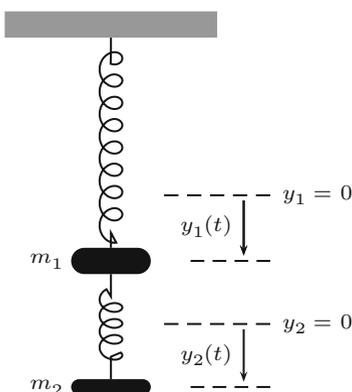# Chapter 4
Linear Constant Coefficient Differential Equations

Two springs systems (without dashpots) are coupled as illustrated in the diagram below:

Let y_1 and y_2 denote the displacements of the bodies of mass m_1 and m_2 from their equilibrium positions, $y_1 = 0$ and $y_2 = 0$, respectively, where distances are measured in the downward direction. In these coordinates, $y_1(t)$ and $y_2(t) - y_1(t)$ represent the length the upper and lower springs are stretched at time t. There are two spring forces acting on the upper body. By Hooke's law, the force of the upper spring is $-k_1 y_1(t)$ while the force of the lower spring is given by $k_2(y_2(t) - y_1(t))$, where k_1 and k_2 are the respective spring constants. Newton's law of motion then implies

$$m_1 y_1''(t) = -k_1 y_1(t) + k_2(y_2(t) - y_1(t)).$$

The lower spring alone exerts a force of $-k_2(y_2(t) - y_1(t))$ on the lower body. Again Newton's law of motion implies

$$m_1 y_2''(t) = -k_2(y_2(t) - y_1(t)).$$

We are thus led to the following differential equations:

$$m_1 y_1''(t) + (k_1 + k_2)y_1(t) = k_2 y_2(t)$$
$$m_2 y_2''(t) + k_2 y_2(t) = k_2 y_1(t). \tag{1}$$

Equation (1) is an example of a ***coupled system***[1] of differential equations. Notice the dependency of each equation on the other: the solution of one equation becomes (up to a multiplicative constant) the forcing function of the other equation. In Sect. 4.4, we will show that both y_1 and y_2 satisfy the fourth order differential equation

$$y^{(4)} + (a + b + c)y'' + cay = 0, \tag{2}$$

where $a = k_1/m_1$, $b = k_2/m_2$, and $c = k_2/m_2$. It will be the work of this chapter to show solution methods for constant coefficient nth order linear differential equations such as given in (2), for $n = 4$. We will return to the coupled spring problem presented above in Sect. 4.4.

Much of the theoretical work that we do here is a simple extension of the work done in Chap. 3 where we focussed on the second order constant coefficient differential equations. Thus, our presentation will be brief.

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[1] In Chap. 9, we will study first order coupled systems in greater detail.

4.1 Notation, Definitions, and Basic Results

Suppose $a_0, \ldots, a_n$ are scalars, $a_n \neq 0$, and f is a function defined on some interval I. A differential equation of the form

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(t), \tag{1}$$

is called a *constant coefficient nth order linear differential equation*. The constants $a_0, \ldots, a_n$ are called the *coefficients*, and a_n is called the *leading coefficient.* By dividing by a_n, if necessary, we can assume that the leading coefficient is 1. When $f = 0$, we call (1) *homogeneous*, otherwise, it is called *nonhomogeneous*.

The left-hand side of (1) is made up of a linear combination of differentiations and multiplications by constants. If $\boldsymbol{D}$ denotes the derivative operator as in Sect. 3.3, then $\boldsymbol{D}^k(y) = y^{(k)}$. Let

$$\boldsymbol{L} = a_n \boldsymbol{D}^n + \cdots + a_1 \boldsymbol{D} + a_0, \tag{2}$$

where $a_0, \ldots a_n$ are the same constants given in (1). Then $\boldsymbol{L}(y) = a_n y^{(n)} + \cdots + a_1 y' + a_0 y$ and (1) can be rewritten

$$\boldsymbol{L}(y) = f.$$

We call $\boldsymbol{L}$ in (2) a *linear constant coefficient differential operator of order* n (assuming, of course, that $a_n \neq 0$). Since $\boldsymbol{L}$ is a linear combination of powers of $\boldsymbol{D}$, we will sometimes refer to it as a *polynomial differential operator*. Specifically, if

$$q(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0,$$

then q is a polynomial of order n. Since L is obtained from q by substituting $\boldsymbol{D}$ for s, and we will write $\boldsymbol{L} = q(\boldsymbol{D})$. The polynomial q is referred as the *characteristic polynomial* of $\boldsymbol{L}$. The operator $\boldsymbol{L}$ takes a function y that has at least n continuous derivatives and produces a continuous function.

Example 1. Suppose $\boldsymbol{L} = \boldsymbol{D}^3 + \boldsymbol{D}$. Find

$$\boldsymbol{L}(\sin t), \qquad \boldsymbol{L}(\cos t), \qquad \boldsymbol{L}(1), \quad \text{and} \quad \boldsymbol{L}(e^t)$$

▶ **Solution.** • $\boldsymbol{L}(\sin t) = \boldsymbol{D}^3(\sin t) + \boldsymbol{D}(\sin t)$
$$= -\cos t + \cos t$$
$$= 0.$$

• $\boldsymbol{L}(\cos t) = \boldsymbol{D}^3(\cos t) + \boldsymbol{D}(\cos t)$
$$= \sin t - \sin t$$
$$= 0.$$

- $L(1) = D^3(1) + D(1)$
   $= 0.$
- $L(e^t) = D^3(e^t) + D(e^t)$
   $= (e^t) + (e^t)$
   $= 2e^t.$ ◄

We say a function y is a ***homogeneous solution*** of L if $Ly = 0$. Example 1 shows that $\sin t$, $\cos t$, and 1 are homogeneous solutions of $L = D^3 + D$.

Linearity and Consequences

It is easy to show by induction that $D^k(y_1 + y_2) = D^k y_1 + D^k y_2$ and $D^k cy = cD^k y$. The proofs of Propositions 2 and 4 of Sect. 3.3 and Theorem 6 of Sect. 3.3 extend to give what we put below in one theorem.

Theorem 2. *1. The operator*

$$L = a_n D^n + \ldots + a_1 D + a_0$$

given by (2) *is linear. Specifically,*

a. If y_1 and y_2 have n derivatives, then

$$L(y_1 + y_2) = L(y_1) + L(y_2).$$

b. If y has n derivatives and c is a scalar, then

$$L(cy) = cL(y).$$

2. If y_1 and y_2 are homogeneous solutions of L and c_1 and c_2 are scalars, then

$$c_1 y_1 + c_2 y_2$$

is a homogeneous solution.

3. Suppose f is a function. If y_p is a fixed particular solution to $Ly = f$ and y_h is any solution to the associated homogeneous differential equation $Ly = 0$, then

$$y_p + y_h$$

is a solution to $Ly = f$. Furthermore, any solution y to $Ly = f$ has the form

$$y = y_p + y_h.$$

Given a constant coefficient linear differential equation $\boldsymbol{L}y = f$, we call $\boldsymbol{L}y = 0$ the ***associated homogeneous differential equation***. The following algorithm follows from Theorem 2 and outlines an effective strategy for finding the solution set to $\boldsymbol{L}y = f$.

Algorithm 3. The general solution to a linear differential equation

$$\boldsymbol{L}y = f$$

can be found as follows:

Solution Method for nth Order Linear Equations

1. Find all the solutions to the associated homogeneous differential equation
   $$\boldsymbol{L}y = 0$$
2. Find one particular solution y_p
3. Add the particular solution to the homogeneous solutions:

   $$y_p + y_h.$$

   As y_h varies over all homogeneous solutions, we obtain all solutions to
   $\boldsymbol{L}y = f$.

You should notice that this is the same strategy as in the second order case discussed in Algorithm 7 of Sect. 3.3. This is the strategy we will follow. Section 4.2 will be devoted to determining solutions to the associated homogeneous differential equation. Section 4.3 will show how to find a particular solution.

Example 4. Use Algorithm 3 and Example 1 to find solutions to

$$y''' + y' = 2e^t.$$

▶ **Solution.** The left-hand side can be written $\boldsymbol{L}y$, where $\boldsymbol{L}$ is the differential operator
$$\boldsymbol{L} = \boldsymbol{D}^3 + \boldsymbol{D}.$$

From Example 1, we found

- $\boldsymbol{L}(e^t) = 2e^t$,
- $\boldsymbol{L}(\sin t) = 0$,
- $\boldsymbol{L}(\cos t) = 0$,
- $\boldsymbol{L}(1) = 0$.

Notice that the first equation tells us that a particular solution is $y_p = e^t$. The second, third, and fourth equations give $\sin t$, $\cos t$, and 1 as solutions to the associated homogeneous differential equation $L y = 0$. Thus, for each scalar c_1, c_2, and c_3, the function $y_h = c_1 \sin t + c_2 \cos t + c_3$ is a homogeneous solution. Now applying Algorithm 3, we have

$$y = y_p + y_h = e^t + c_1 \sin t + c_2 \cos t + c_3$$

is a solution to $L y = 2e^t$, for all scalars c_1, c_2, and c_3. At this point, we cannot say that this is the solution set, but, in fact, it is.   ◄

Initial Value Problems

Suppose L is a constant coefficient linear differential operator of order n and f is a function defined on an interval I. Let $t_0 \in I$. To the equation

$$L y = f$$

we can associate *initial conditions* of the form

$$y(t_0) = y_1, \ y'(t_0) = y_1, \ldots, y^{(n-1)}(t_0) = y_{n-1}.$$

We refer to the initial conditions and the differential equation $L y = f$ as an *initial value problem*, just as in the second order case.

Theorem 5 (The Existence and Uniqueness Theorem). *Suppose p is an nth order real polynomial, $L = p(D)$, and f is a continuous real-valued function on an interval I. Let $t_0 \in I$. Then there is a unique real-valued function y defined on I satisfying*

$$L y = f \quad y(t_0) = y_0, \ y'(t_0) = y_1, \ \ldots, \ y^{(n-1)}(t_0) = y_{n-1}, \tag{3}$$

where y_0, y_1, $\ldots$, $y_{n-1} \in \mathbb{R}$, If $I \supset [0, \infty)$ and f has a Laplace transform then so does the solution y. Furthermore, if f is in $\mathcal{E}$, then y is in $\mathcal{E}$.

When $n = 2$, we get the statement of the existence and uniqueness theorem, Theorem 10 of Sect. 3.1. Theorem 5 will be proved in a more general context in Sect. 9.5.

Example 6. Use Example 4 to find the solution to the following initial value problem:

$$L y = 2e^t, \qquad y(0) = 2, \ y'(0) = 2, \ y''(0) = 3,$$

where $L = D^3 + D$.

► **Solution.** In Example 4, we verified that

$$y = e^t + c_1 \sin t + c_2 \cos t + c_3$$

is a solution to $Ly = 2e^t$ for all scalars c_1, c_2, and c_3. The initial conditions imply the following system:

$$
\begin{aligned}
1 \qquad + c_2 + c_3 &= y(0) = 2 \\
1 + c_1 \qquad\qquad &= y'(0) = 2 \\
1 \qquad - c_2 \qquad &= y''(0) = 3
\end{aligned}
$$

Solving this system gives $c_1 = 1$, $c_2 = -2$, and $c_3 = 3$. It follows that

$$y = e^t + \sin t - 2 \cos t + 3$$

and the existence and uniqueness theorem implies that y is the unique solution. ◄

Exercises

1–4. Determine which of the following are constant coefficient linear differential equations. In these cases, write the equation in the form $Ly = f$ and, determine the order, the characteristic polynomial, and whether they are homogeneous.

1. $y''' - 3y' = e^t$
2. $y^{(4)} + y' + 4y = 0$
3. $y^{(4)} + y^4 = 0$
4. $y^{(5)} + ty'' - 3y = 0$

5–8. For the linear operator L, determine $L(y)$.

5. $L = D^3 - 4D$

   (a) $y = e^{2t}$
   (b) $y = e^{-2t}$
   (c) $y = 2$

6. $L = D - 2$

   (a) $y = e^{-2t}$
   (b) $y = 3e^{2t}$
   (c) $y = \tan t$

7. $L = D^4 + 5D^2 + 4$

   (a) $y = e^{-t}$
   (b) $y = \cos t$
   (c) $y = \sin 2t$

8. $L = D^3 - D^2 + D - 1$

   (a) $y = e^t$
   (b) $y = te^t$
   (c) $y = \cos t$
   (d) $y = \sin t$

9. Suppose L is a polynomial differential operator and

   * $L(te^{2t}) = 8e^{2t}$
   * $L(e^{2t}) = 0$
   * $L(e^{-2t}) = 0$
   * $L(1) = 0$

   Use this information to find other solutions to $Ly = 8e^{2t}$.

10. Suppose L is a polynomial differential operator and

   * $L(\sin t) = -15 \sin t$
   * $L(e^{2t}) = 0$

- $L(e^{-2t}) = 0$
- $L(\sin 2t) = 0$
- $L(\cos 2t) = 0$

Use this information to find other solutions to $Ly = -15 \sin t$.

11. Let L be as in Exercise 9. Use the results there to solve the initial value problem

$$Ly = 8e^{2t},$$

where $y(0) = 2$, $y'(0) = -1$, and $y''(0) = 16$.

