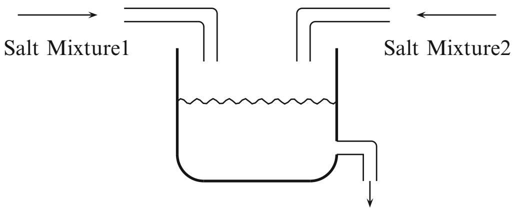
Chapter 6
Discontinuous Functions and the Laplace Transform

Our focus in this chapter is a study of first and second order linear constant coefficient differential equations

$$y' + ay = f(t),$$
$$y'' + ay' + by = f(t),$$

where the input or forcing function $f(t)$ is more general than we have studied so far. These types of forcing functions arise in applications only slightly more complicated than those we have already considered. For example, imagine a mixing problem (see Example 11 of Sect. 1.4 and the discussion that followed it for a review of mixing problems) where there are two sources of incoming salt solutions with different concentrations as illustrated in the following diagram.



Initially, the first source may be flowing for several minutes. Then the second source is turned on at the same time the first source is turned off. Such a situation will result in a differential equation $y' + ay = f(t)$ where the input function has a graph similar to the one illustrated in Fig. 6.1. The most immediate observation is that the input function is discontinuous. Nevertheless, the Laplace transform methods we will develop will easily handle this situation, leading to a formula for the amount of the salt in the tank as a function of time.

As a second example, imagine that a sudden force is applied to a spring-mass dashpot system (see Sect. 3.6 for a discussion of these systems). For example, hit the mass attached to the spring with a hammer, which is a very good idealization of

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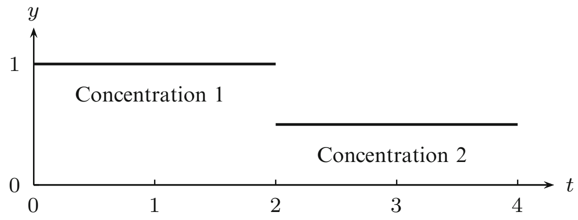


Fig. 6.1 The graph of discontinuous input function $f(t)$ where salt of concentration level 1 enters until time $t = 2$, at which time the concentration switches to a different level

what happens to the shock absorber on a car when the car hits a bump in the road. Modeling this system will lead to a differential equation of the form

$$y'' + ay' + by = f(t),$$

where the forcing function is what we will refer to as an *instantaneous impulse function*. Such a function has a "very large" (or even infinite) value at a single instant $t = t_0$ and is 0 for other times. Such a function is not a true function, but its effect on systems can be analyzed effectively via the Laplace transform methods developed later in this chapter.

This chapter will develop the necessary background on the types of discontinuous functions and impulse functions which arise in basic applications of differential equations. We will start by describing the basic concepts of calculus for these more general classes of functions.

6.1 Calculus of Discontinuous Functions

Piecewise Continuous Functions

A function $f(t)$ has a ***jump discontinuity*** at a point $t = a$ if the left-hand limit $f(a^-) = \lim_{t\to a^-} f(t)$ and the right-hand limit $f(a^+) = \lim_{t\to a^+} f(t)$ both exist (as real numbers, not $\pm\infty$) and

$$f(a^+) \neq f(a^-).$$

The difference $f(a^+)-f(a^-)$ is frequently referred to as the ***jump*** in $f(t)$ at $t=a$. Functions with jump discontinuities are typically described by using different formulas on different subintervals of the domain. For example, the function $f(t)$ defined on the interval [0, 3] by

$$f(t) = \begin{cases} t^3 & \text{if } 0 \le t < 1, \\ 1-t & \text{if } 1 \le t < 2, \\ 1 & \text{if } 2 \le t \le 3 \end{cases}$$

has a jump discontinuity at $t = 1$ since $f(1^-) = 1 \neq f(1^+) = 0$ and at $t = 2$ since $f(2^-) = -1 \neq f(2^+) = 1$. The jump at $t = 1$ is -1 and the jump at $t = 2$ is 2. The graph of $f(t)$ is given in Fig. 6.2. On the other hand, the function

$$g(t) = \begin{cases} 1/(1-t) & \text{if } 0 \le t < 1, \\ t & \text{if } 1 \le t \le 2. \end{cases}$$

defined on the interval [0, 2] has a discontinuity at $t = 1$, but it is not a jump discontinuity since $\lim_{t\to 1^-} g(t) = \infty$ does not exist.

We will say that a function $f(t)$ is ***piecewise continuous on a closed interval*** $[a, b]$ if $f(t)$ is continuous except for possibly finitely many jump discontinuities.

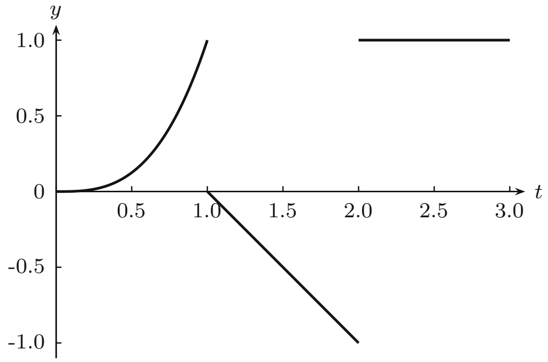


Fig. 6.2 A piecewise continuous function

For convenience, it will not be required that $f(t)$ be defined at the jump discontinuities. Suppose $a_1, \ldots, a_n$ are the locations of the jump discontinuities in the interval $[a, b]$ and assume $a_i < a_{i+1}$, for each i. On the interval (a_i, a_{i+1}), we can extend $f(t)$ to a continuous function on the closed interval $[a_i, a_{i+1}]$ by defining $f(a_i) = \lim_{t \to a_i^+} f(t)$ and $f(a_{i+1}) = \lim_{t \to a_{i+1}^-} f(t)$. Since a continuous function on a closed interval is bounded and there are only finitely many jump discontinuities, we have the following property of piecewise continuous functions.

Proposition 1. *If $f(t)$ is a piecewise continuous function on $[a, b]$, then $f(t)$ is bounded.*

How do we compute the derivative and integral of a piecewise continuous function?

Integration of Piecewise Continuous Functions

If $f(t)$ is a piecewise continuous function on the interval $[a, b]$ and the jump discontinuities are located at $a_1 < \ldots < a_k$, we may let $a_0 = a$ and $a_{k+1} = b$, and, as we observed above, $f(t)$ extends to a continuous function on the each closed interval $[a_i, a_{i+1}]$. Thus, we can define the definite integral of $f(t)$ on $[a, b]$ by the formula

$$\boxed{\int_a^b f(t)\,\mathrm{d}t = \int_{a_0}^{a_1} f(t)\,\mathrm{d}t + \int_{a_1}^{a_2} f(t)\,\mathrm{d}t + \cdots + \int_{a_k}^{a_{k+1}} f(t)\,\mathrm{d}t.}$$

Example 2. Find $\int_0^t f(u)\,\mathrm{d}u$ for all $t \in [0, \infty)$ where $f(t)$ is the piecewise continuous function defined by

$$f(t) = \begin{cases} 1 & \text{if } 0 \le t < 1, \\ 0 & \text{if } 1 \le t < \infty. \end{cases}$$

► **Solution.** The function $f(t)$ is given by different formulas on each of the intervals $[0, 1)$ and $[1, \infty)$. We will therefore break the calculation into two cases. If $t \in [0, 1)$, then

$$\int_0^t f(u)\,\mathrm{d}u = \int_0^t 1\,\mathrm{d}u = t.$$

It $t \in [1, \infty)$, then

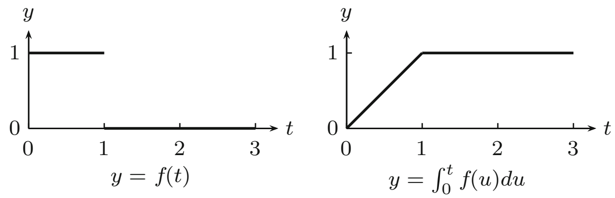


Fig. 6.3 The graph of the piecewise continuous function $f(t)$ and its integral $\int_0^t f(u)\,\mathrm{d}u$

$$\int_0^t f(u)\,\mathrm{d}u = \int_0^1 f(u)\,\mathrm{d}u + \int_1^t f(u)\,\mathrm{d}u$$

$$= 1 + \int_1^t 0\,\mathrm{d}u = 1.$$

Piecing these functions together gives

$$\int_0^t f(u)\,\mathrm{d}u = \begin{cases} t & \text{if } 0 \le t < 1 \\ 1 & \text{if } 1 \le t < \infty. \end{cases}$$

The graph of this function of t is shown in Fig. 6.3. ◀

Notice that the function $\int_0^t f(u)\,\mathrm{d}u$ is a continuous function of t, even though the integrand $f(t)$ is discontinuous. This is always true as long as the function $f(t)$ has only jump discontinuities, which is formalized in the following result.

Proposition 3. *If $f(t)$ is a piecewise continuous function on an interval $[a, b]$ and $c, t \in [a, b]$, then the integral $\int_c^t f(u)\,\mathrm{d}u$ exists and is a continuous function in the variable t.*

Proof. The integral exists as discussed above. Let $F(t) = \int_c^t f(u)\,\mathrm{d}u$. Since $f(t)$ is piecewise continuous on $[a, b]$, it is bounded by Proposition 1. We may then suppose $|f(t)| \le B$, for some $B > 0$. Let $\epsilon > 0$. Then

$$|F(t+\epsilon) - F(t)| \le \int_t^{t+\epsilon} |f(u)|\,\mathrm{d}u \le \int_t^{t+\epsilon} B\,\mathrm{d}u = B\epsilon.$$

Therefore, $\lim_{\epsilon\to 0} F(t+\epsilon) = F(t)$, and hence, $F(t^+) = F(t)$. In a similar way, $F(t^-) = F(t)$. This establishes the continuity of F. □

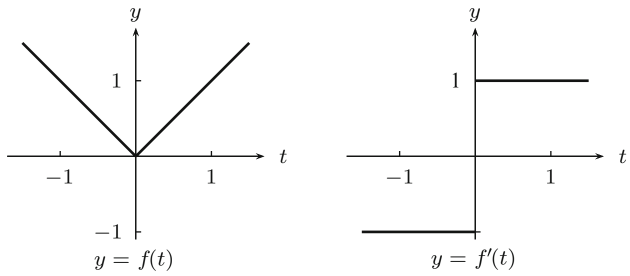


Fig. 6.4 The graph of the piecewise continuous function $f(t)$ and its derivative $f'(t)$

Differentiation of Piecewise Continuous Functions

In the applications, we will consider functions that are differentiable except at finitely many points in any interval $[a, b]$ of finite length. In this case we will use the symbol $f'(t)$ to denote the derivative of $f(t)$ even though it may not be defined at some points. For example, the absolute value function

$$f(t) = |t| = \begin{cases} -t & \text{if } -\infty < t < 0 \\ t & \text{if } 0 \le t < \infty. \end{cases}$$

This function is continuous on $(-\infty, \infty)$ and differentiable at all points except $t = 0$. Then

$$f'(t) = \begin{cases} -1 & \text{if } -\infty < t < 0 \\ 1 & \text{if } 0 < t < \infty. \end{cases}$$

Notice that $f'(t)$ is not defined at $t = 0$, but the derivative of this discontinuous function has produced a function with a jump discontinuity where the derivative does not exist. See Fig. 6.4. Compare this with Fig. 6.3, where we have seen that integrating a function with jump discontinuities produces a continuous function.

Differential Equations and Piecewise Continuous Functions

We now look at some examples of solutions to constant coefficient linear differential equations with piecewise continuous forcing functions. We start with the first order equation $y' + ay = f(t)$ where a is a constant and $f(t)$ is a piecewise continuous function. An equation of this type has a unique solution for each initial condition provided the input function is continuous, which is the situation for $f(t)$ on each subinterval on which it is continuous. To be able to extend the initial condition in a unique manner across each jump discontinuity, we shall define a function $y(t)$ to

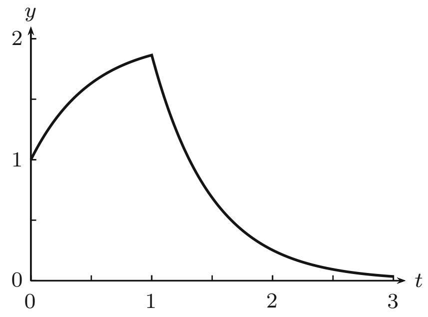


Fig. 6.5 The graph of the solution to Example 4

be a ***solution*** to $y' + ay = f(t)$ if $y(t)$ is *continuous* and satisfies the differential equation except at the jump discontinuities of the of the input function $f(t)$.

Example 4. Find a solution to

$$y' + 2y = f(t) = \begin{cases} 4 & \text{if } 0 \le t < 1 \\ 0 & \text{if } 1 \le t < \infty, \end{cases} \qquad y(0) = 1. \tag{1}$$

► **Solution.** The procedure will be to solve the differential equation separately on each of the subintervals where $f(t)$ is continuous and then piece the solution together to make a continuous solution. Start with the interval [0 1) which includes the initial time $t = 0$. On this first subinterval, $f(t) = 4$, so the differential equation to be solved is $y' + 2y = 4$. The solution uses the integrating factor technique developed in Sect. 1.4. Multiplying both sides of $y' + 2y = 4$ by the integrating factor, e^{2t}, leads to $(e^{2t}y)' = 4e^{2t}$. Integrating and solving for $y(t)$ gives $y(t) = 2 + ce^{-2t}$, and the initial condition $y(0) = 1$ implies that $c = -1$ so that

$$y = 2 - e^{-2t}, \quad 0 \le t < 1.$$

On the interval $[1, \infty)$, the differential equation to solve is $y' + 2y = 0$, which has the general solution $y(t) = ke^{-2t}$. To produce a continuous function, we need to choose k so that this solution will match up with the solution found for the interval $[0, 1)$. To do this, let $y(1) = y(1^-) = 2 - e^{-2}$. This value must match the value $y(1) = ke^{-2}$ computed from the formula on $[1, \infty)$. Thus, $2 - e^{-2} = ke^{-2}$ and solving for k gives $k = 2e^2 - 1$. Therefore, the solution on the interval $[1, \infty)$ is $y(t) = (2e^2 - 1)e^{-2t}$. Putting the two pieces together gives the solution

$$y(t) = \begin{cases} 2 - e^{-2t} & \text{if } 0 \le t < 1 \\ (2e^2 - 1)e^{-2t} & \text{if } 1 \le t < \infty. \end{cases}$$

The graph of this solution is shown in Fig. 6.5, where the discontinuity of the derivative of $y(t)$ at $t = 1$ is evident by the kink at that point. ◄

The method we used here insures that the solution we obtain is continuous and the initial condition at $t = 0$ determines the subsequent initial conditions at the points of discontinuity of f. We also note that the initial condition at $t = 0$, the left-hand endpoint of the domain, was chosen only for convenience; we could have taken the initial value at any point $t_0 \geq 0$ and pieced together a continuous function on both sides of t_0. That this can be done in general is stated in the following theorem.

Theorem 5. *Suppose $f(t)$ is a piecewise continuous function on an interval $[\alpha, \beta]$ and $t_0 \in [\alpha, \beta]$. There is a unique continuous function $y(t)$ which satisfies the initial value problem*

$$y' + ay = f(t), \qquad y(t_0) = y_0.$$

Proof. Follow the method illustrated in the example above to construct a continuous solution. To prove uniqueness, suppose $y_1(t)$ and $y_2(t)$ are two continuous solutions. If $y(t) = y_1(t) - y_2(t)$, then $y(t_0) = 0$ and $y(t)$ is a continuous solution to $y' + ay = 0$. On the interval containing t_0 on which $f(t)$ is continuous, $y(t) = 0$ by the existence and uniqueness theorem. The initial value at the endpoint of adjacent intervals is thus 0. Continuing in this way, we see that $y(t)$ is identically 0 on $[\alpha, \beta]$ and hence $y_1(t) = y_2(t)$. □

Now consider a second order constant coefficient differential equation with a piecewise continuous forcing function. Our method is similar to the one above, however, we demand more out of our solution. Since the solution of a second order equation with continuous input function is determined by the initial values of both $y(t)$ and $y'(t)$, it will be necessary to extend *both* of these values across the jump discontinuity in order to obtain a unique solution with a discontinuous input function. Thus, if $f(t)$ is a piecewise continuous function, then we will say that a function $y(t)$ is a ***solution*** to

$$y'' + ay' + by = f(t),$$

if $y(t)$ is *continuous*, has a *continuous derivative*, and satisfies the differential equation except at the discontinuities of the forcing function $f(t)$.

Example 6. Find a solution y to

$$y'' + y = f(t) = \begin{cases} 1 & \text{if } 0 \leq t < \pi \\ 0 & \text{if } \pi \leq t < \infty, \end{cases} \qquad y(0) = 1,\ y'(0) = 0.$$

► **Solution.** The general solution to the differential equation $y'' + y = 1$ on the interval $[0, \pi)$ is $y(t) = 1 + a\cos t + b\sin t$, and the initial conditions $y(0) = 1$, $y'(0) = 0$ imply $a = 0$, $b = 0$, so the solution on $[0, \pi)$ is $y(t) = 1$. Taking limits as $t \to \pi^-$ gives $y(\pi) = 1$, $y'(\pi) = 0$. On the interval $[\pi, \infty)$, the differential equation $y'' + y = f(t)$ becomes $y'' + y = 0$ with the initial conditions $y(\pi) = 1$, $y'(\pi) = 0$. The general solution on this interval is thus $y(t) = a\cos t + b\sin t$, and taking into account the values at $t = \pi$ gives $a = -1$, $b = 0$. Piecing these two solutions together gives

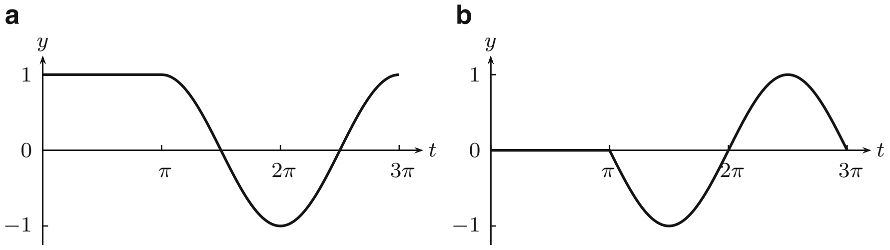


Fig. 6.6 The graph of the solution $y(t)$ to Example 4 is shown in (**a**), while the derivative $y'(t)$ is graphed in (**b**). Note that the derivative $y'(t)$ is continuous, but it is not differentiable at $t = \pi$

$$y(t) = \begin{cases} 1 & \text{if } 0 \le t < \pi \\ -\cos t & \text{if } \pi \le t < \infty. \end{cases}$$

Its derivative is

$$y'(t) = \begin{cases} 0 & \text{if } 0 \le t < \pi \\ \sin t & \text{if } \pi \le t < \infty. \end{cases}$$

Figure 6.6 gives (a) the graph of the solution and (b) the graph of its derivative. The solution is differentiable on the interval $[0, \infty]$, and the derivative is continuous on $[0, \infty)$. However, the kink in the derivative at $t = \pi$ indicates that the second derivative is not continuous. ◀

In direct analogy to the first order case we considered above, we are led to the following theorem. The proof is omitted.

Theorem 7. *Suppose f is a piecewise continuous function on an interval $[\alpha, \beta]$ and $t_0 \in [\alpha, \beta]$. There is a unique continuous function $y(t)$ with continuous derivative which satisfies*

$$y'' + ay' + by = f(t), \qquad y(t_0) = y_0, \; y'(t_0) = y_1.$$

Piecing together solutions in the way that we described above is at best tedious. Later in this chapter, the Laplace transform method for solving differential equations will be extended to provide a simpler alternate method for solving differential equations like the ones above. It is one of the hallmarks of the Laplace transform.

Exercises

1–8. Match the following functions that are given piecewise with their graphs and determine where jump discontinuities occur.

1. $f(t) = \begin{cases} 1 & \text{if } 0 \le t < 4 \\ -1 & \text{if } 4 \le t < 5 \\ 0 & \text{if } 5 \le t \le 6 \end{cases}$

2. $f(t) = \begin{cases} t & \text{if } 0 \le t < 1 \\ 2-t & \text{if } 1 \le t < 2 \\ 1 & \text{if } 2 \le t \le 6 \end{cases}$

3. $f(t) = \begin{cases} t/3 & \text{if } 0 \le t < 3 \\ 2-t/3 & \text{if } 3 \le t \le 6 \end{cases}$

4. $f(t) = t - n$ for $n \le t \le n+1$ and $0 \le n \le 5$

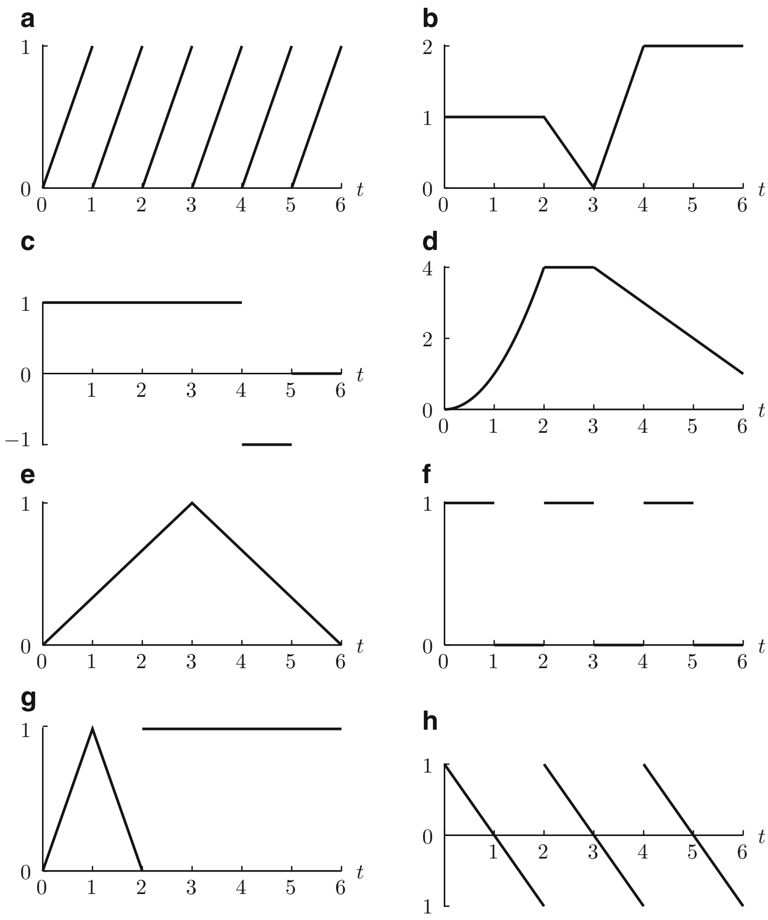
5. $f(t) = \begin{cases} 1 & \text{if } 2n \le t < 2n+1 \\ 0 & \text{if } 2n+1 \le t < 2n+2 \end{cases}$ for $0 \le n \le 2$

6. $f(t) = \begin{cases} t^2 & \text{if } 0 \le t < 2 \\ 4 & \text{if } 2 \le t < 3 \\ 7-t & \text{if } 3 \le t \le 6 \end{cases}$

7. $f(t) = \begin{cases} 1-t & \text{if } 0 \le t < 2 \\ 3-t & \text{if } 2 \le t < 4 \\ 5-t & \text{if } 4 \le t \le 6 \end{cases}$

8. $f(t) = \begin{cases} 1 & \text{if } 0 \le t < 2 \\ 3-t & \text{if } 2 \le t < 3 \\ 2(t-3) & \text{if } 3 \le t < 4 \\ 2 & \text{if } 4 \le t < \infty \end{cases}$

Graphs for problems 1 through 8



9–12. Compute the indicated integral.

9. $\int_0^5 f(t)\,dt$, where $f(t) = \begin{cases} t^2 - 4 & \text{if } 0 \le t < 2 \\ 0 & \text{if } 2 \le t < 3 \\ -t + 3 & \text{if } 3 \le t < 5 \end{cases}$

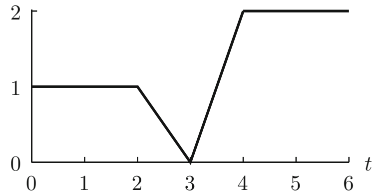
10. $\int_0^2 f(u)\,du$, where $f(u) = \begin{cases} 2 - u & \text{if } 0 \le u < 1 \\ u^3 & \text{if } 1 \le u < 2 \end{cases}$

11. $\int_0^{2\pi} |\sin x|\ dx$

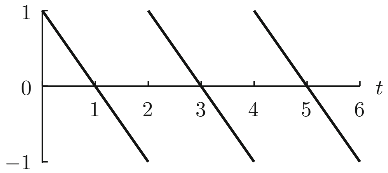
12. $\int_0^3 f(w)\,dw$ where $f(w) = \begin{cases} w & \text{if } 0 \le w < 1 \\ \frac{1}{w} & \text{if } 1 \le w < 2 \\ \frac{1}{2} & \text{if } 2 \le w < \infty \end{cases}$

13–16. Compute the indicated integral (See problems 1–8 for the appropriate formula to match with each graph.)

