# Chapter 1
# First Order Differential Equations

## 1.1 An Introduction to Differential Equations

Many problems of science and engineering require the description of some measurable quantity (position, temperature, population, concentration, electric current, etc.) as a function of time. Frequently, the scientific laws governing such quantities are best expressed as equations that involve the rate at which that quantity changes over time. Such laws give rise to differential equations. Consider the following three examples:

**Example 1 (Newton's Law of Heating and Cooling).** Suppose we are interested in the temperature of an object (e.g., a cup of hot coffee) that sits in an environment (e.g., a room) or space (called, ambient space) that is maintained at a constant temperature $T_{\mathrm{a}}$. ***Newton's law of heating and cooling*** states that the ***rate*** at which the temperature $T(t)$of the object changes is ***proportional*** to the ***temperature difference between*** the object and ambient space. Since rate of change of $T(t)$ is expressed mathematically as the derivative, $T'(t)$,[1] Newton's law of heating and cooling is formulated as the mathematical expression

$$T'(t) = r(T(t) - T_{\mathrm{a}}),$$

where $r$ is the constant of proportionality. Notice that this is an equation that relates the first derivative $T'(t)$ and the function $T(t)$ itself. It is an example of a differential equation. We will study this example in detail in Sect. 1.3.

**Example 2 (Radioactive decay).** Radioactivity results from the instability of the nucleus of certain atoms from which various particles are emitted. The atoms then

[1]In this text, we will generally use the prime notation, that is, $y'$, $y''$, $y'''$ (and $y^{(n)}$ for derivatives of order greater than 3) to denote derivatives, but the Leibnitz notation $\frac{\mathrm{d}y}{\mathrm{d}t}$, $\frac{\mathrm{d}^2 y}{\mathrm{d}t^2}$, etc. will also be used when convenient.

W.A. Adkins and M.G. Davidson, *Ordinary Differential Equations*, Undergraduate Texts in Mathematics, DOI 10.1007/978-1-4614-3618-8_1,

decay into other isotopes or even other atoms. *The **law of radioactive decay** states that the **rate** at which the radioactive atoms disintegrate is **proportional** to the total **number of radioactive atoms** present*. If $N(t)$ represents the number of radioactive atoms at time $t$, then the rate of change of $N(t)$ is expressed as the derivative $N'(t)$. Thus, the law of radioactive decay is expressed as the equation

$$N'(t) = -\lambda N(t).$$

As in the previous example, this is an equation that relates the first derivative $N'(t)$ and the function $N(t)$ itself, and hence is a differential equation. We will consider it further in Sect. 1.3.

As a third example, consider the following:

**Example 3 (Newton's Laws of Motion).** Suppose $s(t)$ is a position function of some body with mass $m$ as measured from some fixed origin. We assume that as time passes, forces are applied to the body so that it moves along some line. Its velocity is given by the first derivative, $s'(t)$, and its acceleration is given by the second derivative, $s''(t)$. ***Newton's second law of motion** states that the **net force** acting on the body is the **product of its mass and acceleration***. Thus,

$$ms''(t) = F_{\text{net}}(t).$$

Now in many circumstances, the net force acting on the body depends on time, the object's position, and its velocity. Thus, $F_{\text{net}}(t) = F(t, s(t), s'(t))$, and this leads to the equation

$$ms''(t) = F(t, s(t), s'(t)).$$

A precise formula for $F$ depends on the circumstances of the given problem. For example, the motion of a body in a spring-body-dashpot system is given by $ms''(t) + \mu s'(t) + ks(t) = f(t)$, where $\mu$ and $k$ are constants related to the spring and dashpot and $f(t)$ is some applied external (possibly) time-dependent force. We will study this example in Sect. 3.6. For now though, we just note that this equation relates the second derivative to the function, its derivative, and time. It too is an example of a differential equation.

Each of these examples illustrates two important points:

- Scientific laws regarding physical quantities are frequently expressed and best understood in terms of how that quantity changes.
- The mathematical model that expresses those changes gives rise to equations that involve derivatives of the quantity, that is, differential equations.

We now give a more formal definition of the types of equations we will be studying. An ***ordinary differential equation*** is an equation relating an unknown function $y(t)$, some of the derivatives of $y(t)$, and the variable $t$, which in many applied problems will represent time. The domain of the unknown function is some interval of the real line, which we will frequently denote by the symbol $I$.[2] The ***order*** of a differential equation is the order of the highest derivative that appears in the differential equation. Thus, the order of the differential equations given in the above examples is summarized in the following table:

| Differential equation | Order |
|---|---|
| $T'(t) = r(T(t) - T_{\mathrm{a}})$ | 1 |
| $N'(t) = -\lambda N(t)$ | 1 |
| $ms''(t) = F(t, s(t), s'(t))$ | 2 |

Note that $y(t)$ is our generic name for an unknown function, but in concrete cases, the unknown function may have a different name, such as $T(t)$, $N(t)$, or $s(t)$ in the examples above. The ***standard form*** for an ordinary differential equation is obtained by solving for the highest order derivative as a function of the unknown function $y = y(t)$, its lower order derivatives, and the independent variable $t$. Thus, a first order ordinary differential equation is expressed in standard form as

$$y'(t) = F(t,\ y(t)), \tag{1}$$

a second order ordinary differential equation in standard form is written

$$y''(t) = F(t,\ y(t),\ y'(t)), \tag{2}$$

and an $n$th order differential equation is expressed in standard form as

$$y^{(n)}(t) = F(t,\ y(t),\ \ldots,\ y^{(n-1)}(t)). \tag{3}$$

The standard form is simply a convenient way to be able to talk about various hypotheses to put on an equation to insure a particular conclusion, such as *existence and uniqueness of solutions* (discussed in Sect. 1.7) and to classify various types of equations (as we do in this chapter, for example) so that you will know which algorithm to apply to arrive at a solution. In the examples given above, the equations

$$T'(t) = r(T(t) - T_{\mathrm{a}}),$$
$$N'(t) = -\lambda N(t)$$

are in standard form while the equation in Example 3 is not. However, simply dividing by $m$ gives

$$s''(t) = \frac{1}{m}F(t,\ s(t),\ s'(t)),$$

a second order differential equation in standard form.

[2] Recall that the standard notations from calculus used to describe an interval $I$ are $(a,\ b)$, $[a,\ b)$, $(a,\ b]$, and $[a,\ b]$ where $a < b$ are real numbers. There are also the infinite length intervals $(-\infty,\ a)$ and $(a,\ \infty)$ where $a$ is a real number or $\pm\infty$.

In differential equations involving the unknown function $y(t)$, the variable $t$ is frequently referred to as the ***independent variable***, while $y$ is referred to as the ***dependent variable***, indicating that $y$ has a functional dependence on $t$. In writing ordinary differential equations, it is conventional to suppress the implicit functional evaluations $y(t)$, $y'(t)$, etc. and write $y$, $y'$, etc. Thus the differential equations in our examples above would be written

$$\begin{aligned} T' &= r(T - T_{\mathrm{a}}), \\ N' &= -\lambda N, \\ \text{and} \quad s'' &= \frac{1}{m} F(t, s, s'), \end{aligned}$$

where the dependent variables are respectively, $T$, $N$, and $s$.

Sometimes we must deal with functions $u = u(t_1, t_2, \ldots, t_n)$ of two or more variables. In this case, a ***partial differential equation*** is an equation relating $u$, some of the partial derivatives of $u$ with respect to the variables $t_1, \ldots, t_n$, and possibly the variables themselves. While there may be a time or two where we need to consider a partial differential equation, the focus of this text is on the study of ordinary differential equations. Thus, when we use the term differential equation without a qualifying adjective, you should assume that we mean *ordinary* differential equation.

**Example 4.** Consider the following differential equations. Determine their order, whether ordinary or partial, and the standard form where appropriate:

1. $y' = 2y$
2. $y' - y = t$
3. $y'' + \sin y = 0$
4. $y^{(4)} - y'' = y$
5. $ay'' + by' + cy = A\cos\omega t \quad (a \neq 0)$
6. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

► **Solution.** Equations (1)–(5) are ordinary differential equations while (6) is a partial differential equation. Equations (1) and (2) are first order, (3) and (5) are second order, and (4) is fourth order. Equation (1) is in standard form. The standard forms for (2)–(5) are as follows:

2. $y' = y + t$
3. $y'' = -\sin y$
4. $y^{(4)} = y'' + y$
5. $y'' = -\dfrac{b}{a}y' - \dfrac{c}{a}y + \dfrac{A}{a}\cos\omega t$ ◄

## *Solutions*

In contrast to algebraic equations, where the given and unknown objects are numbers, differential equations belong to the much wider class of ***functional***

***equations*** in which the given and unknown objects are functions (scalar functions or vector functions) defined on some interval. A ***solution of an ordinary differential equation*** is a function $y(t)$ defined on some specific interval $I \subseteq \mathbb{R}$ such that substituting $y(t)$ for $y$ and substituting $y'(t)$ for $y'$, $y''(t)$ for $y''$, etc. in the equation gives a ***functional identity***. That is, an identity which is satisfied for *all* $t \in I$. For example, if a first order differential equation is given in standard form as $y' = F(t, y)$, then a function $y(t)$ defined on an interval $I$ is a solution if

$$y'(t) = F(t, y(t)) \quad \text{for all } t \in I.$$

More generally, $y(t)$, defined on an interval $I$, is a solution of an $n$th order differential equation expressed in standard form by $y^{(n)} = F(t, y, y', \dots, y^{(n-1)})$ provided

$$y^{(n)}(t) = F(t, y(t), \dots, y^{(n-1)}(t)) \quad \text{for all } t \in I.$$

It should be noted that it is not necessary to express the given differential equation in standard form in order to check that a function is a solution. Simply substitute $y(t)$ and the derivatives of $y(t)$ into the differential equation as it is given. The ***general solution*** of a differential equation is the set of all solutions. As the following examples will show, writing down the general solution to a differential equation can range from easy to difficult.

**Example 5.** Consider the differential equation

$$y' = y - t. \tag{4}$$

Determine which of the following functions defined on the interval $(-\infty, \infty)$ are solutions:

1. $y_1(t) = t + 1$
2. $y_2(t) = e^t$
3. $y_3(t) = t + 1 - 7e^t$
4. $y_4(t) = t + 1 + ce^t$ where $c$ is an arbitrary scalar.

► **Solution.** In each case, we calculate the derivative and substitute the results in (4). The following table summarizes the needed calculations:

| Function | $y'(t)$ | $y(t) - t$ |
|---|---|---|
| $y_1(t) = t + 1$ | $y_1'(t) = 1$ | $y_1(t) - t = t + 1 - t = 1$ |
| $y_2(t) = e^t$ | $y_2'(t) = e^t$ | $y_2(t) - t = e^t - t$ |
| $y_3(t) = t + 1 - 7e^t$ | $y_3'(t) = 1 - 7e^t$ | $y_3(t) - t = t + 1 - 7e^t - t = 1 - 7e^t$ |
| $y_4(t) = t + 1 + ce^t$ | $y_4'(t) = 1 + ce^t$ | $y_4(t) - t = t + 1 + ce^t - t = 1 + ce^t$ |

For $y_i(t)$ to be a solution of (4), the second and third entries in the row for $y_i(t)$ must be the same. Thus, $y_1(t)$, $y_3(t)$, and $y_4(t)$ are solutions while $y_2(t)$ is not a

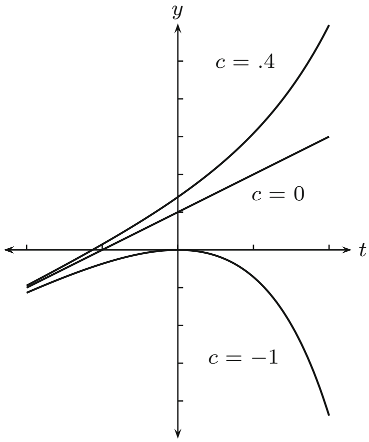



**Fig. 1.1** The solutions $y_g(t) = t + 1 + ce^t$ of $y' = y - t$ for various $c$

solution. Notice that $y_1(t) = y_4(t)$ when $c = 0$ and $y_3(t) = y_4(t)$ when $c = -7$. Thus, $y_4(t)$ actually already contains $y_1(t)$ and $y_3(t)$ by appropriate choices of the constant $c \in \mathbb{R}$, the real numbers. ◀

The differential equation given by (4) is an example of a first order *linear* differential equation. The theory of such equations will be discussed in Sect. 1.4, where we will show that *all* solutions to (4) are included in the function

$$y_4(t) = t + 1 + ce^t, \qquad t \in (-\infty, \infty)$$

of the above example by appropriate choice of the constant $c$. We call this the general solution of (4) and denote it by $y_g(t)$. Figure 1.1 is the graph of $y_g(t)$ for various choices of the constant $c$.

Observe that the general solution is parameterized by the constant $c$, so that there is a solution for each value of $c$ and hence there are infinitely many solutions of (4). This is characteristic of many differential equations. Moreover, the domain is the same for each of the solutions, namely, the entire real line. With the following example, there is a completely different behavior with regard to the domain of the solutions. Specifically, the domain of each solution varies with the parameter $c$ and is not the same interval for all solutions.

**Example 6.** Consider the differential equation

$$y' = -2t(1 + y)^2. \tag{5}$$

Show that the following functions are solutions:

1. $y_1(t) = -1$
2. $y_2(t) = -1 + (t^2 - c)^{-1}$, for any constant $c$

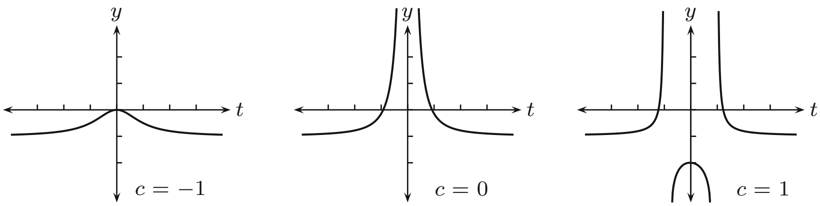



**Fig. 1.2** The solutions $y_2(t) = -1 + (t^2 - c)^{-1}$ of $y' = -2t(1 + y)^2$ for various $c$

► **Solution.** Let $y_1(t) = -1$. Then $y_1'(t) = 0$ and $-2t(1 + y_1(t))^2 = -2t(0) = 0$, which is valid for all $t \in (-\infty, \infty)$. Hence, $y_1(t) = -1$ is a solution.

Now let $y_2(t) = -1 + (t^2 - c)^{-1}$. Straightforward calculations give

$$\begin{aligned} y_2'(t) &= -2t(t^2 - c)^{-2}, \text{ and} \\ -2t(1 + y_2(t))^2 &= -2t(1 + (-1 + (t^2 - c)^{-1}))^2 = -2t(t^2 - c)^{-2}. \end{aligned}$$

Thus, $y_2'(t) = -2t(1 + y_2(t))^2$ so that $y_2(t)$ is a solution for any choice of the constant $c$. ◄

Equation (5) is an example of a *separable* differential equation. The theory of separable equations will be discussed in Sect. 1.3. It turns out that there are no solutions to (5) other than $y_1(t)$ and $y_2(t)$, so that these two sets of functions constitute the general solution $y_g(t)$. Notice that the intervals on which $y_2(t)$ is defined depend on the constant $c$. For example, if $c < 0$, then $y_2(t) = -1 + (t^2 - c)^{-1}$ is defined for all $t \in (-\infty, \infty)$. If $c = 0$, then $y_2(t) = -1 + t^{-2}$ is defined on two intervals: $t \in (-\infty, 0)$ or $t \in (0, \infty)$. Finally, if $c > 0$, then $y_2(t)$ is defined on three intervals: $(-\infty, -\sqrt{c})$, $(-\sqrt{c}, \sqrt{c})$, or $(\sqrt{c}, \infty)$. Figure 1.2 gives the graph of $y_2(t)$ for various choices of the constant $c$.

Note that the interval on which the solution $y(t)$ is defined is not at all apparent from looking at the differential equation (5).

**Example 7.** Consider the differential equation

$$y'' + 16y = 0. \tag{6}$$

Show that the following functions are solutions on the entire real line:

1. $y_1(t) = \cos 4t$
2. $y_2(t) = \sin 4t$
3. $y_3(t) = c_1 \cos 4t + c_2 \sin 4t$, where $c_1$ and $c_2$ are constants.

Show that the following functions are not solutions:

4. $y_4(t) = e^{4t}$
5. $y_5(t) = \sin t$.

► **Solution.** In standard form, (6) can be written as $y'' = -16y$, so for $y(t)$ to be a solution of this equation means that $y''(t) = -16y(t)$ for all real numbers $t$. The following calculations then verify the claims for the functions $y_i(t)$, $(1 \le i \le 5)$:

1. $y_1''(t) = \dfrac{d^2}{dt^2}(\cos 4t) = \dfrac{d}{dt}(-4\sin 4t) = -16\cos 4t = -16y_1(t)$
2. $y_2''(t) = \dfrac{d^2}{dt^2}(\sin 4t) = \dfrac{d}{dt}(4\cos 4t) = -16\sin 4t = -16y_2(t)$
3. $y_3''(t) = \dfrac{d^2}{dt^2}(c_1\cos 4t + c_2\sin 4t) = \dfrac{d}{dt}(-4c_1\sin 4t + 4c_2\cos 4t)$
   $= -16c_1\cos 4t - 16c_2\sin 4t = -16y_3(t)$
4. $y_4''(t) = \dfrac{d^2}{dt^2}(e^{4t}) = \dfrac{d}{dt}(4e^{4t}) = 16e^{4t} \neq -16y_4(t)$
5. $y_5''(t) = \dfrac{d^2}{dt^2}(\sin t) = \dfrac{d}{dt}(\cos t) = -\sin t \neq -16y_5(t)$ ◄

It is true, but not obvious, that letting $c_1$ and $c_2$ vary over all real numbers in $y_3(t) = c_1\cos 4t + c_2\sin 4t$ produces all solutions to $y'' + 16y = 0$, so that $y_3(t)$ is the general solution of (6) . This differential equation is an example of a second order *constant coefficient linear* differential equation. These equations will be studied in Chap. 3.

## *The Arbitrary Constants*

In Examples 5 and 6, we saw that the solution set of the given first order equation was parameterized by an arbitrary constant $c$ (although (5) also had an extra solution $y_1(t) = -1$), and in Example 7, the solution set of the second order equation was parameterized by two constants $c_1$ and $c_2$. To understand why these results are not surprising, consider what is arguably the simplest of all first order differential equations:

$$y' = f(t),$$

where $f(t)$ is some continuous function on some interval $I$. Integration of both sides produces a solution

$$y(t) = \int f(t)\,dt + c, \tag{7}$$

where $c$ is a constant of integration and $\int f(t)\,dt$ is any fixed antiderivative of $f(t)$. The fundamental theorem of calculus implies that all antiderivatives are of this form so (7) is the general solution of $y' = f(t)$. Generally speaking, solving any first order differential equation will implicitly involve integration. A similar calculation for the differential equation

$$y'' = f(t)$$

gives $y'(t) = \int f(t)\,\mathrm{d}t + c_1$ so that a second integration gives

$$\begin{aligned} y(t) = \int y'(t)\,\mathrm{d}t + c_2 &= \int \left( \int f(t)\,\mathrm{d}t + c_1 \right) \mathrm{d}t + c_2 \\ &= \int \left( \int f(t)\,\mathrm{d}t \right) \mathrm{d}t + c_1 t + c_2, \end{aligned}$$

where $c_1$ and $c_2$ are arbitrary scalars. The fact that we needed to integrate twice explains why there are two scalars. It is generally true that *the number of parameters (arbitrary constants) needed to describe the solution set of an ordinary differential equation is the same as the order of the equation.*

## *Initial Value Problems*

As we have seen in the examples of differential equations and their solutions presented in this section, differential equations generally have infinitely many solutions. So to specify a particular solution of interest, it is necessary to specify additional data. What is usually convenient to specify for a first order equation is an initial value $t_0$ of the independent variable and an initial value $y(t_0)$ for the dependent variable evaluated at $t_0$. For a second order equation, one would specify an initial value $t_0$ for the independent variable, together with an initial value $y(t_0)$ and an initial derivative $y'(t_0)$ at $t_0$. There is an obvious extension to higher order equations. When the differential equation and initial values are specified, one obtains what is known as an ***initial value problem***. Thus, a first order initial value problem in standard form is

$$y' = F(t, y), \quad y(t_0) = y_0, \tag{8}$$

while a second order equation in standard form is written

$$y'' = F(t, y, y'), \quad y(t_0) = y_0, \quad y'(t_0) = y_1. \tag{9}$$

**Example 8.** Determine a solution to each of the following initial value problems:

1. $y' = y - t, \quad y(0) = -3$
2. $y'' = 2 - 6t, \quad y(0) = -1,\; y'(0) = 2$

► **Solution.**

1. Recall from Example 5 that for each $c \in \mathbb{R}$, the function $y(t) = t + 1 + ce^t$ is a solution for $y' = y - t$. This is the function $y_4(t)$ from Example 5. Thus, our strategy is just to try to match one of the constants $c$ with the required initial condition $y(0) = -3$. Thus,

$$-3 = y(0) = 1 + ce^0 = 1 + c$$

requires that we take $c = -4$. Hence,

$$y(t) = t + 1 - 4e^t$$

is a solution of the initial value problem.

2. The second equation is asking for a function $y(t)$ whose second derivative is the given function $2 - 6t$. But this is precisely the type of problem we discussed earlier and that you learned to solve in calculus using integration. Integration of $y''$ gives

$$y'(t) = \int y''(t)\,dt + c_1 = \int (2 - 6t)\,dt + c_1 = 2t - 3t^2 + c_1,$$

and evaluating at $t = 0$ gives the equation

$$2 = y'(0) = (2t - 3t^2 + c_1)\big|_{t=0} = c_1.$$

Thus, $c_1 = 2$ and $y'(t) = 2t - 3t^2 + 2$. Now integrate again to get

$$y(t) = \int y'(t)\,dt = \int (2 + 2t - 3t^2)\,dt = 2t + t^2 - t^3 + c_0,$$

and evaluating at $t = 0$ gives the equation

$$-1 = y(0) = (2t + t^2 - t^3 + c_0)\big|_{t=0} = c_0.$$

Hence, $c_0 = -1$ and we get $y(t) = -1 + 2t + t^2 - t^3$ as the solution of our second order initial value problem. ◄

## *Some Concluding Comments*

Because of the simplicity of the second order differential equation in the previous example, we indicated a rather simple technique for solving it, namely, integration repeated twice. This was not possible for the other examples, even of first order equations, due to the functional dependencies between $y$ and its derivatives. In general, there is not a single technique that can be used to solve all differential equations, where by solve we mean to find an explicit functional description of the general solution $y_g(t)$ as an explicit function of $t$, possibly depending on some arbitrary constants. Such a $y_g(t)$ is sometimes referred to as a *closed form solution*. There are, however, solution techniques for certain types or categories of differential equations. In this chapter, we will study categories of first order differential equations such as:

- Separable
- Linear
- Homogeneous
- Bernoulli
- Exact

Each category will have its own distinctive solution technique. For higher order differential equations and systems of first order differential equations, the concept of *linearity* will play a very central role for it allows us to write the general solution in a concise way, and in the *constant coefficient* case, it will allow us to give a precise prescription for obtaining the solution set. This prescription and the role of the Laplace transform will occupy the two main important themes of the text. The role of the Laplace transform will be discussed in Chap. 2. In this chapter, however, we stick to a rather classical approach to first order differential equations and, in particular, we will discuss in the next section *direction fields* which allow us to give a pictorial explanation of solutions.

