy sequence if, for every ϵ, there is a point in the sequence after which the terms are all closer *to each other* than the given ϵ. To spoil the surprise, we will argue in this section that in fact these two definitions are equivalent: Convergent sequences are Cauchy sequences, and Cauchy sequences converge. The significance of the definition of a Cauchy sequence is that there is no mention of a limit. This is somewhat like the situation with the Monotone Convergence Theorem in that we will have another way of proving that sequences converge without having any explicit knowledge of what the limit might be.

Theorem 2.6.2. *Every convergent sequence is a Cauchy sequence.*

Proof. Assume (x_n) converges to x. To prove that (x_n) is Cauchy, we must find a point in the sequence after which we have $|x_n - x_m| < \epsilon$. This can be done using an application of the triangle inequality. The details are requested in Exercise 2.6.1. □

The converse is a bit more difficult to prove, mainly because, in order to prove that a sequence converges, we must have a proposed limit for the sequence to approach. We have been in this situation before in the proofs of the Monotone Convergence Theorem and the Bolzano–Weierstrass Theorem. Our strategy here will be to use the Bolzano–Weierstrass Theorem. This is the reason for the next lemma. (Compare this with Theorem 2.3.2.)

Lemma 2.6.3. *Cauchy sequences are bounded.*

Proof. Given $\epsilon = 1$, there exists an N such that $|x_m - x_n| < 1$ for all $m, n \geq N$. Thus, we must have $|x_n| < |x_N| + 1$ for all $n \geq N$. It follows that

$$M = \max\{|x_1|, |x_2|, |x_3|, \ldots, |x_{N-1}|, |x_N| + 1\}$$

is a bound for the sequence (x_n). □

Theorem 2.6.4 (Cauchy Criterion). *A sequence converges if and only if it is a Cauchy sequence.*

Proof. ($\Rightarrow$) This direction is Theorem 2.6.2.

($\Leftarrow$) For this direction, we start with a Cauchy sequence (x_n). Lemma 2.6.3 guarantees that (x_n) is bounded, so we may use the Bolzano–Weierstrass Theorem to produce a convergent subsequence (x_{n_k}). Set

$$x = \lim x_{n_k}.$$

The idea is to show that the original sequence (x_n) converges to this same limit. Once again, we will use a triangle inequality argument. We know the terms in the subsequence are getting close to the limit x, and the assumption that (x_n) is Cauchy implies the terms in the "tail" of the sequence are close to each other. Thus, we want to make each of these distances less than half of the prescribed ϵ.

Let $\epsilon > 0$. Because (x_n) is Cauchy, there exists N such that

$$|x_n - x_m| < \frac{\epsilon}{2}$$

whenever $m, n \geq N$. Now, we also know that $(x_{n_k}) \to x$, so choose a term in this subsequence, call it x_{n_K}, with $n_K \geq N$ and

$$|x_{n_K} - x| < \frac{\epsilon}{2}.$$

To see that N has the desired property (for the original sequence (x_n)), observe that if $n \geq N$, then

$$\begin{aligned} |x_n - x| &= |x_n - x_{n_K} + x_{n_K} - x| \\ &\leq |x_n - x_{n_K}| + |x_{n_K} - x| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

□

The Cauchy Criterion is named after the French mathematician Augustin Louis Cauchy. Cauchy is a major figure in the history of many branches of mathematics—number theory and the theory of finite groups, to name a few—but he is most widely recognized for his enormous contributions in analysis, especially complex analysis. He is deservedly credited with inventing the ϵ-based definition of limits we use today, although it is probably better to view him as a pioneer of analysis in the sense that his work did not attain the level of refinement that modern mathematicians have come to expect. The Cauchy Criterion, for instance, was devised and used by Cauchy to study infinite series, but he never actually proved it in both directions. The fact that there were gaps in Cauchy's work should not diminish his brilliance in any way. The issues of the day were both difficult and subtle, and Cauchy was far and away the most influential in laying the groundwork for modern standards of rigor. Karl Weierstrass played a major role in sharpening Cauchy's arguments. We will hear a good deal more from Weierstrass, most notably in Chapter 6 when we take up uniform convergence. Bernhard Bolzano was working in Prague and was writing and thinking about many of these same issues surrounding limits and continuity. Because his work was not widely available to the rest of the mathematical community, his historical reputation never achieved the distinction that his impressive accomplishments would seem to merit.

Completeness Revisited

In the first chapter, we established the Axiom of Completeness (AoC) to be the assertion that nonempty sets bounded above have least upper bounds. We then used this axiom as the crucial step in the proof of the Nested Interval Property (NIP). In this chapter, AoC was the central step in the Monotone Convergence Theorem (MCT), and NIP was the key to proving the Bolzano–Weierstrass

Theorem (BW). Finally, we needed BW in our proof of the Cauchy Criterion (CC) for convergent sequences. The list of implications then looks like

$$\text{AoC} \Rightarrow \begin{cases} \text{NIP} & \Rightarrow \quad \text{BW} \quad \Rightarrow \quad \text{CC.} \\ \text{MCT.} \end{cases}$$

But this one-directional list is not the whole story. Recall that in our original discussions about completeness, the fundamental problem was that the rational numbers contained "gaps." The reason for moving from the rational numbers to the real numbers to do analysis is so that when we encounter a sequence that looks as if it is converging to some number—say $\sqrt{2}$—then we can be assured that there is indeed a number there that we can call the limit. The assertion that "nonempty sets bounded above have least upper bounds" is simply one way to mathematically articulate our insistence that there be no "holes" in our ordered field, but it is not the only way. Instead, we could have taken MCT to be our defining axiom and used it to prove NIP and the existence of least upper bounds. This is the content of Exercise 2.4.4.