12. Let L be as in Exercise 10. Use the results there to solve the initial value problem

$$Ly = -15 \sin t,$$

where $y(0) = 0$, $y'(0) = 3$, $y''(0) = -16$, and $y'''(0) = -9$.

4.2   Linear Homogeneous Differential Equations

Suppose $L = q(D)$ where $q(s) = a_n s^n + \cdots + a_1 s + a_0$. As we will see in this section, the characteristic polynomial, q, plays a decisive role in determining the solution set to $L y = 0$.

Lemma 1. *Let $q(s) = a_n s^n + \cdots + a_1 s + a_0$. If y is a function whose nth derivative is of exponential type, then*

$$\mathcal{L}\{q(D)y\} = q(s)\mathcal{L}\{y\} - p(s),$$

where $p(s)$ is a polynomial of degree at most $n - 1$ and depends on the coefficients of q and the initial conditions $y(0), y'(0), \ldots, y^{(n-1)}(0)$.

Proof. If $y^{(n)}$ is of exponential type, then so are all derivatives $y^{(k)}$, $k = 0, 1, \ldots$ $n - 1$ by Lemma 4 of Sect. 2.2. By the nth transform derivative principle, Corollary 8 of Sect. 2.2, we have

$$\mathcal{L}\left\{y^{(k)}\right\}(s) = s^k \mathcal{L}\{y\}(s) - p_k(s),$$

where $p_k(s)$ is a polynomial of order at most $k - 1$ depending on the initial conditions $y(0), y'(0), \ldots, y^{(k-1)}$. Let $\mathcal{L}\{y(t)\} = Y(s)$. Then

$$
\begin{aligned}
\mathcal{L}\{q(D)y\}(s) &= a_n \mathcal{L}\left\{y^{(n)}\right\} + \cdots + a_1 \mathcal{L}\{y'\} + a_0 \mathcal{L}\{y\} \\
&= a_n s^n Y(s) + \cdots + a_1 s Y(s) + a_0 Y(s) - p(s) \\
&= q(s)Y(s) - p(s),
\end{aligned}
$$

where $p(s) = a_n p_n + \cdots + a_0 p_0$ is a polynomial of degree at most $n - 1$ and depends on the initial values $y(0), y'(0), \ldots, y^{(n-1)}(0)$.   □

We now turn our attention to the solution set to

$$L y = 0. \tag{1}$$

By Theorem 5 of Sect. 4.1, any solution to $L\{y\} = 0$ is in $\mathcal{E}$. Suppose y is such a solution. By Lemma 1, we have $\mathcal{L}\{L y\} = q(s)\mathcal{L}\{y\} - p(s) = 0$. Solving for $\mathcal{L}\{y\}$ gives

$$\mathcal{L}\{y\}(s) = \frac{p(s)}{q(s)} \in \mathcal{R}_q.$$

It follows that $y \in \mathcal{E}_q$. Now suppose $y \in \mathcal{E}_q$. Then $\mathcal{L}\{y\}(s) = \frac{p(s)}{q(s)} \in \mathcal{R}_q$ and

$$\mathcal{L}\{L y\}(s) = q(s)\frac{p(s)}{q(s)} - p_1(s) = p(s) - p_1(s),$$

where $p_1(s)$ is a polynomial that depends on the initial conditions. Note, however, that $p(s) - p_1(s)$ is a polynomial in $\mathcal{R}$ and therefore must be identically 0. Thus, $\mathcal{L}\{\mathbf{L}y\} = 0$ and this implies $\mathbf{L}y = 0$. The discussion given above implies the following theorem.

Theorem 2. *Let $q(s)$ be the characteristic polynomial of a linear differential operator $q(\mathbf{D})$ of order n. Then the solution set to*

$$q(\mathbf{D})y = 0$$

is $\mathcal{E}_q$. Thus, if $\mathcal{B}_q = \{y_1, y_2, \ldots, y_n\}$ is the standard basis of $\mathcal{E}_q$, then a solution to $q(\mathbf{D})y = 0$ is of the form

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n,$$

where $c_1, c_2, \ldots, c_n$ are scalars.

The following algorithm codifies the procedure needed to find the solution set.

Algorithm 3.   Given an nth order constant coefficient linear differential equation

$$q(\mathbf{D})y = 0$$

the solution set is determined as follows:

Solution Method for nth order Homogeneous Linear Differential Equations

1. Determine the characteristic polynomial, $q(s)$.
2. Factor $q(s)$ and construct $\mathcal{B}_q = \{y_1, y_2, \ldots, y_n\}$.
3. The solution set $\mathcal{E}_q$ is the set of all linear combinations of the functions in the standard basis $\mathcal{B}_q$. In other words,

$$\mathcal{E}_q = \text{Span } \mathcal{B}_q = \{c_1 y_1(t) + \cdots + c_n y_n(t) : c_1, \ldots, c_n \in \mathbb{R}\}.$$

Example 4.   Find the general solution to the following differential equations:

1. $y''' + y' = 0$
2. $y^{(4)} - y = 0$
3. $y^{(5)} - 8y^{(3)} + 16y' = 0$

▶ **Solution.**   1. The characteristic polynomial for $y''' + y' = 0$ is

$$q(s) = s^3 + s = s(s^2 + 1).$$

The standard basis is $\mathcal{B}_q = \{1, \cos t, \sin t\}$. Thus the solution set is

$$\mathcal{E}_q = \{c_1 + c_2 \cos t + c_3 \sin t : c_1, c_2, c_3 \in \mathbb{R}\}.$$

2. The characteristic polynomial for $y^{(4)} - y = 0$ is

$$q(s) = s^4 - 1 = (s^2 + 1)(s - 1)(s + 1).$$

The standard basis is $\mathcal{B}_q = \{\cos t, \sin t, e^t, e^{-t}\}$. Thus, the solution set is

$$\mathcal{E}_q = \{c_1 \cos t + c_2 \sin t + c_3 e^t + c_4 e^{-t} : c_1, c_2, c_3, c_4 \in \mathbb{R}\}.$$

3. The characteristic polynomial for $y^{(5)} - 8y^{(3)} + 16y' = 0$ is

$$q(s) = s^5 - 8s^3 + 16s = s(s^4 + 8s^2 + 16) = s(s^2 - 4)^2 = s(s - 2)^2(s + 2)^2.$$

The standard basis is $\mathcal{B}_q = \{1, e^{2t}, te^{2t}, e^{-2t}, te^{-2t}\}$. Thus, the solution set is

$$\mathcal{E}_q = \{c_1 + c_2 e^{2t} + c_3 te^{2t} + c_4 e^{-2t} + c_5 te^{-2t} : c_1, \ldots, c_5 \in \mathbb{R}\}. \qquad \blacktriangleleft$$

Initial Value Problems

Let $q(\boldsymbol{D})y = 0$ be a homogeneous constant coefficient differential equation, where $\deg p = n$. Suppose $y(t_0) = y_0, \ldots, y^{(n-1)}(t_0) = y_{n-1}$ are initial conditions. The existence and uniqueness Theorem 5 of Sect. 4.1 states there is a unique solution to the initial value problem. However, the general solution is a linear combination of the n functions in the standard basis $\mathcal{E}_q$. It follows then that the initial conditions uniquely determine the coefficients.

Example 5. Find the solution to

$$y''' - 4y'' + 5y' - 2y = 0; \quad y(0) = 1, \ y'(0) = 2, \ y''(0) = 0.$$

▶ **Solution.** The characteristic polynomial is $q(s) = s^3 - 4s^2 + 5s - 2$ and factors as

$$q(s) = (s - 1)^2(s - 2).$$

The basis for $\mathcal{E}_q$ is $\{e^t, te^t, e^{2t}\}$ and the general solution is

$$y = c_1 e^t + c_2 te^t + c_3 e^{2t}.$$

We first calculate the derivatives and simplify to get

$$y(t) = c_1 e^t + c_2 t e^t + c_3 e^{2t}$$
$$y'(t) = (c_1 + c_2)e^t + c_2 t e^t + 2c_3 e^{2t}$$
$$y''(t) = (c_1 + 2c_2)e^t + c_2 t e^t + 4c_3 e^{2t}.$$

To determine the coefficients c_1, c_2, and c_3, we use the initial conditions. Evaluating at $t = 0$ gives

$$1 = y(0) = c_1 + c_3$$
$$2 = y'(0) = c_1 + c_2 + 2c_3$$
$$0 = y''(0) = c_1 + 2c_2 + 4c_3.$$

Solving these equations gives $c_1 = 4$, $c_2 = 4$, and $c_3 = -3$. The unique solution is thus

$$y(t) = 4e^t + 4te^t - 3e^{2t}. \qquad \blacktriangleleft$$

Abel's Formula

Theorem 6 (Abel's Formula). *Let $q(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$ and suppose $f_1, f_2, \ldots, f_n$ are solutions to $q(D)y = 0$. Then the Wronskian satisfies*

$$w(f_1, f_2, \ldots, f_n) = Ke^{-a_{n-1}t}, \tag{2}$$

for some constant K and $\{f_1, f_2, \ldots, f_n\}$ is linearly independent if and only if $w(f_1, f_2, \ldots, f_n)$ is nonzero.

Proof. The essential idea of the proof is the same as the proof of Abel's Formula for $n = 2$ (see Theorem 8 of Sect. 3.3). First observe that since f_1 is a solution to $q(D)y = 0$, we have $f_1^{(n)} = -a_{n-1}f_1^{(n-1)} - a_{n-2}f_1^{(n-2)} - \cdots - a_1 f_1' - a_0 f_1$ and similarly for $f_2, \ldots, f_n$. To simplify the notation, let $w = w(f_1, f_2, \ldots, f_n)$ and let $F = f_1, f_2, \ldots, f_n$ be the first row of the Wronskian matrix $W(f_1, \ldots, f_n)$. Then $F^{(k)} = f_1^{(k)}, \ldots, f_n^{(k)}$ is the $k + 1$st row of the Wronskian matrix, for all $k = 1, 2, \ldots, n - 1$, and

$$F^{(n)} = -a_{n-1}F^{(n-1)} - a_{n-2}F^{(n-2)} - \cdots - a_1 F' - a_0 F. \tag{3}$$

We can thus write

$$w = \det \begin{bmatrix} F \\ F' \\ F'' \\ \vdots \\ F^{(n-2)} \\ F^{(n-1)} \end{bmatrix}.$$

The derivative of w is the sum of the determinants of the matrices obtained by differentiating each row one at a time. Thus,

$$w' = \det \begin{bmatrix} F' \\ F' \\ F'' \\ \vdots \\ F^{(n-2)} \\ F^{(n-1)} \end{bmatrix} + \det \begin{bmatrix} F \\ F'' \\ F'' \\ \vdots \\ F^{(n-2)} \\ F^{(n-1)} \end{bmatrix} + \cdots + \det \begin{bmatrix} F \\ F' \\ F'' \\ \vdots \\ F^{(n-1)} \\ F^{(n-1)} \end{bmatrix} + \det \begin{bmatrix} F \\ F' \\ F'' \\ \vdots \\ F^{(n-2)} \\ F^{(n)} \end{bmatrix}$$

All determinants but the last are zero because the matrices have two equal rows. Using (3) and linearity of the determinant gives

$$w' = -a_{n-1} \det \begin{bmatrix} F \\ F' \\ \vdots \\ F^{(n-2)} \\ F^{(n-1)} \end{bmatrix} - a_{n-2} \det \begin{bmatrix} F \\ F' \\ \vdots \\ F^{(n-2)} \\ F^{(n-2)} \end{bmatrix} - \cdots - a_0 \det \begin{bmatrix} F \\ F' \\ \vdots \\ F^{(n-2)} \\ F \end{bmatrix}$$

$$= -a_{n-1} w.$$

The first determinant is w and the remaining determinants are zero, again because the matrices have two equal rows. Therefore, w satisfies the differential equation $w' + a_{n-1} w = 0$. By Theorem 2 of Sect. 1.5, there is a constant K so that

$$w(t) = K e^{-a_{n-1} t}.$$

If $w \neq 0$, then it follows from Theorem 13 of Sect. 3.2 that $\{f_1, f_2, \ldots, f_n\}$ is a linearly independent set.

If $w = 0$, then the Wronskian matrix at $t = 0$ is singular and there are scalars $c_1, \ldots, c_n$, not all zero, so that

$$
W(0)
\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}
=
\begin{bmatrix} \boldsymbol{F}(0) \\ \boldsymbol{F}'(0) \\ \boldsymbol{F}''(0) \\ \vdots \\ \boldsymbol{F}^{(n-1)}(0) \end{bmatrix}
\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.
$$

Let $z(t) = c_1 f_1(t) + \ldots c_n f_n(t)$. Then z is a linear combination of $f_1, \ldots, f_n$ and hence is a solution to $q(\boldsymbol{D})y = 0$. Further, from the matrix product given above, $z(0) = 0, z'(0) = 0, \ldots, z^{(n-1)}(0) = 0$. By the uniqueness and existence theorem, Theorem 5 of Sect. 4.1, it follows that $z(t) = 0$ for all $t \in \mathbb{R}$. This implies that $\{f_1, f_2, \ldots, f_n\}$ is linearly dependent. $\square$

Exercises

1–9. Determine the solution set to the following homogeneous differential equations. Write your answer as a linear combination of functions from the standard basis.