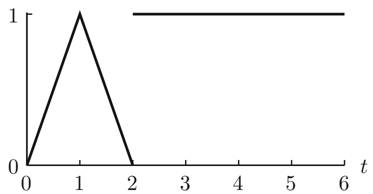
13. $\int_2^5 f(t)\,dt$, where the graph of f is



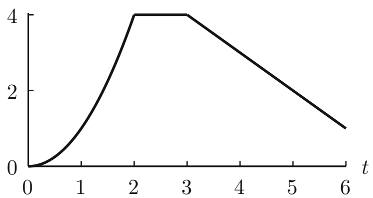
14. $\int_0^6 f(t)\,dt$, where the graph of f is



15. $\int_0^6 f(u)\,du$, where the graph of f is



16. $\int_0^6 f(t)\,dt$, where the graph of f is



17–20. Of the following four piecewise-defined functions, determine which ones

(a) Satisfy the differential equation

$$y' + 4y = f(t) = \begin{cases} 4 & \text{if } 0 \le t < 2 \\ 8t & \text{if } 2 \le t < \infty, \end{cases}$$

except at the point of discontinuity of f

(b) Are continuous

(c) Are continuous solutions to the differential equation with initial condition $y(0) = 2$. Do not solve the differential equation

17. $y(t) = \begin{cases} 1 & \text{if } 0 \le t < 2 \\ 2t - \frac{1}{2} - \frac{5}{2}e^{-4(t-2)} & \text{if } 2 \le t < \infty \end{cases}$

18. $y(t) = \begin{cases} 1 + e^{-4t} & \text{if } 0 \le t < 2 \\ 2t - \frac{1}{2} - \frac{5}{2}e^{-4(t-2)} + e^{-4t} & \text{if } 2 \le t < \infty \end{cases}$

19. $y(t) = \begin{cases} 1 + e^{-4t} & \text{if } 0 \le t < 2 \\ 2t - \frac{1}{2} - \frac{5e^{-4(t-2)}}{2} & \text{if } 2 \le t < \infty \end{cases}$

20. $y(t) = \begin{cases} 2e^{-4t} & \text{if } 0 \le t < 2 \\ 2t - \frac{1}{2} - \frac{5}{2}e^{-4(t-2)} + e^{-4t} & \text{if } 2 \le t < \infty \end{cases}$

21–24. Of the following four piecewise-defined functions, determine which ones

(a) satisfy the differential equation

$$y'' - 3y' + 2y = f(t) = \begin{cases} e^{t} & \text{if } 0 \le t < 1 \\ e^{2t} & \text{if } 1 \le t < \infty, \end{cases}$$

except at the point of discontinuity of f

(b) Are continuous

(c) Have continuous derivatives

(d) Are continuous solutions to the differential equation with initial condition $y(0) = 0$ and $y'(0) = 0$ and have continuous derivatives. Do not solve the differential equation.

21. $y(t) = \begin{cases} -te^t - e^t + e^{2t} & \text{if } 0 \le t < 1 \\ te^{2t} - 2e^t & \text{if } 1 \le t < \infty \end{cases}$

22. $y(t) = \begin{cases} -te^t - e^t + e^{2t} & \text{if } 0 \le t < 1 \\ te^{2t} - 3e^t - \frac{1}{2}e^{2t} & \text{if } 1 \le t < \infty \end{cases}$

23. $y(t) = \begin{cases} -te^t - e^t + e^{2t} & \text{if } 0 \le t < 1 \\ te^{2t} + e^{t+1} - e^t - e^{2t} - e^{2t-1} & \text{if } 1 \le t < \infty \end{cases}$

24. $y(t) = \begin{cases} -te^t + e^t - e^{2t} & \text{if } 0 \le t < 1 \\ te^{2t} + e^{t+1} + e^t - e^{2t-1} - 3e^{2t} & \text{if } 1 \le t < \infty \end{cases}$

25–30. Solve the following differential equations.

25. $y' - y = \begin{cases} 1 & \text{if } 0 \le t < 2, \\ -1 & \text{if } 2 \le t < 4, \\ 0 & \text{if } 4 \le t < \infty \end{cases} \qquad y(0) = 0$

26. $y' + 3y = \begin{cases} t & \text{if } 0 \le t < 1 \\ 1 & \text{if } 1 \le t < \infty, \end{cases} \qquad y(0) = 0$

27. $y' - y = \begin{cases} 0 & \text{if } 0 \le t < 1 \\ t - 1 & \text{if } 1 \le t < 2 \\ 3 - t & \text{if } 2 \le t < 3 \\ 0 & \text{if } 3 \le t < \infty, \end{cases} \qquad y(0) = 0$

28. $y' + y = \begin{cases} \sin t & \text{if } 0 \le t < \pi \\ 0 & \text{if } \pi \le t < \infty \end{cases} \qquad y(\pi) = -1$

29. $y'' - y = \begin{cases} t & \text{if } 0 \le t < 1 \\ 0 & \text{if } 1 \le t < \infty, \end{cases} \qquad y(0) = 0,\ y'(0) = 1$

30. $y'' - 4y' + 4y = \begin{cases} 0 & \text{if } 0 \le t < 2 \\ 4 & \text{if } 2 \le t < \infty \end{cases} \qquad y(0) = 1,\ y'(0) = 0$

31. Suppose f is a piecewise continuous function on an interval $[\alpha, \beta]$. Let $a \in [\alpha, \beta]$ and define $y(t) = y_0 + \int_a^t f(u)\, du$. Show that y is a continuous solution to

$$y' = f(t) \quad y(a) = y_0$$

32. Suppose f is a piecewise continuous function on an interval $[\alpha, \beta]$. Let $a \in [\alpha, \beta]$ and define $y(t) = y_0 + e^{-at} \int_a^t e^{au} f(u)\, du$. Show that y is a continuous solution to

$$y' + ay = f(t) \quad y(a) = y_0$$

33. Let $f(t) = \begin{cases} \sin(1/t) & \text{if } t \neq 0 \\ 0 & \text{if } t = 0 \end{cases}$

    (a) Show that f is bounded.
    (b) Show that f is not continuous at $t = 0$.
    (c) Show that f is not piecewise continuous.

6.2 The Heaviside Class $\mathcal{H}$

In Chap. 2, the Laplace transform was introduced and extensively studied for the class of *continuous* functions of exponential type. We now want to broaden the range of applicability of the Laplace transform method for solving differential equations by allowing for some discontinuous forcing functions. Thus, we say a function $f(t)$ is ***piecewise continuous on*** $[0,\infty)$ if $f(t)$ is piecewise continuous on each closed subinterval $[0,b]$ for all $b>0$. In addition, we will maintain the growth condition so that convergence of the integrals defining the Laplace transform is guaranteed. Thus, we will define the ***Heaviside class*** to be the set $\mathcal{H}$ of all *piecewise continuous functions on* $[0,\ \infty)$ *of exponential type*. Specifically, $f\in\mathcal{H}$ if:

1. f is piecewise continuous on $[0,\infty)$.
2. There are constants K and a such that $|f(t)|\le K\mathrm{e}^{at}$ for all $t\ge 0$.

One can show $\mathcal{H}$ is a linear space, that is, closed under addition and scalar multiplication (see Exercises 43–44). It is to this class $\mathcal{H}$ of functions that we extend the Laplace transform. The first observation is that the argument of Proposition 3 of Sect. 2.2 guaranteeing the existence of the Laplace transform extends immediately to functions in $\mathcal{H}$.

Proposition 1. *For $f\in\mathcal{H}$ of exponential type of order a, the Laplace transform $F(s)=\int_0^\infty \mathrm{e}^{-st}f(t)\,\mathrm{d}t$ exists for $s>a$, and* $\lim_{s\to\infty}F(s)=0$.

Proof. The finite integral $\int_0^N \mathrm{e}^{-st}f(t)\,\mathrm{d}t$ exists because f is piecewise continuous on $[0,N]$. Since f is also of exponential type, there are constants $K\ge 0$ and a such that $|f(t)|\le K\mathrm{e}^{at}$ for all $t\ge 0$. Thus, for all $s>a$,

$$|F(s)|\le\int_0^\infty|\mathrm{e}^{-st}f(t)|\,\mathrm{d}t\le\int_0^\infty|\mathrm{e}^{-st}K\mathrm{e}^{at}|\,\mathrm{d}t=K\int_0^\infty \mathrm{e}^{-(s-a)t}\,\mathrm{d}t=\frac{K}{s-a}.$$

This shows that the integral converges absolutely, and hence, the Laplace transform exists for $s>a$ and $F(s)\le K/(s-a)$. It follows that

$$\lim_{s\to\infty}F(s)=0.$$ □

Many of the properties of the Laplace transform that were discussed in Chap. 2, and collected in Table 2.3, for continuous functions carry over to the Heaviside class without change in statement or proof. Some of these properties are summarized below.

Linearity $\mathcal{L}\{af + bg\} = \mathrm{a}\mathcal{L}\{f\} + b\mathcal{L}\{g\}$.

The first translation principle $\mathcal{L}\{\mathrm{e}^{-at} f(t)\}(s) = \mathcal{L}\{f(t)\}(s-a)$.

Differentiation in transform space

$$\mathcal{L}(-tf(t)) = \frac{\mathrm{d}}{\mathrm{d}s}F(s),$$

$$\mathcal{L}\{(-t)^n f(t)\} = F^{(n)}(s).$$

The dilation principle $\mathcal{L}\{f(bt)\}(s) = \frac{1}{b}\mathcal{L}\{f(t)\}(s/b)$.

The important input derivative formula is not on the above list because of a subtlety to be considered in the next section. For now, we will look at some direct computations of Laplace transforms of functions in $\mathcal{H}$ and a useful tool (the second translation theorem) for avoiding most direct calculations.

As might be expected, computations using the definition to compute Laplace transforms of even simple piecewise continuous functions can be tedious.

Example 2. Use the definition to compute the Laplace transform of

$$f(t) = \begin{cases} t^2 & \text{if } 0 \le t < 1, \\ 2 & \text{if } 1 \le t < \infty. \end{cases}$$

► **Solution.** Clearly, f is piecewise continuous and bounded; hence, it is in the Heaviside class. We can thus proceed with the definition confident, by Proposition 1, that the improper integral will converge. We have

$$\begin{aligned} \mathcal{L}\{f(t)\}(s) &= \int_0^\infty \mathrm{e}^{-st} f(t)\,\mathrm{d}t \\ &= \int_0^1 \mathrm{e}^{-st}\, t^2\,\mathrm{d}t + \int_1^\infty \mathrm{e}^{-st}\, 2\,\mathrm{d}t \end{aligned}$$

For the first integral, we need integration by parts twice:

$$\begin{aligned} \int_0^1 \mathrm{e}^{-st}\, t^2\,\mathrm{d}t &= \left.\frac{t^2\mathrm{e}^{-st}}{-s}\right|_0^1 + \frac{2}{s}\int_0^1 \mathrm{e}^{-st}\, t\,\mathrm{d}t \\ &= \frac{\mathrm{e}^{-s}}{-s} + \frac{2}{s}\left(\left.\frac{t\mathrm{e}^{-st}}{-s}\right|_0^1 + \frac{1}{s}\int_0^1 \mathrm{e}^{-st}\,\mathrm{d}t\right) \end{aligned}$$

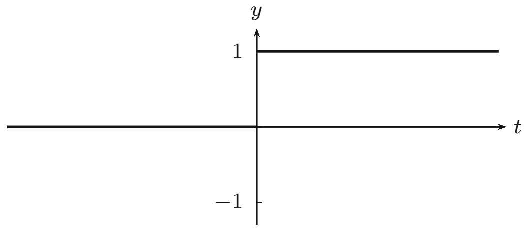


Fig. 6.7 The graph of the Heaviside function $h(t)$ (also called the unit step function)

$$= -\frac{e^{-s}}{s} + \frac{2}{s}\left(-\frac{e^{-s}}{s} - \frac{1}{s^2}e^{-st}\Big|_0^1\right)$$
$$= -\frac{e^{-s}}{s} - \frac{2e^{-s}}{s^2} + \frac{2}{s^3} - \frac{2e^{-s}}{s^3}.$$

The second integral is much simpler, and we get

$$\int_1^\infty e^{-st}\,2\,dt = \frac{2e^{-s}}{s}.$$

Now putting the two pieces together and simplifying gives

$$\mathcal{L}\{f(t)\}(s) = \frac{2}{s^3} + e^{-s}\left(-\frac{2}{s^3} - \frac{2}{s^2} + \frac{1}{s}\right).$$ ◀

As we saw for the Laplace transform of continuous functions, calculations directly from the definition are rarely needed since the Heaviside function that we introduce next will lead to a Laplace transform principle that will allow for the use of our previously derived formulas and make calculations like the one above unnecessary. The ***unit step function*** or ***Heaviside function*** is defined on the real line by

$$h(t) = \begin{cases} 0 & \text{if } t < 0, \\ 1 & \text{if } 0 \le t < \infty. \end{cases}$$

The graph of this function is given in Fig. 6.7.

Clearly, $h(t)$ is piecewise continuous, and it is bounded so it is of exponential type, and hence, $h(t) \in \mathcal{H}$. In addition to $h(t)$ itself, it will be necessary to consider the translations $h(t-c)$ of $h(t)$ for all $c \ge 0$. From the definition of $h(t)$, we see

$$h(t-c) = \begin{cases} 0 & \text{if } -\infty \le t < c, \\ 1 & \text{if } c \le t < \infty. \end{cases}$$

Note that the graph of $h(t-c)$ is just the graph of $h(t)$ translated c units to the right. The graphs of two examples are given in Fig. 6.8.

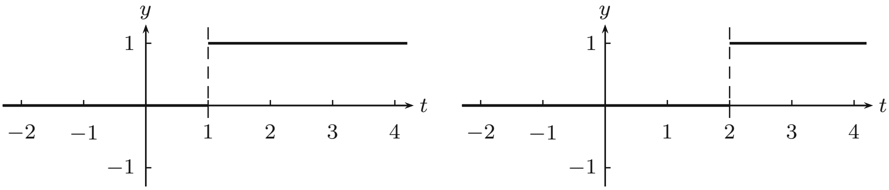


Fig. 6.8 The graphs of the translates $h(t-1)$ and $h(t-2)$

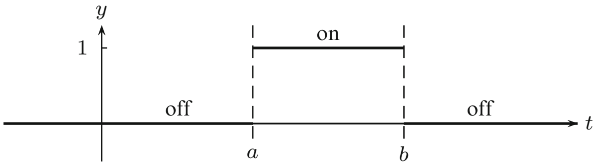


Fig. 6.9 The graph of the on–off switch $\chi_{[a,\,b)}(t)$

More complicated functions can be built from the Heaviside function. The most important building block is the ***characteristic function*** $\chi_{[a,\,b)}(t)$ on the interval $[a,\,b)$ defined by

$$\chi_{[a,\,b)}(t) = \begin{cases} 1 & \text{if } t \in [a,\,b), \\ 0 & \text{if } t \notin [a,\,b). \end{cases}$$

The characteristic function $\chi_{[a,b)}(t)$ serves as the model for an on–off switch at $t = a$ (on) and $t = b$ (off). That is, $\chi_{[a,\,b)}(t)$ is 1 (the *on* state) for t in the interval $[a,\,b)$ and 0 (the *off* state) for t not in $[a,\,b)$. Because of this, we shall also refer to $\chi_{[a,\,b)}(t)$ as an ***on–off switch***. A graph of a typical on–off switch is given in Fig. 6.9.

Here are some useful relationships between the on-off switches $\chi_{[a,\,b)}(t)$ and the Heaviside function $h(t)$. All are obtained by direct comparison of the value of the function on the left with that on the right.

1. $\chi_{[0,\,\infty)}(t) = h(t)$, and if these functions are restricted to the interval $[0,\,\infty)$ (rather than defined on all of $\mathbb{R}$), then

$$\chi_{[0,\,\infty)}(t) = h(t) = 1. \tag{1}$$

2. If $0 \le a < \infty$, then

$$\chi_{[a,\,\infty)}(t) = h(t-a). \tag{2}$$

3. If $0 \le a < b < \infty$, then

$$\chi_{[a,\,b)}(t) = h(t-a) - h(t-b). \tag{3}$$

Using on–off switches, we can easily describe functions defined piecewise. The strategy is to write $f(t)$ as a sum of terms of the form $f_i(t)\chi_{[a_i, a_{i+1})}(t)$ if $f_i(t)$ is the formula used to describe $f(t)$ on the subinterval $[a_i, a_{i+1})$. Then using the relationships listed above, it is possible to write $f(t)$ in terms of translates of the Heaviside function $h(t)$. Here are some examples.

Example 3. Write the piecewise-defined function

$$f(t) = \begin{cases} t^2 & \text{if } 0 \le t < 1, \\ 2 & \text{if } 1 \le t < \infty. \end{cases}$$

in terms of on–off switches and in terms of translates of the Heaviside function.

► **Solution.** In this piecewise function, t^2 is in the on state only in the interval $[0, 1)$ and 2 is in the on state only in the interval $[1, \infty)$. Thus,

$$f(t) = t^2\chi_{[0,1)}(t) + 2\chi_{[1,\infty)}(t).$$

Now rewriting the on–off switches in terms of the Heaviside functions using (1)–(3), we obtain

$$\begin{aligned} f(t) &= t^2(h(t) - h(t-1)) + 2h(t-1) \\ &= t^2h(t) + (2 - t^2)h(t-1) \\ &= t^2 + (2 - t^2)h(t-1). \end{aligned}$$ ◄

Example 4. Write the piecewise-defined function

$$f(t) = \begin{cases} \cos t & \text{if } 0 \le t < \pi, \\ 1 & \text{if } \pi \le t < 2\pi, \\ 0 & \text{if } 2\pi \le t < \infty. \end{cases}$$

in terms of on–off switches and in terms of translates of the Heaviside function.

► **Solution.** The function is defined by different formulas on each of the intervals $[0, \pi)$, $[\pi, 2\pi)$, and $[2\pi, \infty)$. Thus,

$$\begin{aligned} f(t) &= \cos t\,\chi_{[0,\pi)}(t) + 1\chi_{[\pi,2\pi)}(t) + 0\chi_{[2\pi,\infty)}(t) \\ &= \cos t(h(t) - h(t-\pi)) + (h(t-\pi) - h(t-2\pi)) \\ &= (\cos t)h(t) + (1 - \cos t)h(t-\pi) - h(t-2\pi) \\ &= \cos t + (1 - \cos t)h(t-\pi) - h(t-2\pi). \end{aligned}$$ ◄

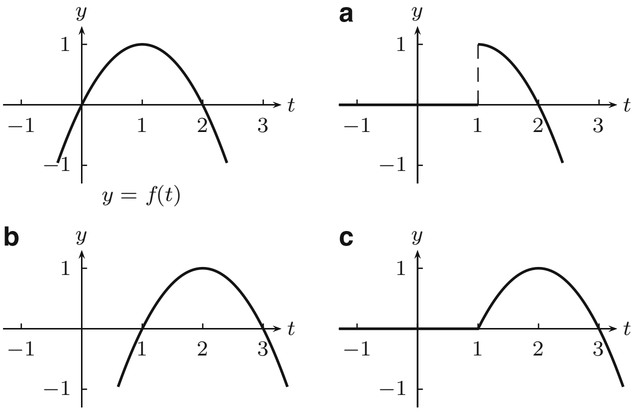


Fig. 6.10 The graphs of (**a**) $h(t-1)f(t)$, (**b**) $f(t-1)$, and (**c**) $h(t-1)f(t-1)$ for $f(t) = 2t-t^2$

In the above descriptions, there are functions of the form $g(t)h(t-c)$ and $g(t)\chi_{[a,\,b)}(t)$. How do the graphs of these functions correspond to the graph of $g(t)$? Here is an example.

Example 5. Compare the graph of each of the following functions to the graph of $f(t) = 2t - t^2$:

(a) $h(t-1)f(t)$ (b) $f(t-1)$ (c) $h(t-1)f(t-1)$

► **Solution.** The graphs are given in Fig. 6.10.
The graph of $f(t)$ is given first. Now, (a) $h(t-1)f(t)$ simply cuts off the graph of $f(t)$ at $t = 1$ and replaces it with the line $y = 0$ for $t < 1$, (b) $f(t-1)$ just shifts the graph of $f(t)$ 1 unit to the right, and (c) $h(t-1)f(t-1)$ shifts the graph of $f(t)$ 1 unit to the right and then cuts off the resulting graph at $t = 1$ and replaces it with the line $y = 0$ for $t < 1$. ◄

Functions of the form $h(t-c)f(t-c)$, namely, translation of f by c and then truncation of the resulting graph for $t < c$, as illustrated for $t = 1$ in the previous example, are precisely the special type of functions in the Heaviside class $\mathcal{H}$ for which it is possible to compute the Laplace transform in an efficient manner. Since any piecewise-defined function will be reducible to functions of this form, it will provide an effective method of computation. Let us start by computing the Laplace transform of a translated Heaviside function $h(t-c)$.

Formula 6. If $c \geq 0$ is any nonnegative real number, verify the Laplace transform formula:

Translates of the Heaviside function

$$\mathcal{L}\{h(t-c)\}(s) = \frac{e^{-sc}}{s}, \qquad s > 0.$$

▼ *Verification.* For the Heaviside function $h(t-c)$, we have

$$\begin{aligned}\mathcal{L}\{h(t-c)\}(s) &= \int_0^\infty e^{-st} h(t-c)\,dt = \int_c^\infty e^{-st}\,dt \\ &= \lim_{r\to\infty} \left.\frac{e^{-ts}}{-s}\right|_c^r \\ &= \lim_{r\to\infty} \frac{e^{-rs}-e^{-sc}}{-s} = \frac{e^{-sc}}{s} \qquad \text{for } s>0.\end{aligned}$$ ▲

Combining Formula 6 with linearity gives the following formula.

Formula 7. If $0 \le a < b < \infty$, then

The on–off switch

$$\mathcal{L}\left\{\chi_{[a,\,b)}(t)\right\}(s) = \frac{e^{-as}}{s} - \frac{e^{-bs}}{s}, \qquad s>0.$$

Formula 6 is a special case of what is known as the second translation principle.

Theorem 8. *Suppose $f(t) \in \mathcal{H}$ is a function with Laplace transform $F(s)$. Then*

Second translation principle

$$\mathcal{L}\{f(t-c)h(t-c)\}(s) = e^{-sc}F(s).$$

In terms of the inverse Laplace transform, this is equivalent to

Inverse second translation principle

$$\mathcal{L}^{-1}\{e^{-sc}F(s)\} = f(t-c)h(t-c).$$

Proof. The calculation is straightforward and involves a simple change of variables:

$$\begin{aligned}\mathcal{L}\{f(t-c)h(t-c)\}(s) &= \int_0^\infty e^{-st} f(t-c)h(t-c)\,dt \\ &= \int_c^\infty e^{-st} f(t-c)\,dt\end{aligned}$$

$$
\begin{aligned}
&= \int_0^\infty e^{-s(t+c)} f(t)\,dt \qquad (t \mapsto t+c) \\
&= e^{-sc} \int_0^\infty e^{-st} f(t)\,dt \\
&= e^{-sc} F(s) \qquad \square
\end{aligned}
$$

As with the notation used with the first translation theorem, it is frequently convenient to express the inverse second translation theorem in the following format:

$$\mathcal{L}^{-1}\left\{e^{-sc}F(s)\right\} = h(t-c)\mathcal{L}^{-1}\left\{F(s)\right\}\big|_{t \to t-c}.$$

That is, take the inverse Laplace transform of $F(s)$, replace t by $t-c$, and then multiply by the translated Heaviside function.

In practice, it is more common to encounter expressions written in the form $g(t)h(t-c)$, rather than the nicely arranged format $f(t-c)h(t-c)$. But if $f(t)$ is replaced by $g(t+c)$ in Theorem 8, then we obtain the (apparently) more general version of the second translation principle.

Corollary 9.

$$\mathcal{L}\left\{g(t)h(t-c)\right\} = e^{-sc}\mathcal{L}\left\{g(t+c)\right\}.$$

A simple example of this occurs when $g = 1$. Then $\mathcal{L}\{h(t-c\} = e^{-sc}\mathcal{L}\{1\} = e^{-sc}/s$, which agrees with Formula 6 found above. When $c = 0$, then $\mathcal{L}\{h(t-0)\} = 1/s$ which is the same as the Laplace transform of the constant function 1. This is consistent since $h(t-0) = h(t) = 1$ for $t \geq 0$.

Now we give some examples of using these formulas.

Example 10. Find the Laplace transform of

$$f(t) = \begin{cases} t^2 & \text{if } 0 \leq t < 1, \\ 2 & \text{if } 1 \leq t < \infty \end{cases}$$

using the second translation principle. This Laplace transform was previously computed directly from the definition in Example 2.

► **Solution.** In Example 3, we found $f(t) = t^2 + (2-t^2)h(t-1)$. By Corollary 9, we get

$$
\begin{aligned}
\mathcal{L}\{f\} &= \frac{2}{s^3} + e^{-s}\mathcal{L}\left\{2-(t+1)^2\right\} \\
&= \frac{2}{s^3} + e^{-s}\mathcal{L}\left\{-t^2-2t+1\right\} \\
&= \frac{2}{s^3} + e^{-s}\left(-\frac{2}{s^3} - \frac{2}{s^2} + \frac{1}{s}\right) \qquad ◄
\end{aligned}
$$

Example 11. Find the Laplace transform of

$$f(t) = \begin{cases} \cos t & \text{if } 0 \le t < \pi, \\ 1 & \text{if } \pi \le t < 2\pi, \\ 0 & \text{if } 2\pi \le t < \infty. \end{cases}$$

► **Solution.** In Example 4, we found

$$f(t) = \cos t + (1 - \cos t)h(t - \pi) - h(t - 2\pi).$$

By Corollary 9, we get

$$\begin{aligned} F(s) &= \frac{s}{s^2+1} + e^{-s\pi}\mathcal{L}\{1 - \cos(t + \pi)\} - \frac{e^{-2s\pi}}{s} \\ &= \frac{s}{s^2+1} + e^{-s\pi}\left(\frac{1}{s} + \frac{s}{s^2+1}\right) - \frac{e^{-2s\pi}}{s}. \end{aligned}$$

In the second line, we have used the fact that $\cos(t + \pi) = -\cos t$. ◄

Uniqueness of the Inverse Laplace Transform

Theorem 8 gives a formula for the inverse Laplace transform of a function $e^{-sc}F(s)$. Such an explicit formula is suggestive of some type of uniqueness for the Laplace transform, that is, $\mathcal{L}\{f(t)\} = \mathcal{L}\{g(t)\} \implies f(t) = g(t)$, which is Theorem 1 of Sect. 2.5 for continuous functions. By expanding the domain of the Laplace transform $\mathcal{L}$ to include the possibly discontinuous functions in the Heaviside class $\mathcal{H}$, the issue of uniqueness is made somewhat more subtle because changing the value of a function at a single point will not change the integral of the function, and hence, the Laplace transform will not change. Therefore, instead of talking about *equality* of functions, we will instead consider the concept of *essential equality* of functions. Two functions $f_1(t)$ and $f_2(t)$ are said to be ***essentially equal*** on $[0, \infty)$ if for each subinterval $[0, N)$ they are equal as functions except at possibly finitely many points. For example, the functions

$$f_1(t) = \begin{cases} 1 & \text{if } 0 \le t < 1, \\ 2 & \text{if } 1 \le t < \infty, \end{cases} \qquad f_2(t) = \begin{cases} 1 & \text{if } 0 \le t < 1, \\ 3 & \text{if } t = 1, \\ 2 & \text{if } 1 < t < \infty, \end{cases} \qquad \text{and}$$

$$f_3(t) = \begin{cases} 1 & \text{if } 0 \le t \le 1, \\ 2 & \text{if } 1 < t < \infty, \end{cases}$$

are essentially equal for they are equal everywhere except at $t=1$, where $f_1(1)=2$, $f_2(1)=3$, and $f_3(1)=1$. Two functions that are essentially equal have the same Laplace transform. This is because the Laplace transform is an integral operator and integration cannot distinguish functions that are essentially equal. The Laplace transforms of $f_1(t)$, $f_2(t)$, and $f_3(t)$ in our example above are all $(1+\mathrm{e}^{-s})/s$. Here is our problem: Given a transform, like $(1+\mathrm{e}^{-s})/s$, how do we decide what "the" inverse Laplace transform is. It turns out that if $F(s)$ is the Laplace transform of two functions $f_1(t), f_2(t) \in \mathcal{H}$, then $f_1(t)$ and $f_2(t)$ are essentially equal. Since for most practical situations it does not matter which one is chosen, we will make a choice by always choosing the function that is right continuous at each point. A function $f(t)$ in the Heaviside class is said to be ***right continuous at a point*** a if we have

$$f(a) = f(a^+) = \lim_{t\to a^+} f(t),$$

and it is ***right continuous on*** $[0,\infty)$ if it is right continuous at each point in $[0,\infty)$. In the example above, $f_1(t)$ is right continuous while $f_2(t)$ and $f_3(t)$ are not. The function $f_3(t)$ is, however, left continuous, using the obvious definition of left continuity. If we decide to use right continuous functions in the Heaviside class, then the correspondence with its Laplace transform is one-to-one. We summarize this discussion as a theorem:

Theorem 12. *If $F(s)$ is the Laplace transform of a function in $\mathcal{H}$, then there is a unique right continuous function $f(t) \in \mathcal{H}$ such that $\mathcal{L}\{f(t)\} = F(s)$. Any two functions in $\mathcal{H}$ with the same Laplace transform are essentially equal.*

All the translates $h(t-c)$ of the Heaviside function $h(t)$ are right continuous, so any piecewise function written as a sum of products of a continuous function and a translated Heaviside function are right continuous. In fact, *the convention of using right continuous functions just means that the inverse transforms of functions in $\mathcal{H}$ will be written as sums of $f(t-c)h(t-c)$, as given in the second translation principle Theorem 8.*

Example 13. Find the inverse Laplace transform of

$$F(s) = \frac{\mathrm{e}^{-s}}{s^2} + \frac{\mathrm{e}^{-3s}}{s-4}$$

and write it as a right continuous piecewise-defined function.