How about NIP? Could this property serve as a starting point for a proper axiomatic treatment of the real numbers? Almost. In Exercise 2.5.4 we showed that NIP implies AoC, but to prevent the argument from making implicit use of AoC we needed an extra assumption that is equivalent to the Archimedean Property (Theorem 1.4.2). This extra hypothesis is unavoidable. Whereas AoC and MCT can both be used to prove that $\mathbf{N}$ is not a bounded subset of $\mathbf{R}$, there is no way to prove this same fact starting from NIP. The upshot is that NIP is a perfectly reasonable candidate to use as the fundamental axiom of the real numbers provided that we also include the Archimedean Property as a second unproven assumption.

In fact, if we assume the Archimedean Property holds, then AoC, NIP, MCT, BW, and CC are equivalent in the sense that once we take any one of them to be true, it is possible to derive the other four. However, because we have an example of an ordered field that is not complete—namely, the set of rational numbers—we know it is impossible to prove any of them using only the field and order properties. Just how we decide which should be the axiom and which then become theorems depends largely on preference and context, and in the end is not especially significant. What is important is that we understand all of these results as belonging to the same family, each asserting the completeness of $\mathbf{R}$ in its own particular language.

One loose end in this conversation is the curious and somewhat unpredictable relationship of the Archimedean Property to these other results. As we have mentioned, the Archimedean Property follows as a consequence of AoC as well as MCT, but not from NIP. Starting from BW, it is possible to prove MCT and thus also the Archimedean Property. On the other hand, the Cauchy Criterion is like NIP in that it cannot be used on its own to prove the Archimedean Property.[1]

[1] A thorough account of the logical dependence between these various results can be found in [23].

Exercises

Exercise 2.6.1. Supply a proof for Theorem 2.6.2.

Exercise 2.6.2. Give an example of each of the following, or argue that such a request is impossible.

(a) A Cauchy sequence that is not monotone.

(b) A Cauchy sequence with an unbounded subsequence.

(c) A divergent monotone sequence with a Cauchy subsequence.

(d) An unbounded sequence containing a subsequence that is Cauchy.

Exercise 2.6.3. If (x_n) and (y_n) are Cauchy sequences, then one easy way to prove that $(x_n + y_n)$ is Cauchy is to use the Cauchy Criterion. By Theorem 2.6.4, (x_n) and (y_n) must be convergent, and the Algebraic Limit Theorem then implies $(x_n + y_n)$ is convergent and hence Cauchy.

(a) Give a direct argument that $(x_n + y_n)$ is a Cauchy sequence that does not use the Cauchy Criterion or the Algebraic Limit Theorem.

(b) Do the same for the product $(x_n y_n)$.

Exercise 2.6.4. Let (a_n) and (b_n) be Cauchy sequences. Decide whether each of the following sequences is a Cauchy sequence, justifying each conclusion.

(a) $c_n = |a_n - b_n|$

(b) $c_n = (-1)^n a_n$

(c) $c_n = [[a_n]]$, where $[[x]]$ refers to the greatest integer less than or equal to x.

Exercise 2.6.5. Consider the following (invented) definition: A sequence (s_n) is *pseudo-Cauchy* if, for all $\epsilon > 0$, there exists an N such that if $n \geq N$, then $|s_{n+1} - s_n| < \epsilon$.

Decide which one of the following two propositions is actually true. Supply a proof for the valid statement and a counterexample for the other.

(i) Pseudo-Cauchy sequences are bounded.

(ii) If (x_n) and (y_n) are pseudo-Cauchy, then $(x_n + y_n)$ is pseudo-Cauchy as well.

Exercise 2.6.6. Let's call a sequence (a_n) *quasi-increasing* if for all $\epsilon > 0$ there exists an N such that whenever $n > m \geq N$ it follows that $a_n > a_m - \epsilon$.

(a) Give an example of a sequence that is quasi-increasing but not monotone or eventually monotone.

(b) Give an example of a quasi-increasing sequence that is divergent and not monotone or eventually monotone.

(c) Is there an analogue of the Monotone Convergence Theorem for quasi-increasing sequences? Give an example of a bounded, quasi-increasing sequence that doesn't converge, or prove that no such sequence exists.

Exercise 2.6.7. Exercises 2.4.4 and 2.5.4 establish the equivalence of the Axiom of Completeness and the Monotone Convergence Theorem. They also show the Nested Interval Property is equivalent to these other two in the presence of the Archimedean Property.