1. $y''' - y = 0$

2. $y''' - 6y'' + 12y' - 8y = 0$

3. $y^{(4)} - y = 0$

4. $y''' + 2y'' + y' = 0$

5. $y^{(4)} - 5y'' + 4y = 0$

6. $(\mathbf{D} - 2)(\mathbf{D}^2 - 25)y = 0$

7. $(\mathbf{D} + 2)(\mathbf{D}^2 + 25)y = 0$

8. $(\mathbf{D}^2 + 9)^3 y = 0$

9. $(\mathbf{D} + 3)(\mathbf{D} - 1)(\mathbf{D} + 3)^2 y = 0$

10–11. Solve the following initial value problems.

10. $y''' + y'' - y' - y = 0$,
$y(0) = 1,\ y'(0) = 4,\ y''(0) = -1$

11. $y^{(4)} - y = 0$,
$y(0) = -1,\ y'(0) = 6,\ y''(0) = -3,\ y'''(0) = 2$

4.3   Nonhomogeneous Differential Equations

In this section, we are concerned with determining a particular solution to

$$q(\boldsymbol{D})y = f,$$

where $q(s)$ is a polynomial of degree n and $f \in \mathcal{E}$. In Chap. 3, we discussed two methods for the second order case: The method of undetermined coefficients and the incomplete partial fraction method. Both of these methods are not dependent on the degree of q so extend quite naturally.

The Method of Undetermined Coefficients

Theorem 1. *Suppose $\boldsymbol{L} = q(\boldsymbol{D})$ is a polynomial differential operator and $f \in \mathcal{E}$. Suppose $\mathcal{L}f = \frac{u}{v}$. Let $\mathcal{B}_q$ denote the standard basis for $\mathcal{E}_q$. Then there is a particular solution y_{p} to*

$$\boldsymbol{L}y = f$$

which is a linear combination of terms in $\mathcal{B}_{qv}$ but not in $\mathcal{B}_q$.

Proof. By Theorem 5 of Sect. 4.1, any solution to $\boldsymbol{L}y = f$ is in $\mathcal{E}$ and hence has a Laplace Transform. Thus,

$$\mathcal{L}\{q(\boldsymbol{D})\}\,y) = \mathcal{L}\{f\} = \frac{u(s)}{v(s)}$$

$$q(s)\mathcal{L}\{y\} - p(s) = \frac{u(s)}{v(s)} \quad \text{by Lemma 1 of Sect. 4.2}$$

$$\mathcal{L}\{y\} = = \frac{p(s)}{q(s)} + \frac{u(s)}{q(s)v(s)} = \frac{u(s) + p(s)v(s)}{q(s)v(s)}.$$

It follows that y is in $\mathcal{E}_{qv}$ and hence a linear combination of terms in $\mathcal{B}_{qv}$. Since $\mathcal{B}_q \subset \mathcal{B}_{qv}$, we can write $y = y_{\mathrm{h}} + y_{\mathrm{p}}$, where y_{h} is the linear combination of terms in $\mathcal{B}_q$ and y_{p} is a linear combination of terms in $\mathcal{B}_{qv}$ but not in $\mathcal{B}_q$. Since y_{h} is a homogeneous solution, it follows that $y_{\mathrm{p}} = y - y_{\mathrm{h}}$ is a particular solution of the required form. □

If $\{\phi_1, \ldots, \phi_m\}$ are the functions in $\mathcal{B}_{qv}$ but not in $\mathcal{B}_q$, then a linear combination

$$a_1\phi_1 + \cdots + a_m\phi_m$$

will be referred to as a ***test function***.

Example 2. Find the general solution to

$$y^{(4)} - y = 4\cos t.$$

► **Solution.** The characteristic polynomial is $q(s) = s^4 - 1 = (s^2 + 1)(s^2 - 1) = (s^2 + 1)(s - 1)(s + 1)$. It follows that $\mathcal{B}_q = \{\cos t, \sin t, e^t, e^{-t}\}$, and hence, the homogeneous solutions take the form

$$y_h = c_1 \cos t + c_2 \sin t + c_3 e^t + c_4 e^{-t}.$$

For the particular solution, note that $\mathcal{L}\{\cos t\} = \frac{s}{s^2+1}$. Let $v(s) = s^2 + 1$ be the denominator. Then $q(s)v(s) = (s^2 + 1)^2(s - 1)(s + 1)$ and

$$\mathcal{B}_{qv} = \{\cos t, t \cos t, \sin t, t \sin t, e^t, e^{-t}\}.$$

The only functions in $\mathcal{B}_{qv}$ that are not in $\mathcal{B}_q$ are $t \cos t$ and $t \sin t$. It follows that a test function takes the form $y_p = a_1 t \cos t + a_2 t \sin t$. The coefficients are determined by substituting y_p into the given differential equation. To that end, observe that

$$y_p = a_1 t \cos t + a_2 t \sin t$$
$$y_p' = a_1(\cos t - t \sin t) + a_2(\sin t + t \cos t)$$
$$y_p'' = a_1(-2 \sin t - t \cos t) + a_2(2 \cos t - t \sin t)$$
$$y_p''' = a_1(-3 \cos t + t \sin t) + a_2(-3 \sin t - t \cos t)$$
$$y_p^{(4)} = a_1(4 \sin t + t \cos t) + a_2(-4 \cos t + t \sin t)$$

and thus,
$$y_p^{(4)} - y_p = 4a_1 \sin t + -4a_2 \cos t = 4 \cos t.$$

The linear independence of $\{\cos t, \sin t\}$ and Theorem 7 of Sect. 3.2 imply

$$4a_1 = 0$$
$$-4a_2 = 4.$$

Hence, $a_1 = 0$ and $a_2 = -1$ from which we get

$$y_p = -t \sin t.$$

The general solution is thus

$$y = y_p + y_h = -t \sin t + c_1 \cos t + c_2 \sin t + c_3 e^t + c_4 e^{-t}. \qquad \blacktriangleleft$$

We can summarize Theorem 1 and the previous example in the following algorithm.

Algorithm 3. If

$$q(\boldsymbol{D})y = f,$$

where q is a polynomial of degree n and $f \in \mathcal{E}$, then the general solution can be obtained as follows:

The Method of Undetermined Coefficients

1. Compute the standard basis, $\mathcal{B}_q$, for $\mathcal{E}_q$.
2. Determine v so that $\mathcal{L}\{f\} = \frac{u}{v}$. That is, $f \in \mathcal{E}_v$.
3. Compute the standard basis, $\mathcal{B}_{vq}$, for $\mathcal{E}_{vq}$.
4. The test function, y_p is the linear combination with arbitrary coefficients of functions in $\mathcal{B}_{vq}$ that are not in $\mathcal{B}_q$.
5. The coefficients in y_p are determined by plugging y_p into the differential equation $q(\boldsymbol{D})y = f$.
6. The general solution is given by

$$y_p + y_h,$$

where $y_h \in \mathcal{E}_q$.

The Incomplete Partial Fraction Method

The description of the incomplete partial fraction method given in Sect. 3.5 was independent of the degree of q and so applies equally well here. To summarize the method, we record the following algorithm.

Algorithm 4. If

$$q(\boldsymbol{D})y = f,$$

where q is a polynomial of degree n and $f \in \mathcal{E}$, then the general solution can be obtained as follows:

The Incomplete Partial Fraction Method

1. Compute $\mathcal{L}\{f\} = \frac{u}{v}$.
2. With trivial initial conditions assumed, write $\mathcal{L}\{y\} = \frac{u}{qv}$.
3. Apply the partial fraction algorithm on $\frac{u}{qv}$ until the denominator of the remainder term is q.
4. A particular solution y_p is obtained by adding up the inverse Laplace transform of the resulting partial fractions.
5. The general solution is given by

$$y_p + y_h,$$

where $y_h \in \mathcal{E}_q$.

To illustrate this general procedure, let us consider two examples.

Example 5. Find the general solution to

$$y''' - 2y'' + y' = te^t.$$

▶ **Solution.** The characteristic polynomial is $q(s) = s^3 - 2s^2 + s = s(s-1)^2$ and $\mathcal{L}\{te^t\} = \frac{1}{(s-1)^2}$. Assuming trivial initial conditions, $y(0) = 0$, $y'(0) = 0$, and $y''(0) = 0$, we get

$$Y(s) = \mathcal{L}\{y\} = \frac{1}{(s-1)^4 s}.$$

The incomplete $(s-1)$-chain for $Y(s)$ is

Incomplete $(s-1)$-chain	
$\dfrac{1}{(s-1)^4 s}$	$\dfrac{1}{(s-1)^4}$
$\dfrac{-1}{(s-1)^3 s}$	$\dfrac{-1}{(s-1)^3}$
$\dfrac{p(s)}{(s-1)^2 s}$	

There is no need to compute $p(s)$ since the denominator is $q(s) = (s-1)^2 s$ and the resulting inverse Laplace transform is a homogeneous solution. Let $Y_p(s) = \frac{1}{(s-1)^4} - \frac{1}{(s-1)^3}$. Then a particular solution is

$$y_p = \mathcal{L}^{-1}\{Y_p\} = \frac{1}{3!}t^3 e^t - \frac{1}{2!}t^2 e^t.$$

Since $\mathcal{B}_q = \{1, e^t, e^{-t}\}$, we get the general solution

$$y = \frac{1}{6}t^3 e^t - \frac{1}{2}t^2 e^t + c_1 + c_2 e^t + c_2 t e^t. \qquad \blacktriangleleft$$

Example 6. Find the general solution to

$$y^{(4)} - y = 4\cos t.$$

▶ **Solution.** The characteristic polynomial is $q(s) = s^4 - 1 = (s^2 - 1)(s^2 + 1)$ and $\mathcal{L}\{4\cos t\} = \frac{4s}{s^2+1}$. Again assume $y(0) = 0$ and $y'(0) = 0$. Then

$$Y(s) = \frac{4s}{(s^2 - 1)(s^2 + 1)^2}.$$

Using the irreducible quadratic partial fraction method, we obtain the incomplete $(s^2 + 1)$-chain for $Y(s)$:

Incomplete $(s^2 + 1)$-chain	
$\dfrac{4s}{(s^2 - 1)(s^2 + 1)^2}$	$\dfrac{-2s}{(s^2 + 1)^2}$
$\dfrac{p(s)}{(s^2 - 1)(s^2 + 1)}$	

By Table 2.9, we have

$$y_p = \mathcal{L}^{-1}\left\{\frac{-2s}{(s^2 + 1)^2}\right\} = -t\sin t.$$

The homogeneous solution as $y_h = c_1 e^t + c_2 e^{-t} + c_3 \cos t + c_4 \sin t$, and thus, the general solution is

$$y = y_p + y_h = -t\sin t + c_1 e^t + c_2 e^{-t} + c_3 \cos t + c_4 \sin t. \qquad \blacktriangleleft$$

Exercises

1–4. Given q and v below, determine the test function y_p for the differential equation $q(\mathbf{D})y = f$, where $\mathcal{L}f = \frac{u}{v}$.

1. $q(s) = s^3 - s \quad v(s) = s + 1$
2. $q(s) = s^3 - s^2 - s + 1 \quad v(s) = s - 1$
3. $q(s) = s^3 - s \quad v(s) = s - 2$
4. $q(s) = s^4 - 81 \quad v(s) = s^2 + 9$

5–8. Use the method of undetermined coefficients to find the general solution for each of the differential equations given below.

5. $y''' - y' = e^t$
6. $y''' - y'' + y' - y = 4\cos t$
7. $y^{(4)} - 5y'' + 4y = e^{2t}$
8. $y^{(4)} - y = e^t + e^{-t}$

9–14. Use the incomplete partial fraction method to solve the following differential equations.

9. $y''' - y' = e^t$
10. $y''' - 4y'' + 4y' = 4te^{2t}$
11. $y''' + 4y' = t$
12. $y^{(4)} - 5y'' + 4y = e^{2t}$
13. $y''' - y'' + y' - y = 4\cos t$
14. $y^{(4)} - y = e^t + e^{-t}$

4.4 Coupled Systems of Differential Equations

Suppose $L_1 = q_1(D)$ and $L_2 = q_2(D)$ are polynomial differential operators of order m and n, respectively. A system of two differential equations of the form

$$L_1 y_1 = \lambda_1 y_2$$
$$L_2 y_2 = \lambda_2 y_1 \tag{1}$$

is an example of a ***coupled system of differential equations***. We assume λ_1 and λ_2 are scalars. Typically, there are initial conditions

$$y_1(0) = a_0, \quad y_1'(0) = a_1, \quad \ldots, \quad y_1^{(m-1)}(0) = a_{m-1}$$
$$y_2(0) = b_0, \quad y_2'(0) = b_1, \quad \ldots, \quad y_2^{(n-1)}(0) = b_{n-1} \tag{2}$$

each given up to one less than the order of the corresponding differential operator. Notice how the solution of one equation becomes the input (up to a scalar multiple) of the other. The spring system given in the introduction provides an example where L_1 and L_2 are both of order 2. It is the goal of this section to show how such systems together with the initial conditions can be solved using higher order constant coefficient differential equations.[2] The result is a uniqueness and existence theorem. We will then return to the coupled spring problem considered in the introduction.

The solution method we describe here involves a basic fact about the algebra of polynomial differential operators.

