# Chapter 7
# Power Series Methods

Thus far in our study of linear differential equations, we have imposed severe restrictions on the coefficient functions in order to find solution methods. Two special classes of note are the constant coefficient and Cauchy–Euler differential equations. The Laplace transform method was also useful in solving some differential equations where the coefficients were linear. Outside of special cases such as these, linear second order differential equations with variable coefficients can be very difficult to solve.

In this chapter, we introduce the use of power series in solving differential equations. Here is the main idea. Suppose a second order differential equation

$$a_2(t)y'' + a_1(t)y' + a_0(t)y = f(t)$$

is given. Under the right conditions on the coefficient functions, a solution can be expressed in terms of a power series which takes the form

$$y(t) = \sum_{n=0}^{\infty} c_n (t - t_0)^n,$$

for some fixed $t_0$. Substituting the power series into the differential equation gives relationships among the coefficients $\{c_n\}_{n=0}^{\infty}$, which when solved gives a power series solution. This technique is called the ***power series method***. While we may not enjoy a closed form solution, as in the special cases thus far considered, power series methods imposes the least restrictions on the coefficient functions.

W.A. Adkins and M.G. Davidson, *Ordinary Differential Equations*,
Undergraduate Texts in Mathematics, DOI 10.1007/978-1-4614-3618-8_7,

## 7.1 A Review of Power Series

We begin with a review of the main properties of power series that are usually learned in a first year calculus course.

### *Definitions and Convergence*

A ***power series centered at $t_0$ in the variable $t$*** is a series of the form

$$\sum_{n=0}^{\infty} c_n(t-t_0)^n = c_0 + c_1(t-t_0) + c_2(t-t_0)^2 + \cdots . \tag{1}$$

The ***center*** of the power series is $t_0$, and the ***coefficients*** are the constants $\{c_n\}_{n=0}^{\infty}$. Frequently, we will simply refer to (1) as a ***power series***. Let $I$ be the set of real numbers where the series converges. Obviously, $t_0$ is in $I$, so $I$ is nonempty. It turns out that $I$ is an interval and is called the ***interval of convergence***. It contains an open interval of the form $(t_0 - R, t_0 + R)$ and possibly one or both of the endpoints. The number $R$ is called the ***radius of convergence*** and can frequently be determined by the ratio test.

***The Ratio Test for Power Series*** *Let $\sum_{n=0}^{\infty} c_n(t-t_0)^n$ be a given power series and suppose $L = \lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right|$. Define $R$ in the following way:*

$$\begin{aligned} R &= 0 && \text{if } L = \infty, \\ R &= \infty && \text{if } L = 0, \\ R &= \tfrac{1}{L} && \text{if } 0 < L < \infty. \end{aligned}$$

*Then*

1. *The power series converges only at $t = t_0$ if $R = 0$.*
2. *The power series converges absolutely for all $t \in \mathbb{R}$ if $R = \infty$.*
3. *The power series converges absolutely when $|t - t_0| < R$ and diverges when $|t - t_0| > R$ if $0 < R < \infty$.*

If $R = 0$, then $I$ is the degenerate interval $[t_0, t_0]$, and if $R = \infty$, then $I = (-\infty, \infty)$. If $0 < R < \infty$, then $I$ is the interval $(t_0 - R, t_0 + R)$ and possibly the endpoints, $t_0 - R$ and $t_0 + R$, which one must check separately using other tests of convergence.

Recall that ***absolute convergence*** means that $\sum_{n=0}^{\infty} |c_n(t-t_0)^n|$ converges and implies the original series converges. One of the important advantages absolute convergence gives us is that we can add up the terms in a series in any order we please and still get the same result. For example, we can add all the even terms and then the odd terms separately. Thus,

$$\begin{aligned}\sum_{n=0}^{\infty} c_n(t-t_0)^n &= \sum_{n \text{ odd}} c_n(t-t_0)^n + \sum_{n \text{ even}} c_n(t-t_0)^n \\ &= \sum_{n=0}^{\infty} c_{2n+1}(t-t_0)^{2n+1} + \sum_{n=0}^{\infty} c_{2n}(t-t_0)^{2n}.\end{aligned}$$

**Example 1.** Find the interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(t-4)^n}{n2^n}.$$

► **Solution.** The ratio test gives

$$\left|\frac{c_{n+1}}{c_n}\right| = \frac{n2^n}{(n+1)2^{n+1}} = \frac{n}{2(n+1)} \to \frac{1}{2}$$

as $n \to \infty$. The radius of convergence is 2. The interval of convergence has 4 as the center, and thus, the endpoints are 2 and 6. When $t = 2$, the power series reduces to $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, which is the alternating harmonic series and known to converge. When $t = 6$, the power series reduces to $\sum_{n=0}^{\infty} \frac{1}{n}$, which is the harmonic series and known to diverge. The interval of convergence is thus $I = [2, 6)$. ◄

**Example 2.** Find the interval of convergence of the power series

$$J_0(t) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{2^{2n}(n!)^2}.$$

► **Solution.** Let $u = t^2$. We apply the ratio test to $\sum_{n=0}^{\infty} \frac{(-1)^n u^n}{2^{2n}(n!)^2}$ to get

$$\left|\frac{c_{n+1}}{c_n}\right| = \frac{2^{2n}(n!)^2}{2^{2(n+1)}((n+1)!)^2} = \frac{1}{4(n+1)^2} \to 0$$

as $n \to \infty$. It follows that $R = \infty$ and the series converges for all $u$. Hence, $\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{2^{2n}(n!)^2}$ converges for all $t$ and $I = (-\infty, \infty)$. ◄

In each example, the power series defines a function on its interval of convergence. In Example 2, the function $J_0(t)$ is known as the ***Bessel function of order*** 0 and plays an important role in many physical problems. More generally, let $f(t) = \sum_{n=0}^{\infty} c_n(t-t_0)^n$ for all $t \in I$. Then $f$ is a function on the interval of convergence $I$, and (1) is its ***power series representation***. A simple example of a power series representation is a polynomial defined on $\mathbb{R}$. In this case, the coefficients are all zero except for finitely many. Other well-known examples from calculus are:

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!} = 1 + t + \frac{t^2}{2} + \frac{t^3}{3!} + \cdots, \tag{2}$$

$$\cos t = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} = 1 - \frac{t^2}{2} + \frac{t^4}{4!} - \cdots, \tag{3}$$

$$\sin t = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!} = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots, \tag{4}$$

$$\frac{1}{1-t} = \sum_{n=0}^{\infty} t^n = 1 + t + t^2 + \cdots, \tag{5}$$

$$\ln t = \sum_{n=1}^{\infty} \frac{(-1)^{(n+1)}(t-1)^n}{n!} = (t-1) - \frac{(t-1)^2}{2} + \frac{(t-1)^3}{3} - \cdots. \tag{6}$$

Equations (2), (3), and (4) are centered at 0 and have interval of convergence $(-\infty, \infty)$. Equation (5), known as the ***geometric series***, is centered at 0 and has interval of convergence $(-1, 1)$. Equation (6) is centered at 1 and has interval of convergence $(0, 2]$.

## *Index Shifting*

In calculus, the variable $x$ in a definite integral $\int_a^b f(x)\,dx$ is called a dummy variable because the value of the integral is independent of $x$. Sometimes it is convenient to change the variable. For example, if we replace $x$ by $x-1$ in the integral $\int_1^2 \frac{1}{x+1}\,dx$, we obtain

$$\int_1^2 \frac{1}{x+1}\,dx = \int_{x-1=1}^{x-1=2} \frac{1}{x-1+1}\,d(x-1) = \int_2^3 \frac{1}{x}\,dx.$$

In like manner, the index $n$ in a power series is referred to as a dummy variable because the sum is independent of $n$. It is also sometimes convenient to make a change of variable, which, for series, is called an ***index shift***. For example, in the series $\sum_{n=0}^{\infty}(n+1)t^{n+1}$, we replace $n$ by $n-1$ to obtain

$$\sum_{n=0}^{\infty}(n+1)t^{n+1} = \sum_{n-1=0}^{n-1=\infty}(n-1+1)t^{n-1+1} = \sum_{n=1}^{\infty} nt^n.$$

The lower limit $n = 0$ is replaced by $n - 1 = 0$ or $n = 1$. The upper limit $n = \infty$ is replace by $n - 1 = \infty$ or $n = \infty$. The terms $(n + 1)t^{n+1}$ in the series go to $(n - 1 + 1)t^{n-1+1} = nt^n$.

When a power series is given is such a way that the index $n$ in the sum is the power of $(t - t_0)$, we say the power series is written in ***standard form***. Thus, $\sum_{n=1}^{\infty} nt^n$ is in standard form while $\sum_{n=0}^{\infty}(n + 1)t^{n+1}$ is not.

**Example 3.** Make an index shift so that the series $\sum_{n=2}^{\infty} \frac{t^{n-2}}{n^2}$ is expressed as a series in standard form.

► **Solution.** We replace $n$ by $n + 2$ and get

$$\sum_{n=2}^{\infty} \frac{t^{n-2}}{n^2} = \sum_{n+2=2}^{n+2=\infty} \frac{t^{n+2-2}}{(n+2)^2} = \sum_{n=0}^{\infty} \frac{t^n}{(n+2)^2}.$$ ◄

## *Differentiation and Integration of Power Series*

If a function can be represented by a power series, then we can compute its derivative and integral by differentiating and integrating each term in the power series as noted in the following theorem.

**Theorem 4.** *Suppose*

$$f(t) = \sum_{n=0}^{\infty} c_n (t - t_0)^n$$

*is defined by a power series with radius of convergence $R > 0$. Then $f$ is differentiable and integrable on $(t_0 - R, t_0 + R)$ and*

$$f'(t) = \sum_{n=1}^{\infty} n c_n (t - t_0)^{n-1} \tag{7}$$

*and*

$$\int f(t)\,\mathrm{d}t = \sum_{n=0}^{\infty} c_n \frac{(t - t_0)^{n+1}}{n + 1} + C. \tag{8}$$

*Furthermore, the radius of convergence for the power series representations of $f'$ and $\int f$ are both $R$.*

We note that the presence of the factor $n$ in $f'(t)$ allows us to write

$$f'(t) = \sum_{n=0}^{\infty} n c_n (t - t_0)^{n-1}$$

since the term at $n = 0$ is zero. This observation is occasionally used. Consider the following examples.

**Example 5.** Find a power series representation for $\frac{1}{(1-t)^2}$ in standard form.

► **Solution.** If $f(t) = \frac{1}{1-t}$, then $f'(t) = \frac{1}{(1-t)^2}$. If follows from Theorem 4 that

$$\frac{1}{(1-t)^2} = \frac{\mathrm{d}}{\mathrm{d}t}\sum_{n=0}^{\infty} t^n = \sum_{n=1}^{\infty} nt^{n-1} = \sum_{n=0}^{\infty}(n+1)t^n. \qquad ◄$$

**Example 6.** Find the power series representation for $\ln(1-t)$ in standard form.

► **Solution.** For $t \in (-1,1)$, $\ln(1-t) = -\int \frac{1}{1-t} + C$. Thus,

$$\ln(1-t) = C - \int \sum_{n=0}^{\infty} t^n \,\mathrm{d}t = C - \sum_{n=0}^{\infty} \frac{t^{n+1}}{n+1} = C - \sum_{n=1}^{\infty} \frac{t^n}{n}.$$

Evaluating both side at $t = 0$ gives $C = 0$. It follows that

$$\ln(1-t) = -\sum_{n=1}^{\infty} \frac{t^n}{n}. \qquad ◄$$

## *The Algebra of Power Series*

Suppose $f(t) = \sum_{n=0}^{\infty} a_n(t-t_0)^n$ and $g(t) = \sum_{n=0}^{\infty} b_n(t-t_0)^n$ are power series representation of $f$ and $g$ and converge on the interval $(t_0 - R, t_0 + R)$ for some $R > 0$. Then

$$f(t) = g(t) \text{ if and only if } a_n = b_n,$$

for all $n = 1, 2, 3, \ldots$. Let $c \in \mathbb{R}$. Then the power series representation of $f \pm g$, $cf$, $fg$, and $f/g$ are given by

$$f(t) \pm g(t) = \sum_{n=0}^{\infty}(a_n \pm b_n)(t-t_0)^n, \tag{9}$$

$$cf(t) = \sum_{n=0}^{\infty} ca_n(t-t_0)^n, \tag{10}$$

$$f(t)g(t) = \sum_{n=0}^{\infty} c_n(t-t_0)^n \quad \text{where } c_n = a_0b_n + a_1b_{n-1} + \cdots + a_nb_0, \tag{11}$$

and

$$\frac{f(t)}{g(t)} = \sum_{n=0}^{\infty} d_n(t-t_0)^n, \quad g(a) \neq 0, \tag{12}$$

where each $d_n$ is determined by the equation $f(t) = g(t)\sum_{n=0}^{\infty} d_n(t-t_0)^n$. In (9), (10), and (11), the series converges on the interval $(t_0-R, t_0+R)$. For division of power series, (12), the radius of convergence is positive but may not be as large as $R$.