► **Solution.** The inverse Laplace transforms of $1/s^2$ and $1/(s-4)$ are, respectively, t and e^{4t}. By Theorem 8, the inverse Laplace transform of $F(s)$ is

$$\begin{aligned}\mathcal{L}^{-1}\{F(s)\} &= h(t-1)\,\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}\bigg|_{t\to t-1} + h(t-3)\,\mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\}\bigg|_{t\to t-3} \\ &= h(t-1)\,(t)|_{t\to t-1} + h(t-3)\,(\mathrm{e}^{4t})\big|_{t\to t-3} \\ &= (t-1)h(t-1) + \mathrm{e}^{4(t-3)}h(t-3).\end{aligned}$$

On the interval $[0, 1)$, both $t - 1$ and $e^{4(t-3)}$ are off. On the interval $[1, 3)$ only $t - 1$ is on. On the interval $[3, \infty)$, both $t - 1$ and $e^{4(t-3)}$ are on. Thus,

$$\mathcal{L}^{-1}\{F(s)\} = \begin{cases} 0 & \text{if } 0 \le t < 1 \\ t - 1 & \text{if } 1 \le t < 3 \\ t - 1 + e^{4(t-3)} & \text{if } 3 \le t < \infty \end{cases} . \quad ◄$$

Exercises

1–8. Graph each of the following functions defined by means of the unit step function $h(t-c)$ and/or the on–off switches $\chi_{[a,\,b)}$.

1. $f(t) = 3h(t-2) - h(t-5)$
2. $f(t) = 2h(t-2) - 3h(t-3) + 4h(t-4)$
3. $f(t) = (t-1)h(t-1)$
4. $f(t) = (t-2)^2 h(t-2)$
5. $f(t) = t^2 h(t-2)$
6. $f(t) = h(t-\pi)\sin t$
7. $f(t) = h(t-\pi)\cos 2(t-\pi)$
8. $f(t) = t^2\chi_{[0,\,1)}(t) + (2-t)\chi_{[1,\,3)}(t) + 3\chi_{[3,\,\infty)}(t)$

9–27. For each of the following functions $f(t)$, (a) express $f(t)$ in terms of on–off switches $\chi_{[a,\,b)}(t)$, (b) express $f(t)$ in terms of translates $h(t-c)$ of the Heaviside function $h(t)$, and (c) compute the Laplace transform $F(s) = \mathcal{L}\{f(t)\}$.

9. $f(t) = \begin{cases} 0 & \text{if } 0 \le t < 2 \\ t-2 & \text{if } 2 \le t < \infty \end{cases}$

10. $f(t) = \begin{cases} 0 & \text{if } 0 \le t < 2 \\ t & \text{if } 2 \le t < \infty \end{cases}$

11. $f(t) = \begin{cases} 0 & \text{if } 0 \le t < 2 \\ t+2 & \text{if } 2 \le t < \infty \end{cases}$

12. $f(t) = \begin{cases} 0 & \text{if } 0 \le t < 4 \\ (t-4)^2 & \text{if } 4 \le t < \infty \end{cases}$

13. $f(t) = \begin{cases} 0 & \text{if } 0 \le t < 4 \\ t^2 & \text{if } 4 \le t < \infty \end{cases}$

14. $f(t) = \begin{cases} 0 & \text{if } 0 \le t < 4 \\ t^2-4 & \text{if } 4 \le t < \infty \end{cases}$

15. $f(t) = \begin{cases} 0 & \text{if } 0 \le t < 2 \\ (t-4)^2 & \text{if } 2 \le t < \infty \end{cases}$

16. $f(t) = \begin{cases} 0 & \text{if } 0 \le t < 4 \\ e^{t-4} & \text{if } 4 \le t < \infty \end{cases}$

17. $f(t) = \begin{cases} 0 & \text{if } 0 \le t < 4 \\ e^{t} & \text{if } 4 \le t < \infty \end{cases}$

18. $f(t) = \begin{cases} 0 & \text{if } 0 \le t < 6 \\ e^{t-4} & \text{if } 6 \le t < \infty \end{cases}$

19. $f(t) = \begin{cases} 0 & \text{if } 0 \le t < 4 \\ te^{t} & \text{if } 4 \le t < \infty \end{cases}$

20. $f(t) = \begin{cases} 1 & \text{if } 0 \le t < 4 \\ -1 & \text{if } 4 \le t < 5 \\ 0 & \text{if } 5 \le t < \infty \end{cases}$

21. $f(t) = \begin{cases} t & \text{if } 0 \le t < 1 \\ 2-t & \text{if } 1 \le t < \infty \end{cases}$

22. $f(t) = \begin{cases} t & \text{if } 0 \le t < 1 \\ 2-t & \text{if } 1 \le t < 2 \\ 1 & \text{if } 2 \le t < \infty \end{cases}$

23. $f(t) = \begin{cases} t^2 & \text{if } 0 \le t < 2 \\ 4 & \text{if } 2 \le t < 3 \\ 7-t & \text{if } 3 \le t < \infty \end{cases}$

24. $f(t) = \begin{cases} 1 & \text{if } 0 \le t < 2 \\ 3-t & \text{if } 2 \le t < 3 \\ 2(t-3) & \text{if } 3 \le t < 4 \\ 2 & \text{if } 4 \le t < \infty \end{cases}$

25. $f(t) = \begin{cases} t & \text{if } 0 \le t < 1 \\ t-1 & \text{if } 1 \le t < 2 \\ t-2 & \text{if } 2 \le t < 3 \\ \vdots & \end{cases}$

26. $f(t) = \begin{cases} 1 & \text{if } 2n \le t < 2n+1 \\ 0 & \text{if } 2n+1 \le t < 2n+2 \end{cases}$

27. $f(t) = \begin{cases} 1-t & \text{if } 0 \le t < 2 \\ 3-t & \text{if } 2 \le t < 4 \\ 5-t & \text{if } 4 \le t < 6 \\ \vdots & \end{cases}$

28–42. Compute the inverse Laplace transform of each of the following functions.

28. $\dfrac{e^{-3s}}{s-1}$

29. $\dfrac{e^{-3s}}{s^2}$

30. $\dfrac{e^{-3s}}{(s-1)^3}$

31. $\dfrac{e^{-\pi s}}{s^2+1}$

32. $\dfrac{s e^{-3\pi s}}{s^2+1}$

33. $\dfrac{e^{-\pi s}}{s^2+2s+5}$

34. $\dfrac{e^{-s}}{s^2} + \dfrac{e^{-2s}}{(s-1)^3}$

35. $\dfrac{e^{-2s}}{s^2+4}$

36. $\dfrac{e^{-2s}}{s^2-4}$

37. $\dfrac{s e^{-4s}}{s^2+3s+2}$

38. $\dfrac{e^{-2s}+e^{-3s}}{s^2-3s+2}$

39. $\dfrac{1 - e^{-5s}}{s^2}$

40. $\dfrac{1 + e^{-3s}}{s^4}$

41. $e^{-\pi s} \dfrac{2s + 1}{s^2 + 6s + 13}$

42. $(1 - e^{-\pi s}) \dfrac{2s + 1}{s^2 + 6s + 13}$

43–44. These exercises show that the Heaviside class $\mathcal{H}$ is a linear space.

43. Suppose f_1 and f_2 are piecewise continuous on $[0, \infty)$ and $c \in \mathbb{R}$. Show $f_1 + cf_2$ is piecewise continuous on $[0, \infty)$.

44. Suppose $f_1, f_2 \in \mathcal{H}$ and $c \in \mathbb{R}$. Show that $f_1 + cf_2 \in \mathcal{H}$ and hence, $\mathcal{H}$ is a linear space.

6.3 Laplace Transform Method for $f(t) \in \mathcal{H}$

The differential equations that we will solve by means of the Laplace transform are first and second order constant coefficient linear differential equations with a forcing function $f(t) \in \mathcal{H}$:

$$y' + ay = f(t),$$
$$y'' + ay' + by = f(t).$$

To employ this method, it will be necessary to compute the Laplace transform of derivatives of functions in $\mathcal{H}$. That is, we need to know the extent to which the input derivative formula

$$\mathcal{L}\left\{f'(t)\right\} = s\mathcal{L}\left\{f(t)\right\} - f(0)$$

is valid when $f(t)$ is in the Heaviside class $\mathcal{H}$. Recall that for $f(t) \in \mathcal{H}$, the symbol $f'(t)$ is used to denote the derivative of $f(t)$ if $f(t)$ is differentiable except possibly at a finite number of points on each interval of the form $[0, N]$. Thus, $f'(t)$ will be (possibly) undefined for finitely many points in $[0, N]$.

Example 1. Verify that the input derivative formula is not valid for the on–off switch function

$$f(t) = \chi_{[0,1)}(t) = \begin{cases} 1 & \text{if } 0 \le t < 1, \\ 0 & \text{if } t \ge 1. \end{cases}$$

▼ *Verification.* From the definition, it is clear that $f'(t) = 0$ for all $t \neq 1$; it is not differentiable, or even continuous, at $t = 1$. Thus, $\mathcal{L}\left\{f'(t)\right\} = 0$. However, by Formula 7 of Sect. 6.2, $\mathcal{L}\left\{f(t)\right\} = F(s) = (1 - e^{-s})/s$ so that

$$sF(s) - f(0) = 1 - e^{-s} - 1 = e^{-s} \neq 0 = \mathcal{L}\left\{f'(t)\right\}. \quad ▲$$

Example 2. Verify that the input derivative formula is valid for the following function

$$f(t) = \begin{cases} 1 & \text{if } 0 \le t < 1, \\ t & \text{if } t \ge 1. \end{cases}$$

See Fig. 6.11

▼ *Verification.* Write $f(t)$ in the standard form using translates of the Heaviside function $h(t)$ to get

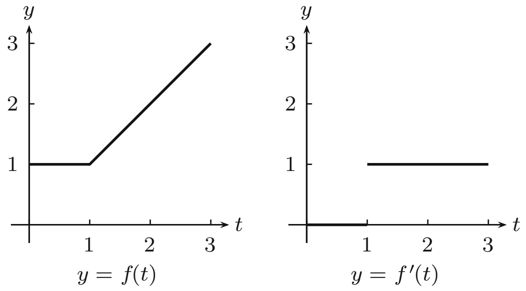
$$f(t) = \chi_{[0,1)}(t) + th(t-1) = 1 + (t-1)h(t-1).$$

Thus,

$$sF(s) - f(0) = s\left(\frac{1}{s} + \frac{e^{-s}}{s^2}\right) - 1 = \frac{e^{-s}}{s} = \mathcal{L}\left\{h(t-1)\right\} = \mathcal{L}\left\{f'(t)\right\},$$

since $f'(t) = h(t-1)$. Therefore, the input derivative principle is satisfied for this function $f(t)$. ▲

Fig. 6.11 A continuous $f(t)$ with discontinuous derivative



Thus, we have an example of a piecewise continuous function (Example 1) for which the input derivative principle fails, and another example (Example 2) for which the input derivative principle holds. What is the significant difference between these two simple examples? For one thing, the second example has a *continuous* $f(t)$, while that of the first example is *discontinuous*. It turns out that this is the feature that needs to be included. That is, the derivative need not be continuous, but the function must.

Theorem 3. *Suppose $f(t)$ is a continuous function on $[0,\infty)$ such that both $f(t)$ and $f'(t)$ are in $\mathcal{H}$. Then*

The input derivative principle

The first derivative

$$\mathcal{L}\{f'(t)\}(s) = s\mathcal{L}\{f(t)\}(s) - f(0).$$

Proof. We begin by computing $\int_0^N e^{-st} f'(t)\,dt$. This computation requires a careful analysis of the points where $f'(t)$ is discontinuous. There are only finitely many such discontinuities on the interval $[0, N)$, which will be labeled $a_1, \ldots, a_k$, and we may assume $a_i < a_{i+1}$. If we let $a_0 = 0$ and $a_{k+1} = N$, then we obtain

$$\int_0^N e^{-st} f'(t)\,dt = \sum_{i=0}^{k} \int_{a_i}^{a_{i+1}} e^{-st} f'(t)\,dt,$$

and integration by parts gives

$$\int_0^N e^{-st} f'(t)\,dt = \sum_{i=0}^{k} \left(f(t)e^{-st}\Big|_{a_i}^{a_{i+1}} + s\int_{a_i}^{a_{i+1}} e^{-st} f(t)\,dt \right)$$

$$= \sum_{i=0}^{k} (f(a_{i+1}^-)e^{-sa_{i+1}} - f(a_i^+)e^{-sa_i}) + s\int_0^N e^{-st} f(t)\,dt$$

$$= \sum_{i=0}^{k} (f(a_{i+1})e^{-sa_{i+1}} - f(a_i)e^{-sa_i}) + s \int_0^N e^{-st} f(t)\, dt$$

$$= f(N)e^{-Ns} - f(0) + s \int_0^N e^{-st} f(t)\, dt.$$

Since $f(t)$ is continuous, we have $f(a_i^+) = f(a_i)$ and $f(a_{i+1}^-) = f(a_{i+1})$ which allows us to go from the second to the third line. The telescoping nature of the sum in line 3 allows it to collapse to $f(a_{k+1})e^{-sa_{k+1}} - f(a_0)e^{-sa_0} = f(N)e^{-Ns} - f(0)$ which produces the final line. Now take the limit as N goes to infinity and the result follows. □

If $f(t) \in \mathcal{H}$, then the definite integral $g(t) = \int_0^t f(u)\, du$ is continuous, in the Heaviside class, and moreover, $g(0) = 0$. Thus, applying the input derivative principle to the function $g(t) \in \mathcal{H}$ gives the input integral principle:

Corollary 4. *Suppose $f(t)$ is a function defined on $[0, \infty)$ such $f(t) \in \mathcal{H}$, and $F(s) = \mathcal{L}\{f(t)\}(s)$, then*

The input integral principle

$$\mathcal{L}\left\{\int_0^t f(u)\, du\right\}(s) = \frac{F(s)}{s}.$$

The second order input derivative principle is an immediate corollary of the first, as long as we are careful to identify the appropriate hypotheses that $f(t)$ must satisfy.

Corollary 5. *If $f(t)$ and $f'(t)$ are continuous and $f(t)$, $f'(t)$, and $f''(t)$ are in $\mathcal{H}$, then*

Input derivative principle

The second derivative

$$\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0).$$

We are now in a position to illustrate the Laplace transform method for solving first and second order constant coefficient differential equations

$$y' + ay = f(t),$$
$$y'' + ay' + by = f(t)$$

with a forcing function $f(t)$ that is possibly discontinuous. In order to apply the Laplace transform method, we will need to take the Laplace transform of a potential solution $y(t)$. To apply the input derivative formula to $y(t)$, it is necessary to know that $y(t)$ is continuous and $y'(t) \in \mathcal{H}$ for the first order equation and both $y(t)$ and $y'(t)$ are continuous while $y''(t) \in \mathcal{H}$ for the second order equation. These facts were proved in Theorems 5 and 7 of Sect. 6.1, and hence, these theorems provide the theoretical underpinnings for applying the Laplace transform method formally. Here are some examples.

Example 6. Solve the following first order differential equation:

$$y' + 2y = f(t), \qquad y(0) = 1,$$

where

$$f(t) = \begin{cases} 0 & \text{if } 0 \le t < 1, \\ t & \text{if } 1 \le t < \infty. \end{cases}$$

► **Solution.** We first rewrite $f(t)$ in terms of Heaviside functions:

$$f(t) = t\,\chi_{[1,\infty)}(t) = t\,h(t-1).$$

By Corollary 9 of Sect. 6.2 of the second translation principle, its Laplace transform is

$$F(s) = \mathcal{L}\{th(t-1)\} = \mathrm{e}^{-s}\mathcal{L}\{t+1\} = \mathrm{e}^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right) = \mathrm{e}^{-s}\left(\frac{s+1}{s^2}\right).$$

Let $Y(s) = \mathcal{L}\{y(t)\}$ where $y(t)$ is the solution to the differential equation. Since the analysis done above shows that $y(t)$ satisfies the hypotheses of the input derivative principle, we can apply the Laplace transform to the differential equation and conclude

$$sY(s) - y(0) + 2Y(s) = \mathrm{e}^{-s}\left(\frac{s+1}{s^2}\right).$$

Solving for $Y(s)$ gives

$$Y(s) = \frac{1}{s+2} + \mathrm{e}^{-s}\frac{s+1}{s^2(s+2)}.$$

A partial fraction decomposition gives

$$\frac{s+1}{s^2(s+2)} = \frac{1}{4}\,\frac{1}{s} + \frac{1}{2}\,\frac{1}{s^2} - \frac{1}{4}\,\frac{1}{s+2},$$

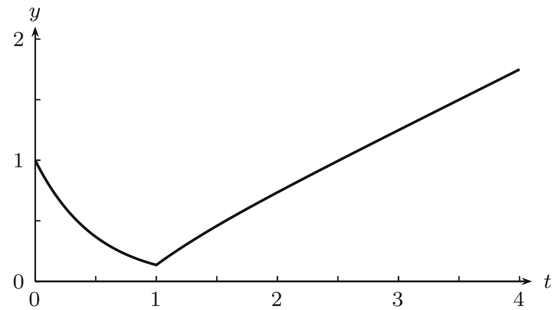


Fig. 6.12 The graph of the solution to Example 6

and the second translation principle (Theorem 8 of Sect. 6.2) gives

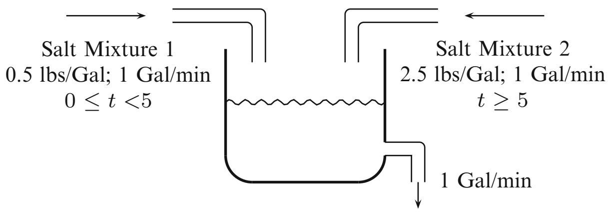
$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + \frac{1}{4}\mathcal{L}^{-1}\left\{e^{-s}\frac{1}{s}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{e^{-s}\frac{1}{s^2}\right\} - \frac{1}{4}\mathcal{L}^{-1}\left\{e^{-s}\frac{1}{s+2}\right\} \\ &= e^{-2t} + \frac{1}{4}h(t-1) + \frac{1}{2}(t-1)h(t-1) - \frac{1}{4}e^{-2(t-1)}h(t-1). \\ &= \begin{cases} e^{-2t} & \text{if } 0 \le t < 1 \\ e^{-2t} + \frac{1}{4}(2t-1) - \frac{1}{4}e^{-2(t-1)} & \text{if } 1 \le t < \infty. \end{cases} \end{aligned}$$

The graph of $y(t)$ is shown in Fig. 6.12, where the discontinuity of the forcing function $f(t)$ at time $t = 1$ is reflected in the abrupt change in the direction of the tangent line of the graph of $y(t)$ at $t = 1$. ◀

We now consider a mixing problem of the type mentioned in the introduction to this chapter.

Example 7. Suppose a tank holds 10 gal of pure water. There are two input sources of brine solution: the first source has a concentration of 2 lbs of salt per gallon while the second source has a concentration of 3 lbs of salt per gallon. The first source flows into the tank at a rate of 1 gal/min for 5 min after which it is turned off and simultaneously the second source is turned on at a rate of 1 gal/min. The well-mixed solution flows out of the tank at a rate of 1 gal/min. Find the amount of salt in the tank at any time t.

► **Solution.** A pictorial representation of the problem is the following diagram.



Letting $y(t)$ denote the amount of salt in the tank at time t, measured in pounds, apply the fundamental balance principle for mixing problems (see Example 11 of Sect. 1.4). This principle states that the rate of change of $y(t)$ comes from the difference between the rate salt is being added and the rate salt is being removed. Symbolically,

$$y'(t) = \text{rate in} - \text{rate out}.$$

Recall that the input and output rates of salt are the product of the concentration of salt and the flow rates of the mixtures. The rate at which salt is being added depends on the interval of time. For the first five minutes, source one adds salt at a rate of 0.5 lbs/min, and after that, source two takes over and adds salt at a rate of 2.5 lbs/min. Since the flow rate in is 1 gal/min, the rate at which salt is being added is given by the function

$$f(t) = \begin{cases} 0.5 & \text{if } 0 \le t < 5, \\ 2.5 & \text{if } 5 \le t < \infty. \end{cases}$$

The concentration of salt at time t is $y(t)/10$ lbs/gal, and since the flow rate out is 1 gal/min, it follows that the rate at which salt is being removed is $y(t)/10$ lbs/min. Since initially there is pure water, it follows that $y(0) = 0$, and therefore, we have that $y(t)$ satisfies the following initial value problem:

$$y' = f(t) - \frac{y(t)}{10}, \qquad y(0) = 0.$$

Rewriting $f(t)$ in terms of translates of the Heaviside function gives

$$\begin{aligned} f(t) &= 0.5\chi_{[0,5)}(t) + 2.5\chi_{[5,\infty)}(t) \\ &= 0.5(h(t) - h(t-5)) + 2.5h(t-5) \\ &= 0.5 + 2h(t-5). \end{aligned}$$

Applying the Laplace transform to the differential equation and solving for $Y(s) = \mathcal{L}\{y(t)\}(s)$ gives

$$\begin{aligned} Y(s) &= \left(\frac{1}{s+\frac{1}{10}}\right)\left(\frac{0.5 + 2e^{-5s}}{s}\right) \\ &= \frac{0.5}{\left(s+\frac{1}{10}\right)s} + e^{-5s}\frac{2}{\left(s+\frac{1}{10}\right)s} \\ &= \frac{5}{s} - \frac{5}{s+\frac{1}{10}} + e^{-5s}\frac{20}{s} - e^{-5s}\frac{20}{s+\frac{1}{10}}. \end{aligned}$$

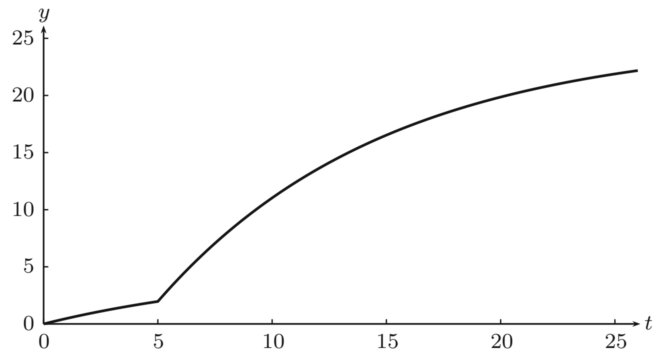


Fig. 6.13 The solution to a mixing problem with discontinuous input function

Taking the inverse Laplace transform of $Y(s)$ gives

$$\begin{aligned} y(t) &= 5 - 5e^{-\frac{t}{10}} + 20h(t-5) - 20e^{-\frac{t-5}{10}}h(t-5) \\ &= \begin{cases} 5 - 5e^{-\frac{t}{10}} & \text{if } 0 \le t < 5, \\ 25 - 5e^{-\frac{t}{10}} - 20e^{-\frac{t-5}{10}} & \text{if } 5 \le t < \infty. \end{cases} \end{aligned}$$

The graph of y is given in Fig. 6.13. As expected, we observe that the solution is continuous, but the kink at $t = 5$ indicates that there is a discontinuity of the derivative at this point. This occurred when the flow of the second source, which had a higher concentration of salt, was turned on. ◀

Here is an example of a second order equation.