(a) Assume the Bolzano–Weierstrass Theorem is true and use it to construct a proof of the Monotone Convergence Theorem without making any appeal to the Archimedean Property. This shows that BW, AoC, and MCT are all equivalent.

(b) Use the Cauchy Criterion to prove the Bolzano–Weierstrass Theorem, and find the point in the argument where the Archimedean Property is implicitly required. This establishes the final link in the equivalence of the five characterizations of completeness discussed at the end of Section 2.6.

(c) How do we know it is impossible to prove the Axiom of Completeness starting from the Archimedean Property?

2.7 Properties of Infinite Series

Given an infinite series $\sum_{k=1}^{\infty} a_k$, it is important to keep a clear distinction between

(i) the sequence of *terms*: $(a_1, a_2, a_3, \ldots)$ and

(ii) the sequence of *partial sums*: $(s_1, s_2, s_3, \ldots)$, where $s_n = a_1 + a_2 + \cdots + a_n$.

The convergence of the series $\sum_{k=1}^{\infty} a_k$ is defined in terms of the sequence (s_n). Specifically, the statement

$$\sum_{k=1}^{\infty} a_k = A \quad \text{means that} \quad \lim s_n = A.$$

It is for this reason that we can immediately translate many of our results from the study of sequences into statements about the behavior of infinite series.

Theorem 2.7.1 (Algebraic Limit Theorem for Series). *If* $\sum_{k=1}^{\infty} a_k = A$ *and* $\sum_{k=1}^{\infty} b_k = B$, *then*

(i) $\sum_{k=1}^{\infty} ca_k = cA$ *for all* $c \in \mathbf{R}$ *and*

(ii) $\sum_{k=1}^{\infty} (a_k + b_k) = A + B$.

Proof. (i) In order to show that $\sum_{k=1}^{\infty} ca_k = cA$, we must argue that the sequence of partial sums

$$t_m = ca_1 + ca_2 + ca_3 + \cdots + ca_m$$

converges to cA. But we are given that $\sum_{k=1}^{\infty} a_k$ converges to A, meaning that the partial sums

$$s_m = a_1 + a_2 + a_3 + \cdots + a_m$$

converge to A. Because $t_m = cs_m$, applying the Algebraic Limit Theorem for sequences (Theorem 2.3.3) yields $(t_m) \to cA$, as desired.

The proof of part (ii) is analogous and is left as an unofficial exercise. □

One way to summarize Theorem 2.7.1 (i) is to say that infinite addition still satisfies the distributive property. Part (ii) verifies that series can be added in the usual way. Missing from this theorem is any statement about the *product* of two infinite series. At the heart of this question is the issue of commutativity, which requires a more delicate analysis and so is postponed until Section 2.8.

Theorem 2.7.2 (Cauchy Criterion for Series). *The series $\sum_{k=1}^{\infty} a_k$ converges if and only if, given $\epsilon > 0$, there exists an $N \in \mathbf{N}$ such that whenever $n > m \geq N$ it follows that*

$$|a_{m+1} + a_{m+2} + \cdots + a_n| < \epsilon.$$

Proof. Observe that

$$|s_n - s_m| = |a_{m+1} + a_{m+2} + \cdots + a_n|$$

and apply the Cauchy Criterion for sequences. □

The Cauchy Criterion leads to economical proofs of several basic facts about series.

Theorem 2.7.3. *If the series $\sum_{k=1}^{\infty} a_k$ converges, then $(a_k) \to 0$.*

Proof. Consider the special case $n = m + 1$ in the Cauchy Criterion for Series. □

Every statement of this result should be accompanied with a reminder to look at the harmonic series (Example 2.4.5) to erase any misconception that the converse statement is true. Knowing (a_k) tends to 0 does not imply that the series converges.

Theorem 2.7.4 (Comparison Test). *Assume (a_k) and (b_k) are sequences satisfying $0 \leq a_k \leq b_k$ for all $k \in \mathbf{N}$.*

(i) *If $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges.*

(ii) *If $\sum_{k=1}^{\infty} a_k$ diverges, then $\sum_{k=1}^{\infty} b_k$ diverges.*

Proof. Both statements follow immediately from the Cauchy Criterion for Series and the observation that

$$|a_{m+1} + a_{m+2} + \cdots + a_n| \leq |b_{m+1} + b_{m+2} + \cdots + b_n|.$$

Alternate proofs using the Monotone Convergence Theorem are requested in the exercises. □

This is a good point to remind ourselves again that statements about convergence of sequences and series are immune to changes in some finite number of initial terms. In the Comparison Test, the requirement that $0 \leq a_k \leq b_k$ does not really need to hold for *all* $k \in \mathbf{N}$ but just needs to be *eventually* true. A weaker, but sufficient, hypothesis would be to assume that there exists some point $M \in \mathbf{N}$ such that the inequality $a_k \leq b_k$ is true for all $k \geq M$.