## Exercises

1–3. In each of these problems, you are asked to model a scientific law by means of a differential equation.

1. *Malthusian Growth Law*. Scientists who study populations (whether populations of people or cells in a Petri dish) observe that over small periods of time, the ***rate of growth*** *of the population is* ***proportional*** *to the* ***population present***. This law is called the ***Malthusian growth law***. Let $P(t)$ represent the number of individuals in the population at time $t$. Assuming the Malthusian growth law, write a differential equation for which $P(t)$ is the solution.
2. *The Logistic Growth Law*. The Malthusian growth law does not account for many factors affecting the growth of a population. For example, disease, overcrowding, and competition for food are not reflected in the Malthusian model. The goal in this exercise is to modify the Malthusian model to take into account the birth rate and death rate of the population. Let $P(t)$ denote the population at time $t$. Let $b(t)$ denote the birth rate and $d(t)$ the death rate at time $t$.

   (a) Suppose the birth rate is proportional to the population. Model this statement in terms of $b(t)$ and $P(t)$.
   (b) Suppose the death rate is proportional to the square of the population. Model this statement in terms of $d(t)$ and $P(t)$.
   (c) The logistic growth law states that the overall growth rate is the difference of the birth rate and death rate, as given in parts (a) and (b). Model this law as a differential equation in $P(t)$.

3. *Torricelli's Law*. Suppose a cylindrical container containing a fluid has a drain on the side. Torricelli's law states that the change in the height of the fluid above the middle of the drain is proportional to the square root of the height. Let $h(t)$ denote the height of the fluid above the middle of the drain. Determine a differential equation in $h(t)$ that models Torricelli's law.

4–11. Determine the order of each of the following differential equations. Write the equation in standard form.

4. $y^2 y' = t^3$
5. $y' y'' = t^3$
6. $t^2 y' + ty = e^t$
7. $t^2 y'' + ty' + 3y = 0$
8. $3y' + 2y + y'' = t^2$
9. $t(y^{(4)})^3 + (y''')^4 = 1$
10. $y' + t^2 y = ty^4$
11. $y''' - 2y'' + 3y' - y = 0$

12–18. Following each differential equation are four functions $y_1, \ldots, y_4$. Determine which are solutions to the given differential equation.

12. $y' = 2y$

    (a) $y_1(t) = 0$
    (b) $y_2(t) = t^2$
    (c) $y_3(t) = 3e^{2t}$
    (d) $y_4(t) = 2e^{3t}$

13. $ty' = y$

    (a) $y_1(t) = 0$
    (b) $y_2(t) = 3t$
    (c) $y_3(t) = -5t$
    (d) $y_4(t) = t^3$

14. $y'' + 4y = 0$

    (a) $y_1(t) = e^{2t}$
    (b) $y_2(t) = \sin 2t$
    (c) $y_3(t) = \cos(2t - 1)$
    (d) $y_4(t) = t^2$

15. $y' = 2y(y - 1)$

    (a) $y_1(t) = 0$
    (b) $y_2(t) = 1$
    (c) $y_3(t) = 2$
    (d) $y_4(t) = \frac{1}{1-e^{2t}}$

16. $2yy' = 1$

    (a) $y_1(t) = 1$
    (b) $y_2(t) = t$
    (c) $y_3(t) = \ln t$
    (d) $y_4(t) = \sqrt{t - 4}$

17. $2yy' = y^2 + t - 1$

    (a) $y_1(t) = \sqrt{-t}$
    (b) $y_2(t) = -\sqrt{e^t - t}$
    (c) $y_3(t) = \sqrt{t}$
    (d) $y_4(t) = -\sqrt{-t}$

18. $y' = \dfrac{y^2 - 4yt + 6t^2}{t^2}$

    (a) $y_1(t) = t$
    (b) $y_2(t) = 2t$
    (c) $y_3(t) = 3t$
    (d) $y_4(t) = \dfrac{3t + 2t^2}{1 + t}$

19–25. Verify that each of the given functions $y(t)$ is a solution of the given differential equation on the given interval $I$. Note that all of the functions depend on an arbitrary constant $c \in \mathbb{R}$.

19. $y' = 3y + 12; \quad y(t) = ce^{3t} - 4, \quad I = (-\infty, \infty)$
20. $y' = -y + 3t; \quad y(t) = ce^{-t} + 3t - 3 \quad I = (-\infty, \infty)$
21. $y' = y^2 - y; \quad y(t) = 1/(1 - ce^t) \quad I = (-\infty, \infty)$ if $c < 0$, $I = (-\ln c, \infty)$ if $c > 0$
22. $y' = 2ty; \quad y(t) = ce^{t^2}, \quad I = (-\infty, \infty)$
23. $y' = -e^y - 1; \quad y(t) = -\ln(ce^t - 1)$ with $c > 0, \quad I = (-\ln c, \infty)$
24. $(t+1)y' + y = 0; \quad y(t) = c(t+1)^{-1}, \quad I = (-1, \infty)$
25. $y' = y^2; \quad y(t) = (c - t)^{-1}, \quad I = (-\infty, c)$

26–31. Solve the following differential equations.

26. $y' = t + 3$
27. $y' = e^{2t} - 1$
28. $y' = te^{-t}$
29. $y' = \dfrac{t+1}{t}$
30. $y'' = 2t + 1$
31. $y'' = 6 \sin 3t$

32–38. Find a solution to each of the following initial value problems. See Exercises 19–31 for the general solutions of these equations.

32. $y' = 3y + 12, \quad y(0) = -2$
33. $y' = -y + 3t, \quad y(0) = 0$
34. $y' = y^2 - y, \quad y(0) = 1/2$
35. $(t+1)y' + y = 0, \quad y(1) = -9$
36. $y' = e^{2t} - 1, \quad y(0) = 4$
37. $y' = te^{-t}, \quad y(0) = -1$
38. $y'' = 6 \sin 3t, \quad y(0) = 1, y'(0) = 2$

## 1.2 Direction Fields

Suppose

$$y' = F(t, y) \tag{1}$$

is a first order differential equation (in standard form), where $F(t, y)$ is defined in some region of the $(t, y)$-plane. The geometric interpretation of the derivative of a function $y(t)$ at $t_0$ as the slope of the tangent line to the graph of $y(t)$ at $(t_0, y(t_0))$ provides us with an elementary and often very effective method for the visualization of the ***solution curves*** (:= graphs of solutions) to (1). The visualization process involves the construction of what is known as a ***direction field*** or ***slope field*** for the differential equation. For this construction, we proceed as follows.

### *Construction of Direction Fields*

1. Solve the given first order differential equation for $y'$ to put it in the standard form $y' = F(t, y)$.
2. Choose a grid of points in a rectangular region

$$\mathcal{R} = \{(t, y) : a \le t \le b;\ c \le y \le d\}$$

   in the $(t, y)$-plane where $F(t, y)$ is defined. This means imposing a graph-paper-like grid of vertical lines $t = t_i$ for $a = t_1 < t_2 < \cdots < t_N = b$ and horizontal lines $y = y_j$ for $c = y_1 < y_1 < \cdots < y_M = d$. The points $(t_i, y_j)$ where the grid lines intersect are the ***grid points***.
3. At each point $(t, y)$, the number $F(t, y)$ represents the slope of a solution curve through this point. For example, if $y' = y^2 - t$ so that $F(t, y) = y^2 - t$, then at the point $(1, 1)$ the slope is $F(1, 1) = 1^2 - 1 = 0$, at the point $(2, 1)$ the slope is $F(2, 1) = 1^2 - 2 = -1$, and at the point $(1, -2)$ the slope is $F(1, -2) = 3$.
4. Through the grid point $(t_i, y_j)$, draw a small line segment having the slope $F(t_i, y_j)$. Thus, for the equation $y' = y^2 - t$, we would draw a small line segment of slope 0 through $(1, 1)$, slope $-1$ through $(2, 1)$, and slope 3 through $(1, -2)$. With a graphing calculator, one of the computer mathematics programs Maple, Mathematica, or MATLAB, or with pencil, paper, and a lot of patience, you can draw line segments of the appropriate slope at all of the points of the chosen grid. The resulting picture is called a ***direction field*** for the differential equation $y' = F(t, y)$.
5. With some luck with respect to scaling and the selection of the $(t, y)$-rectangle $\mathcal{R}$, you will be able to visualize some of the line segments running together to make a graph of one of the solution curves.

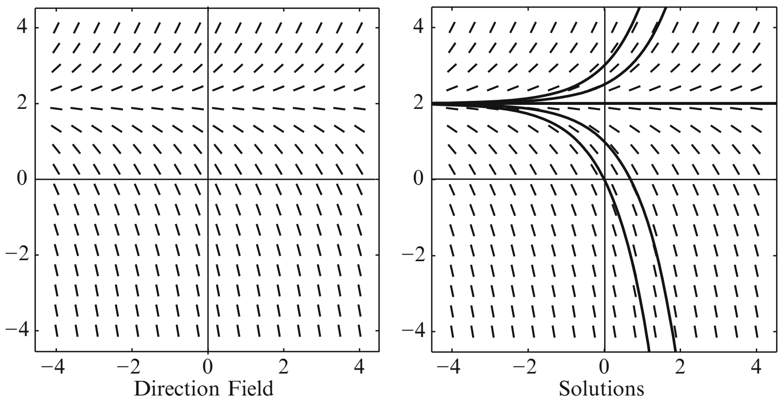



**Fig. 1.3** Direction field and some solutions for $y' = y - 2$

6. *To sketch a solution curve of* $y' = F(t, y)$ *from a direction field, start with a point* $P_0 = (t_0, y_0)$ *on the grid, and sketch a short curve through* $P_0$ *with tangent slope* $F(t_0, y_0)$*. Follow this until you are at or close to another grid point* $P_1 = (t_1, y_1)$*. Now continue the curve segment by using the updated tangent slope* $F(t_1, y_1)$*. Continue this process until you are forced to leave your sample rectangle* $\mathcal{R}$*.* The resulting curve will be an approximate solution to the initial value problem $y' = F(t, y)$, $y(t_0) = y_0$. Generally speaking, more accurate approximations are obtained by taking finer grids. The solutions are sometimes called ***trajectories***.

**Example 1.** Draw the direction field for the differential equation $y' = y - 2$. Draw several solution curves on the direction field.

► **Solution.** We have chosen a rectangle $\mathcal{R} = \{(t, y) : -4 \leq t, \ y \leq 4\}$ for drawing the direction field, and we have chosen to use 16 sample points in each direction, which gives a total of 256 grid points where a slope line will be drawn. Naturally, this is being done by computer and not by hand. Figure 1.3 gives the completed direction field with five solution curves drawn. The solutions that are drawn in are the solutions of the initial value problems

$$y' = y - 2, \qquad y(0) = y_0,$$

where the initial value $y_0$ is 0, 1, 2, 2.5, and 3, reading from the bottom solution to the top. ◄

You will note in this example that the line $y = 2$ is a solution. In general, any solution to (1) of the form $y(t) = y_0$, where $y_0$ is a constant, is called an ***equilibrium solution***. Its graph is called an ***equilibrium line***. Equilibrium solutions are those constant functions $y(t) = y_0$ determined by the constants $y_0$ for which $F(t, y_0) = 0$ for all $t$. For example, Newton's law of heating and cooling

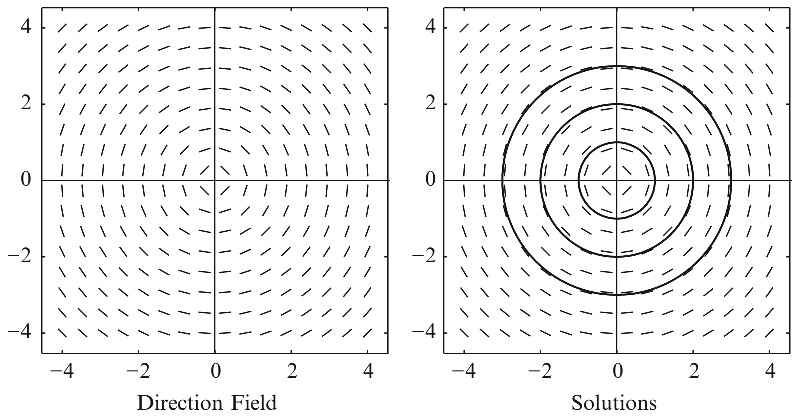



**Fig. 1.4** Direction field and some solutions for $y' = -t/y$

(Example 1 of Sect. 1.1) is modeled by the differential equation $T' = r(T - T_{\rm a})$ which has an equilibrium solution $T(t) = T_{\rm a}$. This conforms with intuition since if the temperature of the object and the temperature of ambient space are the same, then no change in temperature takes place. The object's temperature is then said to be in *equilibrium*.

**Example 2.** Draw the direction field for the differential equation $yy' = -t$. Draw several solution curves on the direction field and deduce the family of solutions.

► **Solution.** Before we can draw the direction field, it is necessary to first put the differential equation $yy' = -t$ into standard form by solving for $y'$. Solving for $y'$ gives the equation

$$y' = -\frac{t}{y}. \tag{2}$$

Notice that this equation is not defined for $y = 0$, even though the original equation is. Thus, we should be alert to potential problems arising from this defect. Again we have chosen a rectangle $\mathcal{R} = \{(t, y) : -4 \le t,\ y \le 4\}$ for drawing the direction field, and we have chosen to use 16 sample points in each direction. Figure 1.4 gives the completed direction field and some solutions. The solutions which are drawn in are the solutions of the initial value problems $yy' = -t$, $y(0) = \pm 1, \pm 2, \pm 3$. The solution curves appear to be circles centered at $(0, 0)$. In fact, the family of such circles is given by $t^2 + y^2 = c$, where $c > 0$. We can verify that functions determined implicity by the family of circles $t^2 + y^2 = c$ are indeed solutions. For, by implicit differentiation of the equation $t^2 + y^2$ (with respect to the $t$ variable), we get $2t + 2yy' = 0$ and solving for $y'$ gives (2). Solving $t^2 + y^2 = c$ implicitly for $y$ gives two families of continuous solutions, specifically, $y_1(t) = \sqrt{c - t^2}$ (upper semicircle) and $y_2(t) = -\sqrt{c - t^2}$ (lower semicircle). For both families of functions, $c$ is a positive constant and the functions are defined on the interval

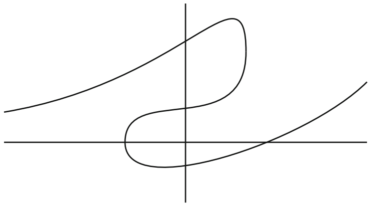


**Fig. 1.5** Graph of $f(t, y) = c$

$(-\sqrt{c},\ \sqrt{c})$. For the solutions drawn in Fig. 1.4, the constant $c$ is 1, $\sqrt{2}$, and $\sqrt{3}$. Notice that, although $y_1$ and $y_2$ are both defined for $t = \pm\sqrt{c}$, they do not satisfy the differential equation at these points since $y_1'$ and $y_2'$ do not exist at these points. Geometrically, this is a reflection of the fact that the circle $t^2 + y^2 = c$ has a vertical tangent at the points $(\pm\sqrt{c},\ 0)$ on the $t$-axis. This is the "defect" that you were warned could occur because the equation $yy' = -t$, when put in standard form $y' = -t/y$, is not defined for $y = 0$. ◀

Note that in the examples given above, *the solution curves do not intersect*. This is no accident. We will see in Sect. 1.7 that under mild smoothness assumptions on the function $F(t,\ y)$, it is absolutely certain that the solution curves (trajectories) of an equation $y' = F(t,\ y)$ can never intersect.

## *Implicitly Defined Solutions*

Example 2 is one of many examples where solutions are sometimes implicitly defined. Let us make a few general remarks when this occurs. Consider a relationship between the two variables $t$ and $y$ determined by the equation

$$f(t, y) = c. \tag{3}$$

We will say that a function $y(t)$ defined on an interval $I$ is ***implicitly defined*** by (3) provided

$$f(t,\ y(t)) = c \quad \text{for all } t \in I. \tag{4}$$

This is a precise expression of what we mean by the statement:

*Solve the equation $f(t,\ y) = c$ for $y$ as a function of $t$.*

To illustrate, we show in Fig. 1.5 a typical graph of the relation $f(t, y) = c$, for a particular $c$. We observe that there are three choices of solutions that are continuous functions. We have isolated these and call them $y_1(t)$, $y_2(t)$, and $y_3(t)$. The graphs of these are shown in Fig. 1.6. Observe that the maximal intervals of definition for $y_1$, $y_2$, and $y_3$ are not necessarily the same.

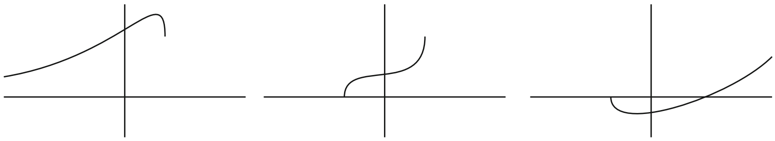



**Fig. 1.6** Graphs of functions implicitly defined by $f(t, y) = c$

By differentiating[3] (4) with respect to $t$ (using the chain rule from multiple variable calculus), we find

$$\frac{\partial f}{\partial t}(t,\ y(t)) + \frac{\partial f}{\partial y}(t,\ y(t))y'(t) = 0.$$

Since the constant $c$ is not present in this equation, we conclude that *every* function implicitly defined by the equation $f(t,\ y) = c$, for any constant $c$, is a solution of the same first order differential equation

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial y}y' = 0. \tag{5}$$

We shall refer to (5) as the ***differential equation for the family of curves*** $f(t,\ y) = c$. One valuable technique that we will encounter in Sect. 1.6 is that of solving a first order differential equation by recognizing it as the differential equation of a particular family of curves.

**Example 3.** Find the first order differential equation for the family of hyperbolas

$$ty = c$$

in the $(t,\ y)$-plane.

► **Solution.** Implicit differentiation of the equation $ty = c$ gives

$$y + ty' = 0$$

as the differential equation for this family. In standard form, this equation is $y' = -y/t$. Notice that this agrees with expectations, since for this simple family $ty = c$, we can solve explicitly to get $y = c/t$ (for $t \neq 0$) so that $y' = -c/t^2 = -y/t$.

◄

It may happen that it is possible to express the solution for the differential equation $y' = F(t,\ y)$ as an explicit formula, but the formula is sufficiently complicated that it does not shed much light on the nature of the solution. In such

[3] In practice, this is just implicit differentiation.

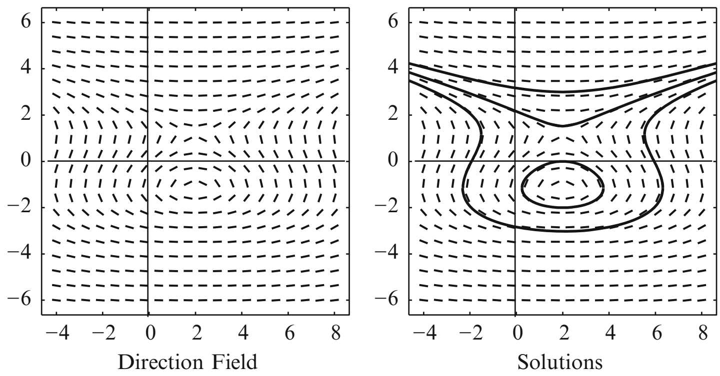



**Fig. 1.7** Direction field and some solutions for $y' = \frac{2t-4}{3y^2-4}$

a situation, constructing a direction field and drawing the solution curves on the direction field can sometimes give useful insight concerning the solutions. The following example is a situation where the picture is more illuminating than the formula.

**Example 4.** Verify that

$$y^3 - 4y - t^2 + 4t = c \tag{6}$$

defines an implicit family of solutions to the differential equation

$$y' = \frac{2t-4}{3y^2-4}.$$

► **Solution.** Implicit differentiation gives

$$3y^2y' - 4y' - 2t + 4 = 0,$$

and solving for $y'$, we get

$$y' = \frac{2t-4}{3y^2-4}.$$

Solving (6) involves a messy cubic equation which does not necessarily shed great light upon the nature of the solutions as functions of $t$. However, if we compute the direction field of $y' = \frac{2t-4}{3y^2-4}$ and use it to draw some solution curves, we see information concerning the nature of the solutions that is not easily deduced from the implicit form given in (6). For example, Fig. 1.7 gives the direction field and some solutions. Some observations that can be deduced from the picture are:

- In the lower part of the picture, the curves seem to be deformed ovals centered about the point $P \approx (2, -1.5)$.
- Above the point $Q \approx (2, 1.5)$, the curves no longer are closed but appear to increase indefinitely in both directions. ◄

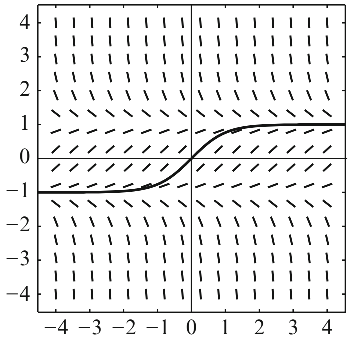
## Exercises

1–3. For each of the following differential equations, use some computer math program to sketch a direction field on the rectangle $\mathcal{R} = \{(t, y) : -4 \le t, y \le 4\}$ with integer coordinates as grid points. That is, $t$ and $y$ are each chosen from the set $\{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$.

1. $y' = t$
2. $y' = y^2$
3. $y' = y(y + t)$

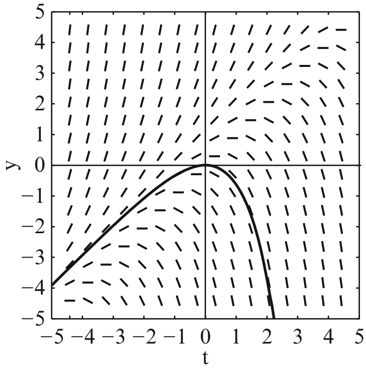
4–9. A differential equation is given together with its direction field. One solution is already drawn in. Draw the solution curves through the points $(t, y)$ as indicated. Keep in mind that the trajectories will not cross each other in these examples.