**Example 7.** Compute the power series representations of

$$\cosh t = \frac{e^t + e^{-t}}{2} \quad \text{and} \quad \sinh t = \frac{e^t - e^{-t}}{2}.$$

► **Solution.** We write out the terms in each series, $e^t$ and $e^{-t}$, and get

$$\begin{aligned} e^t &= 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots, \\ e^{-t} &= 1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} - \cdots, \\ e^t + e^{-t} &= 2 + 2\frac{t^2}{2!} + 2\frac{t^4}{4!} + \cdots, \\ e^t - e^{-t} &= 2t + 2\frac{t^3}{3!} + 2\frac{t^5}{5!} + \cdots. \end{aligned}$$

It follows that

$$\cosh t = \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!} \quad \text{and} \quad \sinh t = \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!}.$$ ◄

**Example 8.** Let $y(t) = \sum_{n=0}^{\infty} c_n t^n$. Compute

$$(1+t^2)y'' + 4ty' + 2y$$

as a power series.

► **Solution.** We differentiate $y$ twice to get

$$y'(t) = \sum_{n=1}^{\infty} c_n n t^{n-1} \quad \text{and} \quad y''(t) = \sum_{n=2}^{\infty} c_n n(n-1)t^{n-2}.$$

In the following calculations, we shift indices as necessary to obtain series in standard form:

$$\begin{aligned} t^2 y'' &= \sum_{n=2}^{\infty} c_n n(n-1)t^n = \sum_{n=0}^{\infty} c_n n(n-1)t^n, \\ y'' &= \sum_{n=2}^{\infty} c_n n(n-1)t^{n-2} = \sum_{n=0}^{\infty} c_{n+2}(n+2)(n+1)t^n, \end{aligned}$$

$$4ty' = \sum_{n=1}^{\infty} 4c_n n t^n = \sum_{n=0}^{\infty} 4c_n n t^n,$$

$$2y = \sum_{n=0}^{\infty} 2c_n t^n.$$

Notice that the presence of the factors $n$ and $n-1$ in the first series allows us to write it with a starting point $n = 0$ instead of $n = 2$, similarly for the third series. Adding these results and simplifying gives

$$(1+t^2)y'' + 4ty' + 2y = \sum_{n=0}^{\infty} ((c_{n+2} + c_n)(n+2)(n+1))\, t^n. \qquad ◀$$

A function $f$ is said to be an ***odd*** function if $f(-t) = -f(t)$ and ***even*** if $f(-t) = f(t)$. If $f$ is odd and has a power series representation with center 0, then all coefficients of even powers of $t$ are zero. Similarly, if $f$ is even, then all the coefficients of odd powers are zero. Thus, $f$ has the following form:

$$f(t) = a_0 + a_2t^2 + a_4t^4 + \ldots = \sum_{n=0}^{\infty} a_{2n}t^{2n} \qquad f\text{ -even,}$$

$$f(t) = a_1t + a_3t^3 + a_5t^5 + \ldots = \sum_{n=0}^{\infty} a_{2n+1}t^{2n+1} \quad f\text{ -odd.}$$

For example, the power series representations of $\cos t$ and $\cosh t$ reflect that they are even, while those of $\sin t$ and $\sinh t$ reflect that they are odd functions.

**Example 9.** Compute the first four nonzero terms in the power series representation of

$$\tanh t = \frac{\sinh t}{\cosh t}.$$

► **Solution.** Division of power series is generally complicated. To make things a little simpler, we observe that $\tanh t$ is an odd function. Thus, its power series expansion is of the form $\tanh t = \sum_{n=1}^{\infty} d_{2n+1}t^{2n+1}$ and satisfies $\sinh t = \cosh t \sum_{n=0}^{\infty} d_{2n+1}t^{2n+1}$. By Example 7, this means

$$\left(t + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots\right) = \left(1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots\right)\left(d_1t + d_3t^3 + d_5t^5 + \cdots\right)$$

$$= \left(d_1t + \left(d_3 + \frac{d_1}{2!}\right)t^3 + \left(d_5 + \frac{d_3}{2!} + \frac{d_1}{4!}\right)t^5 + \cdots\right).$$

We now equate coefficients to get the following sequence of equations:

$$\begin{aligned} d_1 &= 1 \\ d_3 + \frac{d_1}{2!} &= \frac{1}{3!} \\ d_5 + \frac{d_3}{2!} + \frac{d_1}{4!} &= \frac{1}{5!} \\ d_7 + \frac{d_5}{2!} + \frac{d_3}{4!} + \frac{d_1}{6!} &= \frac{1}{7!} \\ &\vdots \end{aligned}$$

Recursively solving these equations gives $d_1 = 1, d_3 = \frac{-1}{3}, d_5 = \frac{2}{15}$, and $d_7 = \frac{-17}{315}$. The first four nonzero terms in the power series expansion for $\tanh t$ is thus

$$\tanh t = t - \frac{1}{3}t^3 + \frac{2}{15}t^5 - \frac{17}{315}t^7 + \cdots . \qquad \blacktriangleleft$$

## *Identifying Power Series*

Given a power series, with positive radius of convergence, it is sometimes possible to identify it with a known function. When we can do this, we will say that it is written in ***closed form***. Usually, such identifications come by using a combination of differentiation, integration, or the algebraic properties of power series discussed above. Consider the following examples.

**Example 10.** Write the power series

$$\sum_{n=0}^{\infty} \frac{t^{2n+1}}{n!}$$

in closed form.

► **Solution.** Observe that we can factor out $t$ and associate the term $t^2$ to get

$$\sum_{n=0}^{\infty} \frac{t^{2n+1}}{n!} = t \sum_{n=0}^{\infty} \frac{(t^2)^n}{n!} = t e^{t^2},$$

from (2). ◄

**Example 11.** Write the power series

$$\sum_{n=1}^{\infty} n(-1)^n t^{2n}$$

in closed form.

► **Solution.** Let $z(t) = \sum_{n=1}^{\infty} n(-1)^n t^{2n}$. Then dividing both sides by $t$ gives

$$\frac{z(t)}{t} = \sum_{n=1}^{\infty} n(-1)^n t^{2n-1}.$$

Integration will now simplify the sum:

$$\begin{aligned}
\int \frac{z(t)}{t}\,\mathrm{d}t &= \sum_{n=1}^{\infty} \int n(-1)^n t^{2n-1}\,\mathrm{d}t \\
&= \sum_{n=1}^{\infty} (-1)^n \frac{t^{2n}}{2} \\
&= \frac{1}{2}\sum_{n=1}^{\infty} (-t^2)^n \\
&= \frac{1}{2}\left(\frac{1}{1+t^2} - 1\right) + c,
\end{aligned}$$

where the last line is obtained from the geometric series by adding and subtracting the $n = 0$ term and $c$ is a constant of integration. We now differentiate this equation to get

$$\begin{aligned}
\frac{z(t)}{t} &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{1+t^2} - 1\right) \\
&= \frac{-t}{(1+t^2)^2}.
\end{aligned}$$

It follows now that

$$z(t) = \frac{-t^2}{(1+t^2)^2}.$$

It is straightforward to check that the radius of convergence is 1 so we get the equality

$$\frac{-t^2}{(1+t^2)^2} = \sum_{n=1}^{\infty} n(-1)^n t^{2n},$$

on the interval $(-1, 1)$. ◄

## *Taylor Series*

Suppose $f(t) = \sum_{n=0}^{\infty} c_n (t - t_0)^n$, with positive radius of converge. Theorem 4 implies that the derivatives, $f^{(n)}$, exist for all $n = 0, 1, \ldots$. Furthermore, it is easy to check that $f^{(n)}(t_0) = n!c_n$ and thus $c_n = \frac{f^{(n)}(t_0)}{n!}$. Therefore, if $f$ is represented by a power series, then it must be that

$$f(t) = \sum_{n=0}^{\infty} \frac{f^{(n)}(t_0)}{n!}(t - t_0)^n. \tag{13}$$

This series is called the ***Taylor series of $f$ centered at $t_0$***.

Now let us suppose that $f$ is a function on some domain $D$ and we wish to find a power series representation centered at $t_0 \in D$. By what we have just argued, $f$ will have to be given by its Taylor Series, which, of course, means that all higher order derivatives of $f$ at $t_0$ must exist. However, it can happen that the Taylor series may not converge to $f$ on any interval containing $t_0$ (see Exercises 28–29 where such an example is considered). When (13) is valid on an open interval containing $t_0$, we call $f$ ***analytic at $t_0$***. The properties of power series listed above shows that the sum, difference, scalar multiple, and product of analytic functions is again analytic. The quotient of analytic functions is likewise analytic at points where the denominator is not zero. Derivatives and integrals of analytic functions are again analytic.

**Example 12.** Verify that the Taylor series of $\sin t$ centered at 0 is that given in (4).

► **Solution.** The first four derivatives of $\sin t$ and their values at 0 are as follows:

| order $n$ | $\sin^{(n)}(t)$ | $\sin^{(n)}(0)$ |
|---|---|---|
| $n = 0$ | $\sin t$ | 0 |
| $n = 1$ | $\cos t$ | 1 |
| $n = 2$ | $-\sin t$ | 0 |
| $n = 3$ | $-\cos t$ | $-1$ |
| $n = 4$ | $\sin t$ | 0 |

The sequence $0, 1, 0, -1$ thereafter repeats. Hence, the Taylor series of $\sin t$ is

$$0 + 1t + 0\frac{t^2}{2!} - \frac{t^3}{3!} + 0\frac{t^4}{4!} + 1\frac{t^5}{5!} + \cdots = \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!},$$

as in (4). ◄

Given a function $f$, it can sometimes be difficult to compute the Taylor series by computing $f^{(n)}(t_0)$ for all $n = 0, 1, \ldots$. For example, compute the first few derivatives of $\tanh t$, considered in Example 9, to see how complicated the derivatives become. Additionally, determining whether the Taylor series converges to $f$ requires some additional information, for example, the Taylor remainder theorem. We will not include this in our review. Rather we will stick to examples where we derive new power series representations from existing ones as we did in Examples 5–9.

## *Rational Functions*

A ***rational function*** is the quotient of two polynomials and is analytic at all points where the denominator is nonzero. Rational functions will arise in many examples. It will be convenient to know what the radius of convergence about a point $t_0$. The following theorem allows us to determine this without going through the work of determining the power series. The proof is beyond the scope of this text.