The Comparison Test is used to deduce the convergence or divergence of one series based on the behavior of another. Thus, for this test to be of any great use, we need a catalog of series we can use as measuring sticks. In Section 2.4, we proved the Cauchy Condensation Test, which led to the general statement that the series $\sum_{n=1}^{\infty} 1/n^p$ converges if and only if $p > 1$.

The next example summarizes the situation for another important class of series.

Example 2.7.5 (Geometric Series). A series is called *geometric* if it is of the form

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + ar^3 + \cdots .$$

If $r = 1$ and $a \neq 0$, the series evidently diverges. For $r \neq 1$, the algebraic identity

$$(1-r)(1 + r + r^2 + r^3 + \cdots + r^{m-1}) = 1 - r^m$$

enables us to rewrite the partial sum

$$s_m = a + ar + ar^2 + ar^3 + \cdots + ar^{m-1} = \frac{a(1-r^m)}{1-r}.$$

Now the Algebraic Limit Theorem (for sequences) and Example 2.5.3 justify the conclusion

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$$

if and only if $|r| < 1$.

Although the Comparison Test requires that the terms of the series be positive, it is often used in conjunction with the next theorem to handle series that contain some negative terms.

Theorem 2.7.6 (Absolute Convergence Test). *If the series $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges as well.*

Proof. This proof makes use of both the necessity (the "if" direction) and the sufficiency (the "only if" direction) of the Cauchy Criterion for Series. Because $\sum_{n=1}^{\infty} |a_n|$ converges, we know that, given an $\epsilon > 0$, there exists an $N \in \mathbf{N}$ such that

$$|a_{m+1}| + |a_{m+2}| + \cdots + |a_n| < \epsilon$$

for all $n > m \geq N$. By the triangle inequality,

$$|a_{m+1} + a_{m+2} + \cdots + a_n| \leq |a_{m+1}| + |a_{m+2}| + \cdots + |a_n|,$$

so the sufficiency of the Cauchy Criterion guarantees that $\sum_{n=1}^{\infty} a_n$ also converges. □

The converse of this theorem is false. In the opening discussion of this chapter, we considered the *alternating harmonic series*

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots .$$

Taking absolute values of the terms gives us the harmonic series $\sum_{n=1}^{\infty} 1/n$, which we have seen diverges. However, it is not too difficult to prove that with the alternating negative signs the series indeed converges. This is a special case of the Alternating Series Test.

Theorem 2.7.7 (Alternating Series Test). *Let (a_n) be a sequence satisfying,*

(i) $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq a_{n+1} \geq \cdots$ *and*

(ii) $(a_n) \to 0$.

Then, the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges.

Proof. A consequence of conditions (i) and (ii) is that $a_n \geq 0$. Several proofs of this theorem are outlined in Exercise 2.7.1. □

Definition 2.7.8. If $\sum_{n=1}^{\infty} |a_n|$ converges, then we say that the original series $\sum_{n=1}^{\infty} a_n$ *converges absolutely.* If, on the other hand, the series $\sum_{n=1}^{\infty} a_n$ converges but the series of absolute values $\sum_{n=1}^{\infty} |a_n|$ does not converge, then we say that the original series $\sum_{n=1}^{\infty} a_n$ *converges conditionally.*

In terms of this newly defined jargon, we have shown that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

converges conditionally, whereas

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}, \quad \sum_{n=1}^{\infty} \frac{1}{2^n} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^n}$$

converge absolutely. In particular, any convergent series with (all but finitely many) positive terms must converge absolutely.

The Alternating Series Test is the most accessible test for conditional convergence, but several others are explored in the exercises. In particular, Abel's Test, outlined in Exercise 2.7.13, will prove useful in our investigations of power series in Chapter 6.

Rearrangements

Informally speaking, a rearrangement of a series is obtained by permuting the terms in the sum into some other order. It is important that all of the original terms eventually appear in the new ordering and that no term gets repeated. In an earlier discussion from Section 2.1, we formed a rearrangement of the alternating harmonic series by taking two positive terms for each negative term:

$$1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\cdots.$$

There are clearly an infinite number of rearrangements of any sum; however, it is helpful to see why neither

$$1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots$$

nor

$$1+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}+\frac{1}{11}-\frac{1}{12}+\cdots$$

is considered a rearrangement of the original alternating harmonic series.

Definition 2.7.9. Let $\sum_{k=1}^{\infty} a_k$ be a series. A series $\sum_{k=1}^{\infty} b_k$ is called a *rearrangement* of $\sum_{k=1}^{\infty} a_k$ if there exists a one-to-one, onto function $f:\mathbf{N}\to\mathbf{N}$ such that $b_{f(k)} = a_k$ for all $k\in\mathbf{N}$.

We now have all the tools and notation in place to resolve an issue raised at the beginning of the chapter. In Section 2.1, we constructed a particular rearrangement of the alternating harmonic series that converges to a limit different from that of the original series. This happens because the convergence is *conditional*.