The Commutativity of Polynomial Differential Operators

Just as polynomials can be multiplied and factored, so too polynomial differential operators. Consider a very simple example. Suppose $q_1(s) = s-1$ and $q_2(s) = s-2$. Then $q_1(D) = D - 1$ and $q_2(D) = D - 2$. Observe,

$$(D - 1)(D - 2)y = (D - 1)(y' - 2y)$$
$$= (D - 1)y' - (D - 1)(2y)$$
$$= y'' - y' - (2y' - 2y)$$

---

[2]There are other methods. For example, in the exercises, a nice Laplace transform approach will be developed. Theoretically, this is a much nicer approach. However, it is not necessarily computationally easier. In Chap. 9, we will consider systems of first order differential equations and show how (1) can fit in that context.

$$= y'' - 3y' + 2y$$
$$= (D^2 - 3D + 2)y = q(D)y,$$

where $q(s) = s^2 - 3s + 2 = (s - 1)(s - 2)$. Therefore, $q(D) = (D - 1)(D - 2)$, and since there was nothing special about the order, we can also write $q(D) = (D - 2)(D - 1)$. More generally, if $q(s) = q_1(s)q_2(s)$, then the corresponding differential operators multiply to give $q(D) = q_1(D)q_2(D)$. This discussion leads to the following commutative principle: If $L_1 = q_1(D)$ and $L_2 = q_2(D)$ are polynomial differential operators, then

$$L_1 L_2 = L_2 L_1.$$

Extending the Initial Conditions

Suppose y_1 and y_2 are solutions to (1) satisfying the initial values given in (2). It turns out then that these equations determine the initial conditions $y_1^{(k)}(0)$ and $y_2^{(k)}(0)$ for all $k = 1, 2, \ldots$. To see this, suppose $L_1 = q_1(D) = c_m D^m + \cdots c_1 D + c_0$. Now evaluate $L_1 y_1 = \lambda_1 y_2$ at $t = 0$ to get $L_1 y_1(0) = \lambda y_2(0)$ or

$$c_m y_1^{(m)}(0) + c_{m-1} y_1^{(m-1)}(0) + \ldots c_0 y_1(0) = \lambda_1 y_2(0).$$

Since all initial values except $y_1^{(m)}(0)$ are known, we can solve for $y_1^{(m)}(0)$. Hence, let $a_m = y_1^{(m)}(0)$ be the unique solution to $L_1 y_1(0) = \lambda_1 y_2(0)$. Now differentiate the equation $L_1 y_1 = \lambda_1 y_2$ and evaluate at $t = 0$ to get

$$c_m y^{(m+1)}(0) + c_{m-1} y^{(m)}(0) + \ldots c_0 y_1'(0) = \lambda_1 y_2'(0).$$

Now, all initial values except $y^{(m+1)}(0)$ are known. We define $a_{m+1} = y^{(m+1)}(0)$ to be the unique solution to $D L_1 y_1(0) = \lambda_1 D y_2(0)$. We can repeat this procedure recursively up to the $(n - 1)$st derivative of y_2 to get $a_m = y_1^{(m)}(0), \ldots, a_{n+m-1} = y_1^{(n+m-1)}(0)$ where a_{n+k} is the unique solution to

$$D^k L_1 y_1(0) = \lambda_1 D^k y_2(0).$$

In a similar way, we can recursively extend the initial values of y_2 to get $b_n = y_2^{(n)}(0), \ldots, b_{n+m-1} = y_2^{(n+m-1)}(0)$ where b_{m+k} is the unique solution to $D^k L_2 y_2(0) = \lambda_2 D^k y_1(0)$. It is important to notice that this procedure does not actually depend on explicitly knowing the functions y_1 and y_2: the values of a_{n+k} and b_{m+k} only depend on the recursive solutions to $D^k L_1 y_1(0) = \lambda_1 D y_2(0)$ and $D^k L_2 y_2(0) = \lambda_2 D y_1(0)$, respectively. It is also useful to observe that since the values of $a_k = y_1^{(k)}(0)$ and $b_k = y_2^{(k)}(0)$ are now known for $k = 0, \ldots, n + m - 1$,

we can repeat the recursion process indefinitely to compute all higher order values of a_k and b_k. The key thing to note is that a_k and b_k are the (recursive) solutions to

$$\boldsymbol{D}^k \boldsymbol{L}_1 y_1(0) = \lambda_1 \boldsymbol{D}^k y_2(0) \quad \text{and} \quad \boldsymbol{D}^k \boldsymbol{L}_1 y_1(0) = \lambda_1 \boldsymbol{D}^k y_2(0), \qquad (3)$$

respectively.

The following example illustrates the process outlined above.

Example 1.   Consider the coupled system of differential equations

$$y_1'(t) + 3y_1(t) = 4y_2(t) \qquad (4)$$

$$y_2''(t) + 3y_2(t) = 2y_1(t), \qquad (5)$$

with initial conditions $y_1(0) = 3$, $y_2(0) = 1$, and $y_2'(0) = 3$. Determine the extended initial values $y_1^{(k)}(0)$ and $y_2^{(k)}(0)$, for $k = 0, \ldots, 4$.

► **Solution.**   We evaluate (4) at $t = 0$ to get $y_1'(0) + 3y_1(0) = 4y_2(0)$. Since $y_1(0) = 3$ and $y_2(0) = 1$, it follows that $y_1'(0) = -5$. Apply $\boldsymbol{D}$ to (4) at $t = 0$ to get $y_1''(0) + 3y_1'(0) = 4y_2'(0)$. Since $y_1'(0) = -5$ and $y_2'(0) = 3$, we get $y_1''(0) = 27$. Now evaluate (5) at $t = 0$ to get $y_2''(0) + 3y_2(0) = 2y_1(0)$. It follows that $y_2''(0) = 3$. Apply $\boldsymbol{D}$ to (5) at $t = 0$ to get $y_2'''(0) + 3y_2'(0) = 2y_1'(0)$ which gives $y_2'''(0) = -19$. Apply $\boldsymbol{D}^2$ to (5) at $t = 0$ to get $y_2^{(4)}(0) + 3y_2''(0) = 2y_1''(0)$ which implies $y_2^{(4)}(0) = 45$. Now apply $\boldsymbol{D}^2$ to (4) at $t = 0$ to get $y_1'''(0) + 3y_1''(0) = 4y_2''(0)$ which implies $y_1'''(0) = -69$. Finally, apply $\boldsymbol{D}^3$ to (4) at $t = 0$ to get $y_1^{(4)}(0) + 3y_1'''(0) = 4y_2'''(0)$ which implies $y_1^{(4)}(0) = 131$. The following table summarizes the data obtained.

$$
\begin{array}{ll}
y_1(0) = 3 & y_2(0) = 1 \\
y_1'(0) = -5 & y_2'(0) = 3 \\
y_1''(0) = 27 & y_2''(0) = 3 \\
y_1'''(0) = -69 & y_2'''(0) = -19 \\
y_1^{(4)}(0) = 131 & y_2^{(4)}(0) = 45. \qquad (6)
\end{array}
$$

With patience, it is possible to compute any initial value for y_1 or y_2 at $t = 0$.   ◄

We are now in a position to state the main theorem.

Theorem 2.   *Suppose $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$ are polynomial differential operators of order m and n, respectively. The unique solution (y_1, y_2) to the coupled system of differential equations*

$$\boldsymbol{L}_1 y_1 = \lambda_1 y_2 \qquad (7)$$

$$\boldsymbol{L}_2 y_2 = \lambda_2 y_1 \qquad (8)$$

with initial conditions

$$y_1(0) = a_0, \quad y_1'(0) = a_1, \quad \ldots, \quad y_1^{(m-1)}(0) = a_{m-1}$$
$$y_2(0) = b_0, \quad y_2'(0) = b_1, \quad \ldots, \quad y_2^{(n-1)}(0) = b_{n-1} \tag{9}$$

is given as follows: Extend the initial values a_k and b_k, for $k = 1, \ldots n + m - 1$, as determined by (3). *Let $\boldsymbol{L} = \boldsymbol{L}_1\boldsymbol{L}_2 - \lambda_1\lambda_2$. Then y_1 and y_2 are the homogeneous solutions to $\boldsymbol{L}y = 0$ with initial conditions $y_1^{(k)}(0) = a_k$ and $y_2^{(k)}(0) = b_k$, $k = 0, \ldots, n + m - 1$, respectively.*

Proof. First, let us suppose that (y_1, y_2) is a solution to the coupled system and satisfies the given initial conditions. Then y_1 and y_2 satisfy the initial conditions given by (3). Now apply $\boldsymbol{L}_2$ to (7) and $\boldsymbol{L}_1$ to (8) to get

$$\boldsymbol{L}_2\boldsymbol{L}_1 y_1 = \lambda_1\boldsymbol{L}_2 y_2 = \lambda_1\lambda_2 y_1$$
$$\boldsymbol{L}_1\boldsymbol{L}_2 y_2 = \lambda_2\boldsymbol{L}_1 y_1 = \lambda_1\lambda_2 y_2.$$

Both of these equations imply that y_1 and y_2 are solutions to $\boldsymbol{L}y = 0$, where $\boldsymbol{L} = \boldsymbol{L}_1\boldsymbol{L}_2 - \lambda_1\lambda_2$, satisfying the initial conditions $a_k = y_1^{(k)}(0)$ and $b_k = y_2^{(k)}(0)$, $k = 0, \ldots, n + m - 1$. By the uniqueness part of Theorem 5 of Sect. 4.1, y_1 and y_2 are uniquely determined. To show that solutions exist, suppose y_1 and y_2 are the solutions to $\boldsymbol{L}y = 0$ with initial conditions (obtained recursively) $a_k = y_1^{(k)}(0)$ and $b_k = y_2^{(k)}(0)$, $k = 1, 2, \ldots, n + m - 1$. We will show that $\boldsymbol{L}_1 y_1 = \lambda_1 y_2$. The argument that $\boldsymbol{L}_2 y_2 = \lambda_1 y_1$ is similar. Let $z_1 = \boldsymbol{L}_1 y_1$ and $z_2 = \lambda_1 y_2$. Then

$$\boldsymbol{L}_1\boldsymbol{L}_2 z_1 = \boldsymbol{L}_1\boldsymbol{L}_1\boldsymbol{L}_2 y_1 = \lambda_1\lambda_2\boldsymbol{L}_1 y_1 = \lambda_1\lambda_2 z_1$$
$$\boldsymbol{L}_1\boldsymbol{L}_2 z_2 = \lambda_1\boldsymbol{L}_1\boldsymbol{L}_2 y_2 = \lambda_1\lambda_1\lambda_2 y_2 = \lambda_1\lambda_2 z_2$$

It follows that z_1 and z_2 are homogeneous solutions to $\boldsymbol{L}y = 0$. By (3), z_1 and z_2 satisfy the same initial values for all $k = 0, \ldots, n + m - 1$. By the existence and uniqueness theorem, Theorem 5 of Sect. 4.1, we have $z_1 = z_2$. □

Algorithm 3. Suppose $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$ are polynomial differential operators of order m and n, respectively. Then the solution to (1) with initial conditions given by (2) is given by the following algorithm:

Solution Method for Coupled Systems

1. Extend (recursively) the initial conditions $y_1^{(k)}(0)$ and $y_2^{(k)}(0)$, for $k = 1, 2, \ldots, n + m - 1$.
2. Let $\boldsymbol{L} = \boldsymbol{L}_1\boldsymbol{L}_2 - \lambda_1\lambda_2$ and solve

$$\boldsymbol{L}y = 0,$$

for each set of initial conditions.

Example 4. Solve the following coupled system:

$$y_1' + 3y_1 = 4y_2$$
$$y_2'' + 3y_2 = 2y_1,$$

$y_1(0) = 3$, $y_2(0) = 1$, and $y_2'(0) = 3$.