4. $y' = 1 - y^2$



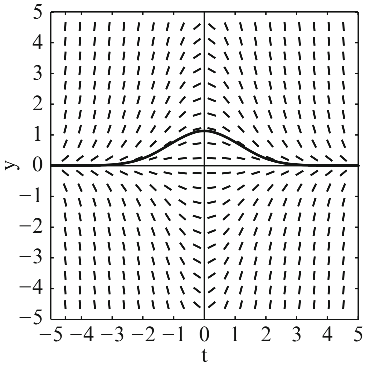

| $t$ | $y$ |
|---|---|
| 0 | 1 |
| 0 | -1 |
| 0 | 2 |
| -2 | 2 |
| -2 | -2 |
| 2 | -2 |

5. $y' = y - t$



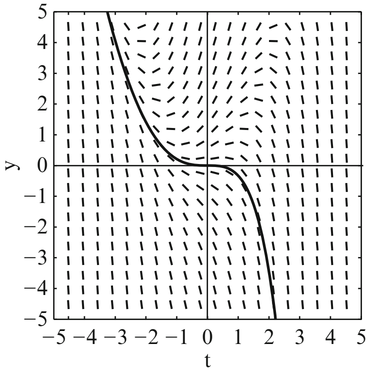

| $t$ | $y$ |
|---|---|
| 1 | 1 |
| 1 | 2 |
| 1 | 3 |
| 0 | -1 |
| 0 | -2 |
| 0 | -3 |

6. $y' = -ty$



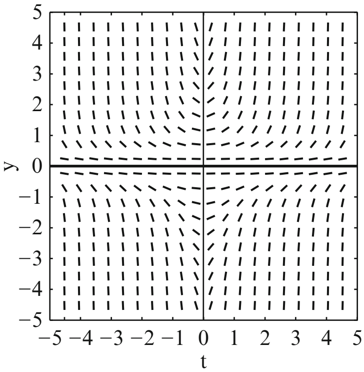

| $t$ | $y$ |
|---|---|
| 0 | 0 |
| 0 | 2 |
| 0 | -1 |
| -4.5 | 1 |

7. $y' = y - t^2$



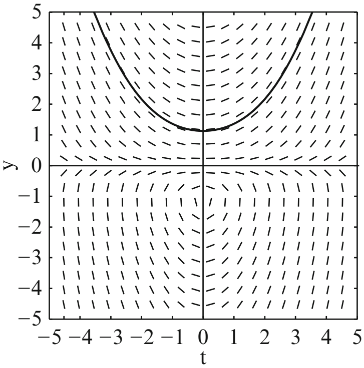

| $t$ | $y$ |
|---|---|
| 0 | 0 |
| 0 | 2 |
| -2 | 0 |
| 2 | 1 |

8. $y' = ty^2$




| $t$ | $y$ |
|---|---|
| -3 | 1 |
| 0 | 1 |
| 0 | 2 |
| 0 | -1 |
| 0 | -3 |

9. $y' = \dfrac{ty}{1+y}$

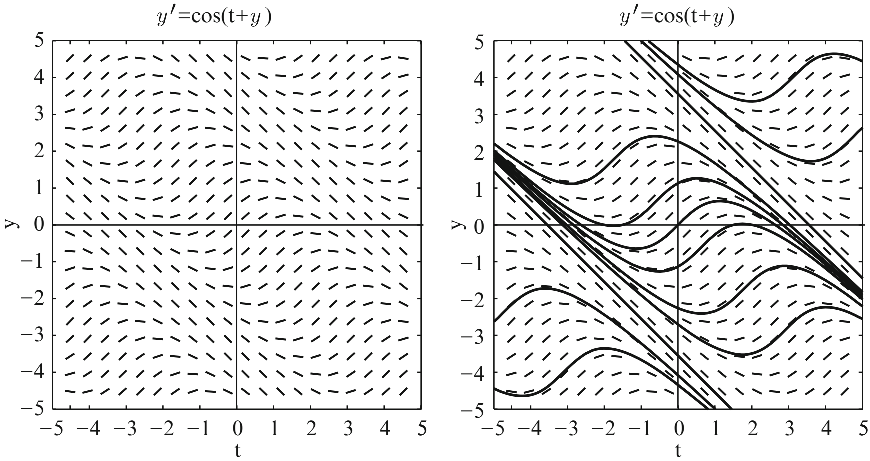


| $t$ | $y$ |
|---|---|
| 0 | 0 |
| 0 | -2 |
| 0 | -3 |

10–13. For the following differential equations, determine the equilibrium solutions, if they exist.

10. $y' = y^2$
11. $y' = y(y + t)$
12. $y' = y - t$
13. $y' = 1 - y^2$

14. The direction field given in Problem 5 for $y' = y - t$ suggests that there may be a linear solution. That is a solution of the form $y = at + b$. Find such a solution.
15. Below is the direction field and some trajectories for $y' = \cos(y + t)$. The trajectories suggest that there are linear solutions that act as asymptotes for the nonlinear trajectories. Find these linear solutions.




Direction Field Solutions

16–19. Find the first order differential equation for each of the following families of curves. In each case, $c$ denotes an arbitrary real constant.

16. $3t^2 + 4y^2 = c$
17. $y^2 - t^2 - t^3 = c$
18. $y = ce^{2t} + t$
19. $y = ct^3 + t^2$

## 1.3 Separable Differential Equations

In the next few sections, we will concentrate on solving particular categories of first order differential equations by means of explicit formulas and algorithms. These categories of equations are described by means of restrictions on the function $F(t, y)$ that appears on the right-hand side of a first order ordinary differential equation given in standard form

$$y' = F(t, y). \tag{1}$$

The first of the standard categories of first order equations to be studied is the class of equations with ***separable variables***, that is, equations of the form

$$y' = h(t)g(y). \tag{2}$$

Such an equation is said to be a ***separable differential equation*** or just ***separable***, for short. Thus, (1) *is separable if the right-hand side $F(t, y)$ can be written as a* ***product*** *of a function of $t$ and a function of $y$*. Most functions of two variables cannot be written as such a product, so being separable is rather special. However, a number of important applied problems turn out to be modeled by separable differential equations. We will explore some of these at the end of this section and in the exercises.

**Example 1.** Identify the separable equations from among the following list of differential equations:

| | | | |
|---|---|---|---|
| 1. | $y' = t^2y^2$ | 2. | $y' = y - y^2$ |
| 3. | $y' = \dfrac{t-y}{t+y}$ | 4. | $y' = \dfrac{t}{y}$ |
| 5. | $(2t-1)(y^2-1)y' + t - y - 1 + ty = 0$ | 6. | $y' = f(t)$ |
| 7. | $y' = p(t)y$ | 8. | $y'' = ty$ |

► **Solution.** Equations (1), (2) and (4)–(7) are separable. For example, in (2), $h(t) = 1$ and $g(y) = y - y^2$; in (4), $h(t) = t$ and $g(y) = 1/y$; and in (6), $h(t) = f(t)$ and $g(y) = 1$. To see that (5) is separable, we bring all terms not containing $y'$ to the other side of the equation, that is,

$$(2t-1)(y^2-1)y' = -t + y + 1 - ty = -t(1+y) + 1 + y = (1+y)(1-t).$$

Solving this equation for $y'$ gives

$$y' = \frac{(1-t)}{(2t-1)} \cdot \frac{(1+y)}{(y^2-1)},$$

which is separable with $h(t) = (1-t)/(2t-1)$ and $g(y) = (1+y)/(y^2-1)$. Equation (3) is not separable because the right-hand side cannot be written as product of a function of $t$ and a function of $y$. Equation (8) is not a separable equation, even though the right-hand side is $ty = h(t)g(y)$, since it is a *second* order equation and our definition of separable applies only to first order equations. ◄

Equation (2) in the previous example is worth emphasizing since it is typical of many commonly occurring separable differential equations. What is special is that it has the form

$$y' = g(y), \tag{3}$$

where the right-hand side depends only on the dependent variable $y$. That is, in (2), we have $h(t) = 1$. Such an equation is said to be ***autonomous*** or time independent. Some concrete examples of autonomous differential equations are the law of radioactive decay, $N' = -\lambda N$; Newton's law of heating and cooling, $T' = r(T - T_{\mathrm{a}})$; and the logistic growth model equation $P' = (a - bP)P$. These examples will be studied later in this section.

To motivate the general algorithm for solving separable differential equations, let us first consider a simple example.

**Example 2.** Solve

$$y' = -2t(1+y)^2. \tag{4}$$

(See Example 6 of Sect. 1.1 where we considered this equation.)

► **Solution.** This is a separable differential equation with $h(t) = -2t$ and $g(y) = (1+y)^2$. We first note that $y(t) = -1$ is an equilibrium solution. (See Sect. 1.2.) To proceed, assume $y \neq -1$. If we use the Leibniz form for the derivative, $y' = \frac{\mathrm{d}y}{\mathrm{d}t}$, then (4) can be rewritten as

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -2t(1+y)^2.$$

Dividing by $(1+y)^2$ and multiplying by $\mathrm{d}t$ give

$$(1+y)^{-2}\,\mathrm{d}y = -2t\,\mathrm{d}t. \tag{5}$$

Now integrate both sides to get

$$-(1+y)^{-1} = -t^2 + c,$$

where $c$ is the combination of the arbitrary constants of integration from both sides. To solve for $y$, multiply both sides by $-1$, take the reciprocal, and then add $-1$.

We then get $y = -1 + (t^2 - c)^{-1}$, where $c$ is an arbitrary scalar. Remember that we have the equilibrium solution $y = -1$ so the solution set is

$$\begin{aligned} y &= -1 + (t^2 - c)^{-1}, \\ y &= -1, \end{aligned}$$

where $c \in \mathbb{R}$. ◀

We note that the $t$ and $y$ variables in (5) have been *separated* by the equal sign, which is the origin of the name of this category of differential equations. The left-hand side is a function of $y$ times $\mathrm{d}y$ and the right-hand side is a function of $t$ times $\mathrm{d}t$. This process allows separate integration to give an implicit relationship between $t$ and $y$. This can be done more generally as outlined in the following algorithm.

**Algorithm 3.** To solve a separable differential equation,

$$y' = h(t)g(y),$$

perform the following operations:

**Solution Method for Separable Differential Equations**

1. *Determine the equilibrium solutions*. These are all of the constant solutions $y = y_0$ and are determined by solving the equation $g(y) = 0$ for $y_0$.
2. *Separate the variables in a form convenient for integration*. That is, we formally write

$$\frac{1}{g(y)}\,\mathrm{d}y = h(t)\,\mathrm{d}t$$

and refer to this equation as the ***differential form*** of the separable differential equation.
3. *Integrate both sides, the left-hand side with respect to $y$ and the right-hand side with respect to $t$.*[4] This yields

$$\int \frac{1}{g(y)}\,\mathrm{d}y = \int h(t)\,\mathrm{d}t,$$

which produces the implicit solution

$$Q(y) = H(t) + c,$$

where $Q(y)$ is an antiderivative of $1/g(y)$ and $H(t)$ is an antiderivative of $h(t)$. Such antiderivatives differ by a constant $c$.
4. (*If possible, solve the implicit relation explicitly for $y$.*) □

Note that Step 3 is valid as long as the antiderivatives exist on an interval. From calculus, we know that an antiderivative exists on an interval as long as the integrand is a continuous function on that interval. Thus, it is sufficient that $h(t)$ and $g(y)$ are continuous on appropriate intervals in $t$ and $y$, respectively, and we will also need $g(y) \neq 0$ in order for $1/g(y)$ to be continuous.

In the following example, please note in the algebra how we carefully track the evolution of the constant of integration and how the equilibrium solution is folded into the general solution set.

**Example 4.** Solve

$$y' = 2ty. \tag{6}$$

► **Solution.** This is a separable differential equation: $h(t) = 2t$ and $g(y) = y$. Clearly, $y = 0$ is an equilibrium solution. Assume now that $y \neq 0$. Then (6) can be rewritten as $\frac{\mathrm{d}y}{\mathrm{d}t} = 2ty$. Separating the variables by dividing by $y$ and multiplying by $\mathrm{d}t$ gives

$$\frac{1}{y}\,\mathrm{d}y = 2t\,\mathrm{d}t.$$

Integrating both sides gives

$$\ln|y| = t^2 + k_0, \tag{7}$$

where $k_0$ is an arbitrary constant. We thus obtain a family of implicitly defined solutions $y$. In this example, we will not be content to leave our answer in this implicit form but rather we will solve explicitly for $y$ as a function of $t$. Carefully note the sequence of algebraic steps we give below. This same algebra is needed in several examples to follow. We first exponentiate both sides of (7) (remembering that $\mathrm{e}^{\ln x} = x$ for all *positive* $x$, and $\mathrm{e}^{a+b} = \mathrm{e}^a\mathrm{e}^b$ for all $a$ and $b$) to get

$$|y| = \mathrm{e}^{\ln|y|} = \mathrm{e}^{t^2+k_0} = \mathrm{e}^{k_0}\mathrm{e}^{t^2} = k_1\mathrm{e}^{t^2}, \tag{8}$$

where $k_1 = \mathrm{e}^{k_0}$ is a *positive* constant, since the exponential function is positive. Next we get rid of the absolute values to get

$$y = \pm|y| = \pm k_1\mathrm{e}^{t^2} = k_2\mathrm{e}^{t^2}, \tag{9}$$

[4]Technically, we are treating $y = y(t)$ as a function of $t$ and both sides are integrated with respect to $t$, but the left-hand side becomes an integral with respect to $y$ using the change of variables $y = y(t)$, $\mathrm{d}y = y'\mathrm{d}t$.

where $k_2 = \pm k_1$ is a *nonzero* real number. Now note that the equilibrium solution $y = 0$ can be absorbed into the family $y = k_2 e^{t^2}$ by allowing $k_2 = 0$. Thus, the solution set can be written

$$y = ce^{t^2}, \tag{10}$$

where $c$ is an *arbitrary* constant. ◀

**Example 5.** Find the solutions of the differential equation

$$y' = \frac{-t}{y}.$$

(This example was considered in Example 2 of Sect. 1.2 via direction fields.)

► **Solution.** We first rewrite the equation in the form $\frac{\mathrm{d}y}{\mathrm{d}t} = -t/y$ and separate the variables to get

$$y\,\mathrm{d}y = -t\,\mathrm{d}t.$$

Integration of both sides gives $\int y\,\mathrm{d}y = -\int t\,\mathrm{d}t$ or $\frac{1}{2}y^2 = -\frac{1}{2}t^2 + c$. Multiplying by 2 and adding $t^2$ to both sides, we get

$$y^2 + t^2 = c,$$

where we write $c$ instead of $2c$ since twice an arbitrary constant $c$ is still an arbitrary constant. This is the standard equation for a circle of radius $\sqrt{c}$ centered at the origin, for $c > 0$. Solving for $y$ gives

$$y = \pm\sqrt{c - t^2},$$

the equations for the half circles we obtained in Example 2 of Sect. 1.2. ◀

It may happen that a formula solution for the differential equation $y' = F(t, y)$ is possible, but the formula is sufficiently complicated that it does not shed much light on the nature of the solutions. In such a situation, it may happen that constructing a direction field and drawing the solution curves on the direction field gives useful insight concerning the solutions. The following example is such a situation.

**Example 6.** Find the solutions of the differential equation

$$y' = \frac{2t - 4}{3y^2 - 4}.$$

► **Solution.** Again we write $y'$ as $\frac{\mathrm{d}y}{\mathrm{d}t}$ and separate the variables to get

$$(3y^2 - 4)\,\mathrm{d}y = (2t - 4)\,\mathrm{d}t.$$

Integration gives

$$y^3 - 4y = t^2 - 4t + c.$$

Solving this cubic equation explicitly for $y$ is possible, but it is complicated and not very revealing so we shall leave our solution in implicit form.[5] The direction field for this example was given in Fig. 1.7. As discussed in Example 6 of Sect. 1.2, the direction field reveals much more about the solutions than the explicit formula derived from the implicit formula given above. ◀

**Example 7.** Solve the initial value problem

$$y' = \frac{y^2+1}{t^2}, \qquad y(1) = 1/\sqrt{3}.$$

Determine the maximum interval on which this solution is defined.

► **Solution.** Since $y^2 + 1 \geq 1$, there are no equilibrium solutions. Separating the variables gives

$$\frac{\mathrm{d}y}{y^2+1} = \frac{\mathrm{d}t}{t^2},$$

and integration of both sides gives $\tan^{-1} y = -\frac{1}{t} + c$. In this case, it is a simple matter to solve for $y$ by applying the tangent function to both sides of the equation. Since $\tan(\tan^{-1} y) = y$, we get

$$y(t) = \tan\left(-\frac{1}{t} + c\right).$$

To find $c$, observe that $1/\sqrt{3} = y(1) = \tan(-1 + c)$, which implies that $c - 1 = \pi/6$, so $c = 1 + \pi/6$. Hence,

$$y(t) = \tan\left(-\frac{1}{t} + 1 + \frac{\pi}{6}\right).$$

To determine the maximum domain on which this solution is defined, note that the tangent function is defined on the interval $(-\pi/2, \pi/2)$, so that $y(t)$ is defined for all $t$ satisfying

$$-\frac{\pi}{2} < -\frac{1}{t} + 1 + \frac{\pi}{6} < \frac{\pi}{2}.$$

[5]The formula for solving a cubic equation is known as Cardano's formula after Girolamo Cardano (1501–1576), who was the first to publish it.

Since $-\frac{1}{t} + 1 + \frac{\pi}{6}$ is increasing and the limit as $t \to \infty$ is $1 + \frac{\pi}{6} < \frac{\pi}{2}$, the second of the above inequalities is valid for all $t > 0$. The first inequality is solved to give $t > 3/(3 + 2\pi)$. Thus, the maximum domain for the solution $y(t)$ is the interval $(3/(3 + 2\pi), \infty)$. ◄

## *Radioactive Decay*

In Example 2 of Sect. 1.1, we discussed the law of radioactive decay, which states: *If $N(t)$ is the quantity*[6] *of radioactive isotopes at time $t$, then the rate of decay is proportional to $N(t)$*. Since rate of change is expressed as the derivative with respect to the time variable $t$, it follows that $N(t)$ is a solution to the differential equation

$$N' = -\lambda N,$$

where $\lambda$ is a constant. You will recognize this as a separable differential equation, which can be written in differential form with variables separated as

$$\frac{\mathrm{d}N}{N} = -\lambda\,\mathrm{d}t.$$

Integrating the left side with respect to $N$ and the right side with respect to $t$ leads to $\ln|N| = -\lambda t + k_0$, where $k_0$ is an arbitrary constant. As in Example 4, we can solve for $N$ as a function of $t$ by applying the exponential function to both sides of $\ln|N| = -\lambda t + k_0$. This gives

$$|N| = \mathrm{e}^{\ln|N|} = \mathrm{e}^{-\lambda t + k_0} = \mathrm{e}^{k_0}\mathrm{e}^{-\lambda t} = k_1\mathrm{e}^{-\lambda t},$$

where $k_1 = \mathrm{e}^{k_0}$ is a positive constant. Since $N = \pm|N| = \pm k_1\mathrm{e}^{-\lambda t}$, we conclude that $N(t) = c\mathrm{e}^{-\lambda t}$, where $c$ is an arbitrary constant. Notice that this family includes the equilibrium solution $N = 0$ when $c = 0$. Further, at time $t = 0$, we have $N(0) = c\mathrm{e}^0 = c$. *The constant $c$ therefore represents the quantity of radioactive isotopes at time $t = 0$ and is denoted $N_0$*. In summary, we find

$$N(t) = N_0\mathrm{e}^{-\lambda t}. \tag{11}$$

The constant $\lambda$ is referred to as the ***decay constant***. However, most scientists prefer to specify the rate of decay by another constant known as the ***half-life*** of the radioactive isotope. The half-life is the time, $\tau$, it takes for half the quantity to decay. Thus, $\tau$ is determined by the equation $N(\tau) = \frac{1}{2}N_0$ or $N_0\mathrm{e}^{-\lambda\tau} = \frac{1}{2}N_0$. Eliminating

[6] $N(t)$ could represent the number of atoms or the mass of radioactive material at time $t$.

$N_0$ gives $e^{-\lambda\tau} = \frac{1}{2}$, which can be solved for $\tau$ by applying the logarithm function ln to both sides of the equation to get $-\lambda\tau = \ln(1/2) = -\ln 2$ so

$$\tau = \frac{\ln 2}{\lambda}, \tag{12}$$

an inverse proportional relationship between the half-life and decay constant. Note, in particular, that the half-life does not depend on the initial amount $N_0$ present. Thus, it takes the same length of time to decay from 2 g to 1 g as it does to decay from 1 g to 1/2 g.

Carbon 14, $^{14}C$, is a radioactive isotope of carbon that decays into the stable nonradioactive isotope nitrogen-14, $^{14}N$, and has a half-life of 5730 years. Plants absorb atmospheric carbon during photosynthesis, so the ratio of $^{14}C$ to normal carbon in plants and animals when they die is approximately the same as that in the atmosphere. However, the amount of $^{14}C$ decreases after death from radioactive decay. Careful measurements allow the date of death to be estimated.[7]

**Example 8.** A sample of wood from an archeological site is found to contain 75% of $^{14}C$ (per unit mass) as a sample of wood found today. What is the age of the wood sample.

► **Solution.** At $t = 0$, the total amount, $N_0$, of carbon-14 in the wood sample begins to decay. Now the quantity is $0.75N_0$. This leads to the equation $0.75N_0 = N_0 e^{-\lambda t}$. Solving for $t$ gives $t = -\frac{\ln 0.75}{\lambda}$. Since the half-life of $^{14}C$ is 5730 years, (12) gives the decay constant $\lambda = \frac{\ln 2}{5730}$. Thus, the age of the sample is

$$t = 5730 - \frac{\ln 0.75}{\ln 2} \approx 2378$$

years. If the wood sample can be tied to the site, then an archeologist might be able to conclude that the site is about 2378 years old. ◄

## *Newton's Law of Heating and Cooling*

Recall that Newton's law of heating and cooling (Example 1 of Sect. 1.1) states that *the rate of change of the temperature, $T(t)$, of a body is proportional to the*

[7]There are limitations, however. The ratio of $^{14}C$ to other forms of carbon in the atmosphere is not constant as originally supposed. This variation is due, among other things, to changes in the intensity of the cosmic radiation that creates $^{14}C$. To compensate for this variation, dates obtained from radiocarbon laboratories are now corrected using standard calibration tables.

*difference between that body and the temperature of ambient space*. This law thus states that $T(t)$ is a solution to the differential equation

$$T' = r(T - T_{\mathrm{a}}),$$

where $T_{\mathrm{a}}$ is the ambient temperature and $r$ is the proportionality constant. This equation is separable and can be written in differential form as

$$\frac{\mathrm{d}T}{T - T_{\mathrm{a}}} = r\,\mathrm{d}t.$$

Integrating both sides leads to $\ln|T - T_{\mathrm{a}}| = rt + k_0$, where $k_0$ is an arbitrary constant. Solving for $T - T_{\mathrm{a}}$ is very similar to the algebra we did in Example 4. We can solve for $T - T_{\mathrm{a}}$ as a function of $t$ by applying the exponential function to both sides of $\ln|T - T_{\mathrm{a}}| = rt + k_0$. This gives

$$|T - T_{\mathrm{a}}| = \mathrm{e}^{\ln|T - T_{\mathrm{a}}|} = \mathrm{e}^{rt+k_0} = \mathrm{e}^{k_0}\mathrm{e}^{rt} = k_1\mathrm{e}^{rt},$$

where $k_1 = \mathrm{e}^{k_0}$ is a positive constant. Since $T - T_{\mathrm{a}} = \pm|T - T_{\mathrm{a}}| = \pm k_1\mathrm{e}^{rt}$, we conclude that $T(t) - T_{\mathrm{a}} = c\mathrm{e}^{rt}$, where $c$ is an arbitrary constant. Notice that this family includes the equilibrium solution $T = T_{\mathrm{a}}$ when $c = 0$. In summary, we find

$$T(t) = T_{\mathrm{a}} + c\mathrm{e}^{rt}. \tag{13}$$

**Example 9.** A turkey, which has an initial temperature of 40° (Fahrenheit), is placed into a 350° oven. After one hour, the temperature of the turkey is 100°. Use Newton's law of heating and cooling to find

1. The temperature of the turkey after 2 h
2. How many hours it takes for the temperature of the turkey to reach 170°

► **Solution.** In this case, the oven is the surrounding medium and has a constant temperature of $T_{\mathrm{a}} = 350°$ so (13) gives

$$T(t) = 350 + c\mathrm{e}^{rt}.$$

The initial temperature is $T(0) = 40°$, and this implies $40 = 350 + c$ and hence $c = -310$. To determine $r$, note that we are given $T(1) = 100$. This implies $100 = T(1) = 350 - 310\mathrm{e}^{r}$, and solving for $r$ gives $r = \ln\frac{25}{31} \approx -0.21511$. To answer question (1), compute $T(2) = 350 - 310\mathrm{e}^{2r} \approx 148.39°$. To answer question (2), we want to find $t$ so that $T(t) = 170$, that is, solve $170 = T(t) = 350 - 310\mathrm{e}^{rt}$. Solving this gives $rt = \ln\frac{18}{31}$ so $t \approx 2.53\,\mathrm{h} \approx 2\,\mathrm{h}\ 32$ min. ◄

## *The Malthusian Growth Model*

Let $P(t)$ represent the number of individuals in a population at time $t$. Two factors influence its growth: the birth rate and the death rate. Let $b(t)$ and $d(t)$ denote the birth and death rates, respectively. Then $P'(t) = b(t) - d(t)$. *In the Malthusian growth model, the birth and death rates are assumed to be proportional to the number in the population*. Thus

$$b(t) = \beta P(t) \quad \text{and} \quad d(t) = \delta P(t), \tag{14}$$

for some real constants $\beta$ and $\delta$. We are thus led to the differential equation

$$P'(t) = (\beta - \delta)P(t) = rP(t), \tag{15}$$

where $r = \beta - \delta$. Except for the notation, this equation is the same as the equation that modeled radioactive decay. The same calculation gives

$$P(t) = P_0 \mathrm{e}^{rt}, \tag{16}$$

for the solution of this differential equation, where $P_0 = P(0)$ is the ***initial population***. The constant $r$ is referred to as the ***Malthusian parameter***.