Theorem 2.7.10. *If a series converges absolutely, then any rearrangement of this series converges to the same limit.*

Proof. Assume $\sum_{k=1}^{\infty} a_k$ converges absolutely to A, and let $\sum_{k=1}^{\infty} b_k$ be a rearrangement of $\sum_{k=1}^{\infty} a_k$. Let's use

$$s_n=\sum_{k=1}^{n} a_k = a_1+a_2+\cdots+a_n$$

for the partial sums of the original series and use

$$t_m=\sum_{k=1}^{m} b_k = b_1+b_2+\cdots+b_m$$

for the partial sums of the rearranged series. Thus we want to show that $(t_m) \to A$.

Let $\epsilon > 0$. By hypothesis, $(s_n) \to A$, so choose N_1 such that

$$|s_n - A| < \frac{\epsilon}{2}$$

for all $n \geq N_1$. Because the convergence is absolute, we can choose N_2 so that

$$\sum_{k=m+1}^{n} |a_k| < \frac{\epsilon}{2}$$

for all $n > m \geq N_2$. Now, take $N = \max\{N_1, N_2\}$. We know that the finite set of terms $\{a_1, a_2, a_3, \ldots, a_N\}$ must all appear in the rearranged series, and we want to move far enough out in the series $\sum_{n=1}^{\infty} b_n$ so that we have included all of these terms. Thus, choose

$$M = \max\{f(k) : 1 \leq k \leq N\}.$$

It should now be evident that if $m \geq M$, then $(t_m - s_N)$ consists of a finite set of terms, the absolute values of which appear in the tail $\sum_{k=N+1}^{\infty} |a_k|$. Our choice of N_2 earlier then guarantees $|t_m - s_N| < \epsilon/2$, and so

$$\begin{aligned} |t_m - A| &= |t_m - s_N + s_N - A| \\ &\leq |t_m - s_N| + |s_N - A| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

whenever $m \geq M$. □

Exercises

Exercise 2.7.1. Proving the Alternating Series Test (Theorem 2.7.7) amounts to showing that the sequence of partial sums

$$s_n = a_1 - a_2 + a_3 - \cdots \pm a_n$$

converges. (The opening example in Section 2.1 includes a typical illustration of (s_n).) Different characterizations of completeness lead to different proofs.

(a) Prove the Alternating Series Test by showing that (s_n) is a Cauchy sequence.

(b) Supply another proof for this result using the Nested Interval Property (Theorem 1.4.1).

(c) Consider the subsequences (s_{2n}) and (s_{2n+1}), and show how the Monotone Convergence Theorem leads to a third proof for the Alternating Series Test.

Exercise 2.7.2. Decide whether each of the following series converges or diverges:

(a) $\sum_{n=1}^{\infty} \frac{1}{2^n+n}$ (b) $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$

(c) $1 - \frac{3}{4} + \frac{4}{6} - \frac{5}{8} + \frac{6}{10} - \frac{7}{12} + \cdots$

(d) $1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} + \frac{1}{8} - \frac{1}{9} + \cdots$

(e) $1 - \frac{1}{2^2} + \frac{1}{3} - \frac{1}{4^2} + \frac{1}{5} - \frac{1}{6^2} + \frac{1}{7} - \frac{1}{8^2} + \cdots$

Exercise 2.7.3. (a) Provide the details for the proof of the Comparison Test (Theorem 2.7.4) using the Cauchy Criterion for Series.

(b) Give another proof for the Comparison Test, this time using the Monotone Convergence Theorem.

Exercise 2.7.4. Give an example of each or explain why the request is impossible referencing the proper theorem(s).

(a) Two series $\sum x_n$ and $\sum y_n$ that both diverge but where $\sum x_n y_n$ converges.

(b) A convergent series $\sum x_n$ and a bounded sequence (y_n) such that $\sum x_n y_n$ diverges.

(c) Two sequences (x_n) and (y_n) where $\sum x_n$ and $\sum (x_n + y_n)$ both converge but $\sum y_n$ diverges.

(d) A sequence (x_n) satisfying $0 \leq x_n \leq 1/n$ where $\sum (-1)^n x_n$ diverges.

Exercise 2.7.5. Now that we have proved the basic facts about geometric series, supply a proof for Corollary 2.4.7.

Exercise 2.7.6. Let's say that a series *subverges* if the sequence of partial sums contains a subsequence that converges. Consider this (invented) definition for a moment, and then decide which of the following statements are valid propositions about subvergent series:

(a) If (a_n) is bounded, then $\sum a_n$ subverges.

(b) All convergent series are subvergent.

(c) If $\sum |a_n|$ subverges, then $\sum a_n$ subverges as well.

(d) If $\sum a_n$ subverges, then (a_n) has a convergent subsequence.

Exercise 2.7.7. (a) Show that if $a_n > 0$ and $\lim(na_n) = l$ with $l \neq 0$, then the series $\sum a_n$ diverges.

(b) Assume $a_n > 0$ and $\lim(n^2 a_n)$ exists. Show that $\sum a_n$ converges.

Exercise 2.7.8. Consider each of the following propositions. Provide short proofs for those that are true and counterexamples for any that are not.

(a) If $\sum a_n$ converges absolutely, then $\sum a_n^2$ also converges absolutely.