**Theorem 13.** *Suppose $\frac{p(t)}{q(t)}$ is a quotient of two polynomials $p$ and $q$. Suppose $q(t_0) \neq 0$. Then the power series expansion for $\frac{p}{q}$ about $t_0$ has radius of convergence equal to the closest distance from $t_0$ to the roots (including complex roots) of $q$.*

**Example 14.** Find the radius of convergence for each rational function about the given point.

1. $\frac{t}{4-t}$ about $t_0 = 1$
2. $\frac{1-t}{9-t^2}$ about $t_0 = 2$
3. $\frac{t^3}{t^2+1}$ about $t_0 = 2$

► **Solution.**

1. The only root of $4 - t$ is 4. Its distance to $t_0 = 1$ is 3. The radius of convergence is 3.
2. The roots of $9-t^2$ are 3 and $-3$. Their distances to $t_0 = 2$ is 1 and 5, respectively. The radius of convergence is 1.
3. The roots of $t^2 + 1$ are i and –i. Their distances to $t_0 = 2$ are $|2 - i| = \sqrt{5}$ and $|2 + i| = \sqrt{5}$. The radius of convergence is $\sqrt{5}$. ◄

## Exercises

1–9. Compute the radius of convergence for the given power series.

1. $\sum_{n=0}^{\infty} n^2(t-2)^n$

2. $\sum_{n=1}^{\infty} \frac{t^n}{n}$

3. $\sum_{n=0}^{\infty} \frac{(t-1)^n}{2^n n!}$

4. $\sum_{n=0}^{\infty} \frac{3^n(t-3)^n}{n+1}$

5. $\sum_{n=0}^{\infty} n!t^n$

6. $\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n+1)!}$

7. $\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n)!}$

8. $\sum_{n=0}^{\infty} \frac{n^n t^n}{n!}$

9. $\sum_{n=0}^{\infty} \frac{n!t^n}{1 \cdot 3 \cdot 5 \cdots (2n+1)}$

10–16. Find the Taylor series for each function with center $t_0 = 0$.

10. $\frac{1}{1+t^2}$

11. $\frac{1}{t-a}$

12. $e^{at}$

13. $\frac{\sin t}{t}$

14. $\frac{e^t - 1}{t}$

15. $\tan^{-1} t$

16. $\ln(1+t^2)$

17–20. Find the first four nonzero terms in the Taylor series with center 0 for each function.

17. $\tan t$

18. $\sec t$

19. $e^t \sin t$

20. $e^t \cos t$

21–25. Find a closed form expression for each power series.

21. $\sum_{n=0}^{\infty} (-1)^n \frac{n+1}{n!} t^n$

22. $\sum_{n=0}^{\infty} \frac{3^n-2}{n!} t^n$

23. $\sum_{n=0}^{\infty} (n+1) t^n$

24. $\sum_{n=0}^{\infty} \frac{t^{2n+1}}{2n-1}$

25. $\sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)(2n-1)}$

26. Redo Exercises 19 and 20 in the following way. Recall Euler's formula $e^{it} = \cos t + i \sin t$ and write $e^t \cos t$ and $e^t \sin t$ as the real and imaginary parts of $e^t e^{it} = e^{(1+i)t}$ expanded as a power series.
27. Use the power series (with center 0) for the exponential function and expand both sides of the equation $e^{at} e^{bt} = e^{(a+b)t}$. What well-known formula arises when the coefficients of $\frac{t^n}{n!}$ are equated?

28–29. A test similar to the ratio test is the root test.

*The Root Test for Power Series. Let* $\sum_{n=0}^{\infty} c_n (t - t_0)^n$ *be a given power series and suppose* $L = \lim_{n\to\infty} \sqrt[n]{|c_n|}$. *Define* $R$ *in the following way:*

$$\begin{aligned} R &= \tfrac{1}{L} && \text{if} \quad 0 < L < \infty, \\ R &= 0 && \text{if} \quad L = \infty, \\ R &= \infty && \text{if} \quad L = 0. \end{aligned}$$

*Then*

i. The power series converges only at $t = t_0$ if $R = 0$.

ii. The power series converges for all $t \in \mathbb{R}$ if $R = \infty$.

iii. The power series converges if $|t - t_0| < R$ and diverges if $|t - t_0| > R$.

28. Use the root test to determine the radius of convergence of $\sum_{n=0}^{\infty} \frac{t^n}{n^n}$.
29. Let $c_n = 1$ if $n$ is odd and $c_n = 2$ if $n$ is even. Consider the power series $\sum_{n=0}^{\infty} c_n t^n$. Show that the ratio test does not apply. Use the root test to determine the radius of convergence.

30–34. In this sequence of exercises, we consider a function that is infinitely differentiable but not analytic. Let

$$f(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ e^{\frac{-1}{t}} & \text{if } t > 0 \end{cases}.$$

30. Compute $f'(t)$ and $f''(t)$ and observe that $f^{(n)}(t) = e^{\frac{-1}{t}} p_n(\frac{1}{t})$ where $p_n$ is a polynomial, $n = 1, 2$. Find $p_1$ and $p_2$.
31. Use mathematical induction to show that $f^{(n)}(t) = e^{\frac{-1}{t}} p_n(\frac{1}{t})$ where $p_n$ is a polynomial.
32. Show that $\lim_{t \to 0^+} f^{(n)}(t) = 0$. To do this, let $u = \frac{1}{t}$ in $f^{(n)}(t) = e^{\frac{-1}{t}} p_n(\frac{1}{t})$ and let $u \to \infty$. Apply L'Hospital's rule.
33. Show that $f^n(0) = 0$ for all $n = 0, 1, \ldots$.
34. Conclude that $f$ is not analytic at $t = 0$ though all derivatives at $t = 0$ exist.

## 7.2 Power Series Solutions About an Ordinary Point

A point $t_0$ is called an ***ordinary point*** of $\boldsymbol{L}y = 0$ if we can write the differential equation in the form

$$y'' + a_1(t)y' + a_0(t)y = 0, \tag{1}$$

where $a_0(t)$ and $a_1(t)$ are analytic at $t_0$. If $t_0$ is not an ordinary point, we call it a ***singular point***.

**Example 1.** Determine the ordinary and singular points for each of the following differential equations:

1. $y'' + \frac{1}{t^2-9}y' + \frac{1}{t+1}y = 0$.
2. $(1-t^2)y'' - 2ty' + n(n+1)y = 0$, where $n$ is an integer.
3. $ty'' + (\sin t)y' + (e^t - 1)y = 0$.

► **Solution.**
1. Here $a_1(t) = \frac{1}{t^2-9}$ is analytic except at $t = \pm 3$. The function $a_0 = \frac{1}{t+1}$ is analytic except at $t = -1$. Thus, the singular points are $-3$, 3, and $-1$. All other points are ordinary.
2. This is Legendre's equation. In standard form, we find $a_1(t) = \frac{-2t}{1-t^2}$ and $a_0(t) = \frac{n(n+1)}{1-t^2}$. They are analytic except at 1 and $-1$. These are the singular points and all other points are ordinary.
3. In standard form, $a_1(t) = \frac{\sin t}{t}$ and $a_0(t) = \frac{e^t-1}{t}$. Both of these are analytic everywhere. (See Exercises 13 and 14 of Sect. 7.1.) It follows that all points are ordinary. ◄

In this section, we restrict our attention to ordinary points. Their importance is underscored by the following theorem. It tells us that there is always a power series solution about ordinary points.

**Theorem 2.** *Suppose $a_0(t)$ and $a_1(t)$ are analytic at $t_0$, both of which converge for $|t - t_0| < R$. Then there is a unique solution $y(t)$, analytic at $t_0$, to the initial value problem*

$$y'' + a_1(t)y' + a_0(t)y = 0, \quad y(t_0) = \alpha,\ y'(t_0) = \beta. \tag{2}$$

*If*

$$y(t) = \sum_{n=0}^{\infty} c_n(t-t_0)^n$$

*then $c_0 = \alpha$, $c_1 = \beta$, and all other $c_k$, $k = 2, 3, \ldots$, are determined by $c_0$ and $c_1$. Furthermore, the power series for $y$ converges for $|t - t_0| < R$.*

Of course the uniqueness and existence theorem, Theorem 6 of Sect. 5.1, implies there is a unique solution. What is new here is that the solution is analytic at $t_0$. Since the solution is necessarily unique, it is not at all surprising that the coefficients are determined by the initial conditions. The only hard part about the proof, which we omit, is showing that the solution converges for $|t - t_0| < R$. Let $y_1$ be the solution with initial conditions $y(t_0) = 1$ and $y'(t_0) = 0$ and $y_2$ the solution with initial condition $y(t_0) = 0$ and $y'(t_0) = 1$. Then it is easy to see that $y_1$ and $y_2$ are independent solutions, and hence, all solutions are of the form $c_1 y_1 + c_2 y_2$. (See Theorem 2 of Sect. 5.2) The ***power series method*** refers to the use of this theorem by substituting $y(t) = \sum_{n=0}^{\infty} c_n (t - t_0)^n$ into (2) and determining the coefficients.

We illustrate the use of Theorem 2 with a few examples. Let us begin with a familiar constant coefficient differential equation.

**Example 3.** Use the power series method to solve

$$y'' + y = 0.$$

► **Solution.** Of course, this is a constant coefficient differential equation. Since $q(s) = s^2 + 1$ and $\mathcal{B}_q = \{\cos t, \sin t\}$, we get solution $y(t) = c_1 \sin t + c_2 \cos t$. Let us see how the power series method gives the same answer. Since the coefficients are constant, they are analytic everywhere with infinite radius of convergence. Theorem 2 implies that the power series solutions converge everywhere. Let $y(t) = \sum_{n=0}^{\infty} c_n t^n$ be a power series about $t_0 = 0$. Then

$$y'(t) = \sum_{n=1}^{\infty} c_n n t^{n-1}$$

$$\text{and} \qquad y''(t) = \sum_{n=2}^{\infty} c_n n(n-1) t^{n-2}.$$

An index shift, $n \to n + 2$, gives $y''(t) = \sum_{n=0}^{\infty} c_{n+2}(n + 2)(n + 1)t^n$. Therefore, the equation $y'' + y = 0$ gives

$$\sum_{n=0}^{\infty} (c_n + c_{n+2}(n + 2)(n + 1))t^n = 0,$$

which implies $c_n + c_{n+2}(n + 2)(n + 1) = 0$, or, equivalently,

$$c_{n+2} = \frac{-c_n}{(n + 2)(n + 1)} \qquad \text{for all } n = 0, 1, \ldots. \tag{3}$$

Equation (3) is an example of a ***recurrence relation***: terms of the sequence are determined by earlier terms. Since the difference in indices between $c_n$ and $c_{n+2}$ is 2, it follows that even terms are determined by previous even terms and odd terms are determined by previous odd terms. Let us consider these two cases separately.