▶ **Solution.** In Example 1, we extended the initial values (see (6), where we will only need the first 3 rows). We let $L_1 = D+3$ and $L_2 = D^2+3$. Then y_1 and y_2 are homogeneous solutions to $L = L_1 L_2 - 8 = D^3 + 3D^2 + 3D + 1$. The characteristic polynomial is $q(s) = s^3 + 3s^2 + 3s + 1 = (s+1)^3$. Since $\mathcal{B}_q = \{e^{-t}, te^{-t}, t^2 e^{-t}\}$ it follows that the homogeneous solutions take the form

$$y = c_1 e^{-t} + c_2 t e^{-t} + c_3 t^2 e^{-t}.$$

To find y_1, we set $y_1 = c_1 e^{-t} + c^2 t e^{-t} + c_3 e^{-t}$. We compute y_1' and y_1'' and substitute the initial conditions $y_1(0) = 3$, $y_1'(0) = -5$, and $y_1''(0) = 27$ to get the following system:

$$
\begin{aligned}
c_1 &&&= 3 \\
-c_1 &+ c_2 &&= -5 \\
c_1 &- 2c_2 &+ 2c_3 &= 27.
\end{aligned}
$$

A short calculation gives $c_1 = 3$, $c_2 = -2$, and $c_3 = 10$. It follows that $y_1(t) = (3 - 2t + 10t^2)e^{-t}$. In a similar way, setting $y_2 = c_1 e^{-t} + c^2 t e^{-t} + c_3 e^{-t}$ leads to the system

$$
\begin{aligned}
c_1 &&&= 1 \\
-c_1 &+ c_2 &&= 3 \\
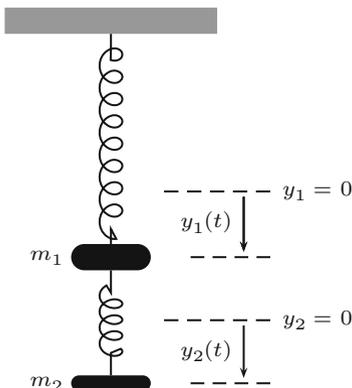c_1 &- 2c_2 &+ 2c_3 &= 3.
\end{aligned}
$$

A short calculation gives $c_1 = 1$, $c_2 = 4$, and $c_3 = 5$, and hence, $y_2(t) = (1 + 4t + 5t^2)e^{-t}$. It follows that

$$y_1(t) = (3 - 2t + 10t^2)e^{-t} \quad \text{and} \quad y_2(t) = (1 + 4t + 5t^2)e^{-t}$$

are the solutions to the coupled system.   ◀

Coupled Spring Systems

We now return to the problem posed at the beginning of this chapter. Two springs systems (without dashpots) are coupled as illustrated in the diagram below:

If y_1 and y_2 denote the displacements of the bodies of mass m_1 and m_2 from their equilibrium positions, $y_1 = 0$ and $y_2 = 0$, then we showed that y_1 and y_2 satisfy

$$m_1 y_1''(t) + (k_1 + k_2)y_1(t) = k_2 y_2(t)$$
$$m_2 y_2''(t) + k_2 y_2(t) = k_2 y_1(t). \tag{10}$$

If we divide each equation by the leading coefficient and let

$$a = \frac{k_1}{m_1}, \quad b = \frac{k_2}{m_1}, \quad \text{and} \quad c = \frac{k_2}{m_2}, \tag{11}$$

then we get the coupled system

$$L_1 y_1 = by_2, \tag{12}$$
$$L_2 y_2 = cy_1, \tag{13}$$

where $L_1 = D^2 + a + b$ and $L_2 = D^2 + c$. We observe that

$$L = L_1 L_2 - bc$$
$$= D^4 + (a + b + c)D^2 + ac.$$

As to initial conditions, suppose we are given that $y_1(0) = A_0$, $y_1'(0) = A_1$, $y_2(0) = B_0$, and $y_2'(0) = B_1$. Then (12) and its derivative imply

$$y_1''(0) = -(a + b)y_1(0) + by_2(0) = -(a + b)A_0 + bB_0$$
$$y_1'''(0) = -(a + b)y_1'(0) + by_2'(0) = -(a + b)A_1 + bB_1.$$

In a similar way, (13) implies

$$y_2''(0) = -cy_2(0) + cy_1(0) = -cB_0 + cA_0$$
$$y_2'''(0) = -cy_2'(0) + cy_1'(0) = -cB_1 + cA_1.$$

Summarizing, we obtain that y_1 and y_2 both satisfy the same 4th order differential equation

$$y^{(4)} + (a + b + c)y'' + acy = 0 \tag{14}$$

with initial conditions

$$
\begin{aligned}
y_1(0) &= A_0 & y_2(0) &= B_0 \\
y_1'(0) &= A_1 & y_2'(0) &= B_1 \\
y_1''(0) &= -(a+b)A_0 + bB_0 & y_2''(0) &= -cB_0 + cA_0 \\
y_1'''(0) &= -(a+b)A_1 + bB_1 & y_2'''(0) &= -cB_1 + cA_1.
\end{aligned}
\tag{15}
$$

Example 5. Consider the coupled spring system with $m_1 = 3$, $m_2 = 4$, $k_1 = 9$, and $k_2 = 12$. At $t = 0$, both masses are pulled downward a distance of 1m from equilibrium and released without imparting any momentum. Determine the motion of the system.

▶ **Solution.** From (11), we get

$$a = \frac{9}{3} = 3, \quad b = \frac{12}{3} = 4, \quad \text{and} \quad c = \frac{12}{4} = 3. \tag{16}$$

Thus, y_1 and y_2 satisfy the following coupled system of differential equations:

$$y_1'' + 7y_1 = 4y_2$$
$$y_2'' + 3y_2 = 3y_1.$$

By Theorem 2, y_1 and y_2 are homogeneous solutions to the 4th degree equation $q(D)y = 0$ where $q(s) = (s^2+7)(s^2+3) - 12 = s^4 + 10s^2 + 9 = (s^2+1)(s^2+9)$. It follows that y_1 and y_2 are linear combinations of $\mathcal{B}_q = \{\cos t, \sin t, \cos 3t, \sin 3t\}$. By (15), the initial conditions of y_1 and y_2 at $t = 0$ up to order 3 are

$$
\begin{aligned}
y_1(0) &= & 1 & \quad y_2(0) = 1 \\
y_1'(0) &= & 0 & \quad y_2'(0) = 0 \\
y_1''(0) &= & -3 & \quad y_2''(0) = 0 \\
y_1'''(0) &= & 0 & \quad y_2'''(0) = 0.
\end{aligned}
$$

Therefore, if $y_1 = c_1 \cos t + c_2 \sin t + c_3 \cos 3t + c_4 \sin 3t$, then the coefficients satisfy

$$\begin{aligned}
c_1 \ + \quad\quad c_3 \quad\quad\quad &= \ 1 \\
c_2 \quad\quad + \ 3c_4 &= \ 0 \\
-c_1 + \quad\quad -9c_3 \quad\quad &= -3 \\
-c_2 \quad\quad - \ 27c_4 &= \ 0
\end{aligned}$$

from which we get $c_1 = \frac{3}{4}$, $c_2 = 0$, $c_3 = \frac{1}{4}$, and $c_4 = 0$. It follows that
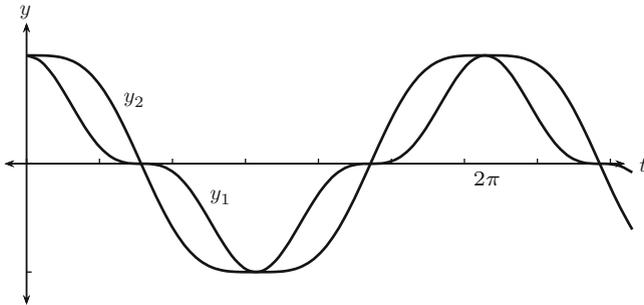
$$y_1(t) = \frac{3}{4}\cos t + \frac{1}{4}\cos 3t.$$

On the other hand, if $y_2 = c_1 \cos t + c_2 \sin t + c_3 \cos 3t + c_4 \sin 3t$, then the coefficients satisfy

$$\begin{aligned}
c_1 \ + \quad\quad c_3 \quad\quad\quad &= \ 1 \\
c_2 \quad\quad + \ 3c_4 &= \ 0 \\
-c_1 + \quad\quad -9c_3 \quad\quad &= \ 0 \\
-c_2 \quad\quad - \ 27c_4 &= \ 0
\end{aligned}$$

from which we get $c_1 = \frac{9}{8}$, $c_2 = 0$, $c_3 = \frac{-1}{8}$, and $c_4 = 0$. It follows that

$$y_2(t) = \frac{9}{8}\cos t - \frac{1}{8}\cos 3t.$$

Since y_1 and y_2 are periodic with period 2π, the motion of the masses (given below) are likewise periodic. Their graphs are simultaneously given below.

◀

Exercises

1–8. Solve the following coupled systems.

1.

$$y_1' - 6y_1 = -4y_2$$
$$y_2' = 2y_1$$

with initial conditions $y_1(0) = 2$ and $y_2(0) = -1$.

2.

$$y_1' - 3y_1 = -4y_2$$
$$y_2' + y_2 = y_1$$

with initial conditions $y_1(0) = 1$ and $y_2(0) = 1$.

3.

$$y_1' = 2y_2$$
$$y_2' = -2y_1$$

with initial conditions $y_1(0) = 1$ and $y_2(0) = -1$.

4.

$$y_1' - 2y_1 = 2y_2$$
$$y_2'' + 2y_2' + y_2 = -2y_1,$$

with initial conditions $y_1(0) = 3$, $y_2(0) = 0$, and $y_2'(0) = 3$.

5.

$$y_1' + 4y_1 = 10y_2$$
$$y_2'' - 6y_2' + 23y_2 = 9y_1,$$

with initial conditions $y_1(0) = 0$, $y_2(0) = 2$, and $y_2'(0) = 2$.

6.

$$y_1' - 2y_1 = -2y_2$$
$$y_2'' + y_2' + 6y_2 = 4y_1,$$

with initial conditions $y_1(0) = 1$, $y_2(0) = 5$, and $y_2'(0) = 4$.

7.

$$y_1'' + 2y_1' + 6y_1 = 5y_2$$
$$y_2'' - 2y_2' + 6y_2 = 9y_1,$$

with initial conditions $y_1(0) = 0$, $y_1'(0) = 0$, $y_2(0) = 6$, and $y_2'(0) = 6$.

8.

$$y_1'' + 2y_1 = -3y_2$$
$$y_2'' + 2y_2' - 9y_2 = 6y_1,$$

with initial conditions $y_1(0) = -1$, $y_1'(0) = -4$, $y_2(0) = 1$, and $y_2'(0) = 2$.

9–10. Solve the coupled spring systems for the given parameters m_1, m_2, k_1, and k_2 and initial conditions.

9. $m_1 = 2$, $m_2 = 1$, $k_1 = 4$, and $k_2 = 2$ with initial conditions $y_1(0) = 3$, $y_1'(0) = 3$, $y_2(0) = 0$, and $y_2'(0) = 0$

10. $m_1 = 4$, $m_3 = 1$, $k_1 = 8$, and $k_2 = 12$ with initial conditions $y_1(0) = 1$, $y_1'(0) = 0$, $y_2(0) = 6$, and $y_2'(0) = 0$

11. *The Laplace Transform Method for Coupled Systems*: In this exercise, we will see how the Laplace transform may be used to solve a coupled system:

$$L_1 y_1 = \lambda_1 y_2$$
$$L_2 y_2 = \lambda_2 y_1$$

with initial conditions

$$y_1(0) = a_0, \quad y_1'(0) = a_1, \quad \ldots, \quad y_1^{(m-1)}(0) = a_{m-1}$$
$$y_2(0) = b_0, \quad y_2'(0) = b_1, \quad \ldots, \quad y_2^{(n-1)}(0) = b_{n-1}.$$

Let $L_1 = q_1(D)$ and $L_2 = q_2(D)$ be polynomial differential operators. Let $Y_1 = \mathcal{L}\{y_1\}$ and $Y_2 = \mathcal{L}\{y_2\}$. Suppose $\mathcal{L}\{L_1 y_1\}(s) = q_1(s)Y_1(s) - p_1(s)$ and $\mathcal{L}\{L_2 y_2\}(s) = q_2(s)Y_2(s) - p_2(s)$, where $p_1(s)$ and $p_2(s)$ are polynomials determined by the initial conditions.

1. Show that Y_1 and Y_2 satisfy the following matrix relation:

$$\begin{pmatrix} q_1(s) & -\lambda_1 \\ -\lambda_2 & q_2(s) \end{pmatrix} \begin{pmatrix} Y_1(s) \\ Y_2(s) \end{pmatrix} = \begin{pmatrix} p_1(s) \\ p_2(s) \end{pmatrix}.$$

2. Show that this system has a solution given by

$$Y_1(s) = \frac{p_1(s)q_2(s) + \lambda_1 p_2(s)}{q_1(s)q_2(s) - \lambda_1\lambda_2},$$

$$Y_2(s) = \frac{p_2(s)q_1(s) + \lambda_2 p_1(s)}{q_1(s)q_2(s) - \lambda_1\lambda_2}.$$

12–16. Use the Laplace transform method developed in Exercise 11 to solve the following coupled systems.

12.

$$y_1' = -y_2$$
$$y_2' - 2y_2 = y_1$$

with initial conditions $y_1(0) = 1$ and $y_2(0) = -1$.

13.

$$y_1' - y_1 = -2y_2$$
$$y_2' - y_2 = 2y_1$$

with initial conditions $y_1(0) = 2$ and $y_2(0) = -2$.

14.

$$y_1' - 2y_1 = -y_2$$
$$y_2'' - y_2' + y_2 = y_1$$

with initial conditions $y_1(0) = 0$, $y_2(0) = -1$, and $y_2'(0) = 2$.

15.

$$y_1' + 2y_1 = 5y_2$$
$$y_2'' - 2y_2' + 5y_2 = 2y_1$$

with initial conditions $y_1(0) = 1$, $y_2(0) = 0$, and $y_2'(0) = 3$.

16.

$$y_1'' + 2y_1 = -3y_2$$
$$y_2'' + 2y_2' - 9y_2 = 6y_1$$

with initial conditions $y_1(0) = 10$, $y_1'(0) = 0$, $y_2(0) = 10$, and $y_2'(0) = 0$.