(b) If $\sum a_n$ converges and (b_n) converges, then $\sum a_n b_n$ converges.

(c) If $\sum a_n$ converges conditionally, then $\sum n^2 a_n$ diverges.

Exercise 2.7.9 (Ratio Test). Given a series $\sum_{n=1}^{\infty} a_n$ with $a_n \neq 0$, the Ratio Test states that if (a_n) satisfies

$$\lim \left| \frac{a_{n+1}}{a_n} \right| = r < 1,$$

then the series converges absolutely.

(a) Let r' satisfy $r < r' < 1$. Explain why there exists an N such that $n \geq N$ implies $|a_{n+1}| \leq |a_n| r'$.

(b) Why does $|a_N| \sum (r')^n$ converge?

(c) Now, show that $\sum |a_n|$ converges, and conclude that $\sum a_n$ converges.

Exercise 2.7.10 (Infinite Products). Review Exercise 2.4.10 about infinite products and then answer the following questions:

(a) Does $\frac{2}{1} \cdot \frac{3}{2} \cdot \frac{5}{4} \cdot \frac{9}{8} \cdot \frac{17}{16} \cdots$ converge?

(b) The infinite product $\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8} \cdot \frac{9}{10} \cdots$ certainly converges. (Why?) Does it converge to zero?

(c) In 1655, John Wallis famously derived the formula

$$\left(\frac{2 \cdot 2}{1 \cdot 3}\right)\left(\frac{4 \cdot 4}{3 \cdot 5}\right)\left(\frac{6 \cdot 6}{5 \cdot 7}\right)\left(\frac{8 \cdot 8}{7 \cdot 9}\right) \cdots = \frac{\pi}{2}.$$

Show that the left side of this identity at least converges to something. (A complete proof of this result is taken up in Section 8.3.)

Exercise 2.7.11. Find examples of two series $\sum a_n$ and $\sum b_n$ both of which diverge but for which $\sum \min\{a_n, b_n\}$ converges. To make it more challenging, produce examples where (a_n) and (b_n) are strictly positive and decreasing.

Exercise 2.7.12 (Summation-by-parts). Let (x_n) and (y_n) be sequences, let $s_n = x_1 + x_2 + \cdots + x_n$ and set $s_0 = 0$. Use the observation that $x_j = s_j - s_{j-1}$ to verify the formula

$$\sum_{j=m}^{n} x_j y_j = s_n y_{n+1} - s_{m-1} y_m + \sum_{j=m}^{n} s_j (y_j - y_{j+1}).$$

Exercise 2.7.13 (Abel's Test). Abel's Test for convergence states that if the series $\sum_{k=1}^{\infty} x_k$ converges, and if (y_k) is a sequence satisfying

$$y_1 \geq y_2 \geq y_3 \geq \cdots \geq 0,$$

then the series $\sum_{k=1}^{\infty} x_k y_k$ converges.

(a) Use Exercise 2.7.12 to show that

$$\sum_{k=1}^{n} x_k y_k = s_n y_{n+1} + \sum_{k=1}^{n} s_k (y_k - y_{k+1}),$$

where $s_n = x_1 + x_2 + \cdots + x_n$.

(b) Use the Comparison Test to argue that $\sum_{k=1}^{\infty} s_k(y_k - y_{k+1})$ converges absolutely, and show how this leads directly to a proof of Abel's Test.

Exercise 2.7.14 (Dirichlet's Test). Dirichlet's Test for convergence states that if the partial sums of $\sum_{k=1}^{\infty} x_k$ are bounded (but not necessarily convergent), and if (y_k) is a sequence satisfying $y_1 \geq y_2 \geq y_3 \geq \cdots \geq 0$ with $\lim y_k = 0$, then the series $\sum_{k=1}^{\infty} x_k y_k$ converges.

(a) Point out how the hypothesis of Dirichlet's Test differs from that of Abel's Test in Exercise 2.7.13, but show that essentially the same strategy can be used to provide a proof.

(b) Show how the Alternating Series Test (Theorem 2.7.7) can be derived as a special case of Dirichlet's Test.

2.8 Double Summations and Products of Infinite Series

Given a doubly indexed array of real numbers $\{a_{ij} : i, j \in \mathbf{N}\}$, we discovered in Section 2.1 that there is a dangerous ambiguity in how we might define $\sum_{i,j=1}^{\infty} a_{ij}$. Performing the sum over first one of the variables and then the other is referred to as an *iterated* summation. In our specific example, summing the rows first and then taking the sum of these totals produced a different result than first computing the sum of each column and adding these sums together. In short,

$$\sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij} \neq \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}.$$

There are still other ways to reasonably define $\sum_{i,j=1}^{\infty} a_{ij}$. One natural idea is to calculate a kind of partial sum by adding together finite numbers of terms in larger and larger "rectangles" in the array; that is, for $m, n \in \mathbf{N}$, set

$$s_{mn} = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}. \tag{1}$$

The order of the sum here is irrelevant because the sum is finite. Of particular interest to our discussion are the sums s_{nn} (sums over "squares"), which form a legitimate sequence indexed by n and thus can be subjected to our arsenal of theorems and definitions. If the sequence (s_{nn}) converges, for instance, we might wish to define

$$\sum_{i,j=1}^{\infty} a_{ij} = \lim_{n\to\infty} s_{nn}.$$

Exercise 2.8.1. Using the particular array (a_{ij}) from Section 2.1, compute $\lim_{n\to\infty} s_{nn}$. How does this value compare to the two iterated values for the sum already computed?