**Example 10.** Suppose 50 bacteria are placed in a Petri dish. After 30 min there are 120 bacteria. Assume the Malthusian growth model. How many bacteria will there be after 2 h? After 6 h?

► **Solution.** The initial population is $P_0 = 50$. After 30 min we have $120 = P(30) = P_0 \mathrm{e}^{rt}|_{t=30} = 50\mathrm{e}^{30r}$. This implies that the Malthusian parameter is $r = \frac{1}{30} \ln \frac{12}{5}$. In 120 min, we have

$$\begin{aligned} P(120) &= P_0 \mathrm{e}^{rt}\big|_{t=120} \\ &= 50\mathrm{e}^{\frac{120}{30} \ln \frac{12}{5}} \approx 1{,}659. \end{aligned}$$

In 6 h or 360 min, we have

$$\begin{aligned} P(360) &= P_0 \mathrm{e}^{rt}\big|_{t=360} \\ &= 50\mathrm{e}^{\frac{360}{30} \ln \frac{12}{5}} \approx 1{,}826{,}017. \end{aligned}$$ ◄

While the Malthusian model may be a reasonable model for short periods of time, it does not take into account growth factors such as disease, overcrowding, and competition for food that come into play for large populations and are not seen in small populations. Thus, while the first calculation in the example above may be plausible, it is likely implausible for the second calculation.

## *The Logistic Growth Model*

Here we generalize the assumptions made in the Malthusian model about the birth and death rates.[8] In a confined environment, resources are limited. Hence, it is reasonable to assume that as a population grows, the birth and death rates will decrease and increase, respectively. From (14), the per capita birth rate in the Malthusian model, $b(t)/P(t) = \beta$, is constant. Now assume it decreases linearly with the population, that is, $b(t)/P(t) = \beta - k_\beta P(t)$, for some positive constant $k_\beta$. Similarly, assume the per capita death rate increases linearly with the population, that is, $d(t)/P(t) = \delta + k_\delta P(t)$, for some positive constant $k_\delta$. We are then led to the following birth and death rate models:

$$b(t) = (\beta - k_\beta P(t))P(t) \quad \text{and} \quad d(t) = (\delta + k_\delta P(t))P(t). \tag{17}$$

Since the rate of change of population is the difference between the birth rate and death rate, we conclude

$$\begin{aligned} P'(t) &= b(t) - d(t) \\ &= ((\beta - \delta) - (k_\beta + k_\delta)P(t))P(t) \\ &= (\beta - \delta)\left(1 - \frac{k_\beta + k_\delta}{\beta - \delta}P(t)\right)P(t) \\ &= r\left(1 - \frac{P(t)}{m}\right)P(t), \end{aligned} \tag{18}$$

where we set $r = \beta - \delta$ and $m = (\beta - \delta)/(k_\beta + k_\delta)$ . Equation (18) shows that under the assumption (17), the population is a solution of the differential equation

$$P' = r\left(1 - \frac{P}{m}\right)P, \tag{19}$$

which is known as the ***logistic differential equation*** or the ***Verhulst population differential equation*** in its classical form. The parameter $r$ is called the ***Malthusian parameter***; it represents the rate at which the population would grow if it were unencumbered by environmental constraints. The number $m$ is called the ***carrying capacity*** of the population, which, as explained below, represents the maximum population possible under the given model.

[8] A special case was discussed in Exercise 2 of Sect. 1.1.

To solve the logistic differential equation (19), first note that the equation is separable since the right-hand side of the equation depends only on the dependent variable $P$. Next observe there are two equilibrium solutions obtained by constant solutions of the equation

$$r\left(1-\frac{P}{m}\right)P = 0.$$

These equilibrium (constant) solutions are $P(t) = 0$ and $P(t) = m$. Now proceed to the separation of variables algorithm. Separating the variables in (19) gives

$$\frac{1}{\left(1-\frac{P}{m}\right)P}\,\mathrm{d}P = r\,\mathrm{d}t.$$

In order to integrate the left-hand side of this equation, it is first necessary to use partial fractions to get

$$\frac{1}{\left(1-\frac{P}{m}\right)P} = \frac{m}{(m-P)P} = \frac{1}{m-P} + \frac{1}{P}.$$

Thus, the separated variables form of (19) suitable for integration is

$$\left(\frac{1}{m-P} + \frac{1}{P}\right)\mathrm{d}P = r\,\mathrm{d}t.$$

Integrating the left side with respect to $P$ and the right side with respect to $t$ gives

$$-\ln|m-P| + \ln|P| = \ln|P/(m-P)| = rt + k,$$

where $k$ is the constant of integration. Using the same algebraic techniques employed in Example 4 and the earlier examples on radioactive decay and Newton's law of cooling, we get

$$\frac{P}{m-P} = c\mathrm{e}^{rt},$$

for some real constant $c$. To solve for $P$, note that

$$P = c\mathrm{e}^{rt}(m-P) = cm\mathrm{e}^{rt} - Pc\mathrm{e}^{rt} \implies P(1+c\mathrm{e}^{rt}) = cm\mathrm{e}^{rt}.$$

This gives

$$\begin{aligned} P(t) &= \frac{cm\mathrm{e}^{rt}}{1+c\mathrm{e}^{rt}} \\ &= \frac{cm}{\mathrm{e}^{-rt}+c}, \end{aligned} \tag{20}$$

where the second equation is obtained from the first by multiplying the numerator and denominator by $e^{-rt}$.

The equilibrium solution $P(t) = 0$ is obtained by setting $c = 0$. The equilibrium solution $P(t) = m$ does not occur for *any* choice of $c$, so this solution is an extra one. Also note that since $r = \beta - \delta > 0$ assuming that the birth rate exceeds the death rate, we have $\lim_{t\to\infty} e^{-rt} = 0$ so

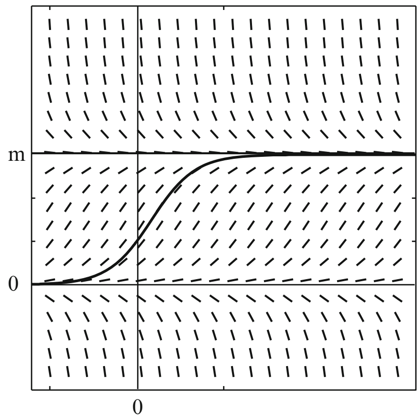
$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty} \frac{cm}{e^{-rt} + c} = \frac{cm}{c} = m,$$

independent of $c \neq 0$. What this means is that if we start with a positive population, then over time, the population will approach a maximum (sustainable) population $m$. This is the interpretation of the carrying capacity of the environment.

When $t = 0$, we get $P(0) = \frac{cm}{1+c}$ and solving for $c$ gives $c = \frac{P(0)}{m-P(0)}$. Substituting $c$ into (20) and simplifying give

$$P(t) = \frac{mP(0)}{P(0) + (m - P(0))e^{-rt}}. \tag{21}$$

Equation (21) is called the ***logistic equation***. That is, the logistic equation refers to the *solution* of the logistic differential equation. Below is its graph. You will note that the horizontal line $P = m$ is an asymptote. You can see from the graph that $P(t)$ approaches the limiting population $m$ as $t$ grows.




The graph of the logistic equation: $P(t) = \dfrac{mP(0)}{P(0) + (m - P(0))e^{-rt}}$

**Example 11.** Suppose biologists stock a lake with 200 fish and estimate the carrying capacity of the lake to be 10,000 fish. After two years, there are 2,500 fish. Assume the logistic growth model and estimate how long it will take for there to be 9,000 fish.

► **Solution.** The initial population is $P(0) = 200$ and the carrying capacity is $m = 10{,}000$. Thus, the logistic equation, (21), is

$$P(t) = \frac{mP(0)}{P(0) + (m - P(0))\mathrm{e}^{-rt}} = \frac{2{,}000{,}000}{200 + 9{,}800\mathrm{e}^{-rt}} = \frac{10{,}000}{1 + 49\mathrm{e}^{-rt}}.$$

Now let $t = 2$ (assume that time is measured in years). Then we get $2{,}500 = 10{,}000/(1 + 49\mathrm{e}^{-2r})$, and solving for $r$ gives $r = \frac{1}{2}\ln\frac{49}{3}$. Now we want to know for what $t$ is $P(t) = 9{,}000$. With $r$ as above the equation $P(t) = 9{,}000$ can be solved for $t$ to give

$$t = \frac{2\ln 441}{\ln\frac{49}{3}} \approx 4.36 \text{ years}.$$

The logistics growth model predicts that the population of fish will reach 9,000 in 4.36 years. ◄

## Exercises

1–9. In each of the following problems, determine whether or not the equation is separable. Do *not* solve the equations!

1. $y' = 2y(5 - y)$
2. $yy' = 1 - y$
3. $t^2 y' = 1 - 2ty$
4. $\dfrac{y'}{y} = y - t$
5. $ty' = y - 2ty$
6. $y' = ty^2 - y^2 + t - 1$
7. $(t^2 + 3y^2)y' = -2ty$
8. $y' = t^2 + y^2$
9. $\mathrm{e}^t y' = y^3 - y$

10–30. Find the general solution of each of the following differential equations. If an initial condition is given, find the particular solution that satisfies this initial condition.

10. $yy' = t$, $y(2) = -1$
11. $(1 - y^2) - tyy' = 0$
12. $y^3 y' = t$
13. $y^4 y' = t + 2$
14. $y' = ty^2$
15. $y' + (\tan t)y = \tan t$, $-\frac{\pi}{2} < t < \frac{\pi}{2}$
16. $y' = t^m y^n$, where $m$ and $n$ are positive integers, $n \neq 1$.
17. $y' = 4y - y^2$
18. $yy' = y^2 + 1$
19. $y' = y^2 + 1$
20. $tyy' + t^2 + 1 = 0$
21. $y + 1 + (y - 1)(1 + t^2)y' = 0$
22. $2yy' = \mathrm{e}^t$
23. $(1 - t)y' = y^2$
24. $\frac{\mathrm{d}y}{\mathrm{d}t} - y = y^2, \quad y(0) = 0.$
25. $y' = 4ty^2, \quad y(1) = 0$
26. $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{xy+2y}{x}, \quad y(1) = e$
27. $y' + 2yt = 0, \quad y(0) = 4$
28. $y' = \frac{\cot y}{t}, \quad y(1) = \frac{\pi}{4}$
29. $\frac{(u^2+1)}{y}\frac{\mathrm{d}y}{\mathrm{d}u} = u, \quad y(0) = 2$
30. $ty - (t + 2)y' = 0$
31. Solve the initial value problem

$$y' = \frac{y^2 + 1}{t^2} \qquad y(1) = \sqrt{3}.$$

Determine the maximum interval $(a, b)$ on which this solution is defined. Show that $\lim_{t \to b^-} y(t) = \infty$.

32–34. *Radioactive Decay*

32. A human bone is found in a melting glacier. The carbon-14 content of the bone is only one-third of the carbon-14 content in a similar bone today. Estimate the age of the bone? The half-life of C-14 is 5,730 years.
33. Potassium-40 is a radioactive isotope that decays into a single argon-40 atom and other particles with a half-life of 1.25 billion years. A rock sample was found that contained 8 times as many potassium-40 atoms as argon-40 atoms. Assume the argon-40 only comes from radioactive decay. Date the rock to the time it contained only potassium-40.
34. Cobalt 60 is a radioactive isotope used in medical radiotherapy. It has a half-life of 5.27 years. How long will it take for a sample of cobalt 60 to reduce to 30% of the original?

35–40. In the following problems, assume *Newton's law of heating and cooling.*

35. A bottle of your favorite beverage is at room temperature, 70°F, and it is then placed in a tub of ice water at time $t = 0$. After 30 min, the temperature is 55°. Assuming Newton's law of heating and cooling, when will the temperature drop to 45°.
36. A cup of coffee, brewed at 180° (Fahrenheit), is brought into a car with inside temperature 70°. After 3 min, the coffee cools to 140°. What is the temperature 2 min later?
37. The temperature outside a house is 90°, and inside it is kept at 65°. A thermometer is brought from the outside reading 90°, and after 2 min, it reads 85°. How long will it take to read 75°? What will the thermometer read after 20 min?
38. A cold can of soda is taken out of a refrigerator with a temperature of 40° and left to stand on the counter top where the temperature is 70°. After 2 h the temperature of the can is 60°. What was the temperature of the can 1 h after it was removed from the refrigerator?
39. A large cup of hot coffee is bought from a local drive through restaurant and placed in a cup holder in a vehicle. The inside temperature of the vehicle is 70° Fahrenheit. After 5 min the driver spills the coffee on himself and receives a severe burn. Doctors determine that to receive a burn of this severity, the temperature of the coffee must have been about 150°. If the temperature of the coffee was 142° six minutes after it was sold what was the temperature at which the restaurant served it.
40. A student wishes to have some friends over to watch a football game. She wants to have cold beer ready to drink when her friends arrive at 4 p.m. According to her taste, the beer can be served when its temperature is 50°. Her experience shows that when she places 80° beer in the refrigerator that is kept at a constant temperature of 40°, it cools to 60° in an hour. By what time should she put the beer in the refrigerator to ensure that it will be ready for her friends?

41–43. In the following problems, assume the *Malthusian growth model.*

41. The population of elk in a region of a national forest was 290 in 1980 and 370 in 1990. Forest rangers want to estimate the population of elk in 2010. They assume the Malthusian growth model. What is their estimate?
42. The time it takes for a culture of 40 bacteria to double its population is 3 h. How many bacteria are there at 30 h?
43. If it takes 5 years for a certain population to triple in size, how long does it take to double in size?

44–47. In the following problems, assume the *logistic growth model.*

44. The population of elk in a region of a national forest was 290 in 1980 and 370 in 1990. Experienced forest rangers estimate that the carrying capacity of the population of elk for the size and location of the region under consideration is 800. They want to estimate the population of elk in 2010. They assume the logistic growth model. What is their estimate?
45. A biologist is concerned about the rat population in a closed community in a large city. Initial readings 2 years ago gave a population of 2,000 rats, but now there are 3,000. The experienced biologist estimates that the community cannot support more than 5,000. The alarmed mayor tells the biologist not to reveal that to the general public. He wants to know how large the population will be at election time two years from now. What can the biologist tell him?
46. Assuming the logistics equation $P(t) = \frac{mP_0}{P_0+(m-P_0)e^{-rt}}$, suppose $P(0) = P_0$, $P(t_0) = P_1$, and $P(2t_0) = P_2$. Show

$$m = \frac{P_1\left(P_1(P_0 + P_2) - 2P_0P_2\right)}{P_1^2 - P_0P_2},$$
$$r = \frac{1}{t_0}\ln\left(\frac{P_2(P_1 - P_0)}{P_0(P_2 - P_1)}\right).$$

47. Four hundred butterflies are introduced into a closed community of the rain-forest and subsequently studied. After 3 years, the population increases to 700, and 3 years thereafter, the population increases to 1,000. What is the carrying capacity?

## 1.4 Linear First Order Equations

A first order differential equation that can be written in the form

$$y' + p(t)y = f(t) \tag{1}$$

is called a ***linear first order differential equation*** or just ***linear***, for short. We will say that (1) is the ***standard form*** for a linear differential equation.[9] We will assume that the ***coefficient function***, $p$, and the ***forcing function***, $f$, are continuous functions on an interval $I$. In later chapters, the continuity restrictions will be removed. Equation (1) is ***homogeneous*** if the forcing function is zero on $I$ and ***nonhomogeneous*** if the forcing function $f$ is not zero. Equation (1) is ***constant coefficient*** provided the coefficient function $p$ is a constant function, that is, $p(t) = p_0 \in \mathbb{R}$ for all $t \in I$.

**Example 1.** Characterize the following list of first order differential equations:

| | |
|---|---|
| 1. $y' = y - t$ | 2. $y' + ty = 0$ |
| 3. $y' = \sec t$ | 4. $y' + y^2 = t$ |
| 5. $ty' + y = t^2$ | 6. $y' - \frac{3}{t}y = t^4$ |
| 7. $y' = 7y$ | 8. $y' = \tan(ty)$ |

► **Solution.** The presence of the term $y^2$ in (4) and the presence of the term $\tan(ty)$ in (8) prevent them from being linear. Equation (1), (5), and (7) can be written $y' - y = -t$, $y' + (1/t)y = t$, and $y' - 7y = 0$, respectively. Thus, all but (4) and (8) are linear. Equations (2) and (7) are homogeneous. Equations (1), (3), (5), and (6) are nonhomogeneous. Equations (1), (3), and (7) are constant coefficient. The interval of continuity for the forcing and coefficient functions is the real line for (1), (2), (7); an interval of the form $(-\frac{\pi}{2} + m\pi, \frac{\pi}{2} + m\pi)$, $m$ an integer, for (3); and $(-\infty, 0)$ or $(0, \infty)$ for (5) and (6). ◄

Notice (2), (3), and (7) are also separable. Generally, this occurs when either the coefficient function, $p(t)$, is zero or the forcing function, $f(t)$, is zero. Thus, there is an overlap between the categories of separable and linear differential equations.

---

[9]This conflicts with the use of the term *standard form*, given in Sect. 1.1, where we meant a first order differential equation written in the form $y' = F(t, y)$. Nevertheless, in the context of first order linear differential equations, we will use the term *standard form* to mean an equation written as in (1).

### *A Solution Method for Linear Equations*

Consider the following two linear differential equations:

$$ty' + 2y = 4, \tag{2}$$

$$t^2y' + 2ty = 4t. \tag{3}$$

They both have the same standard form

$$y' + \frac{2}{t}y = \frac{4}{t} \tag{4}$$

so they both have the same solution set. Further, multiplying (4) by $t$ and $t^2$ gives (2) and (3), respectively. Now (2) is simpler than (3) in that it does not have a needless extra factor of $t$. However, (3) has an important redeeming property that (2) does not have, namely, the left-hand side is a perfect derivative, by which we mean that $t^2y' + 2ty = (t^2y)'$. Just apply the product rule to check this. Thus, (3) can be rewritten in the form

$$(t^2y)' = 4t, \tag{5}$$

and it is precisely this form that leads to the solution: Integrating both sides with respect to $t$ gives $t^2y = 2t^2 + c$ and dividing by $t^2$ gives

$$y = 2 + ct^{-2}, \tag{6}$$

where $c$ is an arbitrary real number. Of course, we could have multiplied (4) by any function, but of all functions, it is only $\mu(t) = t^2$ (up to a multiplicative scalar) that simplifies the left-hand side as in (5).

The process just described generalizes to an arbitrary linear differential equation in *standard form*

$$y' + py = f. \tag{7}$$

That is, we can find a function $\mu$ so that when the left side of (7) is multiplied by $\mu$, the result is a perfect derivative $(\mu y)'$ with respect to the $t$ variable. This will require that

$$\mu y' + \mu py = (\mu y)' = \mu y' + \mu' y. \tag{8}$$

Such a function $\mu$ is called an ***integrating factor***. To find $\mu$, note that simplifying (8) by canceling $\mu y'$ gives $\mu py = \mu' y$, and we can cancel $y$ (assuming that $y(t) \neq 0$) to get

$$\mu' = \mu p, \tag{9}$$

a separable differential equation for the unknown function $\mu$. Separating variables gives $\mu'/\mu = p$ and integrating both sides gives $\ln|\mu| = P$, where $P = \int p\,\mathrm{d}t$ is any antiderivative of $p$. Taking the exponential of both sides of this equation produces a formula for the integrating factor

$$\mu = \mathrm{e}^{P} = \mathrm{e}^{\int p\,\mathrm{d}t}; \tag{10}$$

namely, *$\mu$ is the exponential of any antiderivative of the coefficient function*. For example, the integrating factor $\mu(t) = t^2$ found in the example above is derived as follows: the coefficient function in (4) is $p(t) = 2/t$. An antiderivative is $P(t) = 2\ln t = \ln t^2$ and hence $\mu(t) = \mathrm{e}^{P(t)} = \mathrm{e}^{\ln t^2} = t^2$.

Once an integrating factor is found, we can solve (7) in the same manner as the example above. First multiplying (7) by the integrating factor gives $\mu y' + \mu p y = \mu f$. By (8), $\mu y' + \mu p y = (\mu y)'$ so we get

$$(\mu y)' = \mu f. \tag{11}$$

Integrating both sides of this equation gives $\mu y = \int \mu f\,\mathrm{d}t + c$, and hence,

$$y = \frac{1}{\mu}\int \mu f\,\mathrm{d}t + \frac{c}{\mu},$$

where $c$ is an arbitrary constant.

It is easy to check that the formula we have just derived is a solution. Further, we have just shown that any solution takes on this form. We summarize this discussion in the following theorem:

**Theorem 2.** *Let $p$ and $f$ be continuous functions on an interval $I$. A function $y$ is a solution of the first order linear differential equation $y' + py = f$ on $I$ if and only if*

| ***Solution of First Order Linear Equations*** |
|---|
| $y = \frac{1}{\mu}\int \mu f\,\mathrm{d}t + \frac{c}{\mu},$ |

*where $c \in \mathbb{R}$, $P$ is any antiderivative of $p$ on the interval $I$, and $\mu = \mathrm{e}^{P}$.*

The steps needed to derive the solution when dealing with concrete examples are summarized in the following algorithm.

**Algorithm 3.** The following procedure is used to solve a first order linear differential equation.

**Solution Method for First Order Linear Equations**

1. *Put the given linear equation in standard form:* $y' + py = f$.
2. *Find an integrating factor,* $\mu$*:* To do this, compute an antiderivative $P = \int p \, \mathrm{d}t$ and set $\mu = \mathrm{e}^P$.
3. *Multiply the equation (in standard form) by* $\mu$*:* This yields

$$\mu y' + \mu' y = \mu f.$$

4. *Simplify the left-hand side:* Since $(\mu y)' = \mu y' + \mu' y$, we get

$$(\mu y)' = \mu f.$$

5. *Integrate both sides of the resulting equation:* This yields

$$\mu y = \int \mu f \, \mathrm{d}t + c.$$

6. *Divide by* $\mu$ *to get the solution* $y$*:*

$$y = \frac{1}{\mu} \int \mu f \, \mathrm{d}t + \frac{c}{\mu}. \tag{12}$$

**Remark 4.** You should *not* memorize formula (12). What you should remember instead is the sequence of steps in Algorithm 3, and apply these steps to each concretely presented linear first order differential equation.