4.5   System Modeling

Mathematical modeling involves understanding how a system works mathematically. By a system, we mean something that takes inputs and produces outputs such as might be found in the biological, chemical, engineering, and physical sciences. The core of modeling thus involves expressing how the outputs of a system can be mathematically described as a function of the inputs. The following *system diagram* represents the inputs coming in from the left of the system and outputs going out on the right.

$$f(t) \longrightarrow \boxed{\text{System}} \longrightarrow y(t)$$

For the most part, inputs and outputs will be quantities that are time dependent; they will be represented as functions of t. There are occasions, however, where other parameters such as position or frequency are used in place of time. An input-output pair, $(f(t), y(t))$, implies a relationship which we denote by

$$y(t) = \Phi(f)(t).$$

Notice that Φ is an operation on the input function f and produces the output function y. In a certain sense, understanding Φ is equivalent to understanding the workings of the system. Frequently, we identify the system under study with Φ itself. Our goal in modeling then is to give an explicit mathematical description of Φ.

   In many settings, a mathematical model can be described implicitly by a constant coefficient linear differential equation and its solution gives an explicit description. For example, the mixing problems in Sect. 1.5, spring systems in Sect. 3.6, RCL circuits in Sect. 3.7, and the coupled spring systems in Sect. 4.4 are each modeled by constant coefficient linear differential equations and have common features which we explore in this section. To get a better idea of what we have in mind, let us reconsider the mixing problem as an example.

Example 1. Suppose a tank holds 10 liters of a brine solution with initial concentration $a/10$ grams of salt per liter. (Thus, there are a grams of salt in the tank, initially.) Pure water flows in an intake tube at a rate of b liters per minute and the well-mixed solution flows out of the tank at the same rate. Attached to the intake is a hopper containing salt. The amount of salt entering the intake is controlled by a valve and thus varies as a function of time. Let $f(t)$ be the rate (in grams of salt per minute) at which salt enters the system. Let $y(t)$ represent the amount of salt in the tank at time t. Find a mathematical model that describes y.

▶ **Solution.** As in Chap. 1, our focus is on the way y changes. Observe that

$$y' = \text{input rate} - \text{output rate}.$$
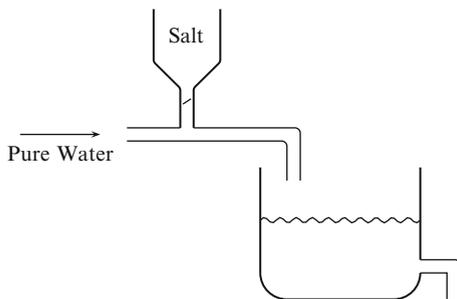
By input rate, we mean the rate at which salt enters the system and this is just $f(t)$.
The output rate is the rate at which salt leaves the system and is given by the product
of the concentration of salt in the tank, $y(t)/10$, and the flow rate b. We are thus led
to the following initial value problem:

$$y' + \frac{b}{10}y = f(t), \quad y(0) = a. \tag{1}$$

In this system, $f(t)$ is the input function and $y(t)$ is the output function. Incorpo-
rating this mathematical model in a system diagram gives the following implicit
description:

$$f(t) \longrightarrow \boxed{\begin{array}{c} \text{Solve } y' + \frac{b}{10}\, y = f(t) \\ y(0) = a \end{array}} \longrightarrow y(t).$$
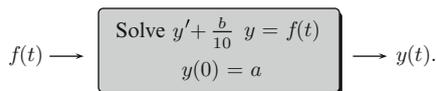
Using Algorithm 3 in Sect. 1.5 to solve (1) gives

$$y(t) = ae^{\frac{-bt}{10}} + e^{-\frac{bt}{10}} \int_0^t f(x)e^{\frac{bx}{10}}\, dx$$

$$= ae^{\frac{-bt}{10}} + \int_0^t f(x)e^{-\frac{b(t-x)}{10}}\, dx$$

$$= ah(t) + f * h(t),$$

where $h(t) = e^{\frac{-bt}{10}}$ and $f * h$ denotes the convolution of f with h. We therefore
arrive at an explicit mathematical model Φ for the mixing system:

$$\Phi(f)(t) = ah(t) + f * h(t). \qquad \blacktriangleleft$$

As we shall see, this simple example illustrates many of the main features shared
by all systems modeled by a constant coefficient differential equation. You notice
that the output consists of two pieces: $ah(t)$ and $f * h(t)$. If $a = 0$, then the initial

state of the system is zero, that is, there is no salt in the tank at time $t = 0$. In this case, the output is

$$y(t) = \Phi(f)(t) = f * h(t). \tag{2}$$

This output is called the ***zero-state response***; it represents the response of the system by purely external forces of the system and not on any nonzero initial condition or state. The zero-state response is the particular solution to $q(\mathbf{D})y = f$, $y(0) = 0$, where $q(s) = s + b/10$.

On the other hand, if $f = 0$, the output is $ah(t)$. This output is called the ***zero-input response***; it represents the response based on purely internal conditions of the system and not on any external inputs. The zero-input response is the homogeneous solution to $q(\mathbf{D})y = 0$ with initial condition $y(0) = a$. The ***total response*** is the sum of the zero-state response and the zero-input response.

The function h is called the ***unit impulse response function***. It is the zero-input response with $a = 1$ and completely characterizes the system. Alternatively, we can understand h in terms of the mixing system by opening the hopper for a very brief moment just before $t = 0$ and letting 1 gram of salt enter the intake. At $t = 0$, the amount of salt in the tank is 1 gram and no salt enters the system thereafter. The expression "unit impulse" reflects the fact that a unit of salt (1 gram) enters the system and does so instantaneously, that is, as an impulse. Such impulsive inputs are discussed more thoroughly in Chap. 6. Over time, the amount of salt in the tank will diminish according to the unit impulse response $h = e^{\frac{-bt}{10}}$ as illustrated in the graph below:
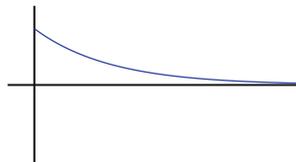
Once h is known, the zero-state response for an input f is completely determined by (2). One of the main points of this section will be to show that in all systems modeled by a constant coefficient differential equation, the zero-state response is given by this same formula, for some h. We will also show how to find h.

With a view to a more general setting, let $q(\mathbf{D})$ be an nth order differential operator with leading coefficient one and let $\mathbf{a} = (a_0, a_1, \ldots, a_{n-1})$ be a vector of n scalars. Suppose Φ is a system. We say Φ is ***modeled by $q(\mathbf{D})$ with initial state a*** if for each input function f, the output function $y = \Phi(f)$ satisfies $q(\mathbf{D})y = f$, and $y(0) = a_0$, $y'(0) = a_1$, $\ldots$, $y^{(n-1)}(0) = a_{n-1}$. Sometimes we say that the ***initial state*** of Φ is ***a***. By the existence and uniqueness theorem, Theorem 5 of Sect. 4.1, y is unique. In terms of a system diagram we have

$$f(t) \longrightarrow \boxed{\begin{array}{c} \text{Solve } q(\boldsymbol{D})y = f \\ \text{with initial state } \mathbf{a} \end{array}} \longrightarrow y(t).$$
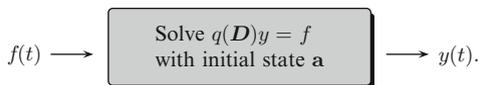
The definitions of zero-state response, zero-input response, and total response given above naturally extend to this more general setting.

The Zero-Input Response

First, we consider the response of the system with no external inputs, that is, $f(t) = 0$. This is the zero-input response and is a result of internal initial conditions of the system only. For example, the initial conditions for a spring system are the initial position and velocity, and for RCL circuits, they are the initial charge and current. In the mixing system we described earlier, the only initial condition is the amount of salt a in the tank at time $t = 0$.
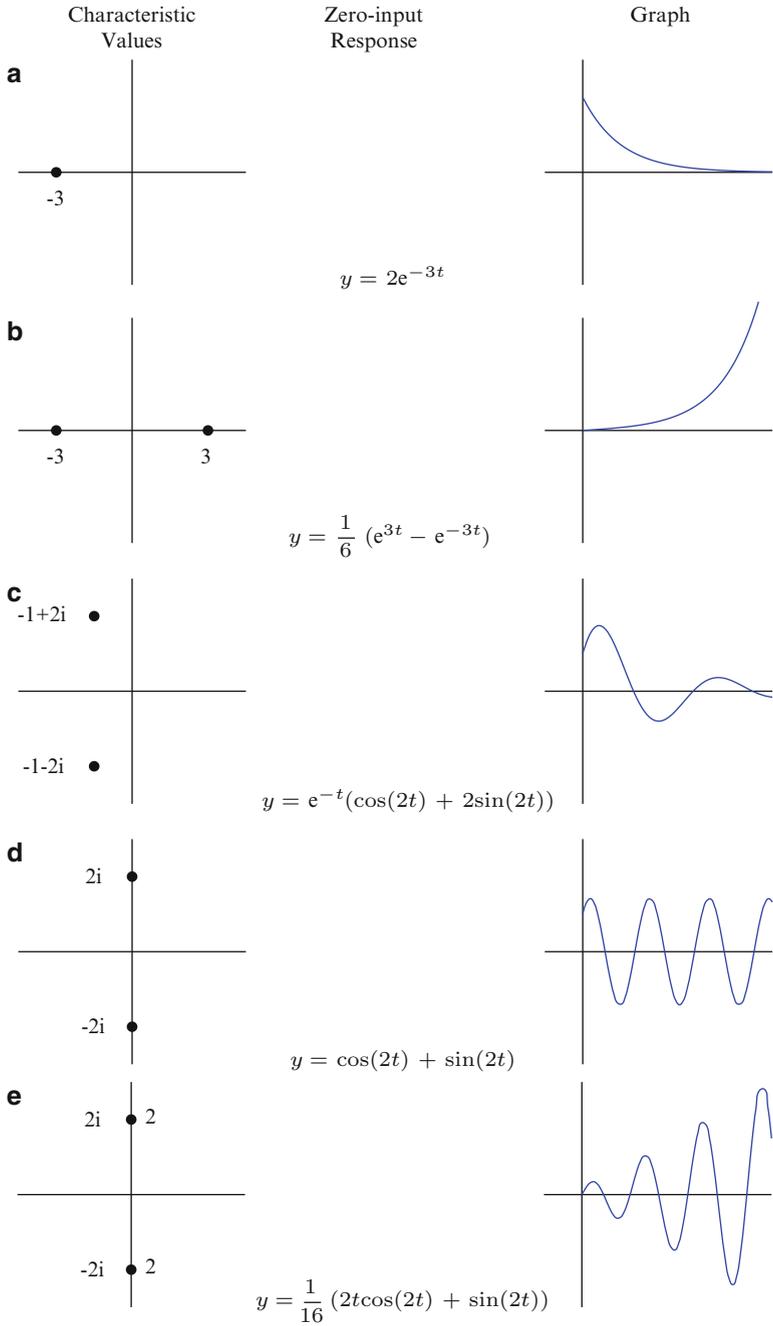
The zero-input response is the solution to

$$q(\boldsymbol{D})y = 0, \quad y(0) = a_0, \ y'(0) = a_1, \dots, y^{(n-1)}(0) = a_{n-1}.$$

By Theorem 2 of Sect. 4.2, y is a linear combination of functions in the standard basis $\mathcal{B}_q$. The roots $\lambda_1, \dots, \lambda_k \in \mathbb{C}$ of q are called the ***characteristic values*** and the functions in the standard basis are called the ***characteristic modes*** of the system. Consider the following example.

Example 2. For each problem below, a system is modeled by $q(\boldsymbol{D})$ with initial state $\mathbf{a}$. Plot the characteristic values in the complex plane, determine the zero-input response, and graph the response.

a. $q(\boldsymbol{D}) = \boldsymbol{D} + 3, \mathbf{a} = 2$
b. $q(\boldsymbol{D}) = (\boldsymbol{D} - 3)(\boldsymbol{D} + 3), \mathbf{a} = (0, 1)$
c. $q(\boldsymbol{D}) = (\boldsymbol{D} + 1)^2 + 4, \mathbf{a} = (1, 3)$
d. $q(\boldsymbol{D}) = \boldsymbol{D}^2 + 4, \mathbf{a} = (1, 2)$
e. $q(\boldsymbol{D}) = (\boldsymbol{D}^2 + 4)^2, \mathbf{a} = (0, 0, 0, 1)$.

▶ **Solution.** The following table summarizes the calculations that we ask the reader to verify.

	Characteristic Values	Zero-input Response	Graph

a

$$y = 2e^{-3t}$$

b

$$y = \frac{1}{6}\left(e^{3t} - e^{-3t}\right)$$

c

$$y = e^{-t}(\cos(2t) + 2\sin(2t))$$

d

$$y = \cos(2t) + \sin(2t)$$

e

$$y = \frac{1}{16}\left(2t\cos(2t) + \sin(2t)\right)$$

In part (e), we have indicated the multiplicity of $\pm 2i$ by a 2 to the right of the characteristic value.   ◀

The location of the characteristic values in the complex plane is related to an important notion called system stability.