*The Even Case*

$$\begin{array}{ll}
n=0 & c_2 = \frac{-c_0}{2\cdot 1} \\
n=2 & c_4 = \frac{-c_2}{4\cdot 3} = \frac{c_0}{4\cdot 3\cdot 2\cdot 1} = \frac{c_0}{4!} \\
n=4 & c_6 = \frac{-c_4}{6\cdot 5} = \frac{-c_0}{6!} \\
n=6 & c_8 = \frac{-c_6}{8\cdot 7} = \frac{c_0}{8!} \\
\vdots & \vdots
\end{array}$$

More generally, we can see that

$$c_{2n} = (-1)^n \frac{c_0}{(2n)!}.$$

*The Odd Case*

$$\begin{array}{ll}
n=1 & c_3 = \frac{-c_1}{3\cdot 2} \\
n=3 & c_5 = \frac{-c_3}{5\cdot 4} = \frac{c_1}{5\cdot 4\cdot 3\cdot 2} = \frac{c_1}{5!} \\
n=5 & c_7 = \frac{-c_5}{7\cdot 6} = \frac{-c_1}{7!} \\
n=7 & c_9 = \frac{-c_7}{9\cdot 8} = \frac{c_1}{9!} \\
\vdots & \vdots
\end{array}$$

Similarly, we see that

$$c_{2n+1} = (-1)^n \frac{c_1}{(2n+1)!}.$$

Now, as we mentioned in Sect. 7.1, we can change the order of absolutely convergent sequences without affecting the sum. Thus, let us rewrite $y(t) = \sum_{n=0}^{\infty} c_n t^n$ in terms of odd and even indices to get

$$\begin{aligned}
y(t) &= \sum_{n=0}^{\infty} c_{2n} t^{2n} + \sum_{n=0}^{\infty} c_{2n+1} t^{2n+1} \\
&= c_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} t^{2n} + c_1 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} t^{2n+1} \\
&= c_0 \cos t + c_1 \sin t,
\end{aligned}$$

where the first power series in the second line is that of $\cos t$ and the second power series is that of $\sin t$ (See (3) and (4) of Sect. 7.1). ◀

**Example 4.** Use the power series method with center $t_0 = 0$ to solve

$$(1+t^2)y'' + 4ty' + 2y = 0.$$

What is a lower bound on the radius of convergence?

► **Solution.** We write the given equation in standard form to get

$$y'' + \frac{4t}{1+t^2} y' + \frac{2}{1+t^2} y = 0.$$

Since the coefficient functions $a_1(t) = \frac{4t}{1+t^2}$ and $a_2(t) = \frac{2}{1+t^2}$ are rational functions with nonzero denominators, they are analytic at all points. By Theorem 13 of Sect. 7.1, it is not hard to see that they have power series expansions about $t_0 = 0$ with radius of convergence 1. By Theorem 2, the radius of convergence for a solution, $y(t) = \sum_{n=0}^{\infty} c_n t^n$ is at least 1. To determine the coefficients, it is easier to substitute $y(t)$ directly into $(1+t^2)y'' + 4ty' + 2y = 0$ instead of its equivalent standard form. The details were worked out in Example 8 of Sect. 7.1. We thus obtain

$$(1+t^2)y'' + 4ty' + 2y = \sum_{n=0}^{\infty} ((c_{n+2} + c_n)(n+2)(n+1))\, t^n = 0.$$

From this equation, we get $c_{n+2} + c_n = 0$ for all $n = 0, 1, \cdots$. Again we consider even and odd cases.

| *The Even Case* | | *The Odd Case* | |
|---|---|---|---|
| $n = 0$ | $c_2 = -c_0$ | $n = 1$ | $c_3 = -c_1$ |
| $n = 2$ | $c_4 = -c_2 = c_0$ | $n = 3$ | $c_5 = -c_3 = c_1$ |
| $n = 4$ | $c_6 = -c_0$ | $n = 5$ | $c_7 = -c_1$ |

More generally, we can see that

$$c_{2n} = (-1)^n c_0.$$

Similarly, we see that

$$c_{2n+1} = (-1)^n c_1.$$

It follows now that

$$y(t) = c_0 \sum_{n=0}^{\infty} (-1)^n t^{2n} + c_1 \sum_{n=0}^{\infty} (-1)^n t^{2n+1}.$$

As we observed earlier, each of these series has radius of convergence at least 1. In fact, the radius of convergence of each is 1. ◄

A couple of observations are in order for this example. First, we can relate the power series solutions to the geometric series, (5) of Sect. 7.1, and write them in closed form. Thus,

$$\sum_{n=0}^{\infty} (-1)^n t^{2n} = \sum_{n=0}^{\infty} (-t^2)^n = \frac{1}{1+t^2},$$

$$\sum_{n=0}^{\infty} (-1)^n t^{2n+1} = t \sum_{n=0}^{\infty} (-t^2)^n = \frac{t}{1+t^2}.$$

It follows now that the general solution is $y(t) = c_0 \frac{1}{1+t^2} + c_1 \frac{t}{1+t^2}$. Second, since $a_1(t)$ and $a_0(t)$ are continuous on $\mathbb{R}$, the uniqueness and existence theorem, Theorem 6 of Sect. 5.1, guarantees the existence of solutions defined on all of $\mathbb{R}$. It is easy to check that these closed forms, $\frac{1}{1+t^2}$ and $\frac{t}{1+t^2}$, are defined on all of $\mathbb{R}$ and satisfy the given differential equation.

This example illustrates that there is some give and take between the uniqueness and existence theorem, Theorem 6 of Sect. 5.1, and Theorem 2 above. On the one hand, Theorem 6 of Sect. 5.1 may guarantee a solution, but it may be difficult or impossible to find without the power series method. The power series method, Theorem 2, on the other hand, may only find a series solution on the interval of convergence, which may be quite smaller than that guaranteed by Theorem 6 of Sect. 5.1. Further analysis of the power series may reveal a closed form solution valid on a larger interval as in the example above. However, it is not always possible to do this. Indeed, some recurrence relations can be difficult to solve and we must be satisfied with writing out only a finite number of terms in the power series solution.

**Example 5.** Discuss the radius of convergence of the power series solution about $t_0 = 0$ to

$$(1-t)y'' + y = 0.$$

Write out the first five terms given the initial conditions

$$y(0) = 1 \quad \text{and} \quad y'(0) = 0.$$

► **Solution.** In standard form, the differential equation is

$$y'' + \frac{1}{1-t}y = 0.$$

Thus, $a_1(t) = 0$, $a_0(t) = \frac{1}{1-t}$, and $t_0 = 0$ is an ordinary point. Since $a_0(t)$ is represented by the geometric series, which has radius of convergence 1, it follows that any solution will have radius of convergence at least 1. Let $y(t) = \sum_{n=0}^{\infty} c_n t^n$. Then $y''$ and $-ty''$ are given by

$$\begin{aligned} y''(t) &= \sum_{n=0}^{\infty} c_{n+2}(n+2)(n+1)t^n, \\ -ty''(t) &= \sum_{n=2}^{\infty} -c_n n(n-1)t^{n-1}, \\ &= \sum_{n=0}^{\infty} -c_{n+1}(n+1)nt^n. \end{aligned}$$

It follows that

$$(1-t)y'' + y = \sum_{n=0}^{\infty} (c_{n+2}(n+2)(n+1) - c_{n+1}(n+1)n + c_n)t^n,$$

which leads to the recurrence relations

$$c_{n+2}(n+2)(n+1) - c_{n+1}(n+1)n + c_n = 0,$$

for all $n = 0, 1, 2, \ldots$. This recurrence relation is not easy to solve generally. We can, however, compute any finite number of terms. First, we solve for $c_{n+2}$:

$$c_{n+2} = c_{n+1}\frac{n}{n+2} - c_n\frac{1}{(n+2)(n+1)}. \tag{4}$$

The initial conditions $y(0) = 1$ and $y'(0) = 0$ imply that $c_0 = 1$ and $c_1 = 0$. Recursively applying (4), we get

$$\begin{array}{ll} n = 0 & c_2 = c_1 \cdot 0 - c_0\frac{1}{2} = -\frac{1}{2}, \\ n = 1 & c_3 = c_2\frac{1}{3} - c_1\frac{1}{6} = -\frac{1}{6}, \\ n = 2 & c_4 = c_3\frac{1}{2} - c_2\frac{1}{12} = -\frac{1}{24}, \\ n = 3 & c_5 = c_4\frac{3}{5} - c_3\frac{1}{20} = -\frac{1}{60}. \end{array}$$

It now follows that the first five terms of $y(t)$ is

$$y(t) = 1 - \frac{1}{2}t^2 - \frac{1}{6}t^3 - \frac{1}{24}t^4 - \frac{1}{60}t^5. \qquad \blacktriangleleft$$

In general, it may not be possible to find a closed form description of $c_n$. Nevertheless, we can use the recurrence relation to find as many terms as we desire. Although this may be tedious, it may suffice to give an approximate solution to a given differential equation.

We note that the examples we gave are power series solutions about $t_0 = 0$. We can always reduce to this case by a substitution. To illustrate, consider the differential equation

$$ty'' - (t-1)y' - ty = 0. \tag{5}$$

It has $t_0 = 1$ as an ordinary point. Suppose we wish to derive a power series solution about $t_0 = 1$. Let $y(t)$ be a solution and let $Y(x) = y(x+1)$. Then $Y'(x) = y'(x+1)$ and $Y''(x) = y''(x+1)$. In the variable $x$, (5) becomes $(x+1)Y''(x) - xY'(x) - (x+1)Y(x) = 0$ and $x_0 = 0$ is an ordinary point. We solve $Y(x) = \sum_{n=0}^{\infty} c_n x^n$ as before. Now let $x = t - 1$. That is, $y(t) = Y(t-1) = \sum_{n=0}^{\infty} c_n (t-1)^n$ is the series solution to (5) about $t_0 = 1$.

## *Chebyshev Polynomials*

We conclude this section with the following two related problems: For a nonnegative integer $n$, expand $\cos nx$ and $\sin nx$ in terms of just $\cos x$ and $\sin x$. It is an easy exercise (see Exercises 12 and 13) to show that we can write $\cos nx$ as a polynomial in $\cos x$ and we can write $\sin nx$ as a product of $\sin x$ and a polynomial in $\cos x$. More specifically, we will find polynomials $T_n$ and $U_n$ such that

$$\begin{aligned}\cos nx &= T_n(\cos x),\\ \sin(n+1)x &= \sin x\, U_n(\cos x).\end{aligned} \tag{6}$$

(The shift by 1 in the formula defining $U_n$ is intentional.) The polynomials $T_n$ and $U_n$ are called the ***Chebyshev polynomials*** of the first and second kind, respectively. They each have degree $n$. For example, if $n = 2$, we have $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1$. Thus, $T_2(t) = 2t^2 - 1$; if $t = \cos x$, we have

$$\cos 2x = T_2(\cos x).$$

Similarly, $\sin 2x = 2\sin x \cos x$. Thus, $U_1(t) = 2t$ and

$$\sin 2x = \sin x U_1(\cos x).$$

More generally, we can use the trigonometric summation formulas

$$\begin{aligned}\sin(x+y) &= \sin x \cos y + \cos x \sin y,\\ \cos(x+y) &= \cos x \cos y - \sin x \sin y,\end{aligned}$$

and the basic identity $\sin^2 x + \cos^2 x = 1$ to expand

$$\cos nx = \cos((n-1)x + x) = \cos((n-1)x)\sin x - \sin((n-1)x)\sin x.$$

Now expand $\cos((n-1)x)$ and $\sin((n-1)x)$ and continue inductively to the point where all occurrences of $\cos kx$ and $\sin kx$, $k > 1$, are removed. Whenever $\sin^2 x$ occurs, replace it by $1 - \cos^2 x$. In the table below, we have done just that for some small values of $n$. We include in the table the resulting Chebyshev polynomials of the first kind, $T_n$.

| $n$ | $\cos nx$ | $T_n(t)$ |
|---|---|---|
| 0 | $\cos 0x = 1$ | $T_0(t) = 1$ |
| 1 | $\cos 1x = \cos x$ | $T_1(t) = t$ |
| 2 | $\cos 2x = 2\cos^2 x - 1$ | $T_2(t) = 2t^2 - 1$ |
| 3 | $\cos 3x = 4\cos^3 x - 3\cos x$ | $T_3(t) = 4t^3 - 3t$ |
| 4 | $\cos 4x = 8\cos^4 x - 8\cos^2 x + 1$ | $T_4(t) = 8t^4 - 8t^2 + 1$ |

In a similar way, we expand $\sin(n+1)x$. The following table gives the Chebyshev polynomials of the second kind, $U_n$, for some small values of $n$.

| $n$ | $\sin(n+1)x$ | $U_n(t)$ |
|---|---|---|
| 0 | $\sin 1x = \sin x$ | $U_0(t) = 1$ |
| 1 | $\sin 2x = \sin x(2\cos x)$ | $U_1(t) = 2t$ |
| 2 | $\sin 3x = \sin x(4\cos^2 x - 1)$ | $U_2(t) = 4t^2 - 1$ |
| 3 | $\sin 4x = \sin x(8\cos^3 x - 4\cos x)$ | $U_3(t) = 8t^3 - 4t$ |
| 4 | $\sin 5x = \sin x(16\cos^4 x - 12\cos^2 x + 1)$ | $U_4(t) = 16t^4 - 12t^2 + 1$ |

The method we used for computing the tables is not very efficient. We will use the interplay between the defining equations, (6), to derive second order differential equations that will determine $T_n$ and $U_n$. This theme of using the interplay between two related families of functions will come up again in Sect. 7.4.