**Example 5.** Find all solutions of the differential equation

$$t^2 y' + ty = 1$$

on the interval $(0, \infty)$.

► **Solution.** We shall follow Algorithm 3 closely. First, we put the given linear differential equation in standard form to get

$$y' + \frac{1}{t} y = \frac{1}{t^2}. \tag{13}$$

The coefficient function is $p(t) = 1/t$ and its antiderivative is $P(t) = \ln |t| = \ln t$ (since we are on the interval $(0, \infty)$, $t = |t|$). It follows that the integrating factor is $\mu(t) = \mathrm{e}^{\ln t} = t$. Multiplying (13) by $\mu(t) = t$ gives

$$ty' + y = \frac{1}{t}.$$

Next observe that the left side of this equality is equal to $\frac{\mathrm{d}}{\mathrm{d}t}(ty)$. Thus,

$$\frac{\mathrm{d}}{\mathrm{d}t}(ty) = \frac{1}{t}.$$

Now take antiderivatives of both sides to get

$$ty = \ln t + c,$$

where $c \in \mathbb{R}$. Now dividing by the integrating factor $\mu(t) = t$ gives the solution

$$y(t) = \frac{\ln t}{t} + \frac{c}{t}.$$ ◄

**Example 6.** Find all solutions of the differential equation

$$y' = (\tan t)y + \cos t$$

on the interval $I = (-\frac{\pi}{2}, \frac{\pi}{2})$.

► **Solution.** We first write the given equation in standard form to get

$$y' - (\tan t)y = \cos t.$$

The coefficient function is $p(t) = -\tan t$ (it is a common mistake to forget the minus sign). An antiderivative is $P(t) = -\ln\sec t = \ln\cos t$. It follows that $\mu(t) = \mathrm{e}^{\ln\cos t} = \cos t$ is an integrating factor. We now multiply by $\mu$ to get $(\cos t)y' - (\sin t)y = \cos^2 t$, and hence,

$$((\cos t)y)' = \cos^2 t.$$

Integrating both sides, taking advantage of double angle formulas, gives

$$\begin{aligned}(\cos t)y &= \int \cos^2 t\,\mathrm{d}t = \int \frac{1}{2}(\cos 2t + 1)\,\mathrm{d}t \\ &= \frac{1}{2}t + \frac{1}{4}\sin 2t + c \\ &= \frac{1}{2}(t + \sin t\cos t) + c.\end{aligned}$$

Dividing by the integrating factor $\mu(t) = \cos t$ gives the solution

$$y = \frac{1}{2}(t \sec t + \sin t) + c \sec t. \qquad ◀$$

**Example 7.** Find the solution of the differential equation

$$y' + y = t,$$

with initial condition $y(0) = 2$.

► **Solution.** The given differential equation is in standard form. The coefficient function is the constant function $p(t) = 1$. Thus, $\mu(t) = \mathrm{e}^{\int 1\,\mathrm{d}t} = \mathrm{e}^t$. Multiplying by $\mathrm{e}^t$ gives $\mathrm{e}^t y' + \mathrm{e}^t y = t\mathrm{e}^t$ which is

$$(\mathrm{e}^t y)' = t\mathrm{e}^t.$$

Integrating both sides gives $\mathrm{e}^t y = \int t\mathrm{e}^t\,\mathrm{d}t = t\mathrm{e}^t - \mathrm{e}^t + c$, where $\int t\mathrm{e}^t\,\mathrm{d}t$ is calculated using integration by parts. Dividing by $\mathrm{e}^t$ gives

$$y = t - 1 + c\mathrm{e}^{-t}.$$

Now the initial condition implies $2 = y(0) = 0 - 1 + c$ and hence $c = 3$. Thus,

$$y = t - 1 + 3\mathrm{e}^{-t}. \qquad ◀$$

## *Initial Value Problems*

In many practical problems, an initial value may be given. As in Example 7 above, the initial condition may be used to determine the arbitrary scalar in the general solution once the general solution has been found. It can be useful, however, to have a single formula that directly encodes the initial value in the solution. This is accomplished by the following corollary to Theorem 2, which also establishes the uniqueness of the solution.

**Corollary 8.** *Let $p$ and $f$ be continuous on an interval $I$, $t_0 \in I$, and $y_0 \in \mathbb{R}$. Then the unique solution of the initial value problem*

$$y' + py = f, \quad y(t_0) = y_0 \tag{14}$$

*is given by*

$$\boxed{y(t) = \mathrm{e}^{-P(t)} \int_{t_0}^{t} \mathrm{e}^{P(u)} f(u)\, \mathrm{d}u + y_0 \mathrm{e}^{-P(t)},} \tag{15}$$

*where* $P(t) = \int_{t_0}^{t} p(u)\, \mathrm{d}u$.

*Proof.* Let $y$ be given by (15). Since $P$ is an antiderivative of $p$, it follows that $\mu = \mathrm{e}^P$ is an integrating factor. Replacing $\mathrm{e}^P$ and $\mathrm{e}^{-P}$ in the formula by $\mu$ and $\mu^{-1}$, respectively, we see that $y$ has the form given in Theorem 2. Hence, $y(t)$ is a solution of the linear first order equation $y' + p(t)y = f(t)$. Moreover, $P(t_0) = \int_{t_0}^{t_0} p(u)\, \mathrm{d}u = 0$, and

$$y(t_0) = y_0 \mathrm{e}^{-P(t_0)} + \mathrm{e}^{-P(t_0)} \int_{t_0}^{t_0} \mathrm{e}^{P(u)} f(u)\, \mathrm{d}u = y_0,$$

so that $y(t)$ is a solution of the initial value problem given by (14). As to uniqueness, suppose that $y_1(t)$ is any other such solution. Set $y_2(t) = y(t) - y_1(t)$. Then

$$\begin{aligned} y_2' + p y_2 &= y' - y_1' + p(y - y_1) \\ &= y' + py - (y_1' + p y_1) \\ &= f - f = 0. \end{aligned}$$

Further, $y_2(t_0) = y(t_0) - y_1(t_0) = 0$. It follows from Theorem 2 that $y_2(t) = c\mathrm{e}^{-\tilde{P}(t)}$ for some constant $c \in \mathbb{R}$ and an antiderivative $\tilde{P}(t)$ of $p(t)$. Since $y_2(t_0) = 0$ and $\mathrm{e}^{-\tilde{P}(t_0)} \neq 0$, it follows that $c = 0$. Thus, $y(t) - y_1(t) = y_2(t) = 0$ for all $t \in I$. This shows that $y_1(t) = y(t)$ for all $t \in I$, and hence, $y(t)$ is the only solution of (14). □

**Example 9.** Use Corollary 8 to find the solution of the differential equation

$$y' + y = t,$$

with initial condition $y(0) = 2$.

► **Solution.** The coefficient function is $p(t) = 1$. Thus, $P(t) = \int_0^t p(u)\, \mathrm{d}u = u|_0^t = t$, $\mathrm{e}^{P(t)} = \mathrm{e}^t$ and $\mathrm{e}^{-P(t)} = \mathrm{e}^{-t}$. Let $y$ be given as in the corollary. Then

$$\begin{aligned} y(t) &= \mathrm{e}^{-P(t)} \int_{t_0}^{t} \mathrm{e}^{P(u)} f(u)\, \mathrm{d}u + y_0 \mathrm{e}^{-P(t)} \\ &= \mathrm{e}^{-t} \int_0^t u\mathrm{e}^u\, \mathrm{d}u + 2\mathrm{e}^{-t} \end{aligned}$$

$$
\begin{aligned}
&= e^{-t}\ (ue^u - e^u)|_0^t + 2e^{-t} \\
&= e^{-t}\left(te^t - e^t - (-1)\right) + 2e^{-t} \\
&= t - 1 + e^{-t} + 2e^{-t} = t - 1 + 3e^{-t}.
\end{aligned}
$$

Rather than memorizing (15), it is generally easier to remember Algorithm 3 and solve such problems as we did in Example 7. ◀

**Example 10.** Find the solution of the initial value problem

$$
y' + y = \frac{1}{1-t},
$$

with $y(0) = 0$ on the interval $(-\infty, 1)$.

► **Solution.** By Corollary 8, the solution is

$$
y(t) = e^{-t} \int_0^t \frac{e^u}{1-u}\, du.
$$

Since the function $\frac{e^u}{1-u}$ has no closed form antiderivative on the interval $(-\infty, 1)$, we might be tempted to stop at this point and say that we have solved the equation. While this is a legitimate statement, the present representation of the solution is of little practical use, and a further detailed study is necessary if you are "really" interested in the solution. Any further analysis (numerical calculations, qualitative analysis, etc.) would be based on what type of information you are attempting to ascertain about the solution. ◀

## *Analysis of the General Solution Set*

For a linear differential equation $y' + py = f$, where $p$ and $f$ are continuous functions on an interval $I$, Theorem 2 gives the general solution

$$
y = \mu^{-1} \int \mu f \, dt + c\mu^{-1},
$$

where $\mu = e^P$, $P = \int p\, dt$, and $c$ is an arbitrary constant. We see that the solution is the sum of two parts.

### The Particular Solution

When we set $c = 0$ in the general solution above, we get a single fixed solution which we denote by $y_{\mathrm{p}}$. Specifically,

$$y_{\mathrm{p}} = \mu^{-1} \int \mu f \, \mathrm{d}t = \mathrm{e}^{-P} \int \mathrm{e}^{P} f \, \mathrm{d}t.$$

This solution is called a ***particular solution***. Keep in mind though that there are many particular solutions depending on the antiderivative chosen for $\int \mathrm{e}^{P} f \, \mathrm{d}t$. Once an antiderivative is fixed, then the particular solution is determined.

### The Homogeneous Solutions

When we set $f = 0$ in $y + py = f$, we get what is called the ***associated homogeneous equation***, $y' + py = 0$. Its solution is then obtained from the general solution. The integral reduces to 0 since $f = 0$ so all that is left is the second summand $c\mathrm{e}^{-P}$. We set

$$y_{\mathrm{h}} = c\mu^{-1} = c\mathrm{e}^{-P}.$$

This solution is called the ***homogeneous solution***. The notation is sometimes used a bit ambiguously. Mostly, we think of $y_{\mathrm{h}}$ as a family of functions parameterized by $c$. Yet, there are times when we will have a specific $c$ in mind and then $y_{\mathrm{h}}$ will be an actual function. It will be clear from the context which is meant.

### The General Solution

The relationship between the *general* solution $y_{\mathrm{g}}$ of $y' + py = f$, a *particular* solution $y_{\mathrm{p}}$ of this equation, and the a *homogeneous* solution $y_{\mathrm{h}}$, is usually expressed as

$$y_{\mathrm{g}} = y_{\mathrm{p}} + y_{\mathrm{h}}. \tag{16}$$

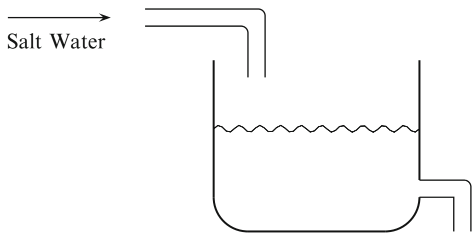
What this means is that *every* solution to $y' + py = f$ can be obtained by starting with a single particular solution $y_{\mathrm{p}}$ and adding to that the homogeneous solution $y_{\mathrm{h}}$. The key observation is the following. Suppose that $y_1$ and $y_2$ are *any* two solutions of $y' + py = f$. Then

$$\begin{aligned}(y_2 - y_1)' + p(y_2 - y_1) &= (y_2' + py_2) - (y_1' + py_1) \\ &= f - f \\ &= 0,\end{aligned}$$

so that $y_2 - y_1$ is a homogeneous solution. Let $y_\mathrm{h} = y_2 - y_1$. Then $y_2 = y_1 + y_\mathrm{h}$. Therefore, given a solution $y_1(t)$ of $y' + p(t)y = f(t)$, any other solution $y_2$ is obtained from $y_1$ by adding a homogeneous solution $y_\mathrm{h}$.

## *Mixing Problems*

**Example 11.** Suppose a tank contains 10 L of a brine solution (salt dissolved in water). Assume the initial concentration of salt is 100 g/L. Another brine solution flows into the tank at a rate of 3 L/min with a concentration of 400 g/L. Suppose the mixture is well stirred and flows out of the tank at a rate of 3 L/min. Let $y(t)$ denote the amount of salt in the tank at time $t$. Find $y(t)$. The following diagram may be helpful to visualize this problem.




► **Solution.** Here again is a situation in which it is best to express how a quantity like $y(t)$ changes. If $y(t)$ represents the amount (in grams) of salt in the tank, then $y'(t)$ represents the total rate of change of salt in the tank with respect to time. It is given by the difference of two rates: the *input rate* of salt in the tank and the *output rate* of salt in the tank. Thus,

$$y' = \text{input rate} - \text{output rate}. \tag{17}$$

Both the input rate and output rate of salt are determined by product of the flow rate of the brine and the concentration of the brine solution. Specifically,

$$\frac{\text{amount of salt}}{\text{unit of time}} = \frac{\text{volume of brine}}{\text{unit of time}} \times \frac{\text{amount of salt}}{\text{volume of brine}}, \tag{18}$$

In this example, the units of measure are liters(L), grams(g), and minutes(min).

**Input Rate of Salt** Since the inflow rate is 3 L/min and the concentration is fixed at 400 g/L we have

$$\text{input rate} = 3\,\frac{\text{L}}{\text{min}} \times 400\,\frac{\text{g}}{\text{L}} = 1200\,\frac{\text{g}}{\text{min}}.$$

**Output Rate of Salt** The outflow rate is likewise 3 L/min. However, the concentration of the solution that leaves the tank varies with time. Since the amount of salt in the tank is $y(t)$ g and the volume of fluid is always 10 L (the inflow and outflow rates are the same (3 L/min) so the volume never changes), the concentration of the fluid as it leaves the tank is $\frac{y(t)}{10}\,\frac{\text{g}}{\text{L}}$. We thus have

$$\text{output rate} = 3\,\frac{\text{L}}{\text{min}} \times \frac{y(t)}{10}\,\frac{\text{g}}{\text{L}} = \frac{3y(t)}{10}\,\frac{\text{g}}{\text{min}}.$$

The initial amount of salt in the tank is $y(0) = 100\,\frac{\text{g}}{\text{L}} \times 10\,\text{L} = 1000$ g. Equation (17) thus leads us to the following linear differential equation

$$y' = 1,200 - \frac{3y}{10}, \tag{19}$$

with initial value $y(0) = 1{,}000$.

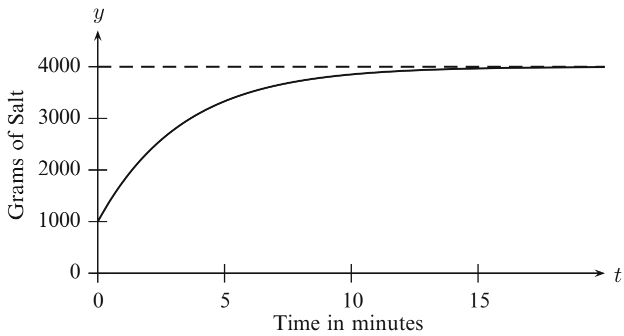
In standard form, (19) becomes $y' + \frac{3}{10}y = 1{,}200$. Multiplying by the integrating factor, $\mathrm{e}^{3t/10}$, leads to $(\mathrm{e}^{3t/10}y)' = 1{,}200\mathrm{e}^{3t/10}$. Integrating and dividing by the integrating factor give

$$y(t) = 4{,}000 + c\mathrm{e}^{-\frac{3}{10}t}.$$

Finally, the initial condition implies $1{,}000 = y(0) = 4{,}000 + c$ so $c = -3{,}000$. Hence,

$$y(t) = 4{,}000 - 3{,}000\mathrm{e}^{-\frac{3}{10}t}. \tag{20}$$

Below we give the graph of (20), representing the amount of salt in the tank at time $t$.




You will notice that $\lim_{t\to\infty}(4{,}000 - 3{,}000e^{-3t/10}) = 4{,}000$ as indicated by the horizontal asymptote in the graph. In the long term, the concentration of the brine in the tank approaches $\frac{4{,}000\text{ g}}{10\text{ L}} = 400$ g/L. This is expected since the brine coming into the tank has concentration 400 g/L. ◀

Problems like Example 11 are called ***mixing problems*** and there are many variations. There is nothing special about the use of a salt solution. Any solvent with a solute will do. For example, one might want to study a pollutant flowing into a lake (the tank) with an outlet feeding a water supply for a town. In Example 11, the inflow and outflow rates were the same but we will consider examples where this is not the case. We will also consider situations where the concentration of salt in the inflow is not constant but varies as a function $f$ of time. This function may even be discontinuous as will be the case in Chap. 6. In Chap. 9, we consider the case where two or more tanks are interconnected. Rather than memorize some general formulas, it is best to derive the appropriate differential equation on a case by case basis. Equations (17) and (18) are the basic principles to model such mixing problems.

**Example 12.** A large tank contains 100 gal of brine in which 50 lbs of salt are dissolved. Brine containing 2 lbs of salt per gallon flows into the tank at the rate of 6 gal/min. The mixture, which is kept uniform by stirring, flows out of the tank at the rate of 4 gal/min. Find the amount of salt in the tank at the end of $t$ minutes. After 50 min, how much salt will be in the tank and what will be the volume of brine? (The units here are abbreviated: gallon(s)=gal, pound(s) =lbs, and minute(s) = min.)

► **Solution.** Let $y(t)$ denote the number of pounds of salt in the tank after $t$ minutes. Note that in this problem, the difference between inflow and outflow rates is 2 gal/min. At time $t$, there will be $V(t) = 100 + 2t$ gal of brine, and so the concentration (in lbs/gal) will then be

$$\frac{y(t)}{V(t)} = \frac{y(t)}{100 + 2t}.$$

We use (18) to compute the input and output rates of salt:

**Input Rate**

$$\text{input rate} = 6\,\frac{\text{gal}}{\text{min}} \times 2\,\frac{\text{lbs}}{\text{gal}} = 12\,\frac{\text{lbs}}{\text{min}}.$$

**Output Rate**

$$\text{output rate} = 4\,\frac{\text{gal}}{\text{min}} \times \frac{y(t)}{100+2t}\,\frac{\text{lbs}}{\text{gal}} = \frac{4y(t)}{100+2t}\,\frac{\text{lbs}}{\text{min}}.$$

Applying (17) yields the initial value problem

$$y'(t) = 12 - \frac{4y(t)}{100+2t}, \qquad y(0) = 50.$$

Simplifying and putting in standard form give $y' + \frac{2}{50+t}y = 12$. The coefficient function is $p(t) = \frac{2}{50+t}$, $P(t) = \int p(t)\,\mathrm{d}t = 2\ln(50+t) = \ln(50+t)^2$, and the integrating factor is $\mu(t) = (50+t)^2$. Thus, $((50+t)^2 y)' = 12(50+t)^2$. Integrating and simplifying give

$$y(t) = 4(50+t) + \frac{c}{(50+t)^2}.$$

The initial condition $y(0) = 50$ implies $c = -3(50)^3$ so

$$y = 4(50+t) - \frac{3(50)^3}{(50+t)^2}.$$

After 50 min, there will be $y(50) = 400 - \frac{3}{8}100 = 362.5$ lbs of salt in the tank and 200 gal of brine. ◀

## Exercises

1–25. Find the general solution of the given differential equation. If an initial condition is given, find the particular solution which satisfies this initial condition.

1. $y' + 3y = e^t, \quad y(0) = -2$
2. $(\cos t)y' + (\sin t)y = 1, \quad y(0) = 5$
3. $y' - 2y = e^{2t}, \quad y(0) = 4$
4. $ty' + y = e^t$
5. $ty' + y = e^t, \quad y(1) = 0$
6. $ty' + my = t\ln(t)$, where $m$ is a constant
7. $y' = -\frac{y}{t} + \cos(t^2)$
8. $y' + 2y = \sin t$
9. $y' - 3y = 25\cos 4t$
10. $t(t+1)y' = 2 + y$
11. $z' = 2t(z - t^2)$
12. $y' + ay = b$, where $a$ and $b$ are constants
13. $y' + y\cos t = \cos t, \quad y(0) = 0$
14. $y' - \dfrac{2}{t+1}y = (t+1)^2$
15. $y' - \dfrac{2}{t}y = \dfrac{t+1}{t}, \quad y(1) = -3$
16. $y' + ay = e^{-at}$, where $a$ is a constant
17. $y' + ay = e^{bt}$, where $a$ and $b$ are constants and $b \neq -a$
18. $y' + ay = t^n e^{-at}$, where $a$ is a constant
19. $y' = y\tan t + \sec t$
20. $ty' + 2y\ln t = 4\ln t$
21. $y' - \dfrac{n}{t}y = e^t t^n$
22. $y' - y = te^{2t}, \quad y(0) = a$
23. $ty' + 3y = t^2, \quad y(-1) = 2$
24. $y' + 2ty = 1, \quad y(0) = 1$
25. $t^2y' + 2ty = 1, \quad y(2) = a$

26–33. *Mixing Problems*

26. A tank contains 100 gal of brine made by dissolving 80 lb of salt in water. Pure water runs into the tank at the rate of 4 gal/min, and the mixture, which is kept uniform by stirring, runs out at the same rate. Find the amount of salt in the tank at any time $t$. Find the concentration of salt in the tank at any time $t$.

27. A tank contains 10 gal of brine in which 2 lb of salt is dissolved. Brine containing 1 lb of salt per gallon flows into the tank at the rate of 3 gal/min, and the stirred mixture is drained off the tank at the rate of 4 gal/min. Find the amount $y(t)$ of salt in the tank at any time $t$.
28. A tank holds 10 L of pure water. A brine solution is poured into the tank at a rate of 1 L/min and kept well stirred. The mixture leaves the tank at the same rate. If the brine solution has a concentration of 10 g of salt per liter, what will the concentration be in the tank after 10 min.
29. A 30-L container initially contains 10 L of pure water. A brine solution containing 20 g of salt per liter flows into the container at a rate of 4 L/min. The well-stirred mixture is pumped out of the container at a rate of 2 L/min.

    (a) How long does it take for the container to overflow?
    (b) How much salt is in the tank at the moment the tank begins to overflow?

30. A 100-gal tank initially contains 10 gal of fresh water. At time $t = 0$, a brine solution containing 0.5 lb of salt per gallon is poured into the tank at the rate of 4 gal/min while the well-stirred mixture leaves the tank at the rate of 2 gal/min.

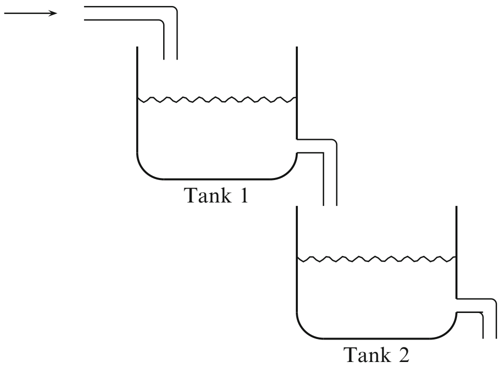
    (a) Find the time $T$ it takes for the tank to overflow.
    (b) Find the amount of salt in the tank at time $T$.
    (c) If $y(t)$ denotes the amount of salt present at time $t$, what is $\lim_{t\to\infty} y(t)$?