Stability

System stability has to do with the long-term behavior of a system. If the zero-input response of a system, for all initial states, tends to zero over time, then we say the system is ***asymptotically stable***. This behavior is seen in the mixing system of Example 1. For any initial state a, the zero-input response $y(t) = ae^{\frac{-bt}{10}}$ has limiting value 0 as $t \to \infty$. In the case $a = 2$ and $b = 30$, the graph is the same as that given in Example 2a. Notice that the system in Example 2c is also asymptotically stable since the zero-input response always takes the form $y(t) = e^{-t}(A \sin(2t) + B \cos(2t))$. The function $t \to A \sin(2t) + B \cos(2t)$ is bounded, so the presence of e^{-t} guarantees that the limit value of $y(t)$ is 0 as $t \to \infty$. In both Example 2a and 2c, the characteristic values lie in the left-half side of the complex plane. More generally, suppose $t^k e^{\alpha t} \cos \beta t$ is a characteristic mode for a system. If $\lambda = \alpha + i\beta$ and $\alpha < 0$, then

$$\lim_{t \to \infty} t^k e^{\alpha t} \cos \beta t = 0,$$

for all nonnegative integers k. A similar statement is true for $t^k e^{\alpha t} \sin \beta t$. On the other hand, if $\alpha > 0$, then the characteristic modes, $t^k e^{\alpha t} \cos \beta t$ and $t^k e^{\alpha t} \sin \beta t$, are unbounded. Thus, a system is asymptotically stable if and only if all characteristic values lie to the left of the imaginary axis.

If a system is not asymptotically stable but the zero-input response is bounded for all possible initial states, then we say the system is ***marginally stable***. Marginal stability is seen in Example 2d and occurs when one or more of the characteristic values lie on the imaginary axis and have multiplicity exactly one. Those characteristic values that are not on the imaginary axis must be to the left of the imaginary axis.

We say a system is ***unstable*** if there is an initial state in which the zero-input response is unbounded over time. This behavior can be seen in Example 2b and 2e. Over time, the response becomes unbounded. Of course, in a real physical system, this cannot happen. The system will break or explode when it passes a certain threshold. Unstable systems occur for two distinct reasons. First, if one of the characteristic values is $\lambda = \alpha + i\beta$ and $\alpha > 0$, then λ lies in the right-half side of the complex plane. In this case, the characteristic mode is of the form $t^k e^{\alpha t} \cos \beta t$ or $t^k e^{\alpha t} \sin \beta t$. This function is unbounded as a function of t. This is what happens in Example 2b. Second, if one of the characteristic values $\lambda = i\beta$ lies on the imaginary axis and, in addition, the multiplicity is greater than one, then the characteristic modes are of the form $t^k \cos(\beta t)$ or $t^k \sin(\beta t)$, $k \geq 1$. These modes oscillate

unboundedly as a function of $t > 0$, as in Example 2e. Remember, it only takes one unbounded characteristic mode for the whole system to be unstable.

Example 3. Determine the stability of each system modeled by $q(D)$ below:

1. $q(D) = (D + 1)^2(D + 3)$
2. $q(D) = (D^2 + 9)(D + 4)$
3. $q(D) = (D + 4)^2(D - 5)$
4. $q(D) = (D^2 + 1)(D^2 + 9)^2$

▶ **Solution.** 1. The characteristic values are $\lambda = -1$ with multiplicity 2 and $\lambda = -3$. The system is asymptotically stable.
2. The characteristic values are $\lambda = \pm 3i$ and $\lambda = -4$. The system is marginally stable.
3. The characteristic values are $\lambda = -4$ with multiplicity 2 and $\lambda = 5$. The system is unstable.
4. The characteristic values are $\lambda = \pm i$ and $\lambda = \pm 3i$ with multiplicity 2. The system is unstable.   ◀

The Unit Impulse Response Function

Suppose Φ is a system modeled by $q(D)$, an nth order constant coefficient differential operator. The ***unit impulse response function*** $h(t)$ is the zero-input response to Φ when the initial state of the system is $\mathbf{a} = (0, \dots, 0, 1)$. More specifically, $h(t)$ is the solution to

$$q(D)y = 0 \qquad y(0) = 0, \dots, y^{(n-2)}(0) = 0, \, y^{(n-1)}(0) = 1.$$

If $n = 1$, then $y(0) = 1$ as in the mixing system discussed in the beginning of this section. In this simple case, $h(t)$ is a multiple of a single characteristic mode. For higher order systems, however, the unit impulse response function is a homogeneous solution to $q(D)y = 0$ and, hence, a linear combination of the characteristic modes of the system.

Example 4. Find the unit impulse response function for a system Φ modeled by

$$q(D) = (D + 1)(D^2 + 1).$$

▶ **Solution.** It is an easy matter to apply the Laplace transform method to $q(D)y = 0$ with initial condition $y(0) = 0$, $y'(0) = 0$, and $y''(0) = 1$. We get $q(s)Y(s) - 1 = 0$. A short calculation gives

$$Y(s) = \frac{1}{q(s)} = \frac{1}{(s+1)(s^2+1)} = \frac{1}{4}\left(\frac{2}{s+1} + \frac{-2s+2}{s^2+1}\right).$$

The inverse Laplace transform is $y(t) = \frac{1}{2}(e^{-t} + \sin t - \cos t)$. The unit impulse response function is thus

$$h(t) = \frac{1}{2}(e^{-t} + \sin t - \cos t).$$ ◀

Observe in this example that $h(t) = \mathcal{L}^{-1}\left\{\frac{1}{q(s)}\right\}(t)$. It is not hard to see that this formula extends to the general case. We record this in the following theorem. The proof is left to the reader.

Theorem 5. *Suppose a system Φ is modeled by a constant coefficient differential operator $q(D)$. The unit impulse response function, $h(t)$, of the system Φ is given by*

$$h(t) = \mathcal{L}^{-1}\left\{\frac{1}{q(s)}\right\}(t).$$

The Zero-State Response

Let us now turn our attention to a system Φ in the zero-state and consider the zero-state response. This occurs precisely when $\mathbf{a} = \mathbf{0}$, that is, all initial conditions of the system are zero. We should think of the system as initially being at rest. We continue to assume that Φ is modeled by an nth order constant coefficient differential operator $q(D)$. Thus, for each input $f(t)$, the output $y(t)$ satisfies $q(D)y = f$ with y and its higher derivatives up to order $n-1$ all zero at $t = 0$. An important feature of Φ in this case is its linearity.

Proposition 6. *Suppose Φ is a system modeled by $q(D)$ in the zero-state. Then Φ is linear. Specifically, if f, f_1, and f_2 are input functions and c is a scalar, then*

1. $\Phi(f_1 + f_2) = \Phi(f_1) + \Phi(f_2)$
2. $\Phi(cf) = c\Phi(f)$.

Proof. If f is any input function, then $\Phi(f) = y$ if and only if $q(D)y = f$ and $y(0) = y'(0) = \cdots = y^{(n-1)}(0) = 0$. If y_1 and y_2 are the zero-state response functions to f_1 and f_2, respectively, then the linearity of $q(D)$ implies

$$q(D)(y_1 + y_2) = q(D)y_1 + q(D)y_2 = f_1 + f_2.$$

Furthermore, since the initial state of both y_1 and y_2 are zero, so is the initial state of $y_1 + y_2$. This implies $\Phi(f_1 + f_2) = y_1 + y_2$. In a similar way,

$$q(\boldsymbol{D})(cy) = cq(\boldsymbol{D})y = cf,$$

by the linearity of $q(\boldsymbol{D})$. The initial state of cy is clearly zero. So $\Phi(cf) = c\Phi(f)$.

$\square$

The following remarkable theorem gives an explicit formula for Φ in terms of convolution with the unit-impulse response function.

Theorem 7. *Suppose Φ is a system modeled by $q(\boldsymbol{D})$ in the zero-state. If f is a continuous input function on an interval which includes zero, then the zero-state response is given by the convolution of f with the unit impulse response function h. That is,*

$$\Phi(f)(t) = f * h(t) = \int_0^t f(x)h(t-x)\,dx.$$

If, in addition, we were to assume that f has a Laplace transform, then the proof would be straightforward. Indeed, if $y(t)$ is the zero-state response, then the Laplace transform method would give $q(s)\mathcal{L}\{y\} = \mathcal{L}\{f\} := F(s)$ and therefore

$$\mathcal{L}\{y\}(s) = \frac{1}{q(s)}F(s).$$

The convolution theorem then gives $y(t) = \Phi(f)(t) = h * f$.

For the more general case, let us introduce the following helpful lemma.

Lemma 8. *Suppose f is continuous on a interval I containing 0. Suppose h is differentiable on I. Then*

$$(f * h)'(t) = f(t)h(0) + f * h'.$$

Proof. Let $y(t) = f * h(t)$. Then

$$\frac{\Delta y}{\Delta t} = \frac{y(t + \Delta t) - y(t)}{\Delta t}$$

$$= \frac{1}{\Delta t}\left(\int_0^{t+\Delta t} f(x)h(t + \Delta t - x)\,dx - \int_0^t f(x)h(t-x)\,dx\right)$$

$$= \int_0^t f(x)\frac{h(t + \Delta t - x) - h(t - x)}{\Delta t}\,dx$$

$$+ \frac{1}{\Delta t}\int_t^{t+\Delta t} f(x)h(t + \Delta t - x)\,dx.$$

We now let Δt go to 0. The first summand has limit $\int_0^t f(x)h'(t - x)\,dx = f * h'(t)$. By the fundamental theorem of calculus, the limit of the second summand is

obtained by evaluating the integrand at $x = t$, thus getting $f(t)h(0)$. The lemma
now follows by adding these two terms.   □

Proof (of Theorem 7). Let $h(t)$ be the unit impulse response function. Then
$h(0) = h'(0) = \cdots = h^{(n-2)}(0) = 0$ and $h^{(n-1)}(0) = 1$. Set $y(t) = f * h(t)$.
Repeated applications of Lemma 8 to y gives

$$y' = f * h' + h(0)f = f * h'$$
$$y'' = f * h'' + h'(0)f = f * h''$$
$$\vdots$$
$$y^{(n-1)} = f * h^{(n-1)} + h^{(n-2)}(0)f = f * h^{(n-1)}$$
$$y^{(n)} = f * h^{(n)} + h^{(n-1)}(0)f = f * h^{(n)} + f.$$

From this it follows that

$$q(D)y = f * q(D)h + f = f,$$

since $q(D)h = 0$. It is easy to check that y is in the zero-state. Therefore,

$$\Phi(f) = y = f * h.   □$$

At this point, let us make a few remarks on what this theorem tells us. The most
remarkable thing is the fact that Φ is precisely determined by the unit impulse
response function h. From a mathematical point of view, knowing h means you
know how the system Φ works in the zero-state. Once h is determined, all output
functions, that is, system responses, are given by the convolution product, $f * h$, for
an input function f. Admittedly, convolution is an unusual product. It is not at all
like the usual product of functions where the value (or state) at time t is determined
by knowing just the value of each factor at time t. Theorem 7 tells us that the state
of a system response at time t depends on knowing the values of the input function
f for all x between 0 and t. The system "remembers" the whole of the input f up
to time t and "meshes" those inputs with the internal workings of the system, as
represented by the impulse response function h, to give $f * h(t)$.

Since the zero-state response, $f * h$, is a solution to $q(D)y = f$, with zero
initial state, it is a particular solution. In practice, computing convolutions can be
time consuming and tedious. In the following examples, we will limit the inputs to
functions in $\mathcal{E}$ and use Table C.7.

Example 9. A system Φ is modeled by $q(D)$. Find the zero-state response for the
given input function:

(a) $q(\boldsymbol{D}) = \boldsymbol{D} + 2$ and $f(t) = 1$
(b) $q(\boldsymbol{D}) = \boldsymbol{D}^2 + 4\boldsymbol{D} + 3$ and $f(t) = e^{-t}$
(c) $q(\boldsymbol{D}) = \boldsymbol{D}^2 + 4$ and $f(t) = \cos(2t)$

► **Solution.** (a) The characteristic polynomial is $q(s) = s + 2$, and therefore, the characteristic mode is e^{-2t}. It follows that $h(t) = Ae^{-2t}$ and with initial condition $h(0) = 1$, we get $h(t) = e^{-2t}$. The system response, $y(t)$, for the input 1 is

$$y(t) = e^{-2t} * 1(t) = \frac{1}{2}(1 - e^{-2t}).$$

(b) The characteristic polynomial is $q(s) = s^2 + 4s + 3 = (s + 1)(s + 3)$. The characteristic modes are e^{-3t} and e^{-t}. Thus, $h(t)$ has the form $h(t) = Ae^{-t} + Be^{-3t}$. The initial conditions $h(0) = 0$ and $h'(0) = 1$ imply

$$h(t) = \frac{1}{2}(e^{-t} - e^{-3t}).$$

The system response to the input function $f(t) = e^{-t}$ is

$$y(t) = \frac{1}{2}(e^{-t} - e^{-3t}) * e^{-t} = \frac{1}{2}te^{-t} - \frac{1}{4}e^{-t} + \frac{1}{4}e^{-3t}.$$

(c) It is easy to verify that the impulse response function is $h(t) = \frac{1}{2}\sin(2t)$. The system response to the input function $f(t) = \cos(2t)$ is

$$y(t) = \frac{1}{2}\sin(2t) * \cos(2t) = \frac{1}{4}t\sin(2t). \qquad ◄$$

Bounded-In Bounded-Out

In Example 9a, we introduce a bounded input, $f(t) = 1$, and the response $y(t) = \frac{1}{2}(1 - e^{-2t})$ is also bounded, by $\frac{1}{2}$, in fact. On the other hand, in Example 9c, we introduce a bounded input, $f(t) = \cos 2t$, yet the response $y(t) = \frac{1}{4}t\sin 2t$ oscillates unboundedly. We say that a system $\varPhi$ is **BIBO-stable** if for every bounded input $f(t)$ the response function $y(t)$ is likewise bounded. (*BIBO* stands for "bounded input bounded output.") Note the following theorem. An outline of the proof is given in the exercises.