Let us begin by differentiating the equations that define $T_n$ and $U_n$ in (6). For the first equation, we get

$$\begin{aligned} \text{LHS:} \quad & \frac{\mathrm{d}}{\mathrm{d}x}\cos nx = -n\sin nx = -n\sin x U_{n-1}(\cos x), \\ \text{RHS:} \quad & \frac{\mathrm{d}}{\mathrm{d}x}T_n(\cos x) = T_n'(\cos x)(-\sin x). \end{aligned}$$

Equating these results, simplifying, and substituting $t = \cos x$ gives

$$T_n'(t) = nU_{n-1}(t). \tag{7}$$

For the second equation, we get

$$\begin{aligned} \text{LHS:} \quad & \frac{\mathrm{d}}{\mathrm{d}x}\sin(n+1)x = (n+1)\cos(n+1)x = (n+1)T_{n+1}(\cos x), \\ \text{RHS:} \quad & \frac{\mathrm{d}}{\mathrm{d}x}\sin x U_n(\cos x) = \cos x U_n(\cos x) + \sin x U_n'(\cos x)(-\sin x) \\ & \qquad = \cos x U_n(\cos x) - (1-\cos^2 x)U_n'(\cos x). \end{aligned}$$

It now follows that $(n+1)T_{n+1}(t) = tU_n(t) - (1-t^2)U_n'(t)$. Replacing $n$ by $n-1$ gives

$$nT_n(t) = tU_{n-1}(t) - (1-t^2)U_{n-1}'(t). \tag{8}$$

We now substitute (7) and its derivative $T_n''(t) = nU_{n-1}'(t)$ into (8). After simplifying, we get that $T_n$ satisfies

$$(1-t^2)T_n''(t) - tT_n' + n^2T_n(t) = 0. \tag{9}$$

By substituting (7) into the derivative of (8) and simplifying we get that $U_n$ satisfies

$$(1-t^2)U_n''(t) - 3tU_n' + n(n+2)U_n(t) = 0. \tag{10}$$

The differential equations

$$(1-t^2)y''(t) - ty' + \alpha^2 y(t) = 0 \tag{11}$$

$$(1-t^2)y''(t) - 3ty' + \alpha(\alpha+2)y(t) = 0 \tag{12}$$

are known as ***Chebyshev's differential equations***. Each have $t_0 = \pm 1$ as singular points and $t_0 = 0$ is an ordinary point. The Chebyshev polynomial $T_n$ is a polynomial solution to (11) and $U_n$ is a polynomial solution to (12), when $\alpha = n$.

**Theorem 6.** *We have the following explicit formulas for $T_n$ and $U_n$;*

$$T_{2n}(t) = n(-1)^n \sum_{k=0}^{n} (-1)^k \frac{(n+k-1)!}{(n-k)!} \frac{(2t)^{2k}}{(2k)!},$$

$$T_{2n+1}(t) = \frac{2n+1}{2}(-1)^n \sum_{k=0}^{n} (-1)^k \frac{(n+k)!}{(n-k)!} \frac{(2t)^{2k+1}}{(2k+1)!},$$

$$U_{2n}(t) = (-1)^n \sum_{k=0}^{n} (-1)^k \frac{(n+k)!}{(n-k)!} \frac{(2t)^{2k}}{(2k)!},$$

$$U_{2n+1}(t) = (-1)^n \sum_{k=0}^{n} (-1)^k \frac{(n+k+1)!}{(n-k)!} \frac{(2t)^{2k+1}}{(2k+1)!}.$$

*Proof.* Let us first consider Chebyshev's first differential equation, for general $\alpha$. Let $y(t) = \sum_{k=0}^{\infty} c_k t^k$. Substituting $y(t)$ into (11), we get the following relation for the coefficients:

$$c_{k+2} = \frac{-(\alpha^2 - k^2)c_k}{(k+2)(k+1)}.$$

Let us consider the even and odd cases.

**The Even Case**

$$
\begin{aligned}
c_2 &= -\frac{\alpha^2 - 0^2}{2 \cdot 1} c_0 \\
c_4 &= -\frac{\alpha^2 - 2^2}{4 \cdot 3} c_2 = \frac{(\alpha^2 - 2^2)(\alpha^2 - 0^2)}{4!} c_0 \\
c_6 &= -\frac{\alpha^2 - 4^2}{6 \cdot 5} c_4 = -\frac{(\alpha^2 - 4^2)(\alpha^2 - 2^2)(\alpha^2 - 0^2)}{6!} c_0 \\
&\vdots
\end{aligned}
$$

More generally, we can see that

$$c_{2k} = (-1)^k \frac{(\alpha^2 - (2k-2)^2) \cdots (\alpha^2 - 0^2)}{(2k)!} c_0.$$

By factoring each expression $\alpha^2 - (2j)^2$ that appears in the numerator into $2^2 \left(\frac{\alpha}{2} + j\right) \left(\frac{\alpha}{2} - j\right)$ and rearranging factors, we can write

$$c_{2k} = (-1)^k \frac{\alpha}{2} \frac{\left(\frac{\alpha}{2} + k - 1\right) \cdots \left(\frac{\alpha}{2} - 1\right) \left(\frac{\alpha}{2}\right) \left(\frac{\alpha}{2} + 1\right) \cdots \left(\frac{\alpha}{2} - k + 1\right)}{(2k)!} 2^{2k} c_0.$$

**The Odd Case**

$$
\begin{aligned}
c_3 &= -\frac{(\alpha^2 - 1^2)}{3 \cdot 2} c_1 \\
c_5 &= -\frac{(\alpha^2 - 3^2)}{5 \cdot 4} c_3 = \frac{(\alpha^2 - 3^2)(\alpha^2 - 1^2)}{5!} c_1 \\
c_7 &= -\frac{(\alpha^2 - 5^2)}{7 \cdot 6} c_5 = -\frac{(\alpha^2 - 5^2)(\alpha^2 - 3^2)(\alpha^2 - 1^2)}{7!} c_1 \\
&\vdots
\end{aligned}
$$

Similarly, we see that

$$c_{2k+1} = (-1)^k \frac{(\alpha^2 - (2k-1)^2) \cdots (\alpha^2 - 1^2)}{(2k+1)!} c_1.$$

By factoring each expression $\alpha^2 - (2j-1)^2$ that appears in the numerator into $2^2 \left(\frac{\alpha-1}{2} + j\right) \left(\frac{\alpha-1}{2} - (j-1)\right)$ and rearranging factors, we get

$$c_{2k+1} = (-1)^k \frac{\left(\frac{\alpha-1}{2} + k\right) \cdots \left(\frac{\alpha-1}{2} - (k-1)\right)}{(2k+1)!} 2^{2k} c_1.$$

Let

$$y_0 = \frac{\alpha}{2} \sum_{k=0}^{\infty} (-1)^k \frac{\left(\frac{\alpha}{2} + k - 1\right) \cdots \left(\frac{\alpha}{2} - k + 1\right)}{(2k)!} (2t)^{2k}$$

and

$$y_1 = \frac{1}{2} \sum_{k=0}^{\infty} (-1)^k \frac{\left(\frac{\alpha-1}{2} + k\right) \cdots \left(\frac{\alpha-1}{2} - (k-1)\right)}{(2k+1)!} (2t)^{2k+1}.$$

Then the general solution to Chebyshev's first differential equation is

$$y = c_0 y_0 + c_1 y_1.$$

It is clear that neither $y_0$ nor $y_1$ is a polynomial if $\alpha$ is not an integer.

### The Case $\alpha = 2n$

In this case, $y_0$ is a polynomial while $y_1$ is not. In fact, for $k > n$, the numerator in the sum for $y_0$ is zero, and hence,

$$\begin{aligned} y_0(t) &= n \sum_{k=0}^{n} (-1)^k \frac{(n+k-1) \cdots (n-k+1)}{(2k)!} (2t)^{2k} \\ &= n \sum_{k=0}^{n} (-1)^k \frac{(n+k-1)!}{(n-k)!} \frac{(2t)^{2k}}{(2k)!}. \end{aligned}$$

It follows $T_{2n}(t) = c_0 y_0(t)$, where $c_0 = T_{2n}(0)$. To determine $T_{2n}(0)$, we evaluate the defining equation $T_{2n}(\cos x) = \cos 2nx$ at $x = \frac{\pi}{2}$ to get $T_{2n}(0) = \cos n\pi = (-1)^n$. The formula for $T_{2n}$ now follows.

### The Case $\alpha = 2n + 1$

In this case, $y_1$ is a polynomial while $y_0$ is not. Further,

$$\begin{aligned} y_1(t) &= \frac{1}{2} \sum_{k=0}^{n} (-1)^k \frac{(n+k) \cdots (n-k+1)}{(2k+1)!} (2t)^{2k+1} \\ &= \frac{1}{2} \sum_{k=0}^{n} (-1)^k \frac{(n+k)!}{(n-k)!} \frac{(2t)^{2k+1}}{(2k+1)!}. \end{aligned}$$

It now follows that $T_{2n+1}(t) = c_1 y_1(t)$. There is no constant coefficient term in $y_1$. However, $y_1'(0) = 1$ is the coefficient of $t$ in $y_1$. Differentiating the defining equation $T_{2n+1}(\cos x) = \cos((2n+1)x)$ at $x = \frac{\pi}{2}$ gives $T'_{2n+1}(0) = (2n+1)(-1)^n$. Let $c_1 = (2n+1)(-1)^n$. The formula for $T_{2n+1}$ now follows. The formulas for $U_n$ follow from (7) which can be written $U_n = \frac{1}{n+1} T'_{n+1}$. $\square$

## Exercises

1–4. Use the power series method about $t_0 = 0$ to solve the given differential equation. Identify the power series with known functions. Since each is constant coefficient, use the characteristic polynomial to solve and compare.

1. $y'' - y = 0$
2. $y'' - 2y' + y = 0$
3. $y'' + k^2 y = 0$, where $k \in \mathbb{R}$
4. $y'' - 3y' + 2y = 0$

5–10. Use the power series method about $t_0 = 0$ to solve each of the following differential equations. Write the solution in the form $y(t) = c_0 y_0(t) + c_1 y_1(t)$, where $y_0(0) = 1$, $y_0'(0) = 0$ and $y_1(0) = 0$, $y_1'(t) = 1$. Find $y_0$ and $y_1$ in closed form.