Example 8. Solve the following second order initial value problem

$$y'' + 4y = f(t), \qquad y(0) = 0, \; y'(0) = 1$$

where

$$f(t) = \begin{cases} 1 & \text{if } 0 \le t < \pi, \\ \sin t & \text{if } t \ge \pi. \end{cases}$$

► **Solution.** First note that

$$\begin{aligned} f(t) &= \chi_{[0,\pi)}(t) + (\sin t)\chi_{[\pi,\infty)}(t) \\ &= 1 - h(t-\pi) + (\sin t)h(t-\pi), \end{aligned}$$

so that

$$\begin{aligned} F(s) &= \mathcal{L}\{f(t)\} \\ &= \frac{1}{s} - \frac{e^{-\pi s}}{s} + e^{-\pi s}\mathcal{L}\{\sin(t+\pi)\} \\ &= \frac{1}{s} - \frac{e^{-\pi s}}{s} + e^{-\pi s}\mathcal{L}\{-\sin t\} \\ &= \frac{1}{s} - \frac{e^{-\pi s}}{s} + e^{-\pi s}\frac{1}{s^2+1}. \end{aligned}$$

Letting $Y(s) = \mathcal{L}\{y(t)\}(s)$, where $y(t)$ is the solution to the initial value problem, and taking the Laplace transform of the differential equation gives

$$s^2Y(s) - 1 + 4Y(s) = \frac{1}{s} - \frac{e^{-\pi s}}{s} + e^{-\pi s}\frac{1}{s^2+1}.$$

Now solve for $Y(s)$ to get

$$Y(s) = \frac{1}{s^2+4} + \frac{1}{s(s^2+4)}(1 - e^{-\pi s}) + e^{-\pi s}\frac{1}{(s^2+1)(s^2+4)}.$$

The solution is completed by taking the inverse Laplace transform, using the second translation theorem:

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\ &= \frac{1}{2}\sin 2t + \mathcal{L}^{-1}\left\{\frac{1}{s(s^2+4)}\right\} - h(t-\pi)\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+4)}\right\}(t-\pi) \\ &\quad + h(t-\pi)\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s^2+4)}\right\}(t-\pi). \end{aligned}$$

Computing the partial fractions in the usual manner

$$\frac{1}{s(s^2+4)} = \frac{\frac{1}{4}}{s} - \frac{\frac{1}{4}s}{s^2+4} \quad \text{and}$$

$$\frac{1}{(s^2+1)(s^2+4)} = \frac{\frac{1}{3}}{s^2+1} - \frac{\frac{1}{3}}{s^2+4}$$

and substituting these into the inverse Laplace transforms gives

$$\begin{aligned} y(t) = \frac{1}{2}\sin 2t &+ \frac{1}{4} - \frac{1}{4}\cos 2t - \frac{1}{4}h(t-\pi)\left(1 - \cos 2(t-\pi)\right) \\ &+ \frac{1}{3}h(t-\pi)\left(\sin(t-\pi) - \frac{1}{2}\sin 2(t-\pi)\right). \end{aligned}$$

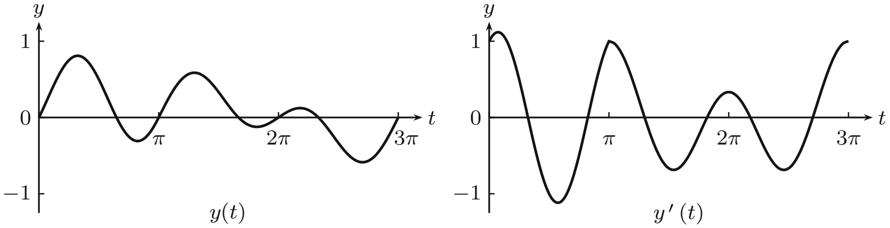


Fig. 6.14 The graph of the solution $y(t)$ to Example 8 is shown along with the graph of the derivative $y'(t)$. Note that the derivative $y'(t)$ is continuous, but it is not differentiable at $t = \pi$, signifying that the second derivative $y''(t)$ is not continuous at $t = \pi$

Evaluating this piecewise gives

$$y(t) = \begin{cases} \frac{1}{2}\sin 2t + \frac{1}{4} - \frac{1}{4}\cos 2t & \text{if } 0 \le t < \pi, \\ -\frac{1}{3}\sin t + \frac{1}{3}\sin 2t & \text{if } t \ge \pi. \end{cases}$$

The graph of the solution $y(t)$ and its derivative $y'(t)$ are given in Fig. 6.14. ◀

Exercises

1–12. Solve each of the following initial value problems.

1. $y' + 2y = \begin{cases} 0 & \text{if } 0 \le t < 1 \\ -3 & \text{if } t \ge 1 \end{cases}, \quad y(0) = 0$

2. $y' + 5y = \begin{cases} -5 & \text{if } 0 \le t < 1 \\ 5 & \text{if } t \ge 1 \end{cases}, \quad y(0) = 1$

3. $y' - 3y = \begin{cases} 0 & \text{if } 0 \le t < 2 \\ 2 & \text{if } 2 \le t < 3 \\ 0 & \text{if } t \ge 3 \end{cases}, \quad y(0) = 0$

4. $y' + 2y = \begin{cases} t & \text{if } 0 \le t < 1 \\ 0 & \text{if } 1 \le t < \infty \end{cases}, \quad y(0) = 0$

5. $y' - 4y = \begin{cases} 12e^t & \text{if } 0 \le t < 1 \\ 12e & \text{if } 1 \le t < \infty \end{cases}, \quad y(0) = 2$

6. $y' + 3y = \begin{cases} 10 \sin t & \text{if } 0 \le t < \pi \\ 0 & \text{if } \pi \le t < \infty \end{cases}, \quad y(0) = -1$

7. $y'' + 9y = h(t-3), \quad y(0) = 0, y'(0) = 0$

8. $y'' - 5y' + 4y = \begin{cases} 1 & \text{if } 0 \le t < 5 \\ 0 & \text{if } t \ge 5 \end{cases}, \quad y(0) = 0, \ y'(0) = 1$

9. $y'' + 5y' + 6y = \begin{cases} 0 & \text{if } 0 \le t < 1 \\ 6 & \text{if } 1 \le t < 3 \\ 0 & \text{if } t \ge 3 \end{cases}, \quad y(0) = 0, \ y'(0) = 0$

10. $y'' + 9y = h(t - 2\pi) \sin t, \quad y(0) = 1, y'(0) = 0$

11. $y'' + 2y' + y = h(t-3), \quad y(0) = 0, y'(0) = 1$

12. $y'' + 2y' + y = \begin{cases} e^{-t} & \text{if } 0 \le t < 4 \\ 0 & \text{if } 4 \le t < \infty \end{cases}, \quad y(0) = 0, y'(0) = 0$

13–15. *Mixing Problems*

13. Suppose a tank holds 4 gal of pure water. There are two input sources of brine solution: the first source has a concentration of 1 lb of salt per gallon while the second source has a concentration of 5 lbs of salt per gallon. The first source flows into the tank at a rate of 2 gal/min for 3 min after which it is turned off, and simultaneously, the second source is turned on at a rate of 2 gal/min. The well-mixed solution flows out of the tank at a rate of 2 gal/min. Find the amount of salt in the tank at any time t.
14. Suppose a tank holds a brine solution consisting of 1 kg salt dissolved in 4 L of water. There are two input sources of brine solution: the first source has a concentration of 2 kg of salt per liter while the second source has a concentration of 3 kg of salt per liter. The first source flows into the tank at a rate of 4 L/min for 5 min after which it is turned off, and simultaneously, the second source is turned on at a rate of 4 L/min. The well-mixed solution flows out of the tank at a rate of 4 L/min. Find the amount of salt in the tank at any time t.
15. Suppose a tank holds a brine solution consisting of 2 kg salt dissolved in 10 L of water. There are two input sources of brine solution: the first source has a concentration of 1 kg of salt per liter while the second source is pure water. The first source flows into the tank at a rate of 3 L/min for 2 min. Thereafter, it is turned off and simultaneously the second source is turned on at a rate of 3 L/min for 2 min. Thereafter, it is turned off and simultaneously the first source is turned back on at a rate of 3 L/min and remains on. The well-mixed solution flows out of the tank at a rate of 3 L/min. Find the amount of salt in the tank at any time t.
16. Suppose $a \neq 0$. Show that the solution to

$$y' + ay = A\chi_{[\alpha,\beta)}, \;\; y(0) = y_0$$

is

$$y(t) = y_0 e^{-at} + \frac{A}{a}\begin{cases} 0 & \text{if } 0 \le t < \alpha, \\ 1 - e^{-a(t-\alpha)} & \text{if } \alpha \le t < \beta, \\ e^{-a(t-\beta)} - e^{-a(t-\alpha)} & \text{if } \beta \le t < \infty. \end{cases}$$

6.4 The Dirac Delta Function

In applications, we may encounter an input into a system we wish to study that is very large in magnitude, but applied over a short period of time. Consider, for example, the following mixing problem:

Example 1. A tank holds 10 gal of a brine solution in which each gallon contains 2 lbs of dissolved salt. An input source begins pouring fresh water into the tank at a rate of 1 gal/min and the thoroughly mixed solution flows out of the tank at the same rate. After 5 min, 3 lbs of salt are poured into the tank where it instantly mixes into the solution. Find the amount of salt in the tank at any time t.

This example introduces a sudden action, namely, the sudden input of 3 lbs of salt at time $t = 5$ min. If we imagine that it actually takes 1 second to do this, then the average rate of input of salt would be $3\,\text{lbs/s} = 180\,\text{lbs/min}$. Thus, we see a high magnitude in the rate of input of salt over a short interval. Moreover, the rate multiplied by the duration of input gives the total input.

More generally, if $r(t)$ represents the rate of input over a time interval $[a, b]$, then $\int_a^b r(t)\,dt$ would represent the total input. A unit input means that this integral is 1. Let $t = c \geq 0$ be fixed and let ϵ be a small positive number. Imagine a constant input rate over the interval $[c, c+\epsilon)$ and 0 elsewhere. The function $d_{c,\epsilon} = \frac{1}{\epsilon}\chi_{[c,c+\epsilon)}$ represents such an input rate with constant input $1/\epsilon$ over the interval $[c, c+\epsilon)$ (cf. Sect. 6.2 where the on–off switch $\chi_{[a,b)}$ is discussed). The constant $1/\epsilon$ is chosen so that the total input is

$$\int_0^\infty d_{c,\epsilon}\,dt = \frac{1}{\epsilon}\int_c^{c+\epsilon} 1\,dt = \frac{1}{\epsilon}\epsilon = 1.$$

For example, if $\epsilon = \frac{1}{60}$ min, then $3d_{5,\epsilon}$ would represent the input of 3 lbs of salt over a 1-s interval beginning at $t = 5$.

Figure 6.15 shows the graphs of $d_{c,\epsilon}$ for $\epsilon = 2,\ 0.5,$ and 0.1. The area of the region under each line segment is 1. The main idea will be to take smaller and smaller values of ϵ, that is, we want to imagine the total input being concentrated at the point c. We would like to define the ***Dirac delta function*** by $\delta_c(t) = \lim_{\epsilon\to 0^+} d_{c,\epsilon}(t)$. However, the pointwise limit would give

$$\delta_c(t) = \begin{cases} \infty & \text{if } t = c, \\ 0 & \text{elsewhere.} \end{cases}$$

In addition, we would like to have the property that

$$\int_0^\infty \delta_c(t)\,dt = \lim_{\epsilon\to 0}\int_0^\infty d_{c,\epsilon}\,dt = 1.$$

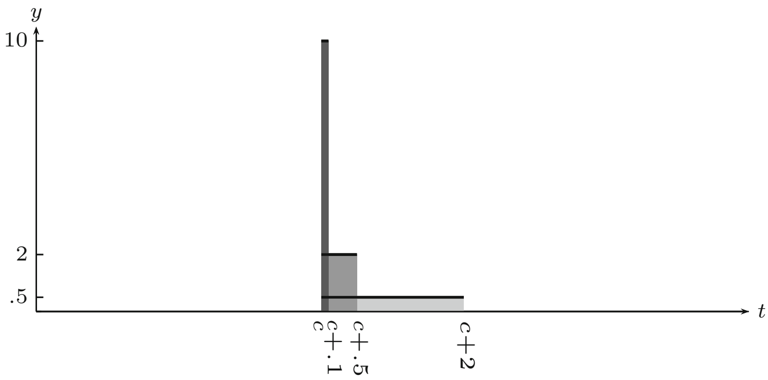


Fig. 6.15 Approximation to a delta function

Of course, there is really no such function with this property. (Mathematically, we can make precise sense out of this idea by extending the Heaviside class to a class that includes ***distributions*** or ***generalized functions***. We will not pursue distributions here as it will take us far beyond the introductory nature of this text.) Nevertheless, this is the idea we want to develop. We will consider first order constant coefficient differential equations of the form

$$y' + ay = f(t),$$

where f involves the Dirac delta function δ_c. It turns out that the main problem lies in the fact that the solution is *not* continuous, so Theorem 3 of Sect. 6.3 does not apply. Nevertheless, we will justify that we can apply the usual Laplace transform method in a formal way to produce the desired solutions. The beauty of doing this is found in the ease in which we can work with the "Laplace transform" of δ_c.

We define the Laplace transform of δ_c by the formula:

$$\mathcal{L}\{\delta_c\} = \lim_{\epsilon \to 0} \mathcal{L}\{d_{c,\epsilon}\}.$$

Theorem 2. *The Laplace transform of* δ_c *is*

$$\mathcal{L}\{\delta_c\} = e^{-cs}.$$

Proof. We begin with $d_{c,\epsilon}$:

$$\begin{aligned} \mathcal{L}\{d_{c,\epsilon}\} &= \frac{1}{\epsilon}\mathcal{L}\{h(t-c) - h(t-c-\epsilon)\} \\ &= \frac{1}{\epsilon}\left(\frac{e^{-cs} - e^{-(c+\epsilon)s}}{s}\right) \\ &= \frac{e^{-cs}}{s}\left(\frac{1 - e^{-\epsilon s}}{\epsilon}\right). \end{aligned}$$

We now take limits as ϵ goes to 0 and use L'Hospital's rule to obtain:

$$\mathcal{L}\{\delta_c\} = \lim_{\epsilon\to 0} \mathcal{L}\{d_{c,\epsilon}\} = \frac{e^{-cs}}{s}\left(\lim_{\epsilon\to 0}\frac{1-e^{-\epsilon s}}{\epsilon}\right) = \frac{e^{-cs}}{s}\cdot s = e^{-cs}. \qquad \square$$

We remark that when $c = 0$ we have $\mathcal{L}\{\delta_0\} = 1$. By Proposition 1 of Sect. 6.2, there is no Heaviside function with this property. Thus, to reiterate, even though $\mathcal{L}\{\delta_c\}$ is a function, δ_c is *not*. We will frequently write $\delta_c(t) = \delta(t-c)$ and $\delta = \delta_0$.

The Dirac delta function allows us to model the mixing problem from Example 1 as a first order linear differential equation.

► **Solution.** *Setting up the differential equation in Example 1*: Let $y(t)$ be the amount of salt in the tank at time t. Then $y(0) = 20$ and y' is the difference of the input rate and the output rate. The only input of salt occurs at $t = 5$. If the salt were input over a small interval, $[5, 5+\epsilon)$ say, then $\frac{3}{\epsilon}\chi_{[5,5+\epsilon)}$ would represent the input of 3 lbs of salt over a period of ϵ minutes. If we let ϵ go to zero, then $3\delta_5$ would represent the input rate. The output rate is $y(t)/10$. We are thus led to the differential equation:

$$y' + \frac{y}{10} = 3\delta_5, \qquad y(0) = 20.$$

The solution to this differential equation will continue below and fall out of the slightly more general discussion we now give. ◄

Differential Equations of the Form $y' + ay = k\delta_c$

We will present progressively three procedures for solving

$$y' + ay = k\delta_c, \qquad y(0) = y_0. \tag{1}$$

The last one, the *formal Laplace transform method*, is the simplest and is, in part, justified by the methods that precede it. The formal method will thereafter be used to solve (1) and will be extended to second order equations.

Limiting Procedure

In our first approach, we solve the equation

$$y' + ay = \frac{k}{\epsilon}\chi_{[c,c+\epsilon)}, \quad y(0) = y_0$$

and call the solution y_ϵ. Since $\delta_c = \lim_{\epsilon\to 0} \frac{1}{\epsilon}\chi_{[c,c+\epsilon)}$, we let $y(t) = \lim_{\epsilon\to 0} y_\epsilon$. Then $y(t)$ will be the solution to $y' + ay = k\delta_c$, $y(0) = y_0$. We will assume $a \neq 0$ and leave the case $a = 0$ to the reader. Recall from Exercise 16 of Sect. 6.3 the solution to

$$y' + ay = A\chi[\alpha, \beta), \quad y(0) = y_0,$$

is

$$y(t) = y_0 e^{-at} + \frac{A}{a}\begin{cases} 0 & \text{if } 0 \le t < \alpha, \\ 1 - e^{-a(t-\alpha)} & \text{if } \alpha \le t < \beta, \\ e^{-a(t-\beta)} - e^{-a(t-\alpha)} & \text{if } \beta \le t < \infty. \end{cases}$$

We let $A = \frac{k}{\epsilon}$, $\alpha = c$, and $\beta = c + \epsilon$ to get

$$y_\epsilon(t) = y_0 e^{-at} + \frac{k}{a\epsilon}\begin{cases} 0 & \text{if } 0 \le t < c, \\ 1 - e^{-a(t-c)} & \text{if } c \le t < c+\epsilon, \\ e^{-a(t-c-\epsilon)} - e^{-a(t-c)} & \text{if } c+\epsilon \le t < \infty. \end{cases}$$

The computation of $\lim_{\epsilon\to 0} y_\epsilon$ is done on each interval separately. If $0 \le t \le c$, then $y_\epsilon = y_0 e^{-at}$ is independent of ϵ, and hence,

$$\lim_{\epsilon\to 0} y_\epsilon(t) = y_0 e^{-at} \qquad 0 \le t \le c.$$

If $c < t < \infty$, then for ϵ small enough, $c + \epsilon < t$ and thus

$$y_\epsilon(t) = y_0 e^{-at} + \frac{k}{a\epsilon}(e^{-a(t-c-\epsilon)} - e^{-a(t-c)}) = y_0 e^{-at} + \frac{k}{a} e^{-a(t-c)} \frac{e^{a\epsilon} - 1}{\epsilon}.$$

Since $\lim_{t\to\epsilon} \frac{e^{a\epsilon}-1}{\epsilon} = a$, we get

$$\lim_{\epsilon\to 0} y_\epsilon(t) = y_0 e^{-at} + k e^{-a(t-c)}, \qquad c < t < \infty.$$

We thus obtain

$$y(t) = \begin{cases} y_0 e^{-at} & \text{if } 0 \le t \le c, \\ y_0 e^{-at} + k e^{-a(t-c)} & \text{if } c < t < \infty. \end{cases}$$

Observe that there is a jump discontinuity in $y(t)$ at $t = c$ with jump k.

Extension of Input Derivative Principle

In this method, we want to focus on the differential equation, $y' + ay = 0$, on the entire interval $[0, \infty)$ with the a priori knowledge that there is a jump discontinuity in $y(t)$ at $t = c$ with jump k. Recall from Theorem 3 of Sect. 6.3 that when y is continuous and both y and y' are in $\mathcal{H}$, we have the formula

$$\mathcal{L}\left\{y'\right\}(s) = sY(s) - y(0).$$

We cannot apply this theorem as stated for y is not continuous. But if y has a single jump discontinuity at $t = c$, we can prove a slight generalization.

Theorem 3. *Suppose y and y' are in $\mathcal{H}$ and y is continuous except for one jump discontinuity at $t = c$ with jump k. Then*

$$\mathcal{L}\left\{y'\right\}(s) = sY(s) - y(0) - k\mathrm{e}^{-cs}.$$

Proof. Let $N > c$. Then integration by parts gives

$$\begin{aligned}\int_0^N \mathrm{e}^{-st} y'(t)\,\mathrm{d}t &= \int_0^c \mathrm{e}^{-st} y'(t)\,\mathrm{d}t + \int_c^N \mathrm{e}^{-st} y'(t)\,\mathrm{d}t \\ &= \mathrm{e}^{-st} y(t)|_0^c + s\int_0^c \mathrm{e}^{-st} y(t)\,\mathrm{d}t \\ &\quad + \mathrm{e}^{-st} y(t)|_c^N + s\int_c^N \mathrm{e}^{-st} y(t)\,\mathrm{d}t \\ &= s\int_0^N \mathrm{e}^{-st} y(t)\,\mathrm{d}t + \mathrm{e}^{-sN} y(N) - y(0) \\ &\quad - \mathrm{e}^{-sc}(y(c^+) - y(c^-)).\end{aligned}$$

We take the limit as N goes to infinity and obtain

$$\mathcal{L}\left\{y'\right\} = s\mathcal{L}\left\{y\right\} - y(0) - k\mathrm{e}^{-cs},$$

where $k = y(c^+) - y(c^-)$ is the jump of $y(t)$ at $t = c$. □

We apply this theorem to the initial value problem

$$y' + ay = 0, \qquad y(0) = y_0$$

with the knowledge that the solution y has a jump discontinuity at $t = c$ with jump k. Apply the Laplace transform to the differential equation to obtain

$$sY(s) - y(0) - k\mathrm{e}^{-cs} + aY(s) = 0.$$

Solving for Y gives

$$Y(s) = \frac{y_0}{s+a} + k\frac{\mathrm{e}^{-cs}}{s+a}.$$

Applying the inverse Laplace transform gives the solution

$$\begin{aligned} y(t) &= y_0\mathrm{e}^{-at} + k\mathrm{e}^{-a(t-c)}h(t-c) \\ &= \begin{cases} y_0\mathrm{e}^{-at} & \text{if } 0 \le t < c, \\ y_0\mathrm{e}^{-at} + k\mathrm{e}^{-a(t-c)} & \text{if } c \le t < \infty. \end{cases} \end{aligned}$$

The Formal Laplace Transform Method

We now return to the differential equation

$$y' + ay = k\delta_c, \qquad y(0) = y_0$$

and apply the Laplace transform method directly. From Theorem 2, the Laplace transform of $k\delta_c$ is $k\mathrm{e}^{-cs}$. This is precisely the term found in Theorem 3 where the assumption of a single jump discontinuity is assumed. Thus, the presence of $k\delta_c$ automatically encodes the jump discontinuity in the solution. Therefore, we can (formally) proceed without any advance knowledge of jump discontinuities. The Laplace transform of

$$y' + ay = k\delta_c, \qquad y(0) = y_0$$

gives

$$sY(s) - y(0) + aY(s) = k\mathrm{e}^{-cs}.$$

Solving for $Y(s)$, we get

$$Y(s) = \frac{y_0}{s+a} + k\frac{\mathrm{e}^{-cs}}{s+a}.$$

Apply the inverse Laplace transform as above to get

$$y(t) = \begin{cases} y_0\mathrm{e}^{-at} & \text{if } 0 \le t < c, \\ y_0\mathrm{e}^{-at} + k\mathrm{e}^{-a(t-c)} & \text{if } c \le t < \infty. \end{cases}$$

Observe that the same result is obtained in each procedure and justifies the formal Laplace transform method, which is thus the preferred method to use. We now use this method to solve the mixing problem given in Example 1.

► **Solution.** We apply the Laplace transform method to

$$y' + \frac{y}{10} = 3\delta_5, \qquad y(0) = 20$$

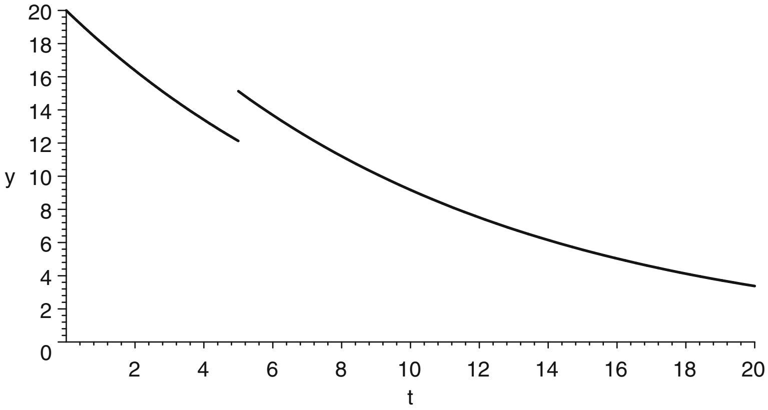


Fig. 6.16 Graph of the solution to the mixing problem

to get

$$sY(s) - 20 + \frac{1}{10}Y(s) = 3e^{-5s}.$$

Solving for $Y(s)$ gives

$$Y(s) = \frac{20}{s + \frac{1}{10}} + \frac{3e^{-5s}}{s + \frac{1}{10}}$$

and Laplace inversion gives

$$\begin{aligned} y(t) &= 20e^{-t/10} + 3e^{-(t-5)/10}h(t-5) \\ &= \begin{cases} 20e^{-t/10} & \text{if } 0 \le t \le 5, \\ 20e^{-t/10} + 3e^{-(t-5)/10} & \text{if } 5 < t < \infty. \end{cases} \end{aligned}$$

The graph is given in Fig. 6.16 where the jump of 3 is clearly seen at $t = 5$. ◀

Impulse Functions

An impulsive force is a force with high magnitude introduced over a short period of time. For example, a bat hitting a ball or a spike in electricity on an electric circuit both involve impulsive forces and are best represented by the Dirac delta function. We consider the effect of the introduction of impulsive forces into spring systems and how they lead to second order differential equations of the form

$$my'' + \mu y' + ky = K\delta_c(t).$$

As we will soon see, the effect of an impulsive force introduces a discontinuity not in y but its derivative y'.

If $F(t)$ represents a force which is 0 outside a time interval $[a, b]$, then $\int_0^\infty F(t)\,\mathrm{d}t = \int_a^b F(t)\,\mathrm{d}t$ represents the ***total impulse*** of the force $F(t)$ over that interval. A unit impulse means that this integral is 1. If F is given by the acceleration of a constant mass, then $F(t) = ma(t) = my''(t)$, where m is the mass and $a(t) = y''(t)$ is the acceleration of the object given by the position function $y(t)$. The total impulse

$$\int_a^b F(t)\,\mathrm{d}t = \int_a^b ma(t)\,\mathrm{d}t = my'(b) - my'(a)$$

represents the change of momentum. (Momentum is the product of mass and velocity). Now imagine a constant force is introduced over a very short period of time with unit impulse. We then model the force by $d_{c,\epsilon} = \frac{1}{\epsilon}\chi_{[c,c+\epsilon)}$. Letting ϵ go to 0 then leads to the Dirac delta function δ_c to represent the instantaneous change of momentum. Since momentum is proportional to velocity, we see that such impacts lead to discontinuities in the derivative y'.