Theorem 10. *Suppose $\varPhi$ is an asymptotically stable system. Then $\varPhi$ is BIBO-stable.*

Unstable systems are of little practical value to an engineer designing a "safe" system. In an unstable system, a set of unintended initial states can lead to an unbounded response that destroys the system entirely. Even marginally stable systems can have bounded input functions that produce unbounded output functions.

This is seen in Example 9c, where the system response to the bounded input $f(t) = \cos(2t)$ is the unbounded function $y(t) = \frac{1}{4}t\sin(2t)$. Asymptotically stable systems are thus the "safest" systems since they are *BIBO*-stable. They produce at worst a bounded response to a bounded input. However, this is not to say that the response cannot be destructive. We will say more about this in the following topic.

Resonance

We now come to a very interesting and important phenomenon called ***resonance***. Loosely speaking, resonance is the phenomenon that occurs when a system reacts very energetically to a relatively mild input. Resonance can sometimes be catastrophic for the system. For example, a wine glass has a characteristic frequency at which it will vibrate. You can hear this frequency by rubbing your moistened finger around the rim of the glass to cause it to vibrate. An opera singer who sings a note at this same frequency with sufficient intensity can cause the wine glass to vibrate so much that it shatters. Resonance can also be used to our advantage as is familiar to a musician tuning a musical instrument to a standard frequency given, for example, by a tuning fork. Resonance occurs when the instrument is "in tune."

The characteristic values of a system Φ are sometimes referred to as the ***characteristic frequencies*** As we saw earlier, the internal workings of a system are governed by these frequencies. A system that is energized tends to operate at these frequencies. Thus, when an input function matches an internal frequency, the system response will generally be quite energetic, even explosive.

A dramatic example of this occurs when a system is marginally stable. Consider the following example.

Example 11. A zero-state system Φ is modeled by the differential equation

$$(\boldsymbol{D}^2 + 1)y = f.$$

Determine the impulse response function h and the system response to the following inputs:

1. $f(t) = \sin(\pi t)$
2. $f(t) = \sin(1.25t)$
3. $f(t) = \sin(t)$.

Discuss the resonance that occurs.

▶ **Solution.** The characteristic values are $\pm i$ with multiplicity one. Thus, the system is marginally stable. The unit impulse response $h(t)$ is the solution to $(\boldsymbol{D}^2 + 1)y = 0$ with initial conditions $y(0) = 0$ and $y'(0) = 1$. The Laplace transform gives $H(s) = \frac{1}{s^2+1}$, and hence, $h(t) = \sin(t)$.
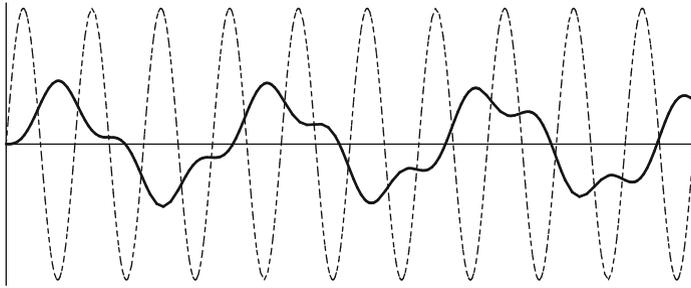
Fig. 4.1 Input and response functions with dissimilar frequencies

The three inputs all have amplitude 1 but different frequencies. We will use the convolution formula

$$\sin(at) * \sin(bt) = \begin{cases} \dfrac{a\sin(bt) - b\sin(at)}{a^2 - b^2} & \text{if } b \neq a \\[2ex] \dfrac{\sin(at) - at\cos(at)}{2a} & \text{if } a = b \end{cases}$$

from Table 2.11.

1. Consider the input function $f(t) = \sin(\pi t)$; its frequency is π. The system response is

$$y(t) = \sin(\pi t) * \sin(t) = \frac{\pi \sin(t) - \sin(\pi t)}{\pi^2 - 1}.$$

   The graph of the input function together with the response is given in Fig. 4.1. The graph of the input function is dashed and has amplitude 1 while the response function has an amplitude less than 1. No resonance is occurring here, and this is reflected in the fact that the inputs characteristic frequency, π, is far from the systems characteristic frequency 1. We also note that the response function is not periodic. This is reflected in the fact that the quotient of the frequencies $\frac{\pi}{1} = \pi$ is not rational. We will say more about periodic functions in Chap. 6.
2. Next we take the input function to be $f(t) = \sin(1.25t)$. Its frequency is 1.25 and is significantly closer to the characteristic frequency. The system response is

$$y(t) = \sin(1.25t) * \sin(t) = \frac{1.25 \sin(t) - \sin(1.1t)}{0.5625}.$$

   The graph of the input function together with the response is given in Fig. 4.2. In this graph, we needed to scale back significantly to see the response function. Notice how the amplitude of the response is significantly higher than that of the input. Also notice how the response comes in pulses. This phenomenon is known
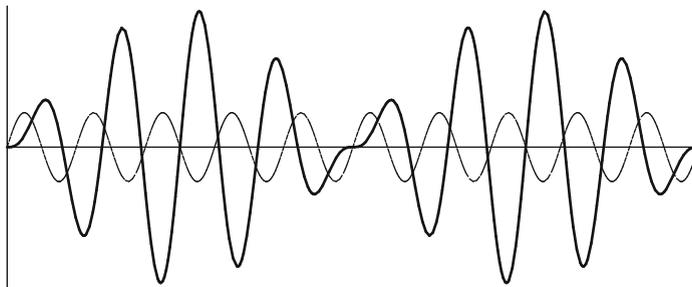
Fig. 4.2   Input and response functions with similar yet unequal frequencies. Beats occur
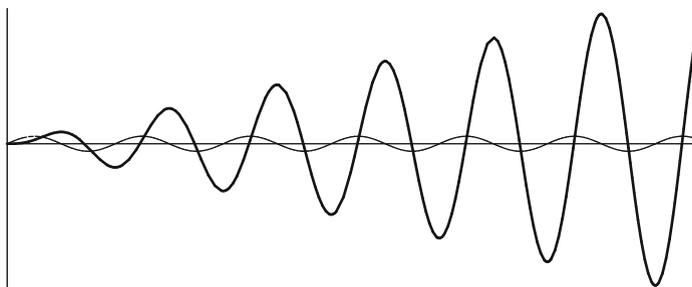

Fig. 4.3   Input and response functions with equal frequencies. Resonance occurs

as **beats** and is familiar to musicians who try to tune an instrument. When the frequency of vibration of the string is close but not exactly equal to that of the tuning fork, one hears a pulsating beat. The instrument is out of tune.

3. We now consider the input function $f(t) = \sin t$. Here the input frequency matches exactly the characteristic frequency. The system response is

$$y(t) = \sin(t) * \sin(t) = \frac{\sin(t) - t\cos(t)}{2}.$$

The presence of t in $t\cos(t)$ implies that the response will oscillate without bound as seen in Fig. 4.3.

Again in this graph, we needed to scale back to see the enormity of the response function. This is resonance in action. In physical systems, resonance can be so energetic that the system may fall apart. Because of the significant damage that can occur, systems designers must be well aware of the internal characteristic values or frequencies of their system and the likely kinds of inputs it may need to deal with.   ◄
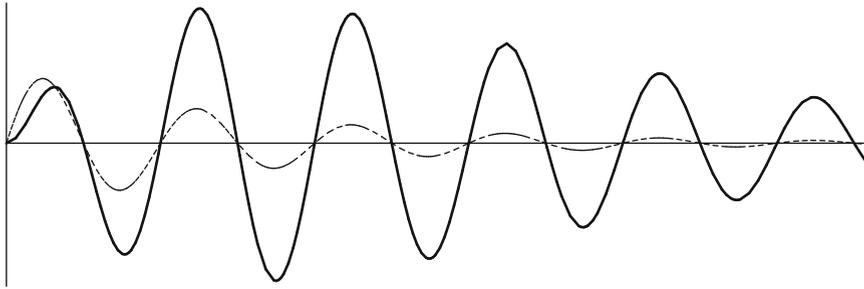
Fig. 4.4   Asymptotically stable system with resonance

As a final example, we consider resonance in an asymptotically stable system.

Example 12.   A zero-state system Φ is modeled by the differential equation

$$((\boldsymbol{D} + 0.1)^2 + 1)y = f.$$

Determine the impulse response function h and the system response to the input function $f(t) = e^{-0.1t} \cos(t)$. Discuss the resonance that occurs.

▶ **Solution.**   The characteristic values are $-0.1 \pm i$ and lie in the left-hand side of the complex plane. Thus, the system is asymptotically stable. The characteristic modes are $\{e^{-0.1t} \sin t, e^{-0.1t} \cos t\}$. A straightforward calculation gives the unit impulse response function

$$h(t)) = e^{-0.1t} \sin(t).$$

The input function $f(t) = e^{-0.1t} \cos(t)$ is a characteristic mode, and the response function is

$$y(t) = h * f(t) = \frac{1}{2} t e^{-0.1t} \sin(t).$$

Figure 4.4 shows the graph. Notice the initial energetic response. This is a manifestation of resonance even though the response dies out in time. If the response passes a certain threshold, the system may break. On the other hand, resonance can be used in a positive way as in tuning a radio to a particular frequency. Again, system designers must be well aware of the resonance effect.   ◀

Exercises

For each problem, a system is modeled by $q(\boldsymbol{D})$ with initial state **a**. Determine the zero-input response. Determine whether the system is asymptotically stable, marginally stable, or unstable.

1. $q(\boldsymbol{D}) = \boldsymbol{D} + 5, \mathbf{a} = 10$
2. $q(\boldsymbol{D}) = \boldsymbol{D} - 2, \mathbf{a} = 2$
3. $q(\boldsymbol{D}) = \boldsymbol{D}^2 - 4\boldsymbol{D} + 3, \mathbf{a} = (2, 4)$
4. $q(\boldsymbol{D}) = \boldsymbol{D}^2 + 5\boldsymbol{D} + 4, \mathbf{a} = (0, 3)$
5. $q(\boldsymbol{D}) = \boldsymbol{D}^2 + 4\boldsymbol{D} + 5, \mathbf{a} = (0, 1)$
6. $q(\boldsymbol{D}) = \boldsymbol{D}^2 + 9, \mathbf{a} = (1, 1)$
7. $q(\boldsymbol{D}) = \boldsymbol{D}^2 + 6\boldsymbol{D} + 9, \mathbf{a} = (1, 1)$
8. $q(\boldsymbol{D}) = \boldsymbol{D}^2 + \boldsymbol{D} - 2, \mathbf{a} = (1, -2)$
9. $q(\boldsymbol{D}) = \boldsymbol{D}^2 - 2\boldsymbol{D} + 2, \mathbf{a} = (1, 2)$
10. $q(\boldsymbol{D}) = \boldsymbol{D}^3 + \boldsymbol{D}^2, \mathbf{a} = (1, -1, 1)$
11. $q(\boldsymbol{D}) = (\boldsymbol{D} + 1)(\boldsymbol{D}^2 + 1), \mathbf{a} = (1, -1, 1)$
12. $q(\boldsymbol{D}) = \boldsymbol{D}^4 - 1, \mathbf{a} = (0, 1, 0, -1)$

For each problem, a system is modeled by $q(\boldsymbol{D})$. Determine the unit impulse response function.

13. $q(\boldsymbol{D}) = D + 1$
14. $q(\boldsymbol{D}) = \boldsymbol{D}^2 + 4$
15. $q(\boldsymbol{D}) = \boldsymbol{D}^2 - 4$
16. $q(\boldsymbol{D}) = \boldsymbol{D}^2 + 2\boldsymbol{D} + 5$
17. $q(\boldsymbol{D}) = \boldsymbol{D}^3 + \boldsymbol{D}$

In this set of problems, we establish that an asymptotically stable system modeled by $q(\boldsymbol{D})$, for some constant coefficient differential operator, is *BIBO*-stable.

18. Let k be a nonnegative integer and $\alpha \in \mathbb{R}$. Show that

$$\int_0^t x^k \mathrm{e}^{\alpha x}\, dx = C + p(t)\mathrm{e}^{\alpha t},$$

where C is a constant and $p(t)$ is a polynomial of degree k. Show that $C + p(t)\mathrm{e}^{\alpha t}$ is a bounded function of $[0, \infty)$ if $\alpha < 0$.

19. Suppose $\lambda = \alpha + i\beta$ is a complex number and $\alpha < 0$. Let k be a nonnegative integer and suppose f is a bounded function on $[0, \infty)$. Show that $t^k \mathrm{e}^{\alpha t} \cos \beta t * f$ and $t^k \mathrm{e}^{\alpha t} \sin \beta t * f$ are bounded functions.

20. Suppose a system modeled by a constant coefficient differential operator is asymptotically stable. Show it is *BIBO*-stable.