5. $(1 - t^2)y'' + 2y = 0 \quad -1 < t < 1$
6. $(1 - t^2)y'' - 2ty' + 2y = 0 \quad -1 < t < 1$
7. $(t - 1)y'' - ty' + y = 0$
8. $(1 + t^2)y'' - 2ty' + 2y = 0$
9. $(1 + t^2)y'' - 4ty' + 6y = 0$
10. $(1 - t^2)y'' - 6ty' - 4y = 0$

11–19. *Chebyshev Polynomials:*

11. Use Euler's formula to derive the following identity known as de Moivre's formula:
$$(\cos x + i \sin x)^n = \cos nx + i \sin nx,$$
for any integer $n$.
12. Assume $n$ is a nonnegative integer. Use the binomial theorem on de Moivre's formula to show that $\cos nx$ is a polynomial in $\cos x$ and that
$$T_n(t) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} t^{n-2k}(1 - t^2)^k.$$
13. Assume $n$ is a nonnegative integer. Use the binomial theorem on de Moivre's formula to show that $\sin(n + 1)x$ is a product of $\sin x$ and a polynomial in $\cos x$ and that
$$U_n(t) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n+1}{2k+1} t^{n-2k}(1 - t^2)^k.$$
14. Show that
$$(1 - t^2)U_n(t) = tT_{n+1}(t) - T_{n+2}(t).$$

15. Show that

$$U_{n+1}(t) = tU_n(t) + T_{n+1}(t).$$

16. Show that

$$T_{n+1}(t) = 2tT_n(t) - T_{n-1}(t).$$

17. Show that

$$U_{n+1}(t) = 2tU_n(t) - U_{n-1}(t).$$

18. Show that

$$T_n(t) = \frac{1}{2}\left(U_n(t) - U_{n-2}(t)\right).$$

19. Show that

$$U_n(t) = \frac{1}{2}\left(T_n(t) - T_{n+2}(t)\right).$$

## 7.3 Regular Singular Points and the Frobenius Method

Recall that the singular points of a differential equation

$$y'' + a_1(t)y' + a_0(t)y = 0 \tag{1}$$

are those points for which either $a_1(t)$ or $a_0(t)$ is not analytic. Generally, they are few in number but tend to be the most important and interesting. In this section, we will describe a modified power series method, called the ***Frobenius Method***, that can be applied to differential equations with certain kinds of singular points.

We say that the point $t_0$ is a ***regular singular point*** of (1) if

1. $t_0$ is a singular point.
2. $A_1(t) = (t - t_0)a_1(t)$ and $A_0(t) = (t - t_0)^2 a_0(t)$ are analytic at $t_0$.

Note that by multiplying $a_1(t)$ by $t - t_0$ and $a_0(t)$ by $(t - t_0)^2$, we "restore" the analyticity at $t_0$. In this sense, a regular singularity at $t_0$ is not too bad. A singular point that is not regular is called ***irregular***.

**Example 1.** Show $t_0 = 0$ is a regular singular point for the differential equation

$$t^2 y'' + t \sin t y' - 2(t + 1)y = 0. \tag{2}$$

► **Solution.** Let us rewrite (2) by dividing by $t^2$. We get

$$y'' + \frac{\sin t}{t} y' - \frac{2(t+1)}{t^2}.$$

While $a_1(t) = \frac{\sin t}{t}$ is analytic at 0 (see Exercise 13, of Sect. 7.1) the coefficient function $a_0(t) = \frac{-2(t+1)}{t^2}$ is not. However, both $t a_1(t) = t\frac{\sin t}{t} = \sin t$ and $t^2 a_0(t) = t^2 \frac{-2(t+1)}{t^2} = -2(1 + t)$ are analytic at $t_0 = 0$. It follows that $t_0 = 0$ is a regular singular point. ◄

In the case of a regular singular point, we will rewrite (1): multiply both sides by $(t - t_0)^2$ and note that

$$(t - t_0)^2 a_1(t) = (t - t_0)A_1(t) \quad \text{and} \quad (t - t_0)^2 a_0(t) = A_0(t).$$

We then get

$$(t - t_0)^2 y'' + (t - t_0)A_1(t)y' + A_0(t)y = 0. \tag{3}$$

We will refer to this equation as the ***standard form*** of the differential equation when $t_0$ is a regular singular point. By making a change of variable, if necessary, we can assume that $t_0 = 0$. We will restrict our attention to this case. Equation (3) then becomes

$$t^2 y'' + tA_1(t)y' + A_0(t)y = 0. \tag{4}$$

When $A_1(t)$ and $A_0(t)$ are constants then (4) is a Cauchy–Euler equation. We would expect that any reasonable adjustment to the power series method should able to handle this simplest case. Before we describe the adjustments let us explore, by an example, what goes wrong when we apply the power series method to a Cauchy–Euler equation. Consider the differential equation

$$2t^2y'' + 5ty' - 2y = 0. \tag{5}$$

Let $y(t) = \sum_{n=0}^{\infty} c_n t^n$. Then

$$\begin{aligned} 2t^2y'' &= \sum_{n=0}^{\infty} 2n(n-1)c_n t^n, \\ 5ty' &= \sum_{n=0}^{\infty} 5nc_n t^n, \\ -2y &= \sum_{n=0}^{\infty} -2c_n t^n. \end{aligned}$$

Thus,

$$2t^2y'' + 5ty' - 2y = \sum_{n=0}^{\infty}(2n(n-1) + 5n - 2)c_n t^n = \sum_{n=0}^{\infty}(2n-1)(n+2)c_n t^n.$$

Equation (5) now implies $(2n-1)(n+2)c_n = 0$, and hence $c_n = 0$, for all $n = 0, 1, \ldots$. The power series method has failed; it has only given us the trivial solution. With a little forethought, we could have seen the problem. The indicial polynomial for (5) is $2s^2 + 5s - 2 = (2s-1)(s+2)$. The roots are $\frac{1}{2}$ and $-2$. Thus, a fundamental set is $\left\{t^{\frac{1}{2}}, t^{-2}\right\}$ and neither of these functions is analytic at $t_0 = 0$. Our assumption that there was a power series solution centered at 0 was wrong!

Any modification of the power series method must take into account that solutions to differential equations about regular singular points can have fractional or negative powers of $t$, as in the example above. It is thus natural to consider solutions of the form

$$y(t) = t^r \sum_{n=0}^{\infty} c_n t^n, \tag{6}$$

where $r$ is a constant to be determined. This is the starting point for the Frobenius method. We may assume that $c_0$ is nonzero for if $c_0 = 0$, we could factor out a power of $t$ and incorporate it into $r$. Under this assumption, we call (6) a ***Frobenius series***. Of course, if $r$ is a nonnegative integer, then a Frobenius series is a power series.

Recall that the fundamental sets for Cauchy–Euler equations take the form

$$\{t^{r_1}, t^{r_2}\}, \quad \{t^r, t^r \ln t\}, \quad \text{and} \quad \{t^\alpha \cos \beta \ln t, t^\alpha \sin \beta \ln t\}.$$

(cf. Sect. 5.3). The power of $t$ depends on the roots of the indicial polynomial. For differential equations with regular singular points, something similar occurs. Suppose $A_1(t)$ and $A_0(t)$ have power series expansions about $t_0 = 0$ given by

$$A_1(t) = a_0 + a_1 t + a_2 t^2 + \cdots \quad \text{and} \quad A_0(t) = b_0 + b_1 t + b_2 t^2 + \cdots.$$

The polynomial

$$q(s) = s(s-1) + a_0 s + b_0 \tag{7}$$

is called the ***indicial polynomial*** associated to (4) and extends the definition given in the Cauchy–Euler case.[1] Its roots are called the ***exponents of singularity*** and, as in the Cauchy–Euler equations, indicate the power to use in the Frobenius series. A Frobenius series that solves (4) is called a ***Frobenius series solution***.

**Theorem 2 (The Frobenius Method).** *Suppose $t_0 = 0$ is a regular singular point of the differential equation*

$$t^2 y'' + t A_1(t) y' + A_0(t) y = 0.$$

*Suppose $r_1$ and $r_2$ are the exponents of singularity.*

***The Real Case:*** *Assume $r_1$ and $r_2$ are real and $r_1 \geq r_2$. There are three cases to consider:*

1. *If $r_1 - r_2$ is not an integer, then there are two Frobenius solutions of the form*

$$y_1(t) = t^{r_1} \sum_{n=0}^{\infty} c_n t^n \quad \text{and} \quad y_2(t) = t^{r_2} \sum_{n=0}^{\infty} C_n t^n.$$

2. *If $r_1 - r_2$ is a positive integer, then there is one Frobenius series solution of the form*

$$y_1(t) = t^{r_1} \sum_{n=0}^{\infty} c_n t^n$$

*and a second independent solution of the form*

$$y_2(t) = \epsilon y_1(t) \ln t + t^{r_2} \sum_{n=0}^{\infty} C_n t^n.$$

[1] If the coefficient of $t^2 y''$ is a number $c$ other than 1, we take the indicial polynomial to be $q(s) = cs(s-1) + a_0 s + b_0$.

*It can be arranged so that $\epsilon$ is either 0 or 1. When $\epsilon = 0$, there are two Frobenius series solution. When $\epsilon = 1$, then a second independent solution is the sum of a Frobenius series and a logarithmic term. We refer to these as the nonlogarithmic and logarithmic cases, respectively.*

3. *If $r_1 - r_2 = 0$, let $r = r_1 = r_2$. Then there is one Frobenius series solution of the form*

$$y_1(t) = t^r \sum_{n=0}^{\infty} c_n t^n$$

*and a second independent solution of the form*

$$y_2(t) = y_1(t)\ln t + t^r \sum_{n=0}^{\infty} C_n t^n.$$

*The second solution $y_2$ is also referred to as a logarithmic case.*

***The Complex Case:*** *If the roots of the indicial polynomial are distinct complex numbers, r and $\overline{r}$ say, then there is a complex-valued Frobenius series solution of the form*

$$y(t) = t^r \sum_{n=0}^{\infty} c_n t^n,$$

*where the coefficients $c_n$ may be complex. The real and imaginary parts of $y(t)$, $y_1(t)$, and $y_2(t)$, respectively, are linearly independent solutions.*

*Each series given for all five different cases has a positive radius of convergence.*

**Remark 3.** You will notice that in each case, there is at least one Frobenius solution. When the roots are real, there is a Frobenius solution for the larger of the two roots. If $y_1$ is a Frobenius solution and there is not a second Frobenius solution, then a second independent solution is the sum of a logarithmic expression $y_1(t)\ln t$ and a Frobenius series. This fact is obtained by applying reduction of order. We will not provide the proof as it is long and not very enlightening. However, we will consider an example of each case mentioned in the theorem. Read these examples carefully. They will reveal some of the subtleties involved in the general proof and, of course, are a guide through the exercises.