31. For this problem, our tank will be a lake and the brine solution will be polluted water entering the lake. Thus, assume that we have a lake with volume $V$ which is fed by a polluted river. Assume that the rate of water flowing into the lake and the rate of water flowing out of the lake are equal. Call this rate $r$, let $c$ be the concentration of pollutant in the river as it flows *into* the lake, and assume perfect mixing of the pollutant in the lake (this is, of course, a very unrealistic assumption).

    (a) Write down and solve a differential equation for the amount $P(t)$ of pollutant in the lake at time $t$ and determine the limiting *concentration* of pollutant in the lake as $t \to \infty$.
    (b) At time $t = 0$, the river is cleaned up, so no more pollutant flows into the lake. Find expressions for how long it will take for the pollution in the lake to be reduced to (i) 1/2 and (ii) 1/10 of the value it had at the time of the cleanup.
    (c) Assuming that Lake Erie has a volume $V$ of 460 km$^3$ and an inflow–outflow rate of $r = 175$ km$^3$/year, give numerical values for the times found in Part (b). Answer the same question for Lake Ontario, where it is assumed that $V = 1640$ km$^3$ and $r = 209$ km$^3$/year.

32. Two tanks, Tank 1 and Tank 2, are arranged so that the outflow of Tank 1 is the inflow of Tank 2. Each tank initially contains 10 L of pure water. A brine solution with concentration 100 g/L flows into Tank 1 at a rate of 4 L/min. The well-stirred mixture flows into Tank 2 at the same rate. In Tank 2, the mixture is

well stirred and flows out at a rate of 4 L/min. Find the amount of salt in Tank 2 at any time $t$. The following diagram may be helpful to visualize this problem.




33. Two large tanks, Tank 1 and Tank 2, are arranged so that the outflow of Tank 1 is the inflow of Tank 2, as in the configuration of Exercise 32. Tank 1 initially contains 10 L of pure water while Tank 2 initially contains 5 L of pure water. A brine solution with concentration 10 g/L flows into Tank 1 at a rate of 4 L/min. The well-stirred mixture flows into Tank 2 at a rate of 2 L/min. In Tank 2, the mixture is well stirred and flows out at a rate of 1 liter per minute. Find the amount of salt in Tank 2 before either Tank 1 or Tank 2 overflows.

## 1.5 Substitutions

Just as a complicated integral may be simplified into a familiar integral form by a judicious substitution, so too may a differential equation. Generally speaking, substitutions take one of two forms. Either we replace $y$ in $y' = F(t, y)$ by some function $y = \phi(t, v)$ to get a differential equation $v' = G(t, v)$ in $v$ and $t$, or we set $z = \psi(t, y)$ for some expression $\psi(t, y)$ that appears in the formula for $F(t, y)$, resulting in a differential equation $z' = H(t, z)$ in $z$ and $t$. Of course, the goal is that this new differential equation will fall into a category where there are known solution methods. Once the unknown function $v$ (or $z$) is determined, then $y$ can be determined by $y = \phi(t, v)$ (or implicitly by $z = \psi(t, y)$). In this section, we will illustrate this procedure by discussing the homogeneous and Bernoulli equations, which are simplified to separable and linear equations, respectively, by appropriate substitutions.

### *Homogeneous Differential Equations*

A ***homogeneous differential equation***[10] is a first order differential equation that can be written in the form

$$y' = f\left(\frac{y}{t}\right), \tag{1}$$

for some function $f$. To solve such a differential equation, we use the substitution $y = tv$, where $v = v(t)$ is some unknown function and rewrite (1) as a differential equation in terms of $t$ and $v$. By the product rule, we have $y' = v + tv'$. Substituting this into (1) gives $v + tv' = f(tv/t) = f(v)$, a separable differential equation for which the variables $t$ and $v$ may be separated to give

$$\frac{\mathrm{d}v}{f(v) - v} = \frac{\mathrm{d}t}{t}. \tag{2}$$

Once $v$ is found we may determine $y$ by the substitution $y = tv$. *It is not useful to try to memorize* (2)*. Rather, remember the substitution method that leads to it.* As an illustration, consider the following example.

**Example 1.** Solve the following differential equation:

$$y' = \frac{t + y}{t - y}.$$

[10]Unfortunately, the term "homogeneous" used here is the same term used in Sect. 1.4 to describe a linear differential equation in which the forcing function is zero. These meanings are different. Usually, context will determine the appropriate meaning.

► **Solution.** This differential equation is neither separable nor linear. However, it is homogeneous since we can write the right-hand side as

$$\frac{t+y}{t-y} = \frac{(1/t)}{(1/t)}\frac{t+y}{t-y} = \frac{1+(y/t)}{1-(y/t)},$$

which is a function of $y/t$. Let $y = tv$, so that $y' = v + tv'$. The given differential equation can then be written as

$$v + tv' = \frac{1+v}{1-v}.$$

Now subtract $v$ from both sides and simplify to get

$$tv' = \frac{1+v}{1-v} - v = \frac{1+v^2}{1-v}.$$

Separating the variables gives

$$\frac{\mathrm{d}v}{1+v^2} - \frac{v\,\mathrm{d}v}{1+v^2} = \frac{\mathrm{d}t}{t}.$$

Integrating both sides gives us $\tan^{-1} v - (1/2)\ln(1+v^2) = \ln t + c$. Now substitute $v = y/t$ to get the implicit solution

$$\tan^{-1}(y/t) - \frac{1}{2}\ln(1+(y/t)^2) = \ln t + c.$$

This may be simplified slightly (multiply by 2 and combine the ln terms) to get

$$2\tan^{-1}(y/t) - \ln(t^2+y^2) = 2c.$$ ◄

A differential equation $y' = F(t, y)$ is homogeneous if we can write $F(t, y) = f(y/t)$, for some function $f$. This is what we did in the example above. To help identify situations where the identification $F(t, y) = f(y/t)$ is possible, we introduce the concept of a homogeneous function. A function $p(t, y)$ is said to be ***homogeneous of degree*** $n$ if $p(\alpha t, \alpha y) = \alpha^n p(t, y)$, for all $\alpha > 0$. To simplify the terminology, we will call a homogeneous function of degree zero just ***homogeneous***. For example, the polynomials $t^3 + ty^2$ and $y^3$ are both homogeneous of degree 3. Their quotient $F(t, y) = (t^3 + ty^2)/y^3$ is a homogeneous function. Indeed, if $\alpha > 0$, then

$$F(\alpha t, \alpha y) = \frac{(\alpha t)^3 + (\alpha t)(\alpha y)^2}{(\alpha y)^3} = \frac{\alpha^3}{\alpha^3}\frac{t^3+ty^2}{y^3} = F(t, y).$$

In fact, it is easy to see that the quotient of any two homogeneous functions of the same degree is a homogeneous function. Suppose $t > 0$ and let $\alpha = 1/t$. For a homogeneous function $F$, we have

$$F(t, y) = F\left(\frac{1}{t}t, \frac{1}{t}y\right) = F\left(1, \frac{y}{t}\right).$$

Similarly, if $t < 0$ and $\alpha = -1/t$ then

$$F(t, y) = F\left(\frac{-1}{t}t, \frac{-1}{t}y\right) = F\left(-1, -\frac{y}{t}\right).$$

Therefore, if $F(t, y)$ is homogeneous, then $F(t, y) = f(t/y)$ where

$$f\left(\frac{y}{t}\right) = \begin{cases} F\left(1, \frac{y}{t}\right) & \text{if } t > 0, \\ F\left(-1, -\frac{y}{t}\right) & \text{if } t < 0 \end{cases}$$

so that we have the following useful criterion for identifying homogeneous functions:

**Lemma 2.** *If $F(t, y)$ is a homogeneous function, then it can be written in the form*

$$F(t, y) = f\left(\frac{y}{t}\right),$$

*for some function $f$. Furthermore, the differential equation $y' = F(t, y)$ is a homogeneous differential equation.*

**Example 3.** Solve the following differential equation

$$y' = \frac{t^2 + 2ty - y^2}{2t^2}.$$

► **Solution.** Since the numerator and denominator are homogeneous of degree 2, the quotient is homogeneous. Letting $y = tv$ and dividing the numerator and denominator of the right-hand side by $t^2$, we obtain

$$v + tv' = \frac{1 + 2v - v^2}{2}.$$

Subtracting $v$ from both sides gives $tv' = \frac{1-v^2}{2}$. There are two equilibrium solutions: $v = \pm 1$. If $v \neq \pm 1$, separate the variables to get

$$\frac{2\,\mathrm{d}v}{1 - v^2} = \frac{\mathrm{d}t}{t},$$

and apply partial fractions to the left-hand side to conclude

$$\frac{\mathrm{d}v}{1-v}+\frac{\mathrm{d}v}{1+v}=\frac{\mathrm{d}t}{t}.$$

Integrating gives $-\ln|1-v|+\ln|1+v|=\ln|t|+c$, exponentiating both sides gives $\frac{1+v}{1-v}=kt$ for $k\neq 0$, and simplifying leads to $v=\frac{kt-1}{kt+1}$, $k\neq 0$. Substituting $y=vt$ gives the solutions

$$y=\frac{kt^2-t}{kt+1},\quad k\neq 0,$$

whereas the equilibrium solutions $v=\pm 1$ produce the solutions $y=\pm t$. Note that for $k=0$, we get the solution $y=-t$, which is one of the equilibrium solutions, but the equilibrium solution $y=t$ does not correspond to any choice of $k$. ◀

## *Bernoulli Equations*

A differential equation of the form

$$y'+p(t)y=f(t)y^n, \tag{3}$$

is called a ***Bernoulli equation***.[11] If $n=0$, this equation is linear, while if $n=1$, the equation is both separable and linear. Thus, we will assume $n\neq 0,1$. Note, also, that if $n>0$, then $y=0$ is a solution. Start by dividing (3) by $y^n$ to get

$$y^{-n}y'+p(t)y^{1-n}=f(t). \tag{4}$$

Use the coefficient of $p(t)$ as a new dependent variable. That is, use the substitution $z=y^{1-n}$. Thus, $z$ is treated as a function of $t$ and the chain rule gives $z'=(1-n)y^{-n}y'$, which is the first term of (4) multiplied by $1-n$. Therefore, substituting for $z$ and multiplying by the constant $(1-n)$ give,

$$z'+(1-n)p(t)z=(1-n)f(t), \tag{5}$$

which is a *linear* first order differential equation in the variables $t$ and $z$. Equation (5) can then be solved by Algorithm 3 of Sect. 1.4, and the solution to (3) is obtained by solving $z=y^{1-n}$ for $y$ (and including $y=0$ in the case $n>0$).

**Example 4.** Solve the Bernoulli equation $y'+y=5(\sin 2t)y^2$.

[11] Named after Jakoub Bernoulli (1654–1705).

► **Solution.** Note first that $y = 0$ is a solution since $n = 2 > 0$. After dividing our equation by $y^2$, we get $y^{-2}y' + y^{-1} = 5\sin 2t$. Let $z = y^{-1}$. Then $z' = -y^{-2}y'$ and substituting gives $-z' + z = 5\sin 2t$. In the standard form for linear equations, this becomes

$$z' - z = -5\sin 2t.$$

We can apply Algorithm 3 of Sect. 1.4 to this linear differential equation. The integrating factor will be $\mathrm{e}^{-t}$. Multiplying by the integrating factor gives $(\mathrm{e}^{-t}z)' = -5\mathrm{e}^{-t}\sin 2t$. Now integrate both sides to get

$$\mathrm{e}^{-t}z = \int -5\mathrm{e}^{-t}\sin 2t \,\mathrm{d}t = (\sin 2t + 2\cos 2t)\mathrm{e}^{-t} + c,$$

where $\int -5\mathrm{e}^{-t}\sin 2t \,\mathrm{d}t$ is computed by using integration by parts twice. Hence,

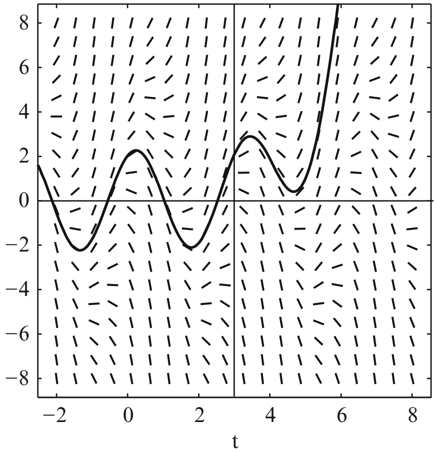
$$z = \sin 2t + 2\cos 2t + c\mathrm{e}^{t}.$$

Now go back to the original function $y$ by solving $z = y^{-1}$ for $y$ to get

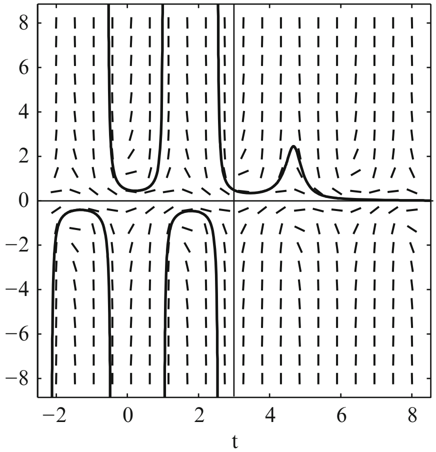
$$y = z^{-1} = (\sin 2t + 2\cos 2t + c\mathrm{e}^{t})^{-1} = \frac{1}{\sin 2t + 2\cos 2t + c\mathrm{e}^{t}}.$$

This function $y$, together with $y = 0$, constitutes the general solution of $y' + y = 5(\sin 2t)y^2$. ◄

**Remark 5.** We should note that in the $z$ variable, the solution $z = \sin t - \cos t + c\mathrm{e}^{t}$ is valid for all $t$ in $\mathbb{R}$. This is to be expected. Since $z' - z = -5\sin 2t$ is linear and the coefficient function, $p(t) = 1$, and the forcing function, $f(t) = -5\sin 2t$, are continuous on $\mathbb{R}$, Theorem 2 of Sect. 1.2 implies that any solution will be defined on all of $\mathbb{R}$. However, in the $y$ variable, things are very different. The solution $y = 1/(\sin t - \cos t + c\mathrm{e}^{t})$ to the given Bernoulli equation is valid only on intervals where the denominator is nonzero. This is precisely because of the inverse relationship between $y$ and $z$: the location of the zeros of $z$ is precisely where $y$ has a vertical asymptote. In the figure below, we have graphed a solution in the $z$ variable and the corresponding solution in the $y = 1/z$ variable. Note that in the $y$ variable, the intervals of definition are determined by the zeros of $z$. Again, it is not at all evident from the differential equation, $y' - y = -2(\sin t)y^2$, that the solution curves would have "chopped up" intervals of definition. The linear case as described in Theorem 2 of Sect. 1.4 is thus rather special.




Solution Curve in $z$ for
$z' - z = -5 \sin 2t$




Solution Curve in $y$ for
$y' + y = 5(\sin 2t)y^2$

## *Linear Substitutions*

Consider a differential equation of the form

$$y' = f(at + by + c), \tag{6}$$

where $a$, $b$, $c$ are real constants. Let $z = at + by + c$. Then $z' = a + by'$ so $y' = (z' - a)/b$. Substituting gives $(z' - a)/b = f(z)$ which in standard form is

$$z' = bf(z) + a,$$

a separable differential equation. Consider the following example.

**Example 6.** Solve the following differential equation:

$$y' = (t + y + 1)^2.$$

► **Solution.** Let $z = t + y + 1$. Then $z' = 1 + y'$ so $y' = z' - 1$. Substituting gives $z' - 1 = z^2$, and in standard form, we get the separable differential equation $z' = 1 + z^2$. Separating variables gives

$$\frac{\mathrm{d}z}{1 + z^2} = \mathrm{d}t.$$

Integrating gives $\tan^{-1} z = t + c$. Thus, $z = \tan(t + c)$. Now substitute $z = t + y + 1$ and simplify to get the solution

$$y = -t - 1 + \tan(t + c). \qquad \blacktriangleleft$$

Substitution methods are a general way to simplify complicated differential equations into familiar forms. These few that we have discussed are just some of many examples that can be solved by the technique of substitution. If you ever come across a differential equation you cannot solve, try to simplify it into a familiar form by finding the right substitution.

## Exercises

1–8. Find the general solution of each of the following homogeneous equations. If an initial value is given, also solve the initial value problem.

1. $t^2 y' = y^2 + yt + t^2, \quad y(1) = 1$
2. $y' = \dfrac{4t - 3y}{t - y}$
3. $y' = \dfrac{y^2 - 4yt + 6t^2}{t^2}, \quad y(2) = 4$
4. $y' = \dfrac{y^2 + 2yt}{t^2 + yt}$
5. $y' = \dfrac{3y^2 - t^2}{2ty}$
6. $y' = \dfrac{t^2 + y^2}{ty}, \quad y(e) = 2e$
7. $ty' = y + \sqrt{t^2 - y^2}$
8. $t^2 y' = yt + y\sqrt{t^2 + y^2}$

9–17. Find the general solution of each of the following Bernoulli equations. If an initial value is given, also solve the initial value problem.

9. $y' - y = ty^2, \quad y(0) = 1$
10. $y' + y = y^2, \quad y(0) = 1$
11. $y' + ty = ty^3$
12. $y' + ty = t^3 y^3$
13. $(1 - t^2)y' - ty = 5ty^2$
14. $y' + \dfrac{y}{t} = y^{2/3}$
15. $yy' + ty^2 = t, \quad y(0) = -2$
16. $2yy' = y^2 + t - 1$
17. $y' + y = ty^3$
18. Write the logistic differential equation $P' = r(1 - \frac{P}{m})P$ ((19) of Sect. 1.3) as a Bernoulli equation and solve it using the technique described in this section.

19–22. Use an appropriate linear substitution to solve the following differential equations:

19. $y' = (2t - 2y + 1)^{-1}$
20. $y' = (t - y)^2$
21. $y' = \dfrac{1}{(t + y)^2}$
22. $y' = \sin(t - y)$

23–26. Use the indicated substitution to solve the given differential equation.

23. $2yy' = y^2 + t - 1$, $z = y^2 + t - 1$ (Compare with Exercise 16.)
24. $y' = \tan y + \dfrac{2\cos t}{\cos y}$, $z = \sin y$
25. $y' + y\ln y = ty$, $z = \ln y$
26. $y' = -e^y - 1$, $z = e^{-y}$

## 1.6 Exact Equations

When $y(t)$ is any function defined implicitly by an equation

$$V(t,\ y) = c,$$

that is,

$$V(t,\ y(t)) = c$$

for all $t$ in an interval $I$, implicit differentiation of this equation with respect to $t$ shows that $y(t)$ satisfies the equation

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}c = \frac{\mathrm{d}}{\mathrm{d}t}V(t, y(t)) = \frac{\partial V}{\partial t}(t, y(t)) + \frac{\partial V}{\partial y}(t, y(t))y'(t),$$

and thus is a solution of the differential equation

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial y}y' = 0.$$

For example, any solution $y(t)$ of the equation $y^2 + 2yt - y^2 = c$ is a solution of the differential equation

$$(2y - 2t) + (2y + 2t)y' = 0.$$

Conversely, suppose we are given a differential equation

$$M + Ny' = 0, \tag{1}$$

where $M$ and $N$ are functions of the two variables $t$ and $y$. If there is a function $V(t,\ y)$ for which

$$M = \frac{\partial V}{\partial t} \quad \text{and} \quad N = \frac{\partial V}{\partial y}, \tag{2}$$

then we can work backward from the implicit differentiation in the previous paragraph to conclude that any implicitly defined solution of

$$V(t,\ y) = c, \tag{3}$$

for $c$ an arbitrary constant, is a solution of the differential equation (1). To summarize, *if the given functions $M$ and $N$ in (1) are such that there is a function $V(t,\ y)$ for which equations (2) are satisfied, then the solution of the differential equation (1) is given by the implicitly defined solutions to (3)*. In this case, we will say that the differential equation $M + Ny' = 0$ is an ***exact differential equation***.

Suppose we are given a differential equation $M + Ny' = 0$, but we are not given a priori that $M = \partial V/\partial t$ and $N = \partial V/\partial t$. How can we determine if there is such a function $V(t,\ y)$, and if there is, how can we find it? That is, is there a *criterion* for determining if a given differential equation is exact, and if so, is there a *procedure* for producing the function $V(t,\ y)$ that determines the solution via $V(t,\ y) = c$. The answer to both questions is yes.

A criterion for exactness comes from the equality of mixed partial derivatives. Recall (from your calculus course) that all functions $V(t,\ y)$ whose second partial derivatives exist and are continuous satisfy[12]

$$\frac{\partial}{\partial y}\left(\frac{\partial V}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{\partial V}{\partial y}\right). \tag{4}$$

If the equation $M + Ny' = 0$ is exact, then, by definition, there is a function $V(t,\ y)$ such that $\partial V/\partial t = M$ and $\partial V/\partial y = N$, so (4) gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial V}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{\partial V}{\partial y}\right) = \frac{\partial N}{\partial t}.$$

Hence, a *necessary condition for exactness* of $M + Ny' = 0$ is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}. \tag{5}$$

We now ask if this condition is also sufficient for the differential equation $M + Ny' = 0$ to be exact. That is, if (5) is true, can we always find a function $V(t,\ y)$ such that $M = \partial V/\partial t$ and $N = \partial V/\partial y$? We can easily find $V(t,\ y)$ such that

$$\frac{\partial V}{\partial t} = M \tag{6}$$

by integrating $M$ with respect to the $t$ variable. After determining $V(t,\ y)$ so that (6) holds, can we also guarantee that $\partial V/\partial y = N$? Integrating (6) with respect to $t$, treating the $y$ variable as a constant, gives

$$V = \int M\,\mathrm{d}t + \phi(y). \tag{7}$$

The function $\phi(y)$ is an arbitrary function of $y$ that appears as the "integration constant" since any function of $y$ goes to 0 when differentiated with respect to $t$. The "integration constant" $\phi(y)$ can be determined by differentiating $V$ in (7) with respect to $y$ and equating this expression to $N$ since $\partial V/\partial y = N$ if the differential

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[12]This equation is known as Clairaut's theorem (after Alexis Clairaut (1713–1765)) on the equality of mixed partial derivatives.

equation is exact. Thus (since integration with respect to $t$ and differentiation with respect to $y$ commute)

$$\frac{\partial V}{\partial y} = \frac{\partial}{\partial y}\int M\,\mathrm{d}t + \frac{\mathrm{d}\phi}{\mathrm{d}y} = \int \frac{\partial M}{\partial y}\,\mathrm{d}t + \frac{\mathrm{d}\phi}{\mathrm{d}y} = N. \tag{8}$$

That is,

$$\frac{\mathrm{d}\phi}{\mathrm{d}y} = N - \int \frac{\partial M}{\partial y}\,\mathrm{d}t. \tag{9}$$

The verification that the function on the right is really a function only of $y$ (as it must be if it is to be the derivative $\frac{\mathrm{d}\phi}{\mathrm{d}y}$ of a function of $y$) is where condition (5) is needed. Indeed, using (5)

$$N - \int \frac{\partial M}{\partial y}\,\mathrm{d}t = N - \int \frac{\partial N}{\partial t}\,\mathrm{d}t = N - (N + \psi(y)) = -\psi(y). \tag{10}$$

Hence, (9) is a valid equation for determining $\phi(y)$ in (7).

We can summarize our conclusions in the following result.