Example 4 (See Sect. 3.6 for a discussion of spring-mass-dashpot systems). A spring is stretched 49 cm when a 1 kg mass is attached. The body is pulled to 10 cm below its spring-body equilibrium and released. We assume the system is frictionless. After 3 s, the mass is suddenly struck by a hammer in a downward direction with total impulse of 1 kg·m/s. Find the motion of the mass.

► **Solution.** *Setting up the differential equation*: We will work in units of kg, m, and s. Thus, the spring constant k is given by $1(9.8) = k\frac{49}{100}$, so that $k = 20$. An external force to the system occurs as an impulse at $t = 3$ which may be represented by the Dirac delta function δ_3. The initial conditions are given by $y(0) = 0.10$ and $y'(0) = 0$, and since the system is frictionless, the initial value problem is

$$y'' + 20y = \delta_3, \qquad y(0) = 0.10, \quad y'(0) = 0.$$

We will return to the solution of this problem after we discuss the more general second order case. ◄

Equations of the Form $y'' + ay' + by = K\delta_c$

Our goal is to solve

$$y'' + ay' + by = K\delta_c, \qquad y(0) = y_0, \quad y'(0) = y_1 \tag{2}$$

using the formal Laplace transform method discussed above for first order differential equations. As we discussed, the effect of $K\delta_c$ is to introduce a single jump discontinuity in y' at $t = c$ with jump K. Therefore, the solution to (2) is equivalent to solving

$$y'' + ay' + by = 0$$

with the advanced knowledge that y' has a jump discontinuity at $t = c$. If we apply Theorem 3 to y', we obtain

$$\begin{aligned}\mathcal{L}\{y''\} &= s\mathcal{L}\{y'\} - y'(0) - K\mathrm{e}^{-sc} \\ &= s^2Y(s) - sy(0) - y'(0) - K\mathrm{e}^{-sc}.\end{aligned}$$

Therefore, the Laplace transform of $y'' + ay' + by = 0$ leads to

$$(s^2 + as + b)Y(s) - sy(0) - y'(0) - K\mathrm{e}^{-sc} = 0.$$

On the other hand, if we (formally) proceed with the Laplace transform of (2) without foreknowledge of discontinuities in y', we obtain the equivalent equation

$$(s^2 + as + b)Y(s) - sy(0) - y'(0) = K\mathrm{e}^{-sc}.$$

Again, the Dirac function δ_c encodes the jump discontinuity automatically. If we proceed as usual, we obtain

$$Y(s) = \frac{sy(0) + y'(0)}{s^2 + as + b} + \frac{K\mathrm{e}^{-sc}}{s^2 + as + b}.$$

The inversion will depend on the way the characteristic polynomial factors.

We now return to Example 4. The equation we wish to solve is

$$y'' + 20y = \delta_3, \qquad y(0) = 0.10, \quad y'(0) = 0.$$

► **Solution.** We apply the formal Laplace transform to obtain

$$Y(s) = \frac{0.1s}{s^2 + 20} + \frac{\mathrm{e}^{-3s}}{s^2 + 20}.$$

The inversion gives

$$\begin{aligned}y(t) &= \frac{1}{10}\cos\left(\sqrt{20}\,t\right) + \frac{1}{\sqrt{20}}\sin\left(\sqrt{20}\,(t-3)\right)h(t-3) \\ &= \frac{1}{10}\cos\left(\sqrt{20}\,t\right) + \begin{cases} 0 & \text{if } 0 \le t < 3, \\ \frac{1}{\sqrt{20}}\sin\left(\sqrt{20}\,(t-3)\right) & \text{if } 3 \le t < \infty. \end{cases}\end{aligned}$$

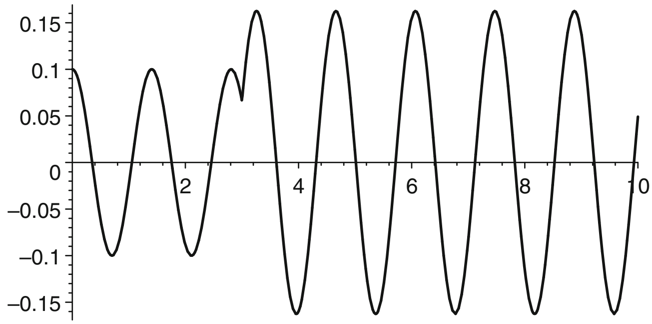


Fig. 6.17 Harmonic motion with impulse function

Figure 6.17 gives the graph of the solution. You will note that y is continuous, but the little kink at $t = 3$ indicates the discontinuity of y'. This is precisely when the impulse to the system was delivered. ◄

Exercises

1–10. Solve each of the following initial value problems.

1. $y' + 2y = \delta_1(t), \quad y(0) = 0$
2. $y' - 3y = 3 + \delta_2(t), \quad y(0) = -1$
3. $y' - 4y = \delta_4(t), \quad y(0) = 2$
4. $y' + y = \delta_1(t) - \delta_3(t), \quad y(0) = 0$
5. $y'' + 4y = \delta_\pi(t), \quad y(0) = 0,\ y'(0) = 1$
6. $y'' - y = \delta_1(t) - \delta_2(t), \quad y(0) = 0,\ y'(0) = 0$
7. $y'' + 4y' + 3y = 2\delta_2(t), \quad y(0) = 1,\ y'(0) = -1$
8. $y'' + 4y = \delta_\pi(t) - \delta_{2\pi}(t), \quad y(0) = 1,\ y'(0) = 0$
9. $y'' + 4y' + 4y = 3\delta_1(t), \quad y(0) = -1,\ y'(0) = 3$
10. $y'' + 4y' + 5y = 3\delta_\pi(t), \quad y(0) = 0,\ y'(0) = 1$

11–13. *Mixing Problems.*

11. Suppose a tank is filled with 12 gal of pure water. A brine solution with concentration 2 lbs salt per gallon flows into the tank at a rate of 3 gal/min and the well-stirred solution flows out of the tank at the same rate. In addition, at $t = 3$ min, 4 lbs of salt is instantly poured into the tank where it immediately dissolves. Find the amount of salt, $y(t)$, in the tank at any time t.
12. A tank holds 10 L of a brine solution in which each liter contains 1 kg of dissolved salt. A brine solution with concentration 0.5 kg/L is poured into the tank at a rate of 2 L/min, and the thoroughly mixed solution flows out of the tank at the same rate. After 2 min, 1 L of salt is poured into the tank where it instantly mixes into the solution. Find the amount of salt, $y(t)$, in the tank at any time t.
13. A tank holds 1 gal of pure water. Pure water flows into the tank at a rate of 1 gal/min, and the well-stirred mixture flows out of the tank at the same rate. At $t = 0, 2, 4, 6$ min, 1 lb of salt is instantly added to the tank where it immediately dissolves. Find the amount, $y(t)$, of salt in the tank at time t. How much salt is in the tank just after the last addition at $t = 6$ min?

14–17. *Spring Problems*

14. A spring is stretched 6 in. when a 1-lb object is attached. The 1-lb object is pulled to 12 in. below its spring-body equilibrium and released. We assume the system is frictionless. After 5π seconds, the mass is suddenly struck by a hammer in a downward direction with total impulse of 1 slug ·ft/s. Find the motion of the object. Determine the amplitudes before and after the hammer impact.
15. A spring is stretched 1 m by a force of 8 N. A body of mass 2 kg is attached to the spring with accompanying dashpot. Suppose the damping force of the dashpot is 8 N when the velocity of the body is 1 m/s. At $t = 0$, the mass is pulled down from its equilibrium position a distance of 10 cm and given an

initial downward velocity of 5 cm/s. After 4 s, the mass is suddenly struck by a hammer in a downward direction with total impulse of 2 kg·m/s. Determine the resulting motion.

16. A 1-Newton force will stretch a spring 1 m. A body of mass 1 kg is attached to the spring and allowed to come to rest. There is no dashpot. At time $t = 0, 2\pi, 4\pi, 6\pi$, a hammer impacts the mass in the downward direction with a magnitude of 1 kg m/s. Find the equation of motion and provide a graph on the interval $[0, 10\pi)$.
17. A 1-Newton force will stretch a spring 1 m. A body of mass 1 kg is attached to the spring and allowed to come to rest. There is no dashpot. At time $t = 0, \pi, 2\pi, 3\pi, 4\pi, 5\pi$, a hammer impacts the mass in the downward direction with a magnitude of 1 kg m/s. Find the equation of motion and provide a graph on the interval $[0, 6\pi)$. Explain.

18–20. In these problems, we justify the Laplace transform method for solving $y' + ay = k\delta_c$, $y(0) = y_0$ in a way different from the limiting procedure and the extension of the input derivative principle introduced in the text.

18. On the interval $[0, c)$, solve $y' + ay = 0$ with initial value $y(0) = y_0$.
19. On the interval $[c, \infty)$, solve $y' + ay = 0$ with initial value $y(c) + k$, where $y(c)$ is the value of $y(t)$ at $t = c$ obtained from Exercise 18 for the interval $[0, c]$.
20. Piece together the solutions obtained from Exercise 18 for the interval $[0, c)$ and from Exercise 19 for the interval $[c, \infty)$ and verify that it is the same obtained from the formal Laplace transform method.

6.5 Convolution

In this section, we extend to the Heaviside class the definition of convolution that we introduced in Sect. 2.8. The importance of convolution is that it is precisely the operation in input space that corresponds via the Laplace transform to the ordinary product in transform space. This is the essence of the convolution principle stated in Theorem 1 of Sect. 2.8 and which will be proved here for the Heaviside class. We will then consider further extensions to the Dirac delta functions δ_c and explore some very pleasant properties.

Given two functions f and g in $\mathcal{H}$, the function

$$u \mapsto f(u)g(t-u)$$

is continuous except for perhaps finitely many points on each interval of the form $[0, t]$. Therefore, the integral

$$\int_0^t f(u)g(t-u)\,\mathrm{d}u$$

exists for each $t > 0$. The ***convolution*** of f and g is given by

$$f * g(t) = \int_0^t f(u)g(t-u)\,\mathrm{d}u.$$

We will not make the argument but it can be shown that $f * g$ is in fact continuous. Since there are numbers K, L, a, and b such that

$$|f(t)| \le K\mathrm{e}^{at} \quad \text{and} \quad |g(t)| \le L\mathrm{e}^{bt},$$

it follows that

$$\begin{aligned} |f * g(t)| &\le \int_0^t |f(u)|\,|g(t-u)|\,\mathrm{d}u \\ &\le KL \int_0^t \mathrm{e}^{au}\mathrm{e}^{b(t-u)}\,\mathrm{d}u \\ &= KL\mathrm{e}^{bt} \int_0^t \mathrm{e}^{(a-b)u}\,\mathrm{d}u \\ &= KL \begin{cases} t\mathrm{e}^{bt} & \text{if } a = b \\ \frac{\mathrm{e}^{at}-\mathrm{e}^{bt}}{a-b} & \text{if } a \ne b \end{cases}. \end{aligned}$$

This shows that $f * g$ is of exponential type since both $t\mathrm{e}^{bt}$ and $\frac{\mathrm{e}^{at}-\mathrm{e}^{bt}}{a-b}$ are exponential polynomials. It follows now that $f * g \in \mathcal{H}$.

Several important properties we listed in Sect. 2.5 extend to $\mathcal{H}$. For convenience, we restate them here: Suppose f, g, and h are in $\mathcal{H}$. Then

1. $(f+h) * g = f * g + h * g$
2. $(cf) * g = c(f * g)$, for c a scalar
3. $f * g = g * f$
4. $(f * g) * h = f * (g * h)$
5. $f * 0 = 0$.

The Sliding Window

When one of the functions, g say, is an on–off switch, then convolution takes a particularly simple form. Suppose $g = \chi_{[a,b)}$. Then $g(t) = 1$ if and only if $a \leq t < b$. Hence, $g(t-u) = 1$ if and only if $a \leq t-u < b$ which is equivalent to $t-b < u \leq t-a$. Thus, $g(t-u) = \chi_{(t-b,t-a]}(u)$ is an on–off switch on the interval $(t-b, t-a]$ which slides to the right as t increases. It follows that

$$f * g(t) = \int_0^t f(u)\chi_{(t-b,t-a]}(u)\,\mathrm{d}u.$$

One can think of $g(t-u)$ as a horizontally *sliding window* by which a portion of f is turned on. That portion is then measured by integration. To illustrate this, consider the following example.

Example 1. Let $f(t) = (t-3)h(t-3)$ and $g(t) = \chi_{[1,2)}(t)$. Find the convolution $f * g$.

► **Solution.** In this case, $g(t-u) = \chi_{[1,2)}(t-u) = \chi_{(t-2,t-1]}(u)$. If $t < 4$, then $t-2 < t-1 < 3$ so $g(t-u)$ turns on that part of f which is 0. Thus,

$$f * g(t) = 0, \quad \text{if } 0 \leq t < 4,$$

as illustrated in Fig. 6.18.

If $4 \leq t < 5$, then $t-2 < 3 \leq t-1$. Thus, $g(t-u)$ turns of that part of f which is 0 on the interval $(t-2, 3)$ and which is $u-3$ on the interval $[3, t-1)$. Thus,

$$\begin{aligned} f * g(t) &= \int_{t-2}^{t-1} f(u)\,\mathrm{d}u = \int_3^{t-1} u-3\,\mathrm{d}u \\ &= \left.\frac{(u-3)^2}{2}\right|_3^{t-1} = \frac{(t-4)^2}{2}, \quad 4 \leq t < 5. \end{aligned}$$

This is illustrated in Fig. 6.19.

Now when $t \geq 5$, then $3 \leq t-2$. Thus, $g(t-u)$ turns on that portion of f which is $u-3$ on the interval $(t-2, t-1]$ and

$$f * g(t) = \int_{t-2}^{t-1} (u-3)\,\mathrm{d}u$$
$$= \left.\frac{(u-3)^2}{2}\right|_{t-2}^{t-1} = \frac{2t-9}{2}, \quad 5 \leq t < \infty.$$

This is illustrated in Fig. 6.20. Finally, we piece together the convolution to get

$$f * g = \begin{cases} 0 & \text{if } 0 \leq t < 4 \\ \frac{(t-2)^2}{2} & \text{if } 4 \leq t < 5 \\ \frac{2t-9}{2} & \text{if } 5 \leq t < \infty \end{cases}. \tag{1}$$

The graph of $f * g$ is given in Fig. 6.21. ◀

Theorem 2 (The Convolution Theorem). *Suppose f and g are in $\mathcal{H}$ and F and G are their Laplace transforms, respectively. Then*

$$\mathcal{L}\{f * g\}(s) = F(s)G(s)$$

or, equivalently,

$$\mathcal{L}^{-1}\{F(s)G(s)\}(t) = (f * g)(t).$$

Proof. For any $f \in \mathcal{H}$, we will define $f(t) = 0$ for $t < 0$. By Theorem 8 of Sect. 6.2,

$$\mathrm{e}^{-st}G(s) = \mathcal{L}\{g(u-t)h(u-t)\}.$$

Therefore,

$$F(s)G(s) = \int_0^\infty \mathrm{e}^{-st} f(t)\,\mathrm{d}t\ G(s)$$
$$= \int_0^\infty \mathrm{e}^{-st} G(s) f(t)\,\mathrm{d}t$$
$$= \int_0^\infty \mathcal{L}\{g(u-t)h(u-t)\}(s) f(t)\,\mathrm{d}t$$
$$= \int_0^\infty \int_0^\infty \mathrm{e}^{-su} g(u-t)h(u-t) f(t)\,\mathrm{d}u\,\mathrm{d}t. \tag{2}$$

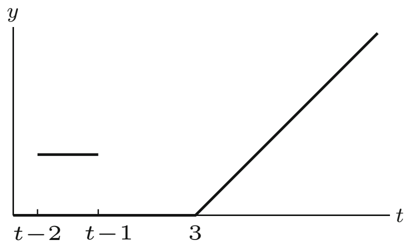


Fig. 6.18 When $t - 1 < 3$, the on–off switch $\chi_{(t-2,t-1]}$ turns on that portion of f that is zero. Hence, $f * g(t) = 0$ for all $0 \le t < 4$. Notice how the on–off switch slides to the right as t increases in the following figures

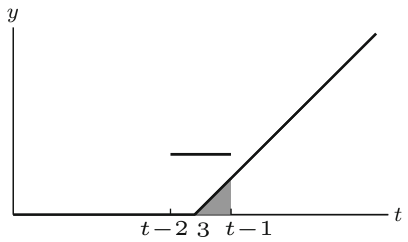


Fig. 6.19 When $4 < t < 5$, then $t - 2 < 3 < t - 1$ and the on–off switch $\chi_{(t-2,t-1]}$ turns on that portion of f that is zero to the left of 3 and the line $u - 3$ to the right of 3. Hence, $f * g(t) = \int_3^{t-1} (u-3)\, \mathrm{d}u = \frac{(t-4)^2}{2}$ for $4 \le t < 5$

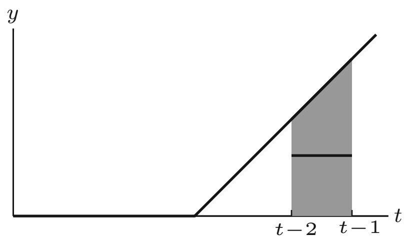


Fig. 6.20 When $5 \le t$, then $3 < t - 2 < t - 1$ and the on–off switch $\chi_{(t-2,t-1]}$ turns on that portion of f that is the line $u - 3$. Hence, $f * g(t) = \int_{t-2}^{t-1} (u-3)\, \mathrm{d}u = \frac{2t-9}{2}$ for $5 \le t < \infty$

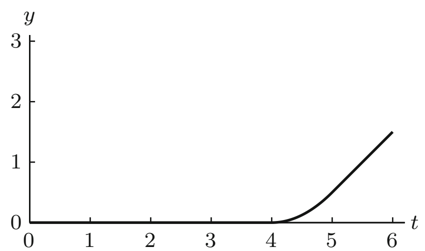


Fig. 6.21 The graph of the convolution $f * g(t) = \begin{cases} 0 & \text{if } 0 \le t < 4 \\ \frac{(t-2)^2}{2} & \text{if } 4 \le t < 5 \\ \frac{2t-9}{2} & \text{if } 5 \le t < \infty \end{cases}$

A theorem in calculus[1] tells us that we can switch the order of integration in (2) when f and g are in $\mathcal{H}$. Thus, we obtain

$$\begin{aligned}F(s)G(s) &= \int_0^\infty \int_0^\infty e^{-su} g(u-t)h(u-t)f(t)\,dt\,du \\ &= \int_0^\infty \int_0^u e^{-su} g(u-t)f(t)\,dt\,du \\ &= \int_0^\infty e^{-su}(f * g)(u)\,du \\ &= \mathcal{L}\{f * g\}(s).\end{aligned}$$

□

There are a variety of uses for the convolution theorem. For one, it is sometimes a convenient way to compute the convolution of two functions f and g using the formula $(f * g)(t) = \mathcal{L}^{-1}\{F(s)G(s)\}$. In the following example, we rework Example 1 in this way.

Example 3. Compute the convolution $f * g$ where

$$f(t) = (t-3)h(t-3) \quad \text{and} \quad g(t) = \chi_{[1,2)}.$$

► **Solution.** The Laplace transforms of f and g are, respectively,

$$F(s) = \frac{e^{-3s}}{s^2} \quad \text{and} \quad G(s) = \frac{e^{-s}}{s} - \frac{e^{-2s}}{s}.$$

The product simplifies to

$$F(s)G(s) = \frac{e^{-4s}}{s^3} - \frac{e^{-5s}}{s^3}.$$

Its inverse Laplace transform is

$$\begin{aligned}(f * g)(t) &= \mathcal{L}^{-1}\{F(s)G(s)\}(t) \\ &= \frac{(t-4)^2}{2}h(t-4) - \frac{(t-5)^2}{2}h(t-5)\end{aligned}$$

[1] cf. *Vector Calculus, Linear Algebra, and Differential Forms*, J.H. Hubbard and B.B Hubbard, page 444.

$$= \begin{cases} 0 & \text{if } 0 \le t < 4 \\ \frac{(t-2)^2}{2} & \text{if } 4 \le t < 5 \\ \frac{2t-9}{2} & \text{if } 5 \le t < \infty \end{cases}. \quad ◀$$

Convolution and the Dirac Delta Function

We would like to extend the definition of convolution to include the Dirac delta functions δ_c, $c \ge 0$. Recall that we formally defined the Dirac delta function by

$$\delta_c(t) = \lim_{\epsilon \to 0} d_{c,\epsilon}(t),$$

where $d_{c,\epsilon} = \frac{1}{\epsilon}\chi_{[c,c+\epsilon)}$. In like manner, for $f \in \mathcal{H}$, we define

$$f * \delta_c(t) = \lim_{\epsilon \to 0} f * d_{c,\epsilon}(t).$$

Theorem 4. *For $f \in \mathcal{H}$,*

$$f * \delta_c(t) = f(t-c)h(t-c),$$

where the equality is understood to mean essentially equal.

Proof. Let $f \in \mathcal{H}$. Then

$$\begin{aligned} f * d_{c,\epsilon}(t) &= \int_0^t f(u) d_{c,\epsilon}(t-u)\,\mathrm{d}t \\ &= \frac{1}{\epsilon}\int_0^t f(u)\chi_{[c,c+\epsilon)}(t-u)\,\mathrm{d}u \\ &= \frac{1}{\epsilon}\int_0^t f(u)\chi_{[t-c-\epsilon,t-c)}(u)\,\mathrm{d}u \end{aligned}$$

Now suppose $t < c$. Then $\chi_{[t-c-\epsilon,t-c)}(u) = 0$, for all $u \in [0,t)$. Thus, $f * d_{c,\epsilon} = 0$. On the other hand, if $t > c$, then for ϵ small enough, we have

$$f * d_{c,\epsilon}(t) = \frac{1}{\epsilon}\int_{t-c-\epsilon}^{t-c} f(u)\,\mathrm{d}u.$$

Let t be such that $t - c$ is a point of continuity of f. Then by the fundamental theorem of calculus

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{t-c-\epsilon}^{t-c} f(u)\,\mathrm{d}u = f(t-c).$$

Since f has only finitely many jump discontinuities on any finite interval, it follows that $f * \delta_c$ is essentially equal to $f(t-c)h(t-c)$. ☐

The special case $c = 0$ produces the following pleasant corollary.

Corollary 5. *For $f \in \mathcal{H}$, we have*

$$f * \delta_0 = f.$$

This corollary tells us that this extension to the Dirac delta function gives an identity under the convolution product. We thus have a correspondence between the multiplicative identities in input space and transform space under the Laplace transform since $\mathcal{L}\{\delta_0\} = 1$.

Remark 6. Notice that when $f(t) = 1$, in Theorem 4, we get

$$1 * \delta_c = h(t-c).$$

This can be reexpressed as

$$\int_0^t \delta_c(u)\,\mathrm{d}u = h(t-c).$$

Thus, the integral of the Dirac delta function is the unit step function shifted by c. Put another way, the derivative of the unit step function, $h(t-c)$, is the Dirac delta function, δ_c.

The Impulse Response Function

Suppose $q(s)$ is a polynomial of degree n. The solution $\zeta(t)$ to the initial value problem

$$q(\boldsymbol{D})y = \delta_0 \quad y(0) = 0,\ y'(0) = 0,\ \ldots,\ y^{(n-1)}(0) = 0, \tag{3}$$

is called the ***unit impulse response function***. In the case where $\deg q(s) = 2$, the unit impulse response function may be viewed as the response to a mass-spring dashpot system initially at rest but hit with a hammer of impulse 1 at $t = 0$ as represented by the Dirac delta function δ_0. Applying the Laplace transform to (3), we get $Y(s) = \frac{1}{q(s)}$, and thus the unit impulse response function is given by the Laplace inversion formula

$$\zeta(t) = \mathcal{L}^{-1}\left\{\frac{1}{q(s)}\right\}.$$

Let $f \in \mathcal{H}$ (or f could be a Dirac delta function) and let us consider the general differential equation

$$q(\boldsymbol{D})y = f, \quad y(0) = y_0,\ y'(0) = y_1,\ y^{(n-1)}(0) = y_{n-1}. \tag{4}$$

Let $F(s) = \mathcal{L}\{f(t)\}$. Applying the Laplace transform to both sides gives

$$q(s)Y(s) - p(s) = F(s),$$

where $p(s)$ is a polynomial that depends on the initial conditions and has degree at most $n-1$. Solving for $Y(s)$ gives

$$Y(s) = \frac{p(s)}{q(s)} + \frac{F(s)}{q(s)}. \tag{5}$$

Let

$$Y_{\mathrm{h}}(s) = \frac{p(s)}{q(s)} \quad \text{and} \quad Y_{\mathrm{p}}(s) = \frac{F(s)}{q(s)}.$$

If $y_{\mathrm{h}}(t) = \mathcal{L}^{-1}\{Y_{\mathrm{h}}(s)\}$, then y_{h} is the homogeneous solution to (4). Specifically, y_{h} is the solution to (4) when $f = 0$ but with the same initial conditions. We sometimes refer to y_{h} as the ***zero-input solution***.

On the other hand, let $y_{\mathrm{p}}(t) = \mathcal{L}^{-1}\{Y_{\mathrm{p}}(s)\}$. Then y_{p} is the solution to (4) when all the initial conditions are zero, sometimes referred to as the ***zero-state***. We refer to y_{p} as the ***zero-state solution***. Since $Y(s) = \frac{1}{q(s)}F(s)$, we get by the convolution theorem

$$y_{\mathrm{p}}(t) = \zeta * f(t). \tag{6}$$

This tells us that the solution to a system in the zero-state is completely determined by convolution of the input function with the unit impulse response function.

We summarize this discussion in the following theorem:

Theorem 7. *Let $f \in \mathcal{H}$ (or a linear combination of Dirac delta functions). The solution to* (4) *can be expressed as*

$$y_{\mathrm{h}} + y_{\mathrm{p}},$$

where y_{h} is the zero-input solution, that is, the homogeneous solution to (4) *(with the same initial conditions), and $y_{\mathrm{p}} = \zeta * f$ is the zero-state solution given by convolution of the unit impulse response function ζ and the input function f.*

Remark 8. In Exercises 15–21, we outline a proof that the homogeneous solution y_{h} can be expressed as a linear combination of

$$\{\zeta,\ \zeta', \ldots, \zeta^{(n-1)}\}$$

and provide formulas for the coefficients in terms of the initial conditions. The reader is encouraged to explore these exercises. It follows then that both y_{h} and y_{p} are directly determined by the unit impulse response function, ζ.