There is a deep similarity between the issue of how to define a double summation and the topic of rearrangements discussed at the end of Section 2.7. Both relate to the commutativity of addition in an infinite setting. For rearrangements, the resolution came with the added hypothesis of *absolute* convergence, and it is not surprising that the same remedy applies for double summations. Under the assumption of absolute convergence, each of the methods discussed for computing the value of a double sum yields the same result.

Exercise 2.8.2. Show that if the iterated series

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |a_{ij}|$$

converges (meaning that for each fixed $i \in \mathbf{N}$ the series $\sum_{j=1}^{\infty} |a_{ij}|$ converges to some real number b_i, and the series $\sum_{i=1}^{\infty} b_i$ converges as well), then the iterated series

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}$$

converges.

Theorem 2.8.1. *Let $\{a_{ij} : i, j \in \mathbf{N}\}$ be a doubly indexed array of real numbers. If*

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |a_{ij}|$$

converges, then both $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}$ and $\sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}$ converge to the same value. Moreover,

$$\lim_{n\to\infty} s_{nn} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij},$$

where $s_{nn} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}$.

Proof. In the same way that we defined the rectangular partial sums s_{mn} above in equation (1), define

$$t_{mn} = \sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|.$$

Exercise 2.8.3. (a) Prove that (t_{nn}) converges.

(b) Now, use the fact that (t_{nn}) is a Cauchy sequence to argue that (s_{nn}) converges.

We can now set

$$S = \lim_{n\to\infty} s_{nn}.$$

In order to prove the theorem, we must show that the two iterated sums converge to this same limit. We will first show that

$$S = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}.$$

Because $\{t_{mn} : m, n \in \mathbf{N}\}$ is bounded above, we can let

$$B = \sup\{t_{mn} : m, n \in \mathbf{N}\}.$$

Exercise 2.8.4. (a) Let $\epsilon > 0$ be arbitrary and argue that there exists an $N_1 \in \mathbf{N}$ such that $m, n \geq N_1$ implies $B - \frac{\epsilon}{2} < t_{mn} \leq B$.

(b) Now, show that there exists an N such that

$$|s_{mn} - S| < \epsilon$$

for all $m, n \geq N$.

For the moment, consider $m \in \mathbf{N}$ to be fixed and write s_{mn} as

$$s_{mn} = \sum_{j=1}^{n} a_{1j} + \sum_{j=1}^{n} a_{2j} + \cdots + \sum_{j=1}^{n} a_{mj}.$$

Our hypothesis guarantees that for each fixed row i, the series $\sum_{j=1}^{\infty} a_{ij}$ converges absolutely to some real number r_i.

Exercise 2.8.5. (a) Show that for all $m \geq N$

$$|(r_1 + r_2 + \cdots + r_m) - S| \leq \epsilon.$$

Conclude that the iterated sum $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}$ converges to S.

(b) Finish the proof by showing that the other iterated sum, $\sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}$, converges to S as well. Notice that the same argument can be used once it is established that, for each fixed column j, the sum $\sum_{i=1}^{\infty} a_{ij}$ converges to some real number c_j. □

One final common way of computing a double summation is to sum along diagonals where $i + j$ equals a constant. Given a doubly indexed array $\{a_{ij} : i, j \in \mathbf{N}\}$, let

$$d_2 = a_{11}, \quad d_3 = a_{12} + a_{21}, \quad d_4 = a_{13} + a_{22} + a_{31},$$

and in general set

$$d_k = a_{1,k-1} + a_{2,k-2} + \cdots + a_{k-1,1}.$$

Then, $\sum_{k=2}^{\infty} d_k$ represents another reasonable way of summing over every a_{ij} in the array.

Exercise 2.8.6. (a) Assuming the hypothesis—and hence the conclusion—of Theorem 2.8.1, show that $\sum_{k=2}^{\infty} d_k$ converges absolutely.

(b) Imitate the strategy in the proof of Theorem 2.8.1 to show that $\sum_{k=2}^{\infty} d_k$ converges to $S = \lim_{n\to\infty} s_{nn}$.