**Theorem 1 (Criterion for exactness).** *A first order differential equation*

$$M(t,\ y) + N(t,\ y)y' = 0$$

*is exact if and only if*

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}.} \tag{11}$$

*If this condition is satisfied, then the general solution of the differential equation is given by $V(t,\ y) = c$, where $V(t,\ y)$ is determined by (7) and (9).*

The steps needed to derive the solution of an exact equation when dealing with concrete examples are summarized in the following algorithm.

**Algorithm 2.** The following procedure is used to solve an exact differential equation.

**Solution Method for Exact Equations**

1. *Check the equation $M + Ny' = 0$ for exactness:* To do this, check if
$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}.$$
2. *Integrate the equation $M = \partial V/\partial t$ with respect to $t$ to get*
$$V(t, y) = \int M(t, y)\,\mathrm{d}t + \phi(y), \tag{12}$$
where the "integration constant" $\phi(y)$ is a function of $y$.
3. *Differentiate this expression for $V(t, y)$ with respect to $y$, and set it equal to $N$:*
$$\frac{\partial}{\partial y}\left(\int M(t, y)\,\mathrm{d}t\right) + \frac{\mathrm{d}\phi}{\mathrm{d}y} = \frac{\partial V}{\partial y} = N(t, y).$$
4. *Solve this equation for $\frac{\mathrm{d}\phi}{\mathrm{d}y}$:*
$$\frac{\mathrm{d}\phi}{\mathrm{d}y} = N(t, y) - \frac{\partial}{\partial y}\left(\int M(t, y)\,\mathrm{d}t\right). \tag{13}$$
5. *Integrate $\frac{\mathrm{d}\phi}{\mathrm{d}y}$ to get $\phi(y)$ and then substitute in (12) to find $V(t, y)$.*
6. *The solution of $M + Ny' = 0$ is then given by*
$$V(t, y) = c, \tag{14}$$
*where $c$ is an arbitrary constant.*

It is worthwhile to emphasize that *the solution to the exact equation $M + Ny' = 0$ is not the function $V(t, y)$ found by the above procedure but rather the functions $y(t)$ defined by the implicit relation $V(t, y) = c$.*

**Example 3.** Determine if the differential equation
$$(3t^2 + 4ty - 2) + (2t^2 + 6y^2)y' = 0$$
is exact. If it is exact, solve it.

► **Solution.** This equation is in the standard form $M + Ny'$ with $M(t, y) = 3t^2 + 4ty - 2$ and $N(t, y) = 2t^2 + 6y^2$. Then
$$\frac{\partial M}{\partial y} = 4t = \frac{\partial N}{\partial t},$$

so the exactness criterion is satisfied and the equation is exact. To solve the equation, follow the solution procedure by computing

$$V(t,\, y) = \int (3t^2 + 4ty - 2)\,\mathrm{d}t + \phi(y) = t^3 + 2t^2 y - 2t + \phi(y),$$

where $\phi(y)$ is a function only of $y$ that is yet to be computed. Now differentiate this expression with respect to $y$ to get

$$\frac{\partial V}{\partial y} = 2t^2 + \frac{\mathrm{d}\phi}{\mathrm{d}y}.$$

But since the differential equation is exact, we also have

$$\frac{\partial V}{\partial y} = N(t,\, y) = 2t^2 + 6y^2,$$

and combining these last two expressions, we conclude

$$\frac{\mathrm{d}\phi}{\mathrm{d}y} = 6y^2.$$

Integrating with respect to $y$ gives $\phi(y) = 2y^3 + c_0$, so that

$$V(t,\, y) = t^3 + 2t^2 y - 2t + 2y^3 + c_0,$$

where $c_0$ is a constant. The solutions of the differential equation are then given by the relation $V(t,\, y) = c$, that is,

$$t^3 + 2t^2 y - 2t + 2y^3 = c,$$

where the constant $c_0$ has been incorporated in the constant $c$. ◀

What happens if we try to solve the equation $M(t,\, y) + N(t,\, y)y' = 0$ by the procedure outlined above without *first* verifying that it is exact? If the equation is not exact, you will discover this fact when you get to (9), since $\frac{\mathrm{d}\phi}{\mathrm{d}y}$ will *not* be a function only of $y$, as the following example illustrates.

**Example 4.** Try to solve the equation $(t - 3y) + (2t + y)y' = 0$ by the solution procedure for exact equations.

► **Solution.** Note that $M(t,\, y) = t - 3y$ and $N(t,\, y) = 2t + y$. First apply (12) to get

$$V(t,\, y) = \int M(t,\, y), \mathrm{d}t = \int (t - 3y)\,\mathrm{d}t = \frac{t^2}{2} - 3ty + \phi(y),$$

and then determine $\phi(y)$ from (13):

$$\frac{d\phi}{dy} = N(t, y) - \frac{\partial}{\partial y}\int M(t, y)\,dt = (2t + y) - \frac{\partial}{\partial y}\left(\frac{t^2}{2} - 3ty + \phi(y)\right) = y - t.$$

But we see that there is a problem since $\frac{d\phi}{dy} = y - t$ involves both $y$ and $t$. This is where it becomes obvious that you are not dealing with an exact equation, and you cannot proceed with this procedure. Indeed, $\partial M/\partial y = -3 \neq 2 = \partial N/\partial t$, so that this equation fails the exactness criterion (11). ◀

## *Integrating Factors*

The differential equation

$$3y^2 + 8t + 2tyy' = 0 \tag{15}$$

can be written in the form $M + Ny' = 0$ where

$$M = 3y^2 + 8t \qquad \text{and} \qquad N = 2ty.$$

Since

$$\frac{\partial M}{\partial y} = 6y, \qquad \frac{\partial N}{\partial t} = 2y,$$

the equation is not exact. However, if we multiply (15) by $t^2$, we arrive at an equivalent differential equation

$$3y^2t^2 + 8t^3 + 2t^3yy' = 0, \tag{16}$$

that is exact. Indeed, in the new equation, $M = 3y^2t^2 + 8t^3$ and $N = 2t^3y$ so that the exactness criterion is satisfied:

$$\frac{\partial M}{\partial y} = 6yt^2 = \frac{\partial N}{\partial t}.$$

Using the solution method for exact equations, we find

$$V(t, y) = \int \left(3y^2t^2 + 8t^3\right)\,dt = y^2t^3 + 2t^4 + \phi(y),$$

where

$$\frac{\partial V}{\partial y} = 2yt^3 + \frac{d\phi}{dy} = N = 2t^3y.$$

Thus, $\frac{d\phi}{dy} = 0$ so that

$$V(t, y) = y^2t^3 + 2t^4 + c_0.$$

Therefore, incorporating the constant $c_0$ into the general constant $c$, we have

$$y^2t^3 + 2t^4 = c,$$

as the solution of (16) and, hence, also of the equivalent equation (15). Thus, multiplication of the nonexact equation (15) by the function $t^2$ has produced an exact equation (16) that is easily solved.

This suggests the following question. Can we find a function $\mu(t, y)$ so that multiplication of a nonexact equation

$$M + Ny' = 0 \tag{17}$$

by $\mu$ produces an equation

$$\mu M + \mu N y' = 0 \tag{18}$$

that is exact? When there is such a $\mu$, we call it an ***integrating factor*** for (17).

The exactness criterion for (18) is

$$\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial t}.$$

Written out, this becomes

$$\mu \frac{\partial M}{\partial y} + M \frac{\partial \mu}{\partial y} = \mu \frac{\partial N}{\partial t} + N \frac{\partial \mu}{\partial t},$$

which implies

$$\frac{\partial \mu}{\partial t} N - \frac{\partial \mu}{\partial y} M = \left( \frac{\partial M}{\partial y} - \frac{\partial N}{\partial t} \right) \mu. \tag{19}$$

Thus, it appears that the search for an integrating factor for the nonexact ordinary differential equation (17) involves the solution of (19), which is a partial differential equation for $\mu$. In general, it is quite difficult to solve (19). However, there are some special situations in which it is possible to use (19) to find an integrating factor $\mu$. One such example occurs if *$\mu$ is a function only of $t$*. In this case, $\partial\mu/\partial t = d\mu/dt$ and $\partial\mu/\partial y = 0$ so (19) can be written as

$$\frac{1}{\mu} \frac{d\mu}{dt} = \frac{\partial M/\partial y - \partial N/\partial t}{N}. \tag{20}$$

Since the left-hand side of this equation depends only on $t$, the same must be true of the right-hand side. We conclude that we can find an integrating factor $\mu = \mu(t)$

consisting of a function only of $t$ provided the right-hand side of (20) is a function only of $t$. If this is true, then we put

$$\frac{\partial M/\partial y - \partial N/\partial t}{N} = p(t), \tag{21}$$

so that (20) becomes the linear differential equation $\mu' = p(t)\mu$, which has the solution

$$\mu(t) = \mathrm{e}^{\int p(t)\,\mathrm{d}t}. \tag{22}$$

After multiplication of (17) by this $\mu(t)$, the resulting equation is exact, and the solution is obtained by the solution method for exact equations.

**Example 5.** Find an integrating factor for the equation

$$3y^2 + 8t + 2tyy' = 0.$$

► **Solution.** This equation is (15). Since

$$\frac{\partial M/\partial y - \partial N/\partial t}{N} = \frac{6y - 2y}{2ty} = \frac{2}{t},$$

is a function of only $t$, we conclude that an integrating factor is

$$\mu(t) = \mathrm{e}^{\int (2/t)\,\mathrm{d}t} = \mathrm{e}^{2\ln t} = t^2.$$

This agrees with what we already observed in (16). ◄

Similar reasoning gives a criterion for (17) to have an integrating factor $\mu(y)$ that involves only the variable $y$. Namely, if the expression

$$\frac{\partial M/\partial y - \partial N/\partial t}{-M} \tag{23}$$

is a function of only $y$, say $q(y)$, then

$$\mu(y) = \mathrm{e}^{\int q(y)\,\mathrm{d}y} \tag{24}$$

is a function only of $y$ that is an integrating factor for (17).

**Remark 6.** A linear differential equation $y' + p(t)y = f(t)$ can be rewritten in the form $M + Ny' = 0$ where $M = p(t)y - f(t)$ and $N = 1$. In this case, $\partial M/\partial y = p(t)$ and $\partial N/\partial t = 0$ so a linear equation is never exact unless $p(t) = 0$. However,

$$\frac{\partial M/\partial y - \partial N/\partial t}{N} = p(t)$$

so there is an integrating factor that depends only on $t$:

$$\mu(t) = \mathrm{e}^{\int p(t)\,\mathrm{d}t}.$$

This is exactly the same function that we called an integrating factor for the linear differential equation in Sect. 1.4. Thus, the integrating factor for a linear differential equation is a special case of the concept of an integrating factor to transform a general equation into an exact one.

## Exercises

1–9. Determine if the equation is exact, and if it is exact, find the general solution.

1. $(y^2 + 2t) + 2tyy' = 0$
2. $y - t + (t + 2y)y' = 0$
3. $2t^2 - y + (t + y^2)y' = 0$
4. $y^2 + 2tyy' + 3t^2 = 0$
5. $(3y - 5t) + 2yy' - ty' = 0$
6. $2ty + (t^2 + 3y^2)y' = 0$, $y(1) = 1$
7. $2ty + 2t^3 + (t^2 - y)y' = 0$
8. $t^2 - y - ty' = 0$
9. $(y^3 - t)y' = y$
10. Find conditions on the constants $a, b, c, d$ which guarantee that the differential equation $(at + by) = (ct + dy)y'$ is exact.

## 1.7 Existence and Uniqueness Theorems

Let us return to the general initial value problem

$$y' = F(t, y), \quad y(t_0) = y_0, \tag{1}$$

introduced in Sect. 1.1. We want to address the existence and uniqueness of solutions to (1). The main theorems we have in mind put certain relatively mild conditions on $F$ to insure that a solution exists, is unique, and/or both. We say that a solution ***exists*** if there is a function $y = y(t)$ defined on an interval containing $t_0$ as an interior point and satisfying (1). We say that a solution is ***unique*** if there is only one such function $y = y(t)$. Why are the concepts of existence and uniqueness important? Differential equations frequently model real-world systems as a function of time. Knowing that a solution exists means that the system has predictable future states. Further, when that solution is also unique, then for each future time there is only *one* possible state, leaving no room for ambiguity.

To illustrate these points and some of the theoretical aspects of the existence and uniqueness theorems that follow, consider the following example.

**Example 1.** Show that the following two functions satisfy the initial value problem

$$y' = 3y^{2/3}, \quad y(0) = 0. \tag{2}$$

1. $y(t) = 0$
2. $y(t) = t^3$

► **Solution.** Clearly, the constant function $y(t) = 0$ for $t \in \mathbb{R}$ is a solution. For the second function $y(t) = t^3$, observe that $y' = 3t^2$ while $3y^{2/3} = 3(t^3)^{2/3} = 3t^2$. Further, $y(0) = 0$. It follows that both of the functions $y(t) = 0$ and $y(t) = t^3$ are solutions of (2). ◄

If this differential equation modeled a real-world system, we would have problems accurately predicting future states. Should we use $y(t) = 0$ or $y(t) = t^3$? It is even worse than this, for further analysis of this differential equation reveals that there are many other solutions from which to choose. (See Example 9.) What is it about (2) that allows multiple solutions? More precisely, what conditions could we impose on (1) to guarantee that a solution exists and is unique? These questions are addressed in Picard's existence and uniqueness theorem, Theorem 5, stated below.

Thus far, our method for proving the existence of a solution to an ordinary differential equation has been to explicitly find one. This has been a reasonable approach for the categories of differential equations we have introduced thus far. However, there are many differential equations that do not fall into any of these categories, and knowing that a solution exists is a fundamental piece of information in the analysis of any given initial value problem.

Suppose $F(t,\ y)$ is a *continuous* function of $(t,\ y)$ in the rectangle

$$\mathcal{R} := \{(t,\ y) : a \le t \le b\,,\ c \le y \le d\}$$

and $(t_0, y_0)$ is an interior point of $\mathcal{R}$. The key to the proof of existence and uniqueness is the fact that a continuously differentiable function $y(t)$ is a solution of (1) if and only if it is a solution of the integral equation

$$y(t) = y_0 + \int_{t_0}^{t} F(u,\ y(u))\,\mathrm{d}u. \tag{3}$$

To see the equivalence between (1) and (3), assume that $y(t)$ is a solution to (1), so that

$$y'(t) = F(t,\ y(t))$$

for all $t$ in an interval containing $t_0$ as an interior point and $y(t_0) = y_0$. Replace $t$ by $u$ in this equation, integrate both sides from $t_0$ to $t$, and use the fundamental theorem of calculus to get

$$\int_{t_0}^{t} F(u,\ y(u))\,\mathrm{d}u = \int_{t_0}^{t} y'(u)\,\mathrm{d}u = y(t) - y(t_0) = y(t) - y_0,$$

which implies that $y(t)$ is a solution of (3). Conversely, if $y(t)$ is a continuously differentiable solution of (3), it follows that

$$g(t) := F(t, y(t))$$

is a continuous function of $t$ since $F(t,\ y)$ is a continuous function of $t$ and $y$. Apply the fundamental theorem of calculus to get

$$\begin{aligned} y'(t) &= \frac{\mathrm{d}}{\mathrm{d}t} y(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(y_0 + \int_{t_0}^{t} F(u,\ y(u))\,\mathrm{d}u\right) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left(y_0 + \int_{t_0}^{t} g(u)\,\mathrm{d}u\right) = g(t) = F(t,\ y(t)), \end{aligned}$$

which is what it means for $y(t)$ to be a solution of (1). Since

$$y(t_0) = y_0 + \int_{t_0}^{t_0} F(u,\ y(u))\,\mathrm{d}u = y_0,$$

$y(t)$ also satisfies the initial value in (1).

We will refer to (3) as the *integral equation* corresponding to the initial value problem (1) and conversely, (1) is referred to as the *initial value problem* corresponding to the integral equation (3). What we have shown is that a solution to the initial value problem is a solution to the corresponding integral equation and vice versa.

**Example 2.** Find the integral equation corresponding to the initial value problem

$$y' = t + y, \quad y(0) = 1.$$

► **Solution.** In this case, $F(t, y) = t + y$, $t_0 = 0$ and $y_0 = 1$. Replace the independent variable $t$ by $u$ in $F(t, y(t))$ to get $F(u, y(u)) = u + y(u)$. Thus, the integral equation (3) corresponding to this initial value problem is

$$y(t) = 1 + \int_0^t (u + y(u))\,\mathrm{d}u. \qquad ◄$$

For any continuous function $y$, define

$$\mathcal{T}y(t) = y_0 + \int_{t_0}^t F(u, y(u))\,\mathrm{d}u.$$

That is, $\mathcal{T}y$ is the right-hand side of (3) for any continuous function $y$. Given a function $y$, $\mathcal{T}$ produces a new function $\mathcal{T}y$. If we can find a function $y$ so that $\mathcal{T}y = y$, we say $y$ is a ***fixed point*** of $\mathcal{T}$. A fixed point $y$ of $\mathcal{T}$ is precisely a solution to (3) since if $y = \mathcal{T}y$, then

$$y = \mathcal{T}y = y_0 + \int_{t_0}^t F(u, y(u)\,\mathrm{d}u,$$

which is what it means to be a solution to (3). To solve equations like the integral equation (3), mathematicians have developed a variety of so-called "fixed point theorems" for operators such as $\mathcal{T}$, each of which leads to an existence and/or uniqueness result for solutions to an integral equation. One of the oldest and most widely used of the existence and uniqueness theorems is due to Émile Picard (1856–1941). Assuming that the function $F(t, y)$ is sufficiently "nice," he first employed the ***method of successive approximations***. This method is an iterative procedure which begins with a crude approximation of a solution and improves it using a step-by-step procedure that brings us as close as we please to an exact and unique solution of (3). The process should remind students of Newton's method where successive approximations are used to find numerical solutions to $f(t) = c$ for some function $f$ and constant $c$. The algorithmic procedure follows.

**Algorithm 3.** Perform the following sequence of steps to produce an *approximate solution* of (3), and hence to the initial value problem, (1).

**Picard Approximations**

1. A rough initial approximation to a solution of (3) is given by the constant function
$$y_0(t) := y_0.$$
2. Insert this initial approximation into the right-hand side of (3) and obtain the first approximation
$$y_1(t) := y_0 + \int_{t_0}^{t} F(u, y_0(u))\,\mathrm{d}u.$$
3. The next step is to generate the second approximation in the same way, that is,
$$y_2(t) := y_0 + \int_{t_0}^{t} F(u, y_1(u))\,\mathrm{d}u.$$
4. At the $n$th stage of the process, we have
$$y_n(t) := y_0 + \int_{t_0}^{t} F(u, y_{n-1}(u))\,\mathrm{d}u,$$
which is defined by substituting the previous approximation $y_{n-1}(t)$ into the right-hand side of (3).

In terms of the operator $\mathcal{T}$ introduced above, we can write

$$\begin{aligned} y_1 &= \mathcal{T} y_0 \\ y_2 &= \mathcal{T} y_1 = \mathcal{T}^2 y_0 \\ y_3 &= \mathcal{T} y_2 = \mathcal{T}^3 y_0 \\ &\vdots \end{aligned}$$

The result is a sequence of functions $y_0(t), y_1(t), y_2(t), \ldots$, defined on an interval containing $t_0$. We will refer to $y_n$ as the ***nth Picard approximation*** and the sequence $y_0(t), y_1(t), \ldots, y_n(t)$ as the ***first n Picard approximations***. Note that the first $n$ Picard approximations actually consist of $n+1$ functions, since the starting approximation $y_0(t) = y_0$ is included.

**Example 4.** Find the first three Picard approximations for the initial value problem

$$y' = t + y, \quad y(0) = 1.$$

► **Solution.** The corresponding integral equation was computed in Example 2:

$$y(t) = 1 + \int_0^t (u + y(u))\,\mathrm{d}u.$$

We have

$$\begin{aligned}
y_0(t) &= 1 \\
y_1(t) &= 1 + \int_0^t (u + y_0(u))\,\mathrm{d}u \\
&= 1 + \int_0^t (u+1)\,\mathrm{d}u \\
&= 1 + \left(\frac{u^2}{2} + u\right)\bigg|_0^t = 1 + \frac{t^2}{2} + t = 1 + t + \frac{t^2}{2}. \\
y_2(t) &= 1 + \int_0^t \left(u + 1 + u + \frac{u^2}{2}\right)\mathrm{d}u = 1 + \int_0^t \left(1 + 2u + \frac{u^2}{2}\right)\mathrm{d}u \\
&= 1 + \left(u + u^2 + \frac{u^3}{6}\right)\bigg|_0^t = 1 + t + t^2 + \frac{t^3}{6}. \\
y_3(t) &= 1 + \int_0^t \left(u + 1 + u + u^2 + \frac{u^3}{6}\right)\mathrm{d}u = 1 + \int_0^t \left(1 + 2u + u^2 + \frac{u^3}{6}\right)\mathrm{d}u \\
&= 1 + \left(u + u^2 + \frac{u^3}{3} + \frac{u^4}{24}\right)\bigg|_0^t = 1 + t + t^2 + \frac{t^3}{3} + \frac{t^4}{24}.
\end{aligned}$$ ◄

It was one of Picard's great contributions to mathematics when he showed that the functions $y_n(t)$ converge to a unique, continuously differentiable solution $y(t)$ of the integral equation, (3), and thus of the initial value problem, (1), under the mild condition that the function $F(t, y)$ and its partial derivative $F_y(t, y) := \frac{\partial}{\partial y} F(t, y)$ are continuous functions of $(t, y)$ on the rectangle $\mathcal{R}$.

**Theorem 5 (Picard's Existence and Uniqueness Theorem).**[13] *Let $F(t, y)$ and $F_y(t, y)$ be continuous functions of $(t, y)$ on a rectangle*

$$\mathcal{R} = \{(t, y) : a \le t \le b\,,\ c \le y \le d\}\,.$$

*If $(t_0, y_0)$ is an interior point of $\mathcal{R}$, then there exists a unique solution $y(t)$ of*

$$y' = F(t, y)\,, \quad y(t_0) = y_0,$$

[13] A proof of this theorem can be found in G.F. Simmons' book *Differential Equations with Applications and Historical Notes*, 2nd edition McGraw-Hill, 1991.

*on some interval* $[a', b']$ *with* $t_0 \in [a', b'] \subset [a, b]$. *Moreover, the sequence of approximations* $y_0(t) := y_0$

$$y_n(t) := y_0 + \int_{t_0}^{t} F(u, y_{n-1}(u))\, du,$$

*computed by Algorithm 3 converges uniformly*[14] *to* $y(t)$ *on the interval* $[a', b']$.

**Example 6.** Consider the initial value problem

$$y' = t + y \quad y(0) = 1.$$

For $n \geq 1$, find the $n$th Picard approximation and determine the limiting function $y = \lim_{n\to\infty} y_n$. Show that this function is a solution and, in fact, the only solution.