As an example of the techniques we have developed in this section, consider the differential equation that solved the mixing problem given in Example 1 of Sect. 6.4.

Example 9. Solve the following differential equation:

$$y' + \frac{1}{10}y = 3\delta_5 \quad y(0) = 20.$$

► **Solution.** The characteristic polynomial is $q(s) = s + 1/10$, and therefore, the unit impulse response function is $\zeta(t) = \mathcal{L}^{-1}\left\{\frac{1}{s+1/10}\right\} = \mathrm{e}^{-1/10t}$. The zero-state solution is $y_{\mathrm{p}} = \zeta * 3\delta_5 = 3\mathrm{e}^{-(t-5)/10}h(t-5)$ by Theorem 4. It is straightforward to see that the homogeneous solution is $y_{\mathrm{h}} = 20\mathrm{e}^{-1/10t}$. We thus get

$$\begin{aligned} y(t) &= 20\mathrm{e}^{-t/10} + 3\mathrm{e}^{-(t-5)/10}h(t-5) \\ &= \begin{cases} 20\mathrm{e}^{-t/10} & \text{if } 0 \le t \le 5, \\ 20\mathrm{e}^{-t/10} + 3\mathrm{e}^{-(t-5)/10} & \text{if } 5 < t < \infty. \end{cases} \end{aligned}$$

◄

Example 10. Solve the following differential equation:

$$y'' + 4y = \chi_{[0,1)} \quad y(0) = 0 \text{ and } y'(0) = 0.$$

► **Solution.** The homogeneous solution to

$$y'' + 4y = 0 \quad y(0) = 0 \text{ and } y'(0) = 0$$

is the trivial solution $y_{\mathrm{h}} = 0$. The unit impulse response function is

$$\zeta(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} = \frac{1}{2}\sin 2t.$$

By Theorem 7, the solution is

$$\begin{aligned} y(t) &= \zeta * \chi_{[0,1)} \\ &= \int_0^t \frac{1}{2}\sin(2u)\chi_{[0,1)}(t-u)\,\mathrm{d}u \end{aligned}$$

$$= \frac{1}{2}\int_0^t \sin(2u)\chi_{(t-1,t]}(u)\,\mathrm{d}u$$

$$= \frac{1}{2}\begin{cases} \int_0^t \sin 2u\,\mathrm{d}u & \text{if } 0 \le t < 1 \\ \int_{t-1}^t \sin 2u\,\mathrm{d}u & \text{if } 1 \le t < \infty \end{cases}$$

$$= \frac{1}{4}\begin{cases} 1 - \cos 2t & \text{if } 0 \le t < 1 \\ \cos 2(t-1) - \cos 2t & \text{if } 1 \le t < \infty \end{cases}. \qquad ◀$$

In the following, we revisit Example 4 of Sect. 6.4.

Example 11. Solve the differential equation

$$y'' + 20y = \delta_3, \quad y(0) = 0.1, \ \ y'(0) = 0,$$

that models the spring problem given in Exercise 4 of Sect. 6.4.

► **Solution.** The characteristic polynomial is $q(s) = s^2 + 20$. So the homogeneous solution is $y_{\mathrm{h}} = c_1 \cos\sqrt{20}t + c_2 \sin\sqrt{20}t$. The initial conditions easily imply that $c_1 = \frac{1}{10}$ and $c_2 = 0$. So $y_{\mathrm{h}} = \frac{1}{10}\cos\sqrt{20}t$. The unit impulse response function is

$$\zeta(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+20}\right\} = \frac{1}{\sqrt{20}}\sin\sqrt{20}t.$$

It follows from Theorem 4 and 7 that

$$\begin{aligned} y_{\mathrm{p}}(t) &= \zeta * \delta_3(t) \\ &= \frac{1}{\sqrt{20}}(\sin\sqrt{20}(t-3))h(t-3). \end{aligned}$$

It follows now that the solution to the spring problem in Example 4 of Sect. 6.4 is

$$y(t) = \frac{1}{10}\cos\left(\sqrt{20}\,t\right) + \frac{1}{\sqrt{20}}\sin\left(\sqrt{20}\,(t-3)\right)h(t-3)$$

$$= \frac{1}{10}\cos\left(\sqrt{20}\,t\right) + \begin{cases} 0 & \text{if } 0 \le t < 3, \\ \frac{1}{\sqrt{20}}\sin\left(\sqrt{20}\,(t-3)\right) & \text{if } 3 \le t < \infty. \end{cases} \qquad ◀$$

Exercises

1–8. Find the convolution of the following pairs of functions.

1. $f(t) = e^t$ and $g(t) = \chi_{[0,1)}(t)$
2. $f(t) = \sin t$ and $g(t) = h(t - \pi)$
3. $f(t) = th(t - 1)$ and $g(t) = \chi_{[3,4)}(t)$
4. $f(t) = t$ and $g(t) = (t - 1)h(t - 1)$
5. $f(t) = \chi_{[0,2)}$ and $g(t) = \chi_{[0,2)}$
6. $f(t) = \cos t$ and $g(t) = \delta_{\pi/2}$
7. $f(t) = \sin t$ and $g(t) = \delta_0 + \delta_\pi$
8. $f(t) = te^{2t}$ and $g(t) = \delta_1 - \delta_2$

9–14. Find the unit impulse response function ζ and use Theorem 7 to solve the following differential equations.

9. $y' - 3y = h(t - 2), \quad y(0) = 2$
10. $y' + 4y = \delta_3, \quad y(0) = 1$
11. $y' + 8y = \chi_{[3,5)}, \quad y(0) = -2$
12. $y'' - y = \delta_1(t) - \delta_2(t), \quad y(0) = 0,\ y'(0) = 0$
13. $y'' + 9y = \chi_{[0,2\pi)}, \quad y(0) = 1, y'(0) = 0.$
14. $y'' - 6y' + 9y = \delta_3, \quad y(0) = -1,\ y'(0) = -3$

15–21. Suppose $q(s) = a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0, a_n \neq 0$. In these exercises, we explore properties of the unit impulse response function ζ for $q(\boldsymbol{D})y = \delta_0$.

15. Show that ζ is the solution to the homogeneous differential equation

$$q(\boldsymbol{D})y = 0$$

with initial conditions

$$\begin{aligned} y(0) &= 0 \\ y'(0) &= 0 \\ &\vdots \\ y^{(n-2)}(0) &= 0 \\ y^{(n-1)}(0) &= 1/a_n. \end{aligned}$$

16. Show that $\mathcal{L}\left\{\zeta^{(k)}\right\} = \frac{s^k}{q(s)}$, $0 \leq k < n$, and hence $\zeta^{(k)} \in \mathcal{E}_q$.
17. Show that $\left\{\zeta,\ \zeta', \ldots, \zeta^{(n-1)}\right\}$ is a linearly independent subset of $\mathcal{E}_q$.

18. Show that $\left\{\zeta,\ \zeta',\ldots,\zeta^{(n-1)}\right\}$ spans $\mathcal{E}_q$.

19. Show that $\left\{\zeta,\ \zeta',\ldots,\zeta^{(n-1)}\right\}$ is a basis of $\mathcal{E}_q$.

20. Show that the Wronskian of $\zeta,\ \zeta',\ldots,\zeta^{(n-1)}$ is given by the following formula:

$$w(\zeta,\ \zeta',\ldots,\zeta^{(n-1)}) = \frac{(-1)^{\lfloor \frac{n}{2} \rfloor}}{a_n^n} e^{\frac{-a_{n-1}}{a_n}t},$$

where $\lfloor x \rfloor$ is the largest integer less than or equal to x.

21. Let y be the solution to $q(\boldsymbol{D})y = 0$ with initial conditions

$$\begin{aligned} y(0) &= y_0 \\ y'(0) &= y_1 \\ &\vdots \\ y^{(n-1)}(0) &= y_{n-1}. \end{aligned}$$

Since $y \in \mathcal{E}_q$, we may write

$$y = c_0\zeta + c_1\zeta' + \cdots + c_{n-1}\zeta^{(n-1)}.$$

Show that

$$c_l = \sum_{k=0}^{n-l-1} a_{k+l+1}y_k.$$

22–26. Use the result of Exercise 21 to find the homogeneous solution to each of the following differential equations. It may be helpful to organize the computation of the coefficients in the following way: Let $0 \le l \le n-1$ and write

$$\begin{array}{llllllll} a_0 & a_1 & \cdots & a_l & a_{l+1} & a_{l+2} & \cdots & a_n \\ & & & & y_0 & y_1 & \cdots\ y_{n-l-1} \cdots & y_n \\ c_0 & c_1 & \cdots & c_l & \cdots & & & \end{array}$$

In the first row, put the coefficients of $q(s)$ starting with the constant coefficient on the left. In the second row, put the initial conditions with y_0 under a_{l+1}, y_1 under a_{l+2}, etc. Multiply terms that overlap in the first two rows and add. Put the result in c_l. Now shift the second row of initial conditions to the right one place and repeat to get c_{l+1}. Repeating this will give all the coefficients $c_0,\ldots,c_{n-1}$ needed in $y = \sum_{l=0}^{n-1} c_l\zeta^{(l)}$.

22. $y'' + 9y = 0, \quad y(0) = 1,\ y'(0) = 2$
23. $y'' - 2y' + y = 0, \quad y(0) = 2,\ y'(0) = -3$
24. $y'' + 4y' + 3y = 0, \quad y(0) = -1,\ y'(0) = 1$
25. $y''' + y' = 0, \quad y(0) = 1,\ y'(0) = 0,\ y''(0) = 4$
26. $q(\boldsymbol{D})y = 0$ where $q(s) = (s-1)^4$ and $y(0) = 0,\ y'(0) = 1,\ y''(0) = 0$, and $y'''(0) = -1$

6.6 Periodic Functions

In modeling mechanical and other systems, it frequently happens that the forcing function repeats over time. Periodic functions best model such repetition.

A function f defined on $[0, \infty)$ is said to be ***periodic*** if there is a positive number p such that $f(t + p) = f(t)$ for all t in the domain of f. We say p is ***a period*** of f. If $p > 0$ is a period of f and there is no smaller period, then we say p is the ***fundamental period*** of f although we will usually just say ***the period***. The interval $[0, p)$ is called the ***fundamental interval***. If there is no such smallest positive p for a periodic function, then the period is defined to be 0. The constant function $f(t) = 1$ is an example of a periodic function with period 0. The sine function is periodic with period 2π: $\sin(t + 2\pi) = \sin(t)$. Knowing the sine on the interval $[0, 2\pi)$ implies knowledge of the function everywhere. Similarly, if we know f is periodic with period $p > 0$ and we know the function on the fundamental interval, then we know the function everywhere. Figure 6.22 illustrates this point.

The Sawtooth Function

A particularly useful periodic function is the ***sawtooth*** function. With it, we may express other periodic functions simply by composition. Let $p > 0$. The saw tooth function is given by

$$\langle t\rangle_p = \begin{cases} t & \text{if } 0 \le t < p \\ t - p & \text{if } p \le t < 2p \\ t - 2p & \text{if } 2p \le t < 3p \\ \vdots & \end{cases}.$$

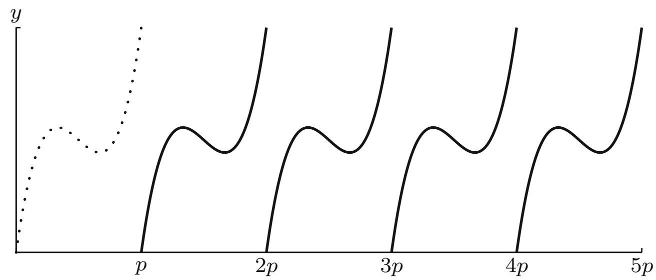


Fig. 6.22 An example of a periodic function with period p. Notice how the interval $[0, p)$ determines the function everywhere

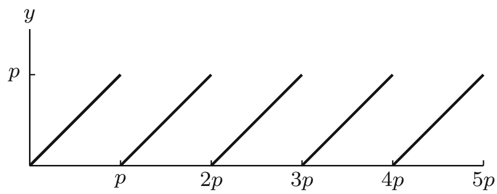


Fig. 6.23 The sawtooth function $\langle t \rangle_p$ with period p

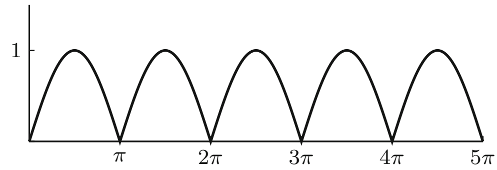


Fig. 6.24 The rectified sine wave: $\sin(\langle t \rangle_\pi)$

It is periodic with period p. Its graph is given in Fig. 6.23. The sawtooth function $\langle t \rangle_p$ is obtained by extending the function $y = t$ on the interval $[0, p)$ periodically to $[0, \infty)$. More generally, given a function f defined on the interval $[0, p)$, then the composition of f and $\langle t \rangle_p$ is the periodic extension of f to $[0, \infty)$. It is given by the formula

$$f(\langle t \rangle_p) = \begin{cases} f(t) & \text{if } 0 \le t < p \\ f(t-p) & \text{if } p \le t < 2p \\ f(t-2p) & \text{if } 2p \le t < 3p \\ \vdots & \vdots \end{cases}.$$

In applications, it is useful to rewrite this piecewise function as

$$\sum_{n=0}^{\infty} f(t - np)\chi_{[np,(n+1)p)}(t).$$

For example, Fig. 6.24 is the graph of $y = \sin(\langle t \rangle_\pi)$. This function, which is periodic with period π, is known as the ***rectified sine wave***.

The Staircase Function

Another function that will be particularly useful is the ***staircase*** function. For $p > 0$, it is defined as follows:

$$[t]_p = \begin{cases} 0 & \text{if } t \in [0, p) \\ p & \text{if } t \in [p, 2p) \\ 2p & \text{if } t \in [2p, 3p) \\ \vdots & \end{cases}.$$

Its graph is given in Fig. 6.25.

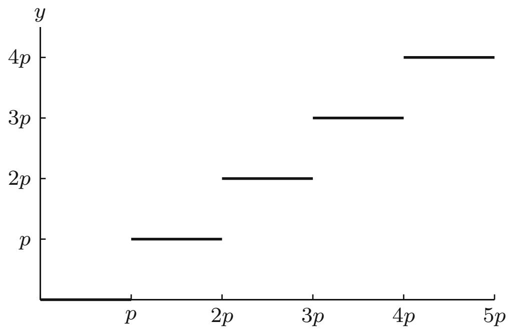


Fig. 6.25 The staircase function: $[t]_p$

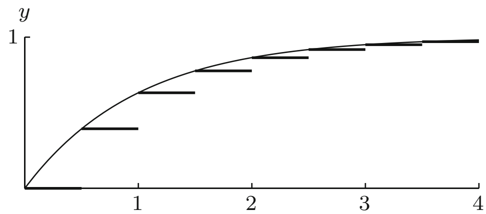


Fig. 6.26 The graph of $1 - e^{-t}$ and $1 - e^{-[t]_{.5}}$

The staircase function is *not* periodic. It is useful in expressing piecewise functions that are like steps on intervals of length p. For example, if f is a function on $[0, \infty)$, then $f([t]_p)$ is a function whose value on $[np, (n+1)p)$ is the constant $f(np)$. Thus,

$$f([t]_p) = \sum_{n=0}^{\infty} f(np)\chi_{[np,(n+1)p)}(t).$$

Figure 6.26 illustrates this idea with the function $f(t) = 1 - e^{-t}$ and $p = 0.5$.

Observe that the staircase function and the sawtooth function are related by

$$< t >_p = t - [t]_p.$$

The Laplace Transform of Periodic Functions

Not surprisingly, the formula for the Laplace transform of a periodic function is determined by the fundamental interval.

Theorem 1. *Let f be a periodic function in $\mathcal{H}$ and $p > 0$ a period of f. Then*

$$\mathcal{L}\{f\}(s) = \frac{1}{1 - e^{-sp}} \int_0^p e^{-st} f(t)\, dt.$$

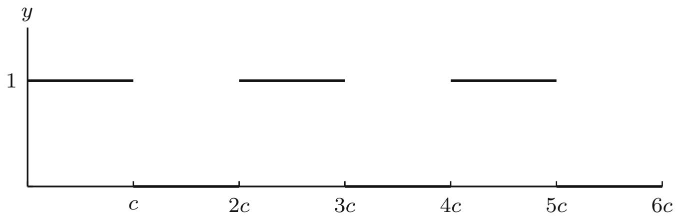


Fig. 6.27 The graph of the square-wave function sw_c

Proof.

$$\begin{aligned}\mathcal{L}\{f\}(s) &= \int_0^\infty e^{-st} f(t)\,dt \\ &= \int_0^p e^{-st} f(t)\,dt + \int_p^\infty e^{-st} f(t)\,dt.\end{aligned}$$

However, the change of variables $t \to t+p$ in the second integral and the periodicity of f gives

$$\begin{aligned}\int_p^\infty e^{-st} f(t)\,dt &= \int_0^\infty e^{-s(t+p)} f(t+p)\,dt \\ &= e^{-sp}\int_0^\infty e^{-st} f(t)\,dt \\ &= e^{-sp}\mathcal{L}\{f\}(s).\end{aligned}$$

Therefore,

$$\mathcal{L}\{f\}(s) = \int_0^p e^{-st} f(t)\,dt + e^{-sp}\mathcal{L}\{f\}(s).$$

Solving for $\mathcal{L}\{f\}$ gives the desired result. □

Example 2. Find the Laplace transform of the ***square-wave*** function sw_c given by

$$\mathrm{sw}_c(t) = \begin{cases} 1 & \text{if } t \in [2nc, (2n+1)c) \\ 0 & \text{if } t \in [(2n+1)c, (2n+2)c) \end{cases} \quad \text{for each integer } n.$$

► **Solution.** The square-wave function sw_c is periodic with period $2c$. Its graph is given in Fig. 6.27 and, by Theorem 1, its Laplace transform is

$$\begin{aligned}
\mathcal{L}\{\mathrm{sw}_c\}(s) &= \frac{1}{1-\mathrm{e}^{-2cs}} \int_0^{2c} \mathrm{e}^{-st}\,\mathrm{sw}_c(t)\,\mathrm{d}t \\
&= \frac{1}{1-\mathrm{e}^{-2cs}} \int_0^{c} \mathrm{e}^{-st}\,\mathrm{d}t \\
&= \frac{1}{1-(\mathrm{e}^{-cs})^2}\,\frac{1-\mathrm{e}^{-cs}}{s} \\
&= \frac{1}{1+\mathrm{e}^{-cs}}\,\frac{1}{s}.
\end{aligned}$$

◀

Example 3. Find the Laplace transform of the sawtooth function $< t >_p$.

► **Solution.** Since the sawtooth function is periodic with period p and since $< t >_p = t$ for $0 \le t < p$, Theorem 1 gives

$$\mathcal{L}\left\{\langle t\rangle_p\right\}(s) = \frac{1}{1-\mathrm{e}^{-sp}} \int_0^{p} \mathrm{e}^{-st}\,t\,\mathrm{d}t.$$

Integration by parts gives

$$\begin{aligned}
\int_0^{p} \mathrm{e}^{-st}\,t\,\mathrm{d}t &= \frac{t\mathrm{e}^{-st}}{-s}\Big|_0^p - \frac{1}{-s}\int_0^{p} \mathrm{e}^{-st}\,\mathrm{d}t \\
&= -\frac{p\mathrm{e}^{-sp}}{s} - \frac{1}{s^2}\mathrm{e}^{-st}\Big|_0^p \\
&= -\frac{p\mathrm{e}^{-sp}}{s} - \frac{\mathrm{e}^{-sp}-1}{s^2}.
\end{aligned}$$

With a little algebra, we obtain

$$\mathcal{L}\left\{< t >_p\right\}(s) = \frac{1}{s^2}\left(1 - \frac{sp\mathrm{e}^{-sp}}{1-\mathrm{e}^{-sp}}\right).$$

◀

As mentioned above, it frequently happens that we build periodic functions by restricting a given function f to the interval $[0, p)$ and then extending it to be periodic with period p: $f(\langle t\rangle_p)$. Suppose now that $f \in \mathcal{H}$. We can then express the Laplace transform of $f(\langle t\rangle_p)$ in terms of the Laplace transform of f. The following corollary expresses this relationship and simplifies unnecessary calculations like the integration by parts that we did in the previous example.

Corollary 4. *Let $p > 0$. Suppose $f \in \mathcal{H}$. Then*

$$\mathcal{L}\left\{f(\langle t\rangle_p)\right\}(s) = \frac{1}{1-\mathrm{e}^{-sp}}\mathcal{L}\left\{f(t) - f(t)h(t-p)\right\}.$$

Proof. The function $f(t) - f(t)h(t-p) = f(t)(1-h(t-p))$ is the same as f on the interval $[0, p)$ and 0 on the interval $[p, \infty)$. Therefore,

$$\int_0^p \mathrm{e}^{-st} f(t)\,\mathrm{d}t = \int_0^\infty \mathrm{e}^{-st}(f(t) - f(t)h(t-p))\,\mathrm{d}t = \mathcal{L}\left\{f(t) - f(t)h(t-p)\right\}.$$

The result now follows from Theorem 1. □

Let us return to the sawtooth function in Example 3 and see how Corollary 4 simplifies the calculation of its Laplace transform.

$$\begin{aligned}\mathcal{L}\left\{\langle t\rangle_p\right\}(s) &= \frac{1}{1-\mathrm{e}^{-sp}}\mathcal{L}\left\{t - th(t-p)\right\} \\ &= \frac{1}{1-\mathrm{e}^{-sp}}\left(\frac{1}{s^2} - \mathrm{e}^{-sp}\mathcal{L}\left\{t+p\right\}\right) \\ &= \frac{1}{1-\mathrm{e}^{-sp}}\left(\frac{1}{s^2} - \mathrm{e}^{-sp}\frac{1+sp}{s^2}\right) \\ &= \frac{1}{s^2}\left(1 - \frac{sp\mathrm{e}^{-sp}}{1-\mathrm{e}^{-sp}}\right).\end{aligned}$$

The last line requires a few algebraic steps.

Example 5. Find the Laplace transform of the rectified sine wave

$$\sin(\langle t\rangle_\pi).$$

See Fig. 6.24.

► **Solution.** Corollary 4 gives

$$\begin{aligned}\mathcal{L}\left\{\sin(\langle t\rangle_\pi)\right\} &= \frac{1}{1-\mathrm{e}^{-\pi s}}\mathcal{L}\left\{\sin t - \sin t\; h(t-\pi)\right\} \\ &= \frac{1}{1-\mathrm{e}^{-\pi s}}\left(\frac{1}{s^2+1} - \mathrm{e}^{-\pi s}\mathcal{L}\left\{\sin(t+\pi)\right\}\right) \\ &= \frac{1}{1-\mathrm{e}^{-\pi s}}\left(\frac{1+\mathrm{e}^{-\pi s}}{s^2+1}\right),\end{aligned}$$

where we use the fact that $\sin(t+\pi) = -\sin(t)$. ◄

Periodic Extensions of the Dirac Delta Function

We will also consider in the applications inputs that are periodic extensions of the Dirac delta function. For example,

$$\delta_c(\langle t\rangle_p) = \delta_c + \delta_{c+p} + \delta_{c+2p} + \delta_{c+3p} + \cdots,$$

where $0 \le c < p$ is the periodic extension of the Dirac delta function, δ_c. An important case is when $c = 0$. Then $\delta_0(\langle t\rangle_p)$, is the periodic extension of δ_0 with period p and represents a unit impulse at each multiple of $t = p$. Another important example is

$$(\delta_0 - \delta_p)(\langle t\rangle_{2p}) = \delta_0 - \delta_p + \delta_{2p} - \delta_{3p} + \cdots,$$

the periodic extension of $\delta_0 - \delta_p$ with period $2p$ which represents a unit impulse at each even multiple of p and a negative unit impulse at odd multiples of p.

Proposition 6. *The Laplace transforms of* $\delta_0(\langle t\rangle_p)$ *and* $(\delta_0 - \delta_p)(\langle t\rangle_{2p})$ *are given by the following formulas:*

$$\mathcal{L}\left\{\delta_0(\langle t\rangle_p)\right\} = \frac{1}{1-\mathrm{e}^{-ps}},$$

$$\mathcal{L}\left\{(\delta_0 - \delta_p)(\langle t\rangle_{2p})\right\} = \frac{1}{1+\mathrm{e}^{-ps}}.$$

Proof. Let r be a fixed real or complex number. Recall that the geometric series

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + \cdots$$

converges to $\frac{1}{1-r}$ when $|r| < 1$. We can now compute the Laplace transforms.

$$\begin{aligned}\mathcal{L}\left\{\delta_0(\langle t\rangle_p)\right\} &= 1 + \mathrm{e}^{-ps} + \mathrm{e}^{-2ps} + \mathrm{e}^{-3ps} + \cdots \\ &= \sum_{n=0}^{\infty} (\mathrm{e}^{-ps})^n \\ &= \frac{1}{1-\mathrm{e}^{-ps}}.\end{aligned}$$

Similarly,

$$\begin{aligned}\mathcal{L}\left\{(\delta_0 - \delta_p)(\langle t\rangle_{2p})\right\} &= 1 - \mathrm{e}^{-ps} + \mathrm{e}^{-2ps} - \mathrm{e}^{-3ps} + \cdots \\ &= \sum_{n=0}^{\infty} (-\mathrm{e}^{-ps})^n \\ &= \frac{1}{1 + \mathrm{e}^{-ps}}.\end{aligned}$$

□

The Inverse Laplace Transform

The inverse Laplace transform of functions of the form

$$\frac{1}{1 - \mathrm{e}^{-sp}} F(s)$$

is not always a straightforward matter to find unless, of course, $F(s)$ is of the form $\mathcal{L}\{f(t) - f(t)h(t-p)\}$ so that Corollary 4 can be used. Usually, though, this is not the case. Since

$$\frac{1}{1 - \mathrm{e}^{-sp}} = \sum_{n=0}^{\infty} \mathrm{e}^{-snp},$$

we can write

$$\frac{1}{1 - \mathrm{e}^{-sp}} F(s) = \sum_{n=0}^{\infty} \mathrm{e}^{-snp} F(s).$$

If $f = \mathcal{L}^{-1}\{F\}$, then a termwise computation gives

$$\mathcal{L}^{-1}\left\{\frac{1}{1 - \mathrm{e}^{-sp}} F(s)\right\} = \sum_{n=0}^{\infty} \mathcal{L}^{-1}\left\{\mathrm{e}^{-snp} F(s)\right\} = \sum_{n=0}^{\infty} f(t - np)h(t - np).$$

For t in an interval of the form $[Np, (N+1)p)$, the function $h(t - np)$ is 1 for $n = 0, \ldots, N$ and 0 otherwise. We thus obtain

$$\mathcal{L}^{-1}\left\{\frac{1}{1 - \mathrm{e}^{-sp}} F(s)\right\} = \sum_{N=0}^{\infty} \left(\sum_{n=0}^{N} f(t - np)\right) \chi_{[Np,(N+1)p)}.$$

A similar argument gives

$$\mathcal{L}^{-1}\left\{\frac{1}{1 + \mathrm{e}^{-sp}} F(s)\right\} = \sum_{N=0}^{\infty} \left(\sum_{n=0}^{N} (-1)^n f(t - np)\right) \chi_{[Np,(N+1)p)}.$$

For reference, we record these results in the following theorem:

Theorem 7. *Let $p > 0$ and suppose $\mathcal{L}\{f(t)\} = F(s)$. Then*

1. $\mathcal{L}^{-1}\left\{\frac{1}{1-\mathrm{e}^{-sp}} F(s)\right\} = \sum_{N=0}^{\infty} \left(\sum_{n=0}^{N} f(t-np)\right) \chi_{[Np,(N+1)p)}$.