**Example 4 (Real Roots Not Differing by an Integer).** Use Theorem 2 to solve the following differential equation:

$$2t^2y'' + 3t(1+t)y' - y = 0. \tag{8}$$

► **Solution.** We can identify $A_1(t) = 3 + 3t$ and $A_0(t) = -1$. It is easy to see that $t_0 = 0$ is a regular singular point and the indicial equation

$$q(s) = 2s(s-1) + 3s - 1 = (2s^2 + s - 1) = (2s-1)(s+1).$$

The exponents of singularity are thus $\frac{1}{2}$ and $-1$, and since their difference is not an integer, Theorem 2 tells us there are two Frobenius solutions: one for each exponent of singularity. Before we specialize to each case, we will first derive the general recurrence relation from which the indicial equation falls out. Let

$$y(t) = t^r \sum_{n=0}^{\infty} c_n t^n = \sum_{n=0}^{\infty} c_n t^{n+r}.$$

Then

$$y'(t) = \sum_{n=0}^{\infty} (n+r) c_n t^{n+r-1} \quad \text{and} \quad y''(t) = \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n t^{n+r-2}.$$

It follows that

$$\begin{aligned}
2t^2 y''(t) &= t^r \sum_{n=0}^{\infty} 2(n+r)(n+r-1) c_n t^n, \\
3t y'(t) &= t^r \sum_{n=0}^{\infty} 3(n+r) c_n t^n, \\
3t^2 y'(t) &= t^r \sum_{n=0}^{\infty} 3(n+r) c_n t^{n+1} = t^r \sum_{n=1}^{\infty} 5(n-1+r) c_{n-1} t^n, \\
-y(t) &= t^r \sum_{n=0}^{\infty} -c_n t^n.
\end{aligned}$$

The sum of these four expressions is $2t^2y'' + 3t(1+t)y' - 1y = 0$. Notice that each term has $t^r$ as a factor. It follows that the sum of each corresponding power series is 0. They are each written in standard form so the sum of the coefficients with the same powers must likewise be 0. For $n = 0$, only the first, second, and fourth series contribute constant coefficients ($t^0$), while for $n \geq 1$, all four series contribute coefficients for $t^n$. We thus get

$$\begin{aligned}
n &= 0 \qquad (2r(r-1) + 3r - 1)c_0 = 0, \\
n &\geq 1 \qquad (2(n+r)(n+r-1) + 3(n+r) - 1)c_n + 3(n-1+r)c_{n-1} = 0.
\end{aligned}$$

Now observe that for $n = 0$, the coefficient of $c_0$ is the indicial polynomial $q(r) = 2r(r-1) + 3r - 1 = (2r-1)(r+1)$, and for $n \geq 1$, the coefficient of $c_n$ is $q(n+r)$. This will happen routinely. We can therefore rewrite these equations in the form

$$\begin{aligned}
n &= 0 \qquad q(r)c_0 = 0 \\
n &\geq 1 \qquad q(n+r)c_n + 3(n-1+r)c_{n-1} = 0. \qquad (9)
\end{aligned}$$

Since a Frobenius series has a nonzero constant term, it follows that $q(r) = 0$ implies $r = \frac{1}{2}$ and $r = -1$, the exponents of singularity derived in the beginning. Let us now specialize to these cases individually. We start with the larger of the two.

**The Case** $r = \frac{1}{2}$. Let $r = \frac{1}{2}$ in the recurrence relation given in (9). Observe that $q(n+\frac{1}{2}) = 2n\,(n+\frac{3}{2}) = n(2n+3)$ and is nonzero for all positive $n$ since the only roots are $\frac{1}{2}$ and $-1$. We can therefore solve for $c_n$ in the recurrence relation and get

$$c_n = \frac{-3(n-\frac{1}{2})}{n(2n+3)}c_{n-1} = \left(\frac{-3}{2}\right)\frac{(2n-1)}{n(2n+3)}c_{n-1}. \tag{10}$$

Recursively applying (10), we get

$$\begin{aligned}
n &= 1 \qquad c_1 = \left(\frac{-3}{2}\right)\frac{1}{5}c_0 = \left(\frac{-3}{2}\right)\frac{3}{1\cdot(5\cdot 3)}c_0,\\
n &= 2 \qquad c_2 = \left(\frac{-3}{2}\right)\frac{3}{2\cdot 7}c_1 = \left(\frac{-3}{2}\right)^2\frac{3}{2\cdot(7\cdot 5)}c_0,\\
n &= 3 \qquad c_3 = \left(\frac{-3}{2}\right)\frac{5}{3\cdot 9}c_2 = \left(\frac{-3}{2}\right)^3\frac{5\cdot 3}{(3\cdot 2)(9\cdot 7\cdot 5)}c_0 = \left(\frac{-3}{2}\right)^3\frac{3}{(3!)(9\cdot 7)}c_0,\\
n &= 4 \qquad c_4 = \left(\frac{-3}{2}\right)\frac{7}{4\cdot 11}c_2 = \left(\frac{-3}{2}\right)^4\frac{3}{(4!)(11\cdot 9)}c_0.
\end{aligned}$$

Generally, we have $c_n = \left(\frac{-3}{2}\right)^n\frac{3}{n!(2n+3)(2n+1)}c_0$. We let $c_0 = 1$ and substitute $c_n$ into the Frobenius series with $r = \frac{1}{2}$ to get

$$y_1(t) = t^{\frac{1}{2}}\sum_{n=0}^{\infty}\left(\frac{-3}{2}\right)^n\frac{3}{n!(2n+3)(2n+1)}t^n.$$

**The Case** $r = -1$. In this case, $q(n+r) = q(n-1) = (2n-3)(n)$ is again nonzero for all positive integers $n$. The recurrence relation in (9) simplifies to

$$c_n = -3\frac{n-2}{(2n-3)(n)}c_{n-1}. \tag{11}$$

Recursively applying (11), we get

$$\begin{aligned}
n &= 1 \qquad c_1 = -3\tfrac{-1}{-1\cdot 1}c_0 = -3c_0\\
n &= 2 \qquad c_2 = 0c_1 = 0\\
n &= 3 \qquad c_3 = 0\\
&\vdots \qquad\quad \vdots
\end{aligned}$$

Again, we let $c_0 = 1$ and substitute $c_n$ into the Frobenius series with $r = -1$ to get

$$y_2(t) = t^{-1}(1-3t) = \frac{1-3t}{t}.$$

Since $y_1$ and $y_2$ are linearly independent, the solutions to (8) is the set of all linear combinations. ◄

Before proceeding to our next example let us make a couple of observations that will apply in general. It is not an accident that the coefficient of $c_0$ is the indicial polynomial $q$. This will happen in general, and since we assumed from the outset that $c_0 \neq 0$, it follows that if a Frobenius series solution exists, then $q(r) = 0$; that is, $r$ must be an exponent of singularity. It is also not an accident that the coefficient of $c_n$ is $q(n+r)$ in the recurrence relation. This will happen in general as well. If we can guarantee that $q(n+r)$ is not zero for all positive integers $n$, then we obtain a consistent recurrence relation, that is, we can solve for $c_n$ to obtain a Frobenius series solution. This will always happen for $r_1$, the larger of the two roots. For the smaller root $r_2$, we need to be more careful. In fact, in the previous example, we were careful to point out that $q(n+r_2) \neq 0$ for $n > 0$. However, if the roots differ by an integer, then the consistency of the recurrence relation comes into question in the case of the smaller root. The next two examples consider this situation.

**Example 5 (Real Roots Differing by an Integer: The Nonlogarithmic Case).** Use Theorem 2 to solve the following differential equation:

$$ty'' + 2y' + ty = 0.$$

► **Solution.** We first multiply both sides by $t$ to put in standard form. We get

$$t^2y'' + 2ty' + t^2y = 0 \tag{12}$$

and it is easy to verify that $t_0 = 0$ is a regular singular point. The indicial polynomial is $q(s) = s(s-1) + 2s = s^2 + s = s(s+1)$. It follows that 0 and $-1$ are the exponents of singularity. They differ by an integer so there may or may not be a second Frobenius solution. Let

$$y(t) = t^r \sum_{n=0}^{\infty} c_n t^n.$$

Then

$$t^2y''(t) = t^r \sum_{n=0}^{\infty} (n+r)(n+r-1)c_n t^n,$$

$$2ty'(t) = t^r \sum_{n=0}^{\infty} 2(n+r)c_n t^n,$$

$$t^2 y(t) = t^r \sum_{n=2}^{\infty} c_{n-2} t^n.$$

The sum of the left side of each of these equations is $t^2 y'' + 2ty' + t^2 y = 0$, and, therefore, the sum of the series is zero. We separate the $n = 0$, $n = 1$, and $n \geq 2$ cases and simplify to get

$$\begin{aligned}
n &= 0 \qquad (r(r+1))c_0 = 0, \\
n &= 1 \qquad ((r+1)(r+2))c_1 = 0, \\
n &\geq 2 \qquad ((n+r)(n+r+1))c_n + c_{n-2} = 0.
\end{aligned} \tag{13}$$

The $n = 0$ case tells us that $r = 0$ or $r = -1$, the exponents of singularity.

**The Case** $r = 0$. If $r = 0$ is the larger of the two roots, then the $n = 1$ case in (13) implies $c_1 = 0$. Also $q(n + r) = q(n) = n(n + 1)$ is nonzero for all positive integers $n$. The recurrence relation simplifies to

$$c_n = \frac{-c_{n-2}}{(n+1)n}.$$

Since the difference in indices in the recurrence relation is 2 and $c_1 = 0$, it follows that all the odd terms, $c_{2n+1}$, are zero. For the even terms, we get

$$\begin{aligned}
n &= 2 \qquad c_2 = \frac{-c_0}{3 \cdot 2} \\
n &= 4 \qquad c_4 = \frac{-c_2}{5 \cdot 4} = \frac{c_0}{5!} \\
n &= 6 \qquad c_6 = \frac{-c_4}{7 \cdot 6} = \frac{-c_0}{7!},
\end{aligned}$$

and generally, $c_{2n} = \frac{(-1)^n}{(2n+1)!} c_0$. If we choose $c_0 = 1$, then

$$y_1(t) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n+1)!}$$

is a Frobenius solution (with exponent of singularity 0).

**The Case** $r = -1$. In this case, we see something different in the recurrence relation. For in the $n = 1$ case, (13) gives the equation

$$0 \cdot c_1 = 0.$$

This equation is satisfied for all $c_1$. There is no restriction on $c_1$ so we will choose $c_1 = 0$, as this is most convenient. The recurrence relation becomes

$$c_n = \frac{-c_{n-2}}{(n-1)n}, \quad n \geq 2.$$

A calculation similar to what we did above gives all the odd terms $c_{2n+1}$ zero and

$$c_{2n} = \frac{(-1)^n}{(2n)!} c_0.$$

If we set $c_0 = 1$, we find the Frobenius solution with exponent of singularity $-1$

$$y_2(t) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n-1}}{(2n)!}.$$

It is easy to verify that $y_1(t) = \frac{\sin t}{t}$ and $y_2(t) = \frac{\cos t}{t}$. Since $y_1$ and $y_2$ are linearly independent, the solutions to (12) is the set of all linear combinations of $y_1$ and $y_2$. ◀

The main difference that we saw in the previous example from that of the first example was in the case of the smaller root $r = -1$. We had $q(n-1) = 0$ when $n = 1$, and this leads to the equation $c_1 \cdot 0 = 0$. We were fortunate in that any $c_1$ is a solution and choosing $c_1 = 0$ leads to a second Frobenius solution. The recurrence relation remained consistent. In the next example, we will not be so fortunate. (If $c_1$ were chosen to be a fixed nonzero number, then the odd terms would add up to a multiple of $y_1$; nothing is gained.)

**Example 6 (Real Roots Differing by an Integer: The Logarithmic Case).** Use Theorem 2 to solve the following differential equation:

$$t^2 y'' - t^2 y' - (3t + 2)y = 0. \tag{14}$$

► **Solution.** It is easy to verify that $t_0 = 0$ is a regular singular point. The indicial polynomial is $q(s) = s(s-1) - 2 = s^2 - s - 2 = (s-2)(s+1)$. It follows that 2 and $-1$ are the exponents of singularity. They differ by an integer so there may or may not be a second Frobenius solution. Let

$$y(t) = t^r \sum_{n=0}^{\infty} c_n t^n.$$

Then

$$t^2 y''(t) = t^r \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n t^n,$$

$$-t^2 y'(t) = t^r \sum_{n=1}^{\infty} -(n-1+r)c_{n-1}t^n,$$

$$-3ty(t) = t^r \sum_{n=1}^{\infty} -3c_{n-1}t^n,$$

$$-2y(t) = t^r \sum_{n=0}^{\infty} -2c_n t^n.$$

As in the previous examples, the sum of the series is zero. We separate the $n = 0$ and $n \geq 1$ cases and simplify to get

$$\begin{array}{ll} n = 0 & (r-2)(r+1)c_0 = 0, \\ n \geq 1 & ((n+r-2)(n+r+1))c_n - (n+r+2)c_{n-1} = 0. \end{array} \qquad (15)$$

The $n = 0$ case tells us that $r = 2$ or $r = -1$, the exponents of singularity.