► **Solution.** In Example 4, we computed the first three Picard approximations:

$$\begin{aligned} y_1(t) &= 1 + t + \frac{t^2}{2}, \\ y_2(t) &= 1 + t + t^2 + \frac{t^3}{3!}, \\ y_3(t) &= 1 + t + t^2 + \frac{t^3}{3} + \frac{t^4}{4!}. \end{aligned}$$

It is not hard to verify that

$$\begin{aligned} y_4(t) &= 1 + t + t^2 + \frac{t^3}{3} + \frac{t^4}{12} + \frac{t^5}{5!} \\ &= 1 + t + 2\left(\frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!}\right) + \frac{t^5}{5!} \end{aligned}$$

and inductively,

$$y_n(t) = 1 + t + 2\left(\frac{t^2}{2!} + \frac{t^3}{3!} + \cdots + \frac{t^n}{n!}\right) + \frac{t^{n+1}}{(n+1)!}.$$

---

[14]*Uniform convergence* is defined as follows: for all $\epsilon > 0$, there exists $n_0$ such that the maximal distance between the graph of the functions $y_n(t)$ and the graph of $y(t)$ (for $t \in [a', b']$) is less than $\epsilon$ for all $n \geq n_0$. We will not explore in detail this kind of convergence, but we will note that it implies *pointwise convergence*. That is, for each $t \in [\alpha', \beta']$ $y_n(t) \to y(t)$.

Recall from calculus that $\mathrm{e}^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}$, so the part in parentheses in the expression for $y_n(t)$ is the first $n$ terms of the expansion of $\mathrm{e}^t$ minus the first two:

$$\frac{t^2}{2!} + \frac{t^3}{3!} + \cdots + \frac{t^n}{n!} = \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots + \frac{t^n}{n!}\right) - (1 + t).$$

Thus,

$$y(t) = \lim_{n\to\infty} y_n(t) = 1 + t + 2(\mathrm{e}^t - (1 + t)) = -1 - t + 2\mathrm{e}^t.$$

It is easy to verify by direct substitution that $y(t) = -1 - t + 2\mathrm{e}^t$ is a solution to $y' = t + y$ with initial value $y(0) = 1$. Moreover, since the equation $y' = t + y$ is a first order linear differential equation, the techniques of Sect. 1.4 show that $y(t) = -1 - t + 2\mathrm{e}^t$ is the *unique* solution since it is obtained by an explicit formula. Alternatively, Picard's theorem may be applied as follows. Consider any rectangle $\mathcal{R}$ about the point $(0, 1)$. Let $F(t, y) = t + y$. Then $F_y(t, y) = 1$. Both $F(t, y)$ and $F_y(t, y)$ are continuous functions on the whole $(t, y)$-plane and hence continuous on $\mathcal{R}$. Therefore, Picard's theorem implies that $y(t) = \lim_{n\to\infty} y_n(t)$ is the unique solution of the initial value problem

$$y' = t + y \quad y(0) = 1.$$

Hence, $y(t) = -1 - t + 2\mathrm{e}^t$ is the only solution. ◀

**Example 7.** Consider the Riccati equation

$$y' = y^2 - t$$

with initial condition $y(0) = 0$. Determine whether Picard's theorem applies on a rectangle containing $(0, 0)$. What conclusions can be made? Determine the first three Picard approximations.

► **Solution.** Here, $F(t, y) = y^2 - t$ and $F_y(t, y) = 2y$ are continuous on all of $\mathbb{R}^2$ and hence on *any* rectangle that contains the origin. Thus, by Picard's Theorem, the initial value problem

$$y' = y^2 - t\,, \quad y(0) = 0$$

has a unique solution on some interval $I$ containing 0. Picard's theorem does not tell us on what interval the solution is defined, only that there is *some* interval. The direction field for $y' = y^2 - t$ with the unique solution through the origin is given below and suggests that the maximal interval $I_{\max}$ on which the solution exists should be of the form $I_{\max} = (a, \infty)$ for some $-\infty \leq a < -1$. However, without further analysis of the problem, we have no precise knowledge about the maximal domain of the solution.

Next we show how Picard's method of successive approximations works in this example. To use this method, we rewrite the initial value problem as an integral equation. In this example, $F(t, y) = y^2 - t$, $t_0 = 0$ and $y_0 = 0$. Thus, the corresponding integral equation is

$$y(t) = \int_0^t (y(u)^2 - u)\,\mathrm{d}u. \tag{4}$$

We start with our initial approximation $y_0(t) = 0$, plug it into (4), and obtain our first approximation

$$y_1(t) = \int_0^t (y_0(u)^2 - u)\,\mathrm{d}u = -\int_0^t u\,\mathrm{d}u = -\frac{1}{2}t^2.$$

The second iteration yields

$$y_2(t) = \int_0^t (y_1(u)^2 - u)\,\mathrm{d}u = \int_0^t \left(\frac{1}{4}u^4 - u\right)\mathrm{d}u = \frac{1}{4\cdot 5}t^5 - \frac{1}{2}t^2.$$

Since $y_2(0) = 0$ and

$$y_2(t)^2 - t = \frac{1}{4^2\cdot 5^2}t^{10} - \frac{1}{4\cdot 5}t^7 + \frac{1}{4}t^4 - t = \frac{1}{4^2\cdot 5^2}t^{10} - \frac{1}{4\cdot 5}t^7 + y_2'(t) \approx y_2'(t)$$

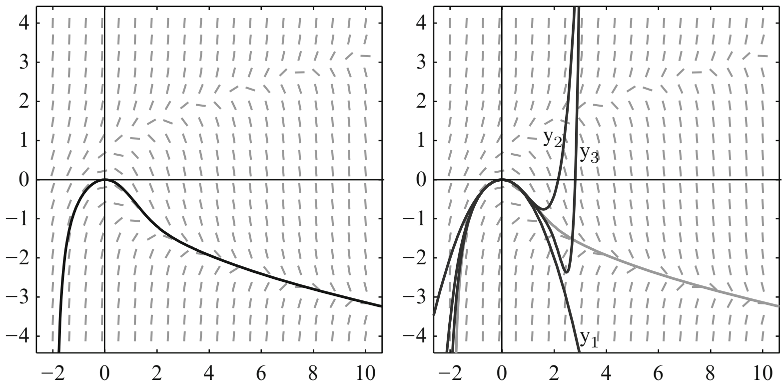
if $t$ is close to 0, it follows that the second iterate $y_2(t)$ is already a "good" approximation of the exact solution for $t$ close to 0. Since

$$y_2(t)^2 = \frac{1}{4^2\cdot 5^2}t^{10} - \frac{1}{4\cdot 5}t^7 + \frac{1}{4}t^4,$$

it follows that

$$y_3(t) = \int_0^t \left( \frac{1}{4^2 \cdot 5^2} u^{10} - \frac{1}{4 \cdot 5} u^7 + \frac{1}{4} u^4 - u \right) \mathrm{d}u$$
$$= \frac{1}{11 \cdot 4^2 \cdot 5^2} t^{11} - \frac{1}{4 \cdot 5 \cdot 8} t^8 + \frac{1}{4 \cdot 5} t^5 - \frac{1}{2} t^2.$$

According to Picard's theorem, the successive approximations $y_n(t)$ converge toward the exact solution $y(t)$, so we expect that $y_3(t)$ is an even better approximation of $y(t)$ for $t$ close enough to 0. The graphs of $y_1$, $y_2$, and $y_3$ are given below.




Direction Field and Solution for $y' = y^2 - t$, $y(0) = 0$

Solution $y(t)$ and Picard Approximations $y_1$, $y_2$, and $y_3$

Although the Riccati equation looks rather simple in form, its solution cannot be obtained by methods developed in this chapter. In fact, the solution is not expressible in terms of elementary functions but requires special functions such as the Bessel functions. The calculation of the Picard approximations does not reveal a pattern by which we might guess what the $n$th term might be. This is rather typical. Only in special cases can we expect to find a such a general formula for $y_n$. ◀

If one only assumes that the function $F(t, y)$ is continuous on the rectangle $\mathcal{R}$, but makes no assumptions about $F_y(t, y)$, then Guiseppe Peano (1858–1932) showed that the initial value problem (1) still has a solution on some interval $I$ with $t_0 \in I \subset [a, b]$. However, in this case, the solutions are not necessarily unique.

**Theorem 8 (Peano's Existence Theorem[15]).** *Let $F(t, y)$ be a continuous functions of $(t, y)$ on a rectangle*

[15] For a proof see, for example, A.N. Kolmogorov and S.V. Fomin, *Introductory Real Analysis*, Chap. 3, Sect. 11, Dover 1975.

$$\mathcal{R} = \{(t,\ y) : a \le t \le b\,,\ c \le y \le d\}\,.$$

*If $(t_0, y_0)$ is an interior point of $\mathcal{R}$, then there exists a solution $y(t)$ of*

$$y' = F(t,\ y)\,, \quad y(t_0) = y_0,$$

*on some interval $[a', b']$ with $t_0 \in [a', b'] \subset [a, b]$.*

Let us reconsider the differential equation introduced in Example 1.

**Example 9.** Consider the initial value problem

$$y' = 3y^{2/3}\,, \quad y(t_0) = y_0. \tag{5}$$

Discuss the application of Picard's existence and uniqueness theorem and Peano's existence theorem.

► **Solution.** The function $F(t,\ y) = 3y^{2/3}$ is continuous for all $(t,\ y)$, so Peano's existence theorem shows that the initial value problem (5) has a solution for all possible initial values $y(t_0) = y_0$. Moreover, $F_y(t,\ y) = \frac{2}{y^{1/3}}$ is continuous on any rectangle not containing a point of the form $(t, 0)$. Thus, Picard's existence and uniqueness theorem tells us that the solutions of (5) are unique *as long as the initial value $y_0$ is nonzero*. Assume that $y_0 \neq 0$. Since the differential equation $y' = 3y^{2/3}$ is separable, we can rewrite it in the differential form

$$\frac{1}{y^{2/3}}\,\mathrm{d}y = 3\mathrm{d}t,$$

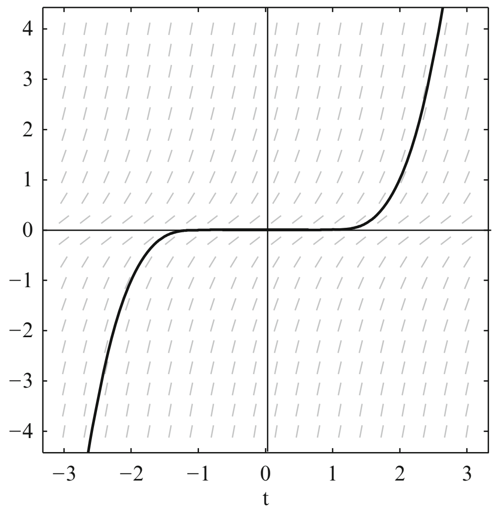
and integrate the differential form to get

$$3y^{1/3} = 3t + c.$$

Thus, the functions $y(t) = (t + c)^3$ for $t \in \mathbb{R}$ are solutions for $y' = 3y^{2/3}$. Clearly, the equilibrium solution $y(t) = 0$ does not satisfy the initial condition. The constant $c$ is determined by $y(t_0) = y_0$. We get $c = y_0^{1/3} - t_0$, and thus $y(t) = (t + y_0^{1/3} - t_0)^3$ is the unique solution of *if* $y_0 \neq 0$. If $y_0 = 0$, then (5) admits more than one solution. Two of them are given in Example 1. However, there are many more. In fact, the following functions are all solutions:

$$y(t) = \begin{cases} (t-\alpha)^3 & \text{if } t < \alpha \\ 0 & \text{if } \alpha \le t \le \beta \\ (t-\beta)^3 & \text{if } t > \beta, \end{cases} \tag{6}$$

where $t_0 \in [\alpha, \beta]$. The graph of one of these functions (where $\alpha = -1$, $\beta = 1$) is depicted below. What changes among the different functions is the length of the straight line segment joining $\alpha$ to $\beta$ on the $t$-axis.




Graph of Equation (6)
$\alpha = -1$ and $\beta = 1$

◀

Picard's theorem, Theorem 5, is called a local existence and uniqueness theorem because it guarantees the existence of a unique solution in some subinterval $I \subset [a, b]$. In contrast, the following important variant of Picard's theorem yields a unique solution on the whole interval $[a, b]$.

**Theorem 10.** *Let $F(t, y)$ be a continuous function of $(t, y)$ that satisfies a Lipschitz condition on a strip $\mathcal{S} = \{(t, y) : a \le t \le b\,, -\infty < y < \infty\}$. That is, assume that*

$$|F(t, y_1) - F(t, y_2)| \le K|y_1 - y_2|$$

*for some constant $K > 0$ and for all $(t, y_1)$ and $(t, y_2)$ in $\mathcal{S}$. If $(t_0, y_0)$ is an interior point of $\mathcal{S}$, then there exists a unique solution of*

$$y' = F(t, y)\,, \quad y(t_0) = y_0,$$

*on the interval $[a, b]$.*

**Example 11.** Show that the following differential equations have unique solutions on all of $\mathbb{R}$:

1. $y' = e^{\sin ty}$, $y(0) = 0$
2. $y' = |ty|$, $y(0) = 0$

► **Solution.** For each differential equation, we will show that Theorem 10 applies on the strip $\mathcal{S} = \{(t, y) : -a \le t \le a, -\infty < y < \infty\}$ and thus guarantees a unique solution on the interval $[-a, a]$. Since $a$ is arbitrary, the solution exists on $\mathbb{R}$.

1. Let $F(t, y) = e^{\sin ty}$. Here we will use the fact that the partial derivative of $F$ with respect to $y$ exists so we can apply the mean value theorem:

$$F(t, y_1) - F(t, y_2) = F_y(t, y_0)(y_1 - y_2), \tag{7}$$

where $y_1$ and $y_2$ are real numbers with $y_0$ between $y_1$ and $y_2$. Now focus on the partial derivative $F_y(t, y) = e^{\sin ty} t \cos(ty)$. Since the exponential function is increasing, the largest value of $e^{\sin ty}$ occurs when sin is at its maximum value of 1. Since $|\cos ty| \le 1$ and $|t| \le a$, we have $\left|F_y(t, y)\right| = \left|e^{\sin ty} t \cos ty\right| \le e^1 a = ea$. Now take the absolute value of (7) to get

$$|F(t, y_1) - F(t, y_2)| = \left|e^{\sin ty_1} - e^{\sin ty_2}\right| \le ea\,|y_1 - y_2|\,.$$

It follows that $F(t, y) = e^{\sin ty}$ satisfies the Lipschitz condition with $K = ea$. Theorem 10 now implies that $y' = e^{\sin ty}$, with $y(0) = 0$ has a unique solution on the interval $[-a, a]$. Since $a$ is arbitrary, a solution exists and is unique on all of $\mathbb{R}$.

2. Let $F(t, y) = |ty|$. Here $F$ does not have a partial derivative at $(0, 0)$. Nevertheless, it satisfies a Lipschitz condition for

$$|F(t, y_1) - F(t, y_2)| = ||ty_1| - |ty_2|| \le |t|\,|y_1 - y_2| \le a\,|y_1 - y_2|\,,$$

since the maximum value of $t$ on $[-a, a]$ is $a$. It follows that $F(t, y) = |ty|$ satisfies the Lipschitz condition with $K = a$. Theorem 10 now implies that

$$y' = |ty| \quad y(0) = 0,$$

has a unique solution on the interval $[-a, a]$. Since $a$ is arbitrary, a solution exists and is unique on all of $\mathbb{R}$. ◄

**Remark 12.**

1. When Picard's theorem is applied to the initial value problem $y' = e^{\sin ty}$, $y(0) = 0$, we can only conclude that there is a unique solution in an interval about the origin. Theorem 10 thus tells us much more, namely, that the solution is in fact defined on the entire real line.
2. In the case of $y' = |ty|$, $y(0) = 0$, Picard's theorem does not apply at all since the absolute value function is not differentiable at 0. Nevertheless, Theorem 10 tells us that a unique solution exists on all of $\mathbb{R}$. Now that you know this, can you guess what that unique solution is?

## *The Geometric Meaning of Uniqueness*

The theorem on existence and uniqueness of solutions of differential equations, Theorem 5, has a particularly useful geometric interpretation. Suppose that $y' = F(t, y)$ is a first order differential equation for which Picard's theorem applies. If $y_1(t)$ and $y_2(t)$ denote two different solutions of $y' = F(t, y)$, then the graphs of $y_1(t)$ and $y_2(t)$ can *never* intersect. The reason for this is just that if $(t_0, y_0)$ is a point of the plane which is common to both the graph of $y_1(t)$ and that of $y_2(t)$, then *both* of these functions will satisfy the initial value problem

$$y' = F(t, y), \quad y(t_0) = y_0.$$

But if $y_1(t)$ and $y_2(t)$ are different functions, this will violate the uniqueness provision of Picard's theorem.

To underscore this point, consider the following contrasting example: The differential equation

$$ty' = 3y \tag{8}$$

is linear (and separable). Thus, it is easy to see that $y(t) = ct^3$ is its general solution. In standard form, (8) is

$$y' = \frac{3y}{t} \tag{9}$$

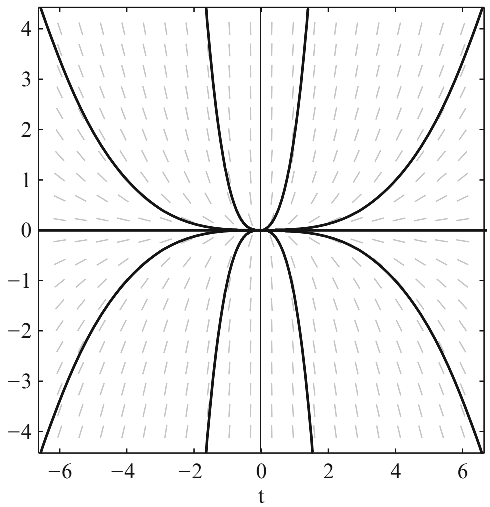
and the right-hand side, $F(t, y) = 3y/t$, is continuous provided $t \neq 0$. Thus, assuming $t \neq 0$, Picard's theorem applies to give the conclusion that the initial value problem $y' = 3y/t$ , $y(t_0) = y_0$ has a unique local solution, given by $y(t) = (y_0/t_0^3)t^3$. However, if $t_0 = 0$, Picard's theorem provides no information about the existence and uniqueness of solutions. Indeed, in its standard form, (9), it is not meaningful to talk about solutions of this equation at $t = 0$ since $F(t, y) = 3y/t$ is not even defined for $t = 0$. But in the originally designated form, (8), where the $t$ appears as multiplication on the left side of the equation, then an initial value problem starting at $t = 0$ makes sense, and moreover, the initial value problem

$$ty' = 3y\,, \quad y(0) = 0$$

has infinitely many solutions of the form $y(t) = ct^3$ for *any* $c \in \mathbb{R}$, whereas the initial value problem

$$ty' = 3y\,, \quad y(0) = y_0$$

has no solution if $y_0 \neq 0$. See the figure below, where we have graphed the function $y(t) = ct^3$ for several values of $c$. Notice that all of them pass through the origin (i.e., $y(0) = 0$), but none pass through any other point on the $y$-axis.




Thus, the situation depicted above where several solutions of the same differential equation go through the same point (in this case $(0, 0)$) can never occur for a differential equation which satisfies the hypotheses of Theorem 5.

The above remark can be exploited in the following way. The constant function $y_1(t) = 0$ is clearly a solution to the differential equation $y' = y^3 + y$. Since $F(t, y) = y^3 + y$ has continuous partial derivatives, Picard's theorem applies. Hence, if $y_2(t)$ is a solution of the equation for which $y_2(0) = 1$, the above observation takes the form of stating that $y_2(t) > 0$ for *all* $t$. This is because, in order for $y(t)$ to ever be negative, it must first cross the $t$-axis, which is the graph of $y_1(t)$, and we have observed that two solutions of the same differential equation can never cross.

## Exercises

1–4. Write the corresponding integral equation for each of the following initial value problems.

1. $y' = ty, \quad y(1) = 1$
2. $y' = y^2 \quad y(0) = -1$
3. $y' = \dfrac{t - y}{t + y}, \quad y(0) = 1$
4. $y' = 1 + t^2, \quad y(0) = 0$

5–9. Find the first $n$ Picard approximations for the following initial value problems.

5. $y' = ty, \quad y(1) = 1, \quad n = 3$
6. $y' = t - y, \quad y(0) = 1, \quad n = 4$
7. $y' = t + y^2, \quad y(0) = 0, \quad n = 3$
8. $y' = y^3 - y, \quad y(0) = 1, \quad n = 3$
9. $y' = 1 + (t - y)^2, \quad y(0) = 0, \quad n = 5$

10–14. Which of the following initial value problems are guaranteed a unique solution by Picard's theorem (Theorem 5)? Explain.

10. $y' = 1 + y^2, \quad y(0) = 0$
11. $y' = \sqrt{y}, \quad y(1) = 0$
12. $y' = \sqrt{y}, \quad y(0) = 1$
13. $y' = \dfrac{t - y}{t + y}, \quad y(0) = -1$
14. $y' = \dfrac{t - y}{t + y}, \quad y(1) = -1$
15. Determine a formula for the $n$th Picard approximation for the initial value problem
$$y' = ay, \quad y(0) = 1,$$
where $a \in \mathbb{R}$. What is the limiting function $y(t) = \lim_{n\to\infty} y_n(t)$. Is it a solution? Are there other solutions that we may have missed?
16. (a) Find the exact solution of the initial value problem
$$y' = y^2, \quad y(0) = 1.$$
(b) Calculate the first three Picard approximations $y_1(t)$, $y_2(t)$, and $y_3(t)$ and compare these results with the exact solution.
17. Determine whether the initial value problem
$$y' = \cos(t + y), \quad y(t_0) = y_0,$$
has a unique solution defined on all of $\mathbb{R}$.
18. Consider the linear differential equation $y' + p(t)y = f(t)$, with initial condition $y(t_0) = y_0$, where $p(t)$ and $f(t)$ are continuous on an interval

$I = [a, b]$ containing $t_0$ as an interior point. Use Theorem 10 to show that there is unique solution defined on $[a, b]$.

19. (a) Find the general solution of the differential equation

$$ty' = 2y - t.$$

Sketch several specific solutions from this general solution.

(b) Show that there is no solution satisfying the initial condition $y(0) = 2$. Why does this not contradict Theorem 5?

20. (a) Let $t_0$, $y_0$ be arbitrary and consider the initial value problem

$$y' = y^2, \quad y(t_0) = y_0.$$

Explain why Theorem 5 guarantees that this initial value problem has a solution on some interval $|t - t_0| \le h$.

(b) Since $F(t, y) = y^2$ and $F_y(t, y) = 2y$ are continuous on all of the $(t, y)$−plane, one might hope that the solutions are defined for all real numbers $t$. Show that this is not the case by finding a solution of $y' = y^2$ which is defined for all $t \in \mathbb{R}$ and another solution which is *not* defined for all $t \in \mathbb{R}$. (Hint: Find the solutions with $(t_0, y_0) = (0, 0)$ and $(0, 1)$.)

21. Is it possible to find a function $F(t, y)$ that is continuous and has a continuous partial derivative $F_y(t, y)$ on a rectangle containing the origin such that the two functions $y_1(t) = t$ and $y_2(t) = t^2 - 2t$ are both solutions to $y' = F(t, y)$ on an interval containing 0?

22. Is it possible to find a function $F(t, y)$ that is continuous and has a continuous partial derivative $F_y(t, y)$ on a rectangle containing $(0, 1)$ such that the two functions $y_1(t) = (t + 1)^2$ and the constant function $y_2(t) = 1$ are both solutions to $y' = F(t, y)$ on an interval containing 0?

23. Show that the function

$$y_1(t) = \begin{cases} 0, & \text{for } t < 0 \\ t^3 & \text{for } t \ge 0 \end{cases}$$

is a solution of the initial value problem $ty' = 3y$, $y(0) = 0$. Show that $y_2(t) = 0$ for all $t$ is a second solution. Explain why this does not contradict Theorem 5.