2. $\mathcal{L}^{-1}\left\{\frac{1}{1+\mathrm{e}^{-sp}} F(s)\right\} = \sum_{N=0}^{\infty} \left(\sum_{n=0}^{N} (-1)^n f(t-np)\right) \chi_{[Np,(N+1)p)}$.

Example 8. Find the inverse Laplace transform of

$$\frac{1}{(1-\mathrm{e}^{-2s})s}.$$

► **Solution.** If $f(t) = 1$, then $F(s) = \frac{1}{s}$ is its Laplace transform. We thus have

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{1}{(1-\mathrm{e}^{-2s})s}\right\} &= \sum_{N=0}^{\infty} \left(\sum_{n=0}^{N} f(t-2n)\right) \chi_{[2N,2(N+1))} \\
&= \sum_{N=0}^{\infty} (N+1)\chi_{[2N,2(N+1))} \\
&= 1 + \frac{1}{2}\sum_{N=0}^{\infty} 2N\chi_{[2N,2(N+1))} \\
&= 1 + \frac{1}{2}[t]_2.
\end{aligned}$$ ◄

Remark 9. Generally, it will not be possible to express the final answer in a nice closed form as in Example 8; one may have to settle for an infinite sum as given in Theorem 7.

Exercises

1–5. Reexpress each sum in terms of the sawtooth function $\langle t\rangle_p$ and/or the staircase function $[t]_p$. (Since $t - \langle t\rangle_p = [t]_p$, there are more than one equivalent answer.)

1. $\sum_{n=0}^{\infty}(t-n)^2\chi_{[n,(n+1))}(t)$
2. $\sum_{n=0}^{\infty}(t-n)^2\chi_{[2n,2(n+1))}(t)$
3. $\sum_{n=0}^{\infty} n^2\chi_{[3n,3(n+1))}(t)$
4. $\sum_{n=0}^{\infty} \mathrm{e}^{8n}\chi_{[4n,4(n+1))}(t)$
5. $\sum_{n=0}^{\infty}(t+n)\chi_{[2n,2(n+1))}(t)$

6–10. Find the Laplace transform of each periodic function.

6. $f(\langle t\rangle_2)$ where $f(t) = t^2$
7. $f(\langle t\rangle_3)$ where $f(t) = \mathrm{e}^t$
8. $f(\langle t\rangle_2)$ where $f(t) = \begin{cases} t & \text{if } 0 \le t < 1 \\ 2-t & \text{if } 1 \le t < 2 \end{cases}$
9. $f(\langle t\rangle_{2p})$ where $f(t) = \begin{cases} 1 & \text{if } 0 \le t < p \\ -1 & \text{if } p \le t < 2p \end{cases}$
10. $f(\langle t\rangle_\pi)$ where $f(t) = \cos t$

11–14. Find the Laplace transform of $f([t]_p)$, where $[t]_p$ is the staircase function.

11. $f([t]_p)$ where $f(t) = t$. That is, find the Laplace transform of the staircase function $[t]_p$.
12. $f([t]_1)$ where $f(t) = \mathrm{e}^t$
13. $f([t]_2)$ where $f(t) = \mathrm{e}^{-t}$
14. $f([t]_3)$ where $f(t) = t^2$. Hint: Use the fact that $\sum_{n=0}^{\infty} n^2x^n = \frac{x(1+x)}{(1-x)^3}$ for $|x| < 1$.
15. Suppose $f \in \mathcal{H}$. Show that

$$\mathcal{L}\{f([t]_p)\} = \frac{1-\mathrm{e}^{-ps}}{s}\sum_{n=0}^{\infty} f(np)\mathrm{e}^{-nps}.$$

16–18. Find the inverse Laplace transform of each function.

16. $\dfrac{e^{-2s}}{s(1-e^{-2s})}$

17. $\dfrac{1-e^{-4(s-2)}}{(1-e^{-4s})(s-2)}$

18. $\dfrac{1}{(1-e^{-4s})(s-2)}$

19. Let $F(s) = \frac{1}{(s+a)(1+e^{-ps})}$. Show that

$$\mathcal{L}^{-1}\{F(s)\} = e^{-at}\begin{cases} \frac{1+e^{a(N+1)p}}{1+e^{ap}} & \text{if } t \in [Np, (N+1)p), \ (N \text{ even}) \\ \frac{1-e^{a(N+1)p}}{1+e^{ap}} & \text{if } t \in [Np, (N+1)p), \ (N \text{ odd}) \end{cases}$$

$$= e^{-at}\left(\frac{1+(-1)^{\frac{[t]_p}{p}} e^{a([t]_p+p)}}{1+e^{ap}}\right).$$

(Use the fact that $1 + x + x^2 + \cdots + x^N = \frac{1-x^{N+1}}{1-x}$.)

20. In the text, we stated that the constant function 1 is periodic with period 0. Here is another example: Let $\mathbb{Q}$ denote the set of rational numbers. Let

$$\chi_{\mathbb{Q}}(t) = \begin{cases} 1 & \text{if } t \in \mathbb{Q} \\ 0 & \text{if } t \notin \mathbb{Q} \end{cases}.$$

Show that χ is periodic with period q for each positive $q \in \mathbb{Q}$. Conclude that $\chi_{\mathbb{Q}}$ has fundamental period 0.

6.7 First Order Equations with Periodic Input

We now turn our attention to two examples of mixing problems with periodic input functions. Each example will be modeled by a first order differential equation of the form

$$y' + ay = f(t),$$

where $f(t)$ is a periodic function.

Example 1. Suppose a tank contains 10 gal of pure water. Two input sources alternately flow into the tank for 1-min intervals. The first input source begins flowing at $t = 0$. It consists of a brine solution with concentration 1 lb salt per gallon and flows (when on) at a rate of 5 gal/min. The second input source is pure water and flows (when on) at a rate of 5 gal/min. The tank has a drain with a constant outflow of 5 gal/min. Let $y(t)$ denote the total amount of salt at time t. Find $y(t)$ and for large values of t determine how $y(t)$ fluctuates.

► **Solution.** The input rate of salt is given piecewise by the formula

$$5\,\mathrm{sw}_1(t) = \begin{cases} 5 & \text{if } 2n \le t < 2n+1) \\ 0 & \text{if } 2n+1 \le t < 2n+2 \end{cases}.$$

The output rate is given by

$$\frac{y(t)}{10} \cdot 5.$$

This leads to the first order differential equation

$$y' + \frac{1}{2}y = 5\,\mathrm{sw}_1(t) \qquad y(0) = 0.$$

A calculation using Example 2 of Sect. 6.6 shows that the Laplace transform is

$$Y(s) = 5\frac{1}{1+\mathrm{e}^{-s}}\,\frac{1}{s\left(s+\frac{1}{2}\right)},$$

and a partial fraction decomposition of $\frac{1}{s(s+1/2)}$ gives

$$Y(s) = 10\frac{1}{1+\mathrm{e}^{-s}}\,\frac{1}{s} - 10\frac{1}{1+\mathrm{e}^{-s}}\,\frac{1}{s+\frac{1}{2}}.$$

Now apply the inverse Laplace transform. To simplify the calculations, let

$$Y_1(s) = 10\frac{1}{1+\mathrm{e}^{-s}}\,\frac{1}{s},$$

$$Y_2(s) = 10\frac{1}{1+e^{-s}}\,\frac{1}{s+\frac{1}{2}}.$$

Then $Y(s) = Y_1(s) - Y_2(s)$. By Example 2 of Sect. 6.6, we have

$$\mathcal{L}^{-1}\{Y_1(s)\} = 10\,\mathrm{sw}_1(t).$$

By Theorem 7 of Sect. 6.6, the inverse Laplace transform of the second expression is

$$\begin{aligned}
\mathcal{L}^{-1}\{Y_2(s)\} &= 10\sum_{N=0}^{\infty}\sum_{n=0}^{N}(-1)^n e^{-\frac{1}{2}(t-n)}\chi_{[N,N+1)} \\
&= 10e^{-\frac{1}{2}t}\sum_{N=0}^{\infty}\sum_{n=0}^{N}\left(-e^{\frac{1}{2}}\right)^n \chi_{[N,N+1)} \\
&= 10e^{-\frac{1}{2}t}\sum_{N=0}^{\infty}\frac{1-\left(-e^{\frac{1}{2}}\right)^{N+1}}{1+e^{\frac{1}{2}}}\chi_{[N,N+1)} \\
&= \frac{10e^{-\frac{1}{2}t}}{1+e^{\frac{1}{2}}}\begin{cases}1+e^{\frac{N+1}{2}} & \text{if } t\in[N,N+1) \text{ (N even)}\\ 1-e^{\frac{N+1}{2}} & \text{if } t\in[N,N+1) \text{ (N odd)}\end{cases}.
\end{aligned}$$

Finally, we put these two expression together to get our solution

$$y(t) = 10\,\mathrm{sw}_1(t) - \frac{10e^{-\frac{1}{2}t}}{1+e^{\frac{1}{2}}}\begin{cases}1+e^{\frac{N+1}{2}} & \text{if } t\in[N,N+1) \text{ (N even)}\\ 1-e^{\frac{N+1}{2}} & \text{if } t\in[N,N+1) \text{ (N odd)}\end{cases}$$

$$= \begin{cases}10-10\dfrac{e^{-\frac{1}{2}t}+e^{\frac{-t+N+1}{2}}}{1+e^{\frac{1}{2}}} & \text{if } t\in[N,N+1) \text{ (N even)}\\[2ex] -10\dfrac{e^{-\frac{1}{2}t}-e^{\frac{-t+N+1}{2}}}{1+e^{\frac{1}{2}}} & \text{if } t\in[N,N+1) \text{ (N odd)}\end{cases}.$$

The graph of $y(t)$, obtained with the help of a computer, is presented in Fig. 6.28. The solution is sandwiched in between a lower and upper curve. The lower curve, $l(t)$, is obtained by setting $t = m$ to be an even integer in the formula for the solution and then continuing it to all reals. We obtain

$$l(m) = 10 - 10\frac{e^{-\frac{1}{2}m}+e^{\frac{-m+m+1}{2}}}{1+e^{\frac{1}{2}}} = 10 - 10\frac{e^{-\frac{1}{2}m}+e^{\frac{1}{2}}}{1+e^{\frac{1}{2}}}$$

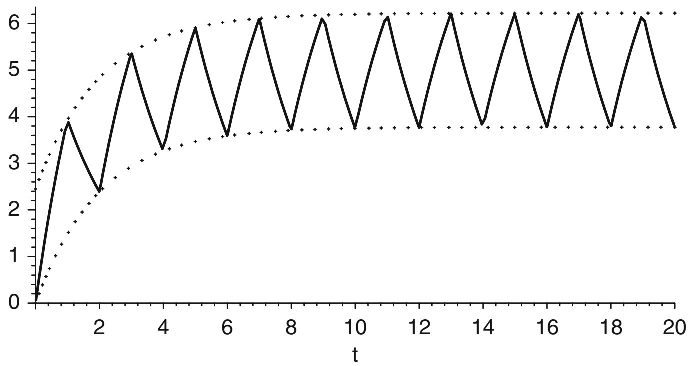


Fig. 6.28 A mixing problem with square-wave input function

and thus

$$l(t) = 10 - 10\frac{e^{-\frac{1}{2}t} + e^{\frac{1}{2}}}{1 + e^{\frac{1}{2}}}.$$

In a similar way, the upper curve, $u(t)$, is obtained by setting $t = m^-$ to be slight smaller than an odd integer and continuing to all reals. We obtain

$$u(t) = -10\frac{e^{-\frac{1}{2}t} - e^{\frac{1}{2}}}{1 + e^{\frac{1}{2}}}.$$

An easy calculation gives

$$\lim_{t\to\infty} l(t) = 10 - \frac{10e^{\frac{1}{2}}}{1+e^{\frac{1}{2}}} \simeq 3.78 \text{ and } \lim_{t\to\infty} u(t) = \frac{10e^{\frac{1}{2}}}{1+e^{\frac{1}{2}}} \simeq 6.22.$$

This means that the salt fluctuation in the tank varies between 3.78 and 6.22 lbs for large values of t. ◀

In practice, it is not always possible to know the input function, $f(t)$, precisely. Suppose though that it is known that f is periodic with period p. Then the total input on all intervals of the form $[np, (n+1)p)$ is $\int_{np}^{(n+1)p} f(t)\,dt = h$, a constant. On the interval $[0, p)$, we could model the input with a Dirac delta function concentrated at a point, c say, and then extend it periodically. We would then obtain a sum of Dirac delta functions of the form

$$a(t) = h(\delta_c + \delta_{c+p} + \delta_{c+2p} + \cdots)$$

that may adequately represent the input for the system we are trying to model. Additional information may justify distributing the total input over two or more points in the interval and extend periodically. Whatever choices are made, the solution will need to be analyzed in the light of empirical data known about the system. Consider the example above. Suppose that it is known that the input is periodic with period 2 and total input 5 on the fundamental interval. Suppose additionally that you are told that the distribution of the input of salt is on the first half of each interval. We might be led to try to model the input on $[0, 2)$ by $\frac{5}{2}\delta_0 + \frac{5}{2}\delta_1$ and then extend periodically to obtain

$$a(t) = \frac{5}{2}\sum_{n=0}^{\infty}\delta_n.$$

Of course, the solution modeled by the input function $a(t)$ will differ from the solution obtained using input function $5\,\mathrm{sw}_1$ as given in Example 1. What is true though is that both exhibit similar long-term behavior. This can be observed in the following example.

Example 2. Suppose a tank contains 10 gal of pure water. Pure water flows into the tank at a rate of 5 gal/min. The tank has a drain with a constant outflow of 5 gal/min. Suppose $\frac{5}{2}$ lbs of salt is put in the tank each minute whereupon it instantly and uniformly dissolves. Assume the level of fluid in the tank is always 10 gal. Let $y(t)$ denote the total amount of salt at time t. Find $y(t)$ and for large values of t determine how $y(t)$ fluctuates.

► **Solution.** As discussed above, the input function is $\frac{5}{2}\sum_{n=1}^{\infty}\delta_n$, and therefore, the differential equation that models this system is

$$y' + \frac{1}{2}y = \frac{5}{2}\sum_{n=1}^{\infty}\delta_n, \quad y(0) = 0.$$

The Laplace transform leads to

$$Y(s) = \frac{5}{2}\sum_{n=0}^{\infty}\mathrm{e}^{-sn}\frac{1}{s+\frac{1}{2}},$$

and inverting the Laplace transform and using Theorem 7 of Sect. 6.6 give

$$\begin{aligned} y(t) &= \frac{5}{2}\mathrm{e}^{-\frac{1}{2}t}\sum_{N=0}^{\infty}\left(\sum_{n=0}^{N}\left(\mathrm{e}^{-\frac{1}{2}(t-n)}\right)\right)\chi_{[N,N+1)} \\ &= \frac{5}{2}\mathrm{e}^{-\frac{1}{2}t}\sum_{N=0}^{\infty}\left(\sum_{n=0}^{N}\left(\mathrm{e}^{\frac{1}{2}}\right)^n\right)\chi_{[N,N+1)} \\ &= \frac{5}{2}\mathrm{e}^{-\frac{1}{2}t}\sum_{N=0}^{\infty}\frac{1-\mathrm{e}^{\frac{N+1}{2}}}{1-\mathrm{e}^{\frac{1}{2}}}\chi_{[N,N+1)} \\ &= \frac{5\left(\mathrm{e}^{-\frac{1}{2}t}-\mathrm{e}^{-\frac{1}{2}(t-[t]_1-1)}\right)}{2\left(1-\mathrm{e}^{\frac{1}{2}}\right)}. \end{aligned}$$

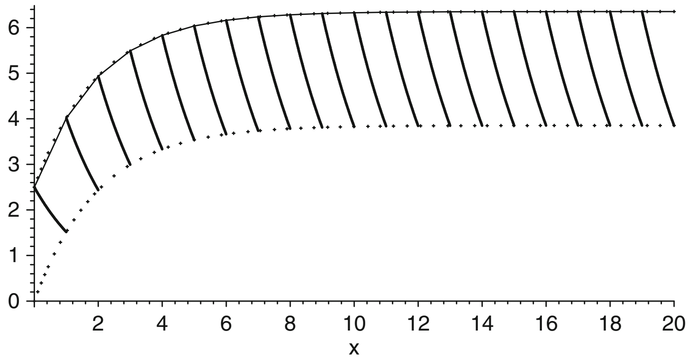


Fig. 6.29 A mixing problem with a periodic Dirac delta function: The solution to the differential equation $y' + \frac{1}{2}y = \frac{5}{2}\sum_{n=1}^{\infty}\delta_n, \quad y(0) = 0$

The graph of this equation is given in Fig. 6.29. The solution is sandwiched in between a lower and upper curve. The upper curve, $u(t)$, is obtained by setting $t = m$ to be an integer in the formula for the solution and then continuing it to all reals. We obtain

$$u(m) = \frac{5}{2\left(1 - e^{-\frac{1}{2}}\right)}\left(e^{-\frac{m}{2}} - e^{\frac{-m+m+1}{2}}\right) = \frac{5}{2\left(1 - e^{-\frac{1}{2}}\right)}\left(e^{-\frac{m}{2}} - e^{\frac{1}{2}}\right)$$

and thus

$$u(t) = \frac{5}{2\left(1 - e^{-\frac{1}{2}}\right)}\left(e^{-\frac{t}{2}} - e^{\frac{1}{2}}\right).$$

In a similar way, the upper curve, $l(t)$, is obtained by setting $t = (m+1)^-$ (slightly less than the integer $m+1$) and continuing to all reals. We obtain

$$l(t) = \frac{5}{2\left(1 - e^{-\frac{1}{2}}\right)}\left(e^{-\frac{t}{2}} - 1\right).$$

An easy calculation gives

$$\lim_{t\to\infty} u(t) = \frac{-5e^{\frac{1}{2}}}{2\left(1-e^{\frac{1}{2}}\right)} \simeq 6.35 \text{ and } \lim_{t\to\infty} l(t) = \frac{-5}{2\left(1-e^{\frac{1}{2}}\right)} \simeq 3.85.$$

This means that the salt fluctuation in the tank varies between 3.85 and 6.35 lbs for large values of t. ◀

A comparison of the solutions in these examples reveals similar long-term behavior in the fluctuation of the salt content in the tank. Remember though that each problem that is modeled must be weighed against hard empirical data to determine if the model is appropriate or not. Also, we could have modeled the instantaneous input by assuming the input was concentrated at a single point, rather than two points. The results are not as favorable. These other possibilities are explored in the exercises.

Exercises

1–3. Solve the following mixing problems.

1. Suppose a tank contains 10 gal of pure water. Two input sources alternately flow into the tank for 2-min intervals. The first input source begins flowing at $t = 0$. It is a brine solution with concentration 2 lbs salt per gallon and flows (when on) at a rate of 4 gal/min. The second input source is a brine solution with concentration 1 lb salt per gallon and flows (when on) at a rate of 4 gal/min. The tank has a drain with a constant outflow of 4 gal/min. Let $y(t)$ denote the total amount of salt at time t. Find $y(t)$ and for large values of t determine how $y(t)$ fluctuates.
2. Suppose a tank contains 10 gal of brine in which 20 lbs of salt are dissolved. Two input sources alternately flow into the tank for 1-min intervals. The first input source begins flowing at $t = 0$. It is a brine solution with concentration 1 lb salt per gallon and flows (when on) at a rate of 2 gal/min. The second input source is a pure water and flows (when on) at a rate of 2 gal/min. The tank has a drain with a constant outflow of 2 gal/min. Let $y(t)$ denote the total amount of salt at time t. Find $y(t)$ and for large values of t determine how $y(t)$ fluctuates.
3. Suppose a tank contains 10 gal of pure water. Pure water flows into the tank at a rate of 5 gal/min. The tank has a drain with a constant outflow of 5 gal/min. Suppose 5 lbs of salt is put in the tank every other minute beginning at $t = 0$ whereupon it instantly and uniformly dissolves. Assume the level of fluid in the tank is always 10 gal. Let $y(t)$ denote the total amount of salt at time t. Find $y(t)$ and for large values of t determine how $y(t)$ fluctuates.

4–5. Solve the following harvesting problems.

4. In a certain area of the Louisiana swamp, a population of 2,500 alligators is observed. Given adequate amounts of food and space, their population will follow the Malthusian growth model. After 12 months, scientists observe that there are 3,000 alligators. Alarmed by their rapid growth, the Louisiana Wildlife and Fisheries institutes the following hunting policy for a specialized group of alligator hunters: Hunting is allowed in only an odd-numbered month, and the total number of alligators taken is limited to 80. Assuming the limit is attained in each month allowed and uniformly over the month, determine a model that gives the population of alligators. Solve that model. How many alligators are there at the beginning of the fifth year? (Assume a population of 3,000 alligators at the beginning of the initial year.)
5. Assume the premise of Exercise 4, but instead of the stated hunting policy in odd-numbered months, assume that the Louisiana Wildlife and Fisheries contracts an elite force of Cajun alligator hunters to take out 40 alligators at the beginning of each month. (You may assume this is done instantly on the first day of each month.) Determine a model that gives the population of alligators. Solve that model. How many alligators are there at the beginning of the fifth year?

6.8 Undamped Motion with Periodic Input

In Sect. 3.6, we discussed various kinds of motion of a spring-body-dashpot system modeled by the differential equation

$$my'' + \mu y' + ky = f(t).$$

Undamped motion led to the differential equation

$$my'' + ky = f(t). \tag{1}$$

In particular, we explored the case where $f(t) = F_0 \cos \omega t$ and were led to the solution

$$y(t) = \begin{cases} \frac{F_0}{a(\beta^2-\omega^2)} (\cos \omega t - \cos \beta t) & \text{if } \beta \neq \omega, \\ \\ \frac{F_0}{2a\omega} t \sin \omega t & \text{if } \beta = \omega, \end{cases} \tag{2}$$

where $\beta = \sqrt{\frac{k}{m}}$. The case where β is close to but not equal to ω gave rise to the notion of beats, while the case $\beta = \omega$ gave us resonance. Since $\cos \omega t$ is periodic, the system that led to (1) is an example of ***undamped motion with periodic input***. In this section, we will explore this phenomenon with two further examples: a square-wave periodic function, sw_c, and a periodic impulse function, $\delta_0(\langle t \rangle_c) = \sum_{n=0}^{\infty} \delta_{nc}$. Both examples are algebraically tedious, so you will be asked to fill in some of the algebraic details in the exercises. To simplify the notation, we will rewrite (1) as

$$y'' + \beta^2 y = g(t)$$

and assume $y(0) = y'(0) = 0$.

Undamped Motion with Square-Wave Forcing Function

Example 1. A constant force of r units for c units of time is applied to a mass-spring system with no damping force that is initially at rest. The force is then released for c units of time. This on–off force is extended periodically to give a periodic forcing function with period $2c$. Describe the motion of the mass.

► **Solution.** The differential equation which describes this system is

$$y'' + \beta^2 y = r\, \mathrm{sw}_c(t), \quad y(0) = 0,\ y'(0) = 0, \tag{3}$$

where sw_c is the square-wave function with period $2c$ and β^2 is the spring constant. By Example 2 of Sect. 6.6, the Laplace transform leads to the equation

$$\begin{aligned} Y(s) &= r\frac{1}{1+\mathrm{e}^{-sc}}\frac{1}{s(s^2+\beta^2)} = \frac{r}{\beta^2}\frac{1}{1+\mathrm{e}^{-sc}}\left(\frac{1}{s}-\frac{s}{s^2+\beta^2}\right). \\ &= \frac{r}{\beta^2}\frac{1}{1+\mathrm{e}^{-sc}}\frac{1}{s} - \frac{r}{\beta^2}\frac{1}{1+\mathrm{e}^{-sc}}\frac{s}{s^2+\beta^2}. \end{aligned} \tag{4}$$

Let

$$F_1(s) = \frac{r}{\beta^2}\frac{1}{1+\mathrm{e}^{-sc}}\frac{1}{s} \quad \text{and} \quad F_2(s) = \frac{r}{\beta^2}\frac{1}{1+\mathrm{e}^{-sc}}\frac{s}{s^2+\beta^2}.$$

Then $Y(s) = F_1(s) - F_2(s)$. Again, by Example 2 of Sect. 6.6, we have

$$f_1(t) = \frac{r}{\beta^2}\,\mathrm{sw}_c(t). \tag{5}$$

By Theorem 7 of Sect. 6.6, we have

$$f_2(t) = \frac{r}{\beta^2}\sum_{N=0}^{\infty}\left(\sum_{n=0}^{N}(-1)^n\cos(\beta t - n\beta c)\right)\chi_{[Nc,(N+1)c)}. \tag{6}$$

We consider two cases.