**The Case** $r = 2$. Since $r = 2$ is the larger of the two roots, the $n \geq 1$ case in (15) implies $c_n = \frac{n+4}{n(n+3)}c_{n-1}$. We then get

$$\begin{array}{ll} n = 1 & c_1 = \dfrac{5}{1 \cdot 4}c_0 \\ n = 2 & c_2 = \dfrac{6}{2 \cdot 5}c_1 = \dfrac{6}{2! \cdot 4}c_0 \\ n = 3 & c_3 = \dfrac{7}{3 \cdot 6}c_2 = \dfrac{7}{3! \cdot 4}c_0, \end{array}$$

and generally, $c_n = \frac{n+4}{n! \cdot 4}c_0$. If we choose $c_0 = 4$, then

$$y_1(t) = t^2 \sum_{n=0}^{\infty} \frac{n+4}{n!}t^n = \sum_{n=0}^{\infty} \frac{n+4}{n!}t^{n+2} \qquad (16)$$

is a Frobenius series solution; the exponent of singularity is 2. (It is easy to see that $y_1(t) = (t^3 + 4t^2)e^t$ but we will not use this fact.)

**The Case** $r = -1$. The recurrence relation in (15) simplifies to

$$n(n-3)c_n = -(n+1)c_{n-1}.$$

In this case, there is a problem when $n = 3$. Observe

$$\begin{array}{llll} n = 1 & -2c_1 = 2c_0 & \text{hence} & c_1 = -c_0, \\ n = 2 & -2c_2 = 3c_1 & \text{hence} & c_2 = \frac{3}{2}c_0, \\ n = 3 & 0 \cdot c_3 = 4c_2 = 6c_0, & \Rightarrow\Leftarrow & \end{array}$$

In the $n = 3$ case, there is no solution since $c_0 \neq 0$. The recurrence relation is inconsistent and there is no second Frobenius series solution. However, Theorem 2 tells us there is a second independent solution of the form

$$y(t) = y_1(t)\ln t + t^{-1}\sum_{n=0}^{\infty} c_n t^n. \tag{17}$$

Although the calculations that follow are straightforward, they are more involved than in the previous examples. The idea is simple though: substitute (17) into (14) and solve for the coefficients $c_n$, $n = 0, 1, 2, \ldots$. If $y(t)$ is as in (17), then a calculation gives

$$\begin{aligned}
t^2y'' &= t^2y_1''\ln t + 2ty_1' - y_1 + t^{-1}\sum_{n=0}^{\infty}(n-1)(n-2)c_nt^n,\\
-t^2y' &= -t^2y_1'\ln t - ty_1 + t^{-1}\sum_{n=1}^{\infty}-(n-2)c_{n-1}t^n,\\
-3ty &= -3ty_1\ln t + t^{-1}\sum_{n=1}^{\infty}-3c_{n-1}t^n,\\
-2y &= -2y_1\ln t + t^{-1}\sum_{n=0}^{\infty}-2c_nt^n.
\end{aligned}$$

The sum of the terms on the left is zero since we are assuming a $y$ is a solution. The sum of the terms with $\ln t$ as a factor is also zero since $y_1$ is the solution, (16), we found in the case $r = 2$. Observe also that the $n = 0$ term only occurs in the first and fourth series. In the first series, the constant term is $(-1)(-2)c_0 = 2c_0$, and in the fourth series, the constant term is $(-2)c_0 = -2c_0$. Since the $n = 0$ terms cancel, we can thus start all the series at $n = 1$. Adding these terms together and simplifying gives

$$\begin{aligned}
0 &= 2ty_1' - (t+1)y_1\\
&\quad + t^{-1}\sum_{n=1}^{\infty}(n(n-3))c_n - (n+1)c_{n-1})\,t^n.
\end{aligned} \tag{18}$$

Now let us calculate the power series for $2ty_1' - (t+1)y_1$ and factor $t^{-1}$ out of the sum. A short calculation and some index shifting gives

$$2ty_1' = \sum_{n=0}^{\infty}\frac{2(n+4)(n+2)}{n!}t^{n+2} = t^{-1}\sum_{n=3}^{\infty}\frac{2(n+1)(n-1)}{(n-3)!}t^n,$$

$$-y_1 = \sum_{n=0}^{\infty} -\frac{n+4}{n!}t^{n+2} = t^{-1}\sum_{n=3}^{\infty} -\frac{n+1}{(n-3)!}t^n,$$

$$-ty_1 = \sum_{n=0}^{\infty} -\frac{n+4}{n!}t^{n+3} = t^{-1}\sum_{n=3}^{\infty} -\frac{n(n-3)}{(n-3)!}t^n.$$

Adding these three series and simplifying gives

$$2ty_1' - (t+1)y_1 = t^{-1}\sum_{n=3}^{\infty} \frac{(n+3)(n-1)}{(n-3)!}t^n.$$

We now substitute this calculation into (18), cancel out the common factor $t^{-1}$, and get

$$\sum_{n=3}^{\infty} \frac{(n+3)(n-1)}{(n-3)!}t^n + \sum_{n=1}^{\infty}(n(n-3)c_n - (n+1)c_{n-1})t^n = 0.$$

We separate out the $n = 1$ and $n = 2$ cases to get:

$$\begin{array}{lccl} n = 1 & -2c_1 - 2c_0 = 0 & \text{hence} & c_1 = -c_0, \\ n = 2 & -2c_2 - 3c_1 = 0 & \text{hence} & c_2 = \frac{3}{2}c_0. \end{array}$$

For $n \geq 3$, we get

$$\frac{(n+3)(n-1)}{(n-3)!} + n(n-3)c_n - (n+1)c_{n-1} = 0$$

which we can rewrite as

$$n(n-3)c_n = (n+1)c_{n-1} - \frac{(n+3)(n-1)}{(n-3)!} \qquad n \geq 3. \tag{19}$$

You should notice that the coefficient of $c_n$ is zero when $n = 3$. As observed earlier, this led to an inconsistency of the recurrence relation for a Frobenius series. However, the additional term $\frac{(n+3)(n-1)}{(n-3)!}$ that comes from the logarithmic term results in consistency but only for a specific value of $c_2$. To see this, let $n = 3$ in (19) to get $0 = 0c_3 = 4c_2 - 12$ and hence $c_2 = 3$. Since $c_2 = \frac{3}{2}c_0$, we have that $c_0 = 2$ and $c_1 = -2$. We now have $0c_3 = 4c_2 - 12 = 0$, and we can choose $c_3$ to be any number. It is convenient to let $c_3 = 0$. For $n \geq 4$, we can write (19) as

$$c_n = \frac{n+1}{n(n-3)}c_{n-1} - \frac{(n+3)(n-1)}{n(n-3)(n-3)!} \qquad n \geq 4. \tag{20}$$

Such recurrence relations are generally very difficult to solve in a closed form. However, we can always solve any finite number of terms. In fact, the following terms are easy to check:

| | | | |
|---|---|---|---|
| $n = 0$ | $c_0 = 2,$ | $n = 4$ | $c_4 = \frac{-21}{4}$ |
| $n = 1$ | $c_1 = -2,$ | $n = 5$ | $c_5 = \frac{-19}{4}$ |
| $n = 2$ | $c_2 = 3,$ | $n = 6$ | $c_6 = \frac{-163}{72}$ |
| $n = 3$ | $c_3 = 0,$ | $n = 7$ | $c_7 = \frac{-53}{72}.$ |

We substitute these values into (17) to obtain (an approximation to) a second linearly independent solution

$$y_2(t) = y_1(t)\ln t + t^{-1}\left(2 - 2t + 3t^2 - \frac{21}{4}t^4 - \frac{19}{4}t^5 - \frac{163}{72}t^6 - \frac{53}{72}t^7 \cdots\right). \quad ◀$$

A couple of remarks are in order. By far the logarithmic cases are the most tedious. In the case just considered, the difference in the roots is $2 - (-1) = 3$ and it was precisely at $n = 3$ in the recurrence relation where $c_0$ is determined in order to achieve consistency. After $n = 3$, the coefficients are nonzero so the recurrence relation is consistent. In general, it is at the difference in the roots where this junction occurs. If we choose $c_3$ to be a nonzero fixed constant, the terms that would arise with $c_3$ as a coefficient would be a multiple of $y_1$, and thus, nothing is gained. Choosing $c_3 = 0$ does not exclude any critical part of the solution.

**Example 7 (Real roots: Coincident).** Use Theorem 2 to solve the following differential equation:

$$t^2 y'' - t(t+3)y' + 4y = 0. \tag{21}$$

► **Solution.** It is easy to verify that $t_0 = 0$ is a regular singular point. The indicial polynomial is $q(s) = s(s-1) - 3s + 4 = s^2 - 4s + 4 = (s-2)^2$. It follows that 2 is a root with multiplicity two. Hence, $r = 2$ is the only exponent of singularity. There will be only one Frobenius series solution. A second solution will involve a logarithmic term. Let

$$y(t) = t^r \sum_{n=0}^{\infty} c_n t^n.$$

Then

$$
\begin{aligned}
t^2 y''(t) &= t^r \sum_{n=0}^{\infty} (n+r)(n+r-1)c_n t^n, \\
-t^2 y'(t) &= t^r \sum_{n=1}^{\infty} (n+r-1)c_{n-1} t^n, \\
-3t y'(t) &= t^r \sum_{n=0}^{\infty} -3c_n (n+r) t^n, \\
4y(t) &= t^r \sum_{n=0}^{\infty} 4c_n t^n.
\end{aligned}
$$

As in previous examples, the sum of the series is zero. We separate the $n = 0$ and $n \geq 1$ cases and simplify to get

$$
\begin{aligned}
n &= 0 \qquad (r-2)^2 c_0 = 0, \\
n &\geq 1 \qquad (n+r-2)^2 c_n = (n+r-1)c_{n-1}.
\end{aligned}
\tag{22}
$$

**The Case** $r = 2$. Equation (22) implies $r = 2$ and

$$
c_n = \frac{n+1}{n^2} c_{n-1} \quad n \geq 1.
$$

We then get

$$
\begin{aligned}
n &= 1 \qquad c_1 = \tfrac{2}{1^2} c_0, \\
n &= 2 \qquad c_2 = \tfrac{3}{2^2} c_1 = \tfrac{3\cdot 2}{2^2\cdot 1^2} c_0, \\
n &= 3 \qquad c_3 = \tfrac{4}{3^2} c_2 = \tfrac{4\cdot 3\cdot 2}{3^2\cdot 2^2\cdot 1^2} c_0,
\end{aligned}
$$

and generally,

$$
c_n = \frac{(n+1)!}{(n!)^2} c_0 = \frac{n+1}{n!} c_0.
$$

If we choose $c_0 = 1$, then

$$
y_1(t) = t^2 \sum_{n=0}^{\infty} \frac{n+1}{n!} t^n \tag{23}
$$

is a Frobenius series solution (with exponent of singularity 2). This is the only Frobenius series solution.

A second independent solution takes the form

$$
y(t) = y_1(t) \ln t + t^2 \sum_{n=0}^{\infty} c_n t^n. \tag{24}
$$

The ideas and calculations are very similar to the previous example. A straightforward calculation gives

$$
\begin{aligned}
t^2y'' &= t^2y_1''\ln t + 2ty_1' - y_1 + t^2\sum_{n=0}^{\infty}(n+2)(n+1)c_nt^n,\\
-t^2y' &= -t^2y_1'\ln t - ty_1 + t^2\sum_{n=1}^{\infty}-(n+1)c_{n-1}t^n,\\
-3ty' &= -3ty_1'\ln t - 3y_1 + t^2\sum_{n=0}^{\infty}-3