Products of Series

Conspicuously missing from the Algebraic Limit Theorem for Series (Theorem 2.7.1) is any statement about the product of two convergent series. One way to formally carry out the algebra on such a product is to write

$$\begin{aligned} \left(\sum_{i=1}^{\infty} a_i\right)\left(\sum_{j=1}^{\infty} b_j\right) &= (a_1 + a_2 + a_3 + \cdots)(b_1 + b_2 + b_3 + \cdots) \\ &= a_1b_1 + (a_1b_2 + a_2b_1) + (a_3b_1 + a_2b_2 + a_1b_3) + \cdots \\ &= \sum_{k=2}^{\infty} d_k, \end{aligned}$$

where

$$d_k = a_1b_{k-1} + a_2b_{k-2} + \cdots + a_{k-1}b_1.$$

This particular form of the product, examined earlier in Exercise 2.8.6, is called the *Cauchy product* of two series. Although there is something algebraically natural about writing the product in this form, it may very well be that computing the value of the sum is more easily done via one or the other iterated summation. The question remains, then, as to how the value of the Cauchy product—if it exists—is related to these other values of the double sum. If the two series being multiplied converge absolutely, it is not too difficult to prove that the sum may be computed in whatever way is most convenient.

Exercise 2.8.7. Assume that $\sum_{i=1}^{\infty} a_i$ converges absolutely to A, and $\sum_{j=1}^{\infty} b_j$ converges absolutely to B.

(a) Show that the iterated sum $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|a_ib_j|$ converges so that we may apply Theorem 2.8.1.

(b) Let $s_{nn} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_ib_j$, and prove that $\lim_{n\to\infty} s_{nn} = AB$. Conclude that

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_ib_j = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_ib_j = \sum_{k=2}^{\infty} d_k = AB,$$

where, as before, $d_k = a_1b_{k-1} + a_2b_{k-2} + \cdots + a_{k-1}b_1$.

2.9 Epilogue

Theorems 2.7.10 and 2.8.1 make it clear that absolute convergence is an extremely desirable quality to have when manipulating series. On the other hand, the situation for conditionally convergent series is delightfully pathological. In the case of rearrangements, not only are they no longer guaranteed to converge to the same limit, but in fact if $\sum_{n=1}^{\infty} a_n$ converges conditionally, then for *any* $r \in \mathbf{R}$ there exists a rearrangement of $\sum_{n=1}^{\infty} a_n$ that converges to r. To see why, let's look again at the alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots .$$

The negative terms taken alone form the series $\sum_{n=1}^{\infty}(-1)/2n$. The partial sums of this series are precisely $-1/2$ the partial sums of the harmonic series, and so march off (at half speed) to negative infinity. A similar argument shows that the sum of positive terms $\sum_{n=1}^{\infty} 1/(2n-1)$ also diverges to infinity. It is not too difficult to argue that this situation is *always* the case for conditionally convergent series. Now, let r be some proposed limit, which, for the sake of this argument, we take to be positive. The idea is to take as many positive terms as necessary to form the first partial sum greater than r. We then add negative terms until the partial sum falls below r, at which point we switch back to positive terms. The fact that there is no bound on the sums of either the positive terms or the negative terms allows this process to continue indefinitely. The fact that the terms themselves tend to zero is enough to guarantee that the partial sums, when constructed in this manner, indeed converge to r as they oscillate around this target value.

Perhaps the best way to summarize the situation is to say that the hypothesis of absolute convergence essentially allows us to treat infinite sums as though they were finite sums. This assessment extends to double sums as well, although there are a few subtleties to address. In the case of products, we showed in Exercise 2.8.7 that the Cauchy product of two absolutely convergent infinite series converges to the product of the two factors, but in fact the same conclusion follows if we only have absolute convergence in one of the two original series. In the notation of Exercise 2.8.7, if $\sum a_n$ converges absolutely to A, and if $\sum b_n$ converges (perhaps conditionally) to B, then the Cauchy product $\sum d_k = AB$.

On the other hand, if both $\sum a_n$ and $\sum b_n$ converge conditionally, then it is possible for the Cauchy product to diverge. Squaring $\sum(-1)^n/\sqrt{n}$ provides an example of this phenomenon. Of course, it is also possible to find $\sum a_n = A$ conditionally and $\sum b_n = B$ conditionally whose Cauchy product $\sum d_k$ converges. If this is the case, then the convergence is to the right value, namely $\sum d_k = AB$. A proof of this last fact will be offered in Chapter 6 (Exercise 6.5.9), where we undertake the study of *power series*. Here is the connection. A power series has the form $a_0 + a_1 x + a_2 x^2 + \cdots$. If we multiply two power series together as though they were polynomials, then when we collect common powers of x the result is

$$
\begin{aligned}
&(a_0 + a_1 x + a_2 x^2 + \cdots)(b_0 + b_1 x + b_2 x^2 + \cdots) \\
&\quad = \quad a_0 b_0 + (a_0 b_1 + a_1 b_0)x + (a_0 b_2 + a_1 b_1 + a_2 b_0)x^2 + \cdots \\
&\quad = \quad d_0 + d_1 x + d_2 x^2 + \cdots ,
\end{aligned}
$$

which is the Cauchy product of $\sum a_n x^n$ and $\sum b_n x^n$. (The index starts with $n = 0$ rather than $n = 1$.) Upcoming results about the good behavior of power series will lead to a proof that convergent Cauchy products sum to the proper value. In the other direction, Exercise 2.8.7 will be useful in establishing a theorem about the product of two power series.