βc Is not an Odd Multiple of π

Lemma 2. *Suppose v is not an odd multiple of π and let $\alpha = \frac{\sin(v)}{1+\cos(v)}$. Then*

1. $$\begin{aligned}\sum_{n=0}^{N}(-1)^n\cos(u+nv) &= \frac{1}{2}(\cos u + \alpha\sin u) \\ &\quad + \frac{(-1)^N}{2}(\cos(u+Nv) - \alpha\sin(u+Nv)).\end{aligned}$$
2. $$\begin{aligned}\sum_{n=0}^{N}(-1)^n\sin(u+nv) &= \frac{1}{2}(\sin u - \alpha\cos(u)) \\ &\quad + \frac{(-1)^N}{2}(\sin(u+Nv) + \alpha\cos(u+Nv)).\end{aligned}$$

Proof. The proof of the lemma is left as an exercise. □

Let $u = \beta t$ and $v = -\beta c$. Then $\alpha = \frac{-\sin(\beta c)}{1+\cos(\beta c)}$. In this case, we can apply part (1) of the lemma to (6) to get

$$\begin{aligned} f_2(t) &= \frac{r}{2\beta^2}\sum_{N=0}^{\infty}(\cos\beta t + \alpha\sin\beta t)\,\chi_{[Nc,N+1)c} \\ &\quad + \frac{r}{2\beta^2}\sum_{N=0}^{\infty}(-1)^N\,(\cos\beta(t-Nc) - \alpha\sin\beta(t-Nc))\,\chi_{[Nc,N+1)c} \\ &= \frac{r}{2\beta^2}(\cos\beta t + \alpha\sin\beta t) \\ &\quad + \frac{r}{2\beta^2}(-1)^{[t/c]_1}(\cos\beta\langle t\rangle_c - \alpha\sin\beta\langle t\rangle_c). \end{aligned}$$

Let

$$\begin{aligned} y_1(t) &= \frac{r}{\beta^2}\,\mathrm{sw}_c(t) - \frac{r}{2\beta^2}(-1)^{[t/c]_1}(\cos\beta\langle t\rangle_c - \alpha\sin\beta\langle t\rangle_c) \\ &= \frac{r}{2\beta^2}\left(2\,\mathrm{sw}_c(t) - (-1)^{[t/c]_1}(\cos\beta\langle t\rangle_c - \alpha\sin\beta\langle t\rangle_c)\right) \end{aligned}$$

and

$$y_2(t) = -\frac{r}{2\beta^2}(\cos\beta t + \alpha\sin\beta t).$$

Then

$$\begin{aligned} y(t) &= f_1(t) - f_2(t) = y_1(t) + y_2(t) \\ &= \frac{r}{2\beta^2}\left(2\,\mathrm{sw}_c(t) - (-1)^{[t/c]_1}(\cos\beta\langle t\rangle_c - \alpha\sin\beta\langle t\rangle_c)\right) \\ &\quad - \frac{r}{2\beta^2}(\cos\beta t + \alpha\sin\beta t). \end{aligned} \tag{7}$$

A quick check shows that y_1 is periodic with period $2c$ and y_2 is periodic with period $\frac{2\pi}{\beta}$. Clearly, y_2 is continuous, and since the solution $y(t)$ is continuous by Theorem 7 of Sect. 6.1, so is y_1. The following lemma will help us determine when y is a periodic solution.

Lemma 3. *Suppose g_1 and g_2 are continuous periodic functions with periods $p_1 > 0$ and $p_2 > 0$, respectively. Then $g_1 + g_2$ is periodic if and only if $\frac{p_1}{p_2}$ is a rational number.*

Proof. If $\frac{p_1}{p_2} = \frac{m}{n}$ is rational, then $np_1 = mp_2$ is a common period of g_1 and g_2 and hence is a period of $g_1 + g_2$. It follows that $g_1 + g_2$ is periodic. The opposite implication, namely, that the periodicity of $g_1 + g_2$ implies $\frac{p_1}{p_2}$ is rational, is a nontrivial fact. We do not include a proof. □

Using this lemma, we can determine precisely when the solution $y = y_1 + y_2$ is periodic. Namely, y is periodic precisely when $\frac{2c}{2\pi/\beta} = \frac{c\beta}{\pi}$ is rational. Consider

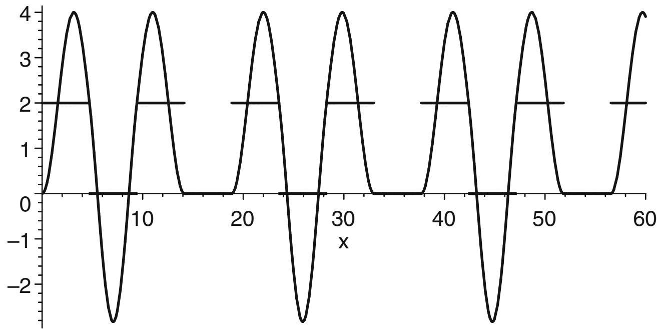


Fig. 6.30 The graph of (8) with $c = \frac{3\pi}{2}$: a periodic solution

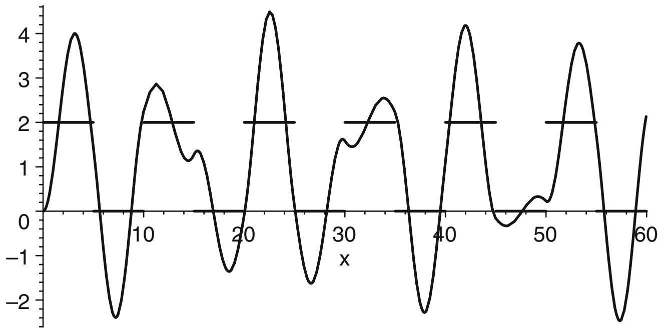


Fig. 6.31 The graph of (9): a nonperiodic solution

the following illustrative example. Set $r = 2$, $c = \frac{3\pi}{2}$, and $\beta = 1$. Then $\frac{c\beta}{\pi} = \frac{3}{2}$ is rational. Further, α, defined in Lemma 2, is 1 and

$$y(t) = 2\,\mathrm{sw}_c(t) - (-1)^{[t/c]_1}(\cos\langle t\rangle_c - \sin\langle t\rangle_c) - (\cos t + \sin t). \tag{8}$$

This function is graphed simultaneously with the forcing function in Fig. 6.30. The solution is periodic with period $4c = 6\pi$. Notice that there is an interval where the motion of the mass is stopped. This occurs in the interval $[3c, 4c)$. The constant force applied on the interval $[2c, 3c)$ gently stops the motion of the mass by the time $t = 3c$. Since the force is 0 on $[3c, 4c)$, there is no movement. At $t = 4c$, the force is reapplied and the process thereafter repeats itself. This phenomenon occurs in all cases where the solution y is periodic.

Now consider the following example that illustrates a nonperiodic solution. Set $r = 2$, $c = 5$, and $\beta = 1$ in (7). Then

$$y(t) = 2\,\mathrm{sw}_5(t) - (-1)^{[t/5]_1}\,(\cos\langle t\rangle_5 - \alpha\sin\langle t\rangle_5) - \cos t - \alpha\sin t, \tag{9}$$

where $\alpha = \frac{-\sin 5}{1+\cos 5}$. Further, $\frac{c\beta}{\pi} = \frac{5}{\pi}$ is irrational so $y(t)$ is not periodic. This is clearly seen in the rather erratic motion given by the graph of $y(t)$ in Fig. 6.31.

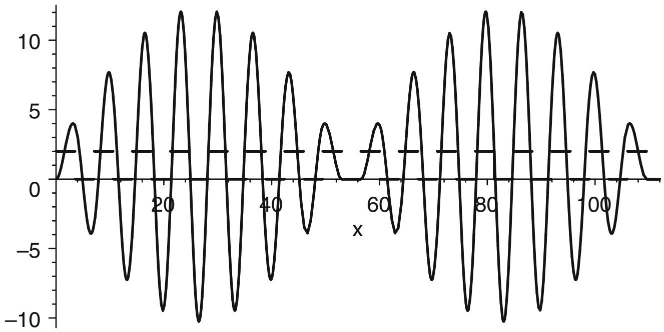


Fig. 6.32 The graph of (8) with $c = \frac{9\pi}{8}$: the beats are evident here

In Sect. 3.6, we observed that when the characteristic frequency of the spring is close to but not equal to the frequency of the forcing function, $\cos(\omega t)$, then one observes vibrations that exhibit a beat. This phenomenon likewise occurs for the square-wave forcing function. Let $r = 2$, $c = \frac{9\pi}{8}$, and $\beta = 1$. Recall that ***frequency*** is merely the reciprocal of the period so when these frequencies are close, so are their periods. The period of the spring is $\frac{2\pi}{\beta} = 2\pi$ while the period of the forcing function is $2c = \frac{9\pi}{4}$: their periods are close and likewise their frequencies. Figure 6.32 gives a graph of y in this case. Again it is evident that the motion of the mass stops on the last subinterval before the end of its period. More interesting is the fact that y oscillates with an amplitude that varies with time and produces “beats”.

βc Is an Odd Multiple of π

We now return to equation (6) in the case βc is an odd multiple of π. Things reduce substantially because $\cos(\beta t - N\beta c) = (-1)^N \cos(\beta t)$ and we get

$$\begin{aligned} f_2(t) &= \frac{r}{\beta^2} \sum_{N=0}^{\infty} \sum_{n=0}^{N} \cos(\beta t) \chi_{[Nc,(N+1)c)} \\ &= \frac{r}{\beta^2} \sum_{N=0}^{\infty} (N+1) \chi_{[Nc,(N+1)c)} \cos(\beta t) \\ &= \frac{r}{\beta^2} ([t/c]_1 + 1) \cos(\beta t). \end{aligned}$$

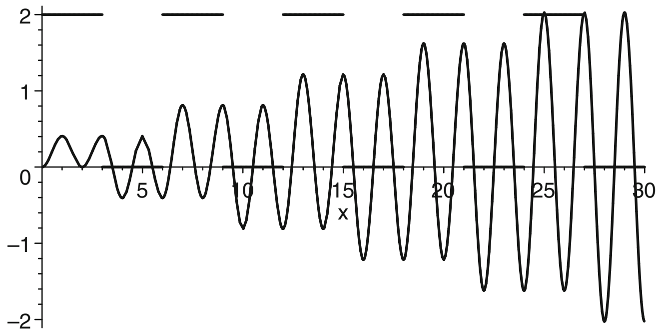


Fig. 6.33 The graph of (10) with $r = 2$, $\beta = \pi$, and $c = 3$: resonance is evident here

The solution now is

$$\begin{aligned} y(t) &= f_1(t) - f_2(t) \\ &= \frac{r}{\beta^2} \left(\mathrm{sw}_c(t) - [t/c]_1 \cos(\beta t) - \cos(\beta t) \right) \\ &= \frac{r}{\beta^2} \begin{cases} 1 - (n+1)\cos(\beta t) & \text{if } t \in [cn, c(n+1)),\ n \text{ even} \\ -(n+1)\cos(\beta t) & \text{if } t \in [cn, c(n+1)),\ n \text{ odd}. \end{cases} \end{aligned} \tag{10}$$

The presence of the factor $n + 1$ implies that $y(t)$ is unbounded. Figure 6.33 gives the graph of this in the case where $r = 2$, $\beta = \pi$, and $c = 3$. Resonance becomes clearly evident. Of course, this is an idealized situation; the system would eventually fail. ◄

Undamped Motion with Periodic Impulses

Example 4. A mass-spring system with no damping force is acted upon at rest by an impulse force of r units at all multiples of c units of time starting at $t = 0$. (Imagine a hammer exerting blows to the mass at regular intervals.) Describe the motion of the mass.

► **Solution.** The differential equation that describes this system is given by

$$y'' + \beta^2 y = r \sum_{n=0}^{\infty} \delta_{nc} \qquad y(0) = 0,\ y'(0) = 0,$$

where, again, β^2 is the spring constant. The Laplace transform gives

$$Y(s) = \frac{r}{\beta} \sum_{n=0}^{\infty} e^{-ncs} \frac{\beta}{s^2 + \beta^2}.$$

By Theorem 7 of Sect. 6.6,

$$y(t) = \frac{r}{\beta} \sum_{N=0}^{\infty} \sum_{n=0}^{N} \sin(\beta t - n\beta c) \chi_{[Nc,(N+1)c)}. \tag{11}$$

Again we will consider two cases. ◀

βc Is not a Multiple of 2π

Lemma 5. *Suppose v is not a multiple of 2π. Let $\gamma = \frac{\sin v}{1-\cos v}$. Then*

1. $\sum_{n=0}^{N} \sin(u + nv) = \frac{1}{2}(\sin u + \gamma \cos u + \sin(u + Nv) - \gamma \cos(u + Nv))$.

2. $\sum_{n=0}^{N} \cos(u + nv) = \frac{1}{2}(\cos u - \gamma \sin u + \cos(u + Nv) + \gamma \sin(u + Nv))$.

Let $u = \beta t$ and $v = -\beta c$. By the first part of Lemma 5, we get

$$\begin{aligned} y(t) &= \frac{r}{2\beta} \sum_{N=0}^{\infty} (\sin \beta t + \gamma \cos \beta t)\, \chi_{[Nc,(N+1)c)} \\ &\quad + \frac{r}{2\beta} \sum_{N=0}^{\infty} (\sin \beta(t - Nc) - \gamma \cos \beta(t - Nc))\, \chi_{[Nc,(N+1)c)} \\ &= \frac{r}{2\beta} (\sin \beta t + \gamma \cos \beta t + \sin \beta \langle t \rangle_c - \gamma \cos \beta \langle t \rangle_c), \end{aligned} \tag{12}$$

where $\gamma = \frac{-\sin \beta c}{1-\cos \beta c}$. Lemma 3 implies that the solution will be periodic when $\frac{c}{2\pi/\beta} = \frac{\beta c}{2\pi}$ is rational. Consider the following example. Let $r = 2$, $\beta = 1$, and $c = \frac{3\pi}{2}$. Equation (12) becomes

$$y(t) = \sin t + \cos t + \sin \langle t \rangle_c - \cos \langle t \rangle_c \tag{13}$$

and its graph is given in Fig. 6.34. The period is $6\pi = 4c$. Observe that on the interval $[3c, 4c)$, the motion of the mass is completely stopped. At $t = 3c$, the hammer strikes and imparts a velocity that stops the mass dead in its track. At $t = 4c$, the process begins to repeat itself. As in the previous example, this phenomenon occurs in all cases where the solution y is periodic, that is, when $\frac{c}{2\pi/(\beta)} = \frac{\beta c}{2\pi}$ is rational.

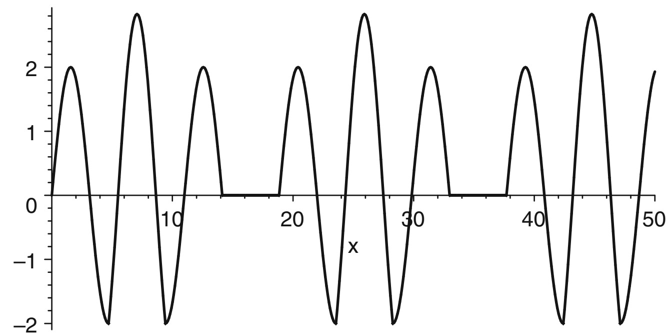


Fig. 6.34 The graph of (13): $c = \frac{3\pi}{2}$

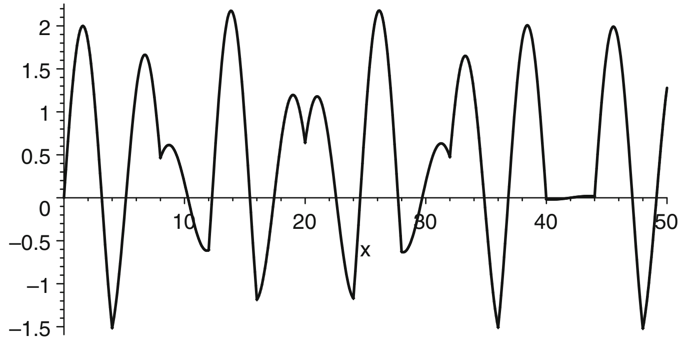


Fig. 6.35 The graph of (14): a nonperiodic solution

Now consider the following example that illustrates a nonperiodic solution. Set $r = 2$, $c = 4$, and $\beta = 1$ in (12). Then

$$y(t) = \sin t - \gamma \cos t + \sin\langle t\rangle_4 - \gamma \cos\langle t\rangle_4, \tag{14}$$

where $\gamma = \frac{-\sin 4}{1-\cos 4}$. Further, $\frac{c\beta}{2\pi} = \frac{2}{\pi}$ is irrational so $y(t)$ is not periodic. The graph of $y(t)$ in is given Fig. 6.35. Observe that the impulses given every 4 units suddenly changes the direction of the motion in a most erratic way.

When the period of the forcing function is close to that of the period of the spring, the beats in the solution can again be seen. For example, if $c = \frac{9}{8}(2\pi) = \frac{9\pi}{4}$, $\beta = 1$, and $r = 2$, then (12) becomes

$$y(t) = \sin t - \left(\sqrt{2} + 1\right) \cos t + \sin\langle t\rangle_c + \left(\sqrt{2} + 1\right) \cos\langle t\rangle_c \tag{15}$$

and Fig. 6.36 shows its graph.

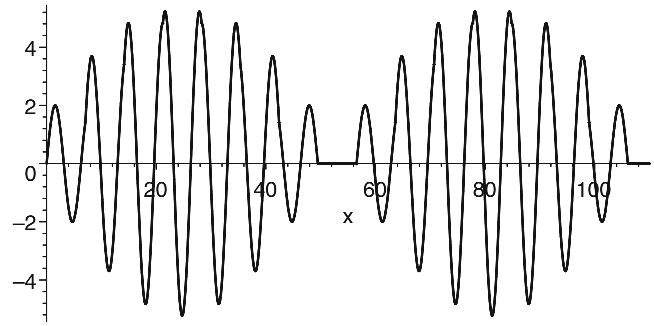


Fig. 6.36 The graph of (15) with $c = \frac{9\pi}{4}$. A solution that demonstrates beats

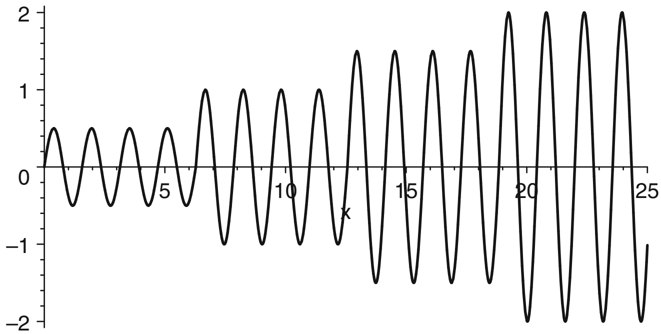


Fig. 6.37 The graph of (16) with $r = 2$, $c = 2\pi$, and $\beta = 4$. Resonance is evident

βc Is a Multiple of 2π

In this case, (11) simplifies to

$$y(t) = \frac{r}{\beta}\left(\sin\beta t + [t/c]_1 \sin\beta t\right) \tag{16}$$

$$= \frac{r}{\beta}(n+1)\sin\beta t \quad t \in [cn, c(n+1)). \tag{17}$$

The presence of the factor $n+1$ implies that $y(t)$ is unbounded; resonance is present. Figure 6.37 gives a graph of the solution when $c = 2\pi$, $\beta = 4$, and $r = 2$.

Exercises

1–6. For the parameters β, r, and c given in each problem below, determine the solution $y(t)$ to the differential equation

$$y'' + \beta^2 y = r\,\mathrm{sw}_c(t), \quad y(0) = 0,\ y'(0) = 0,$$

which models undamped motion with square-wave forcing function. Is the solution periodic or nonperiodic? Does it exhibit resonance?

1. $r = 4, c = 1, \beta = \sqrt{2}$
2. $r = 2, c = 2\pi, \beta = 1/\pi$
3. $r = 2, c = 2, \beta = \pi$
4. $r = 4, c = 1, \beta = \pi/2$
5. $r = 2, c = 1, \beta = \pi$
6. $r = 3, c = 5\pi, \beta = 1$

7–12. For the parameters β, r and c given in each problem below determine the solution $y(t)$ to the differential equation

$$y'' + \beta^2 y = r \sum_{n=0}^{\infty} \delta_{nc}, \quad y(0) = 0,\ y'(0) = 0,$$

which models undamped motion with a periodic impulse function. Is the solution periodic or non periodic? Does it exhibit resonance?

7. $r = 2, c = \pi, \beta = 1$
8. $r = 1, c = 2, \beta = \pi/4$
9. $r = 2, c = 1, \beta = 1$
10. $r = 2, c = \sqrt{2}, \beta = \pi/2$
11. $r = 2, c = 2\pi, \beta = 1$
12. $r = 2, c = 4, \beta = \pi$

13–16. Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

is very useful in establishing Lemmas 2 and 5. These exercises guide you through the verifications.

13. Suppose θ is not a multiple of 2π. Let $\gamma = \gamma(\theta) = \frac{\sin\theta}{1-\cos\theta}$. Use Euler's formula to show that

$$\sum_{n=0}^{N} \cos n\theta = \frac{1}{2}\left(1 + \cos N\theta + \gamma \sin N\theta\right),$$

$$\sum_{n=0}^{N} \sin n\theta = \frac{1}{2}\left(\sin N\theta + \gamma(1 - \cos N\theta)\right).$$

(Hint: Expand $\sum_{n=0}^{N} \left(e^{i\theta}\right)^n$ in two different ways and equate real and imaginary parts, use the formula $1 + x + x^2 + \cdots x^N = \frac{x^{N+1}-1}{x-1}$, and use the trigonometric sum and difference formulas.)

14. Suppose θ is not an odd multiple of π. Let $\alpha = \alpha(\theta) = \frac{\sin\theta}{1+\cos\theta}$. Show that

$$\sum_{n=0}^{N} (-1)^n \cos n\theta = \frac{1}{2}\left(1 + (-1)^N (\cos N\theta - \alpha \sin N\theta)\right),$$

$$\sum_{n=0}^{N} (-1)^n \sin n\theta = \frac{1}{2}\left(-\alpha + (-1)^N (\sin N\theta + \alpha \cos N\theta)\right).$$

15. Prove Lemma 5. Namely, suppose v is not a multiple of 2π. Let $\gamma = \frac{\sin v}{1-\cos v}$. Then

    1. $\sum_{n=0}^{N} \cos(u + nv) = \frac{1}{2}(\cos u - \gamma \sin u + \cos(u + Nv) + \gamma \sin(u + Nv))$.
    2. $\sum_{n=0}^{N} \sin(u + nv) = \frac{1}{2}(\sin u + \gamma \cos u + \sin(u + Nv) - \gamma \cos(u + Nv))$.

16. Prove Lemma 2. Namely, suppose v is not an odd multiple of π and let $\alpha = \frac{\sin(v)}{1+\cos(v)}$. Then

    1. $$\sum_{n=0}^{N} (-1)^n \cos(u + nv) = \frac{1}{2}(\cos u + \alpha \sin u) + \frac{(-1)^N}{2}(\cos(u + Nv) - \alpha \sin(u + Nv)).$$
    2. $$\sum_{n=0}^{N} (-1)^n \sin(u + nv) = \frac{1}{2}(\sin u - \alpha \cos(u)) + \frac{(-1)^N}{2}(\sin(u + Nv) + \alpha \cos(u + Nv)).$$

6.9 Summary of Laplace Transforms

Laplace transforms and convolutions presented in Chap. 6 are summarized in Tables 6.1–6.3.

Table 6.1 Laplace transform rules

$f(t)$	$F(s)$	Page
Second translation principle		
1. $f(t-c)h(t-c)$	$\mathrm{e}^{-sc}F(s)$	405
Corollary to the second translation principle		
2. $g(t)h(t-c)$	$\mathrm{e}^{-sc}\mathcal{L}\{g(t+c)\})$	405
Periodic functions		
3. $f(t)$, periodic with period p	$\dfrac{1}{1-\mathrm{e}^{-sp}}\int_0^p \mathrm{e}^{-st}f(t)\,\mathrm{d}t$	455
4. $f(\langle t\rangle_p)$	$\dfrac{1}{1-\mathrm{e}^{-sp}}\mathcal{L}\{f(t)-f(t)h(t-p)\}$	458
Staircase functions		
5. $f([t]_p)$	$\dfrac{1-\mathrm{e}^{-ps}}{s}\sum_{n=0}^{\infty} f(np)\mathrm{e}^{-nps}$	463
Transforms involving $\dfrac{1}{1\pm\mathrm{e}^{-sp}}$		
6. $\sum_{N=0}^{\infty}\sum_{n=0}^{N} f(t-np)\chi_{[Np,(N+1)p)}$	$\dfrac{1}{1-\mathrm{e}^{-sp}}F(s)$	461
7. $\sum_{N=0}^{\infty}\sum_{n=0}^{N}(-1)^n f(t-np)\chi_{[Np,(N+1)p)}$	$\dfrac{1}{1+\mathrm{e}^{-sp}}F(s)$	461

Table 6.2 Laplace transforms

$f(t)$	$F(s)$	Page
The Heaviside function		
1. $h(t-c)$	$\frac{e^{-sc}}{s}$	404
The on–off switch		
2. $\chi_{[a,b)}$	$\frac{e^{-as}}{s} - \frac{e^{-bs}}{s}$	405
The Dirac delta function		
3. δ_c	e^{-cs}	428
The square-wave function		
4. sw_c	$\frac{1}{1+e^{-cs}}\frac{1}{s}$	456
The sawtooth function		
5. $\langle t \rangle_p$	$\frac{1}{s^2}\left(1 - \frac{spe^{-sp}}{1-e^{-sp}}\right)$	457
Periodic Dirac delta functions		
6. $\delta_0(\langle t \rangle_p)$	$\frac{1}{1-e^{-ps}}$	459
Alternating periodic Dirac delta functions		
7. $(\delta_0 - \delta_p)(\langle t \rangle_{2p})$	$\frac{1}{1+e^{-ps}}$	459

Table 6.3 Convolutions

$f(t)$	$g(t)$	$(f * g)(t)$	Page
1. $f(t)$	$g(t)$	$f * g(t) = \int_0^t f(u)g(t-u)\,du$	439
2. f	$\delta_c(t)$	$f(t-c)h(t-c)$	444
3. f	$\delta_0(t)$	$f(t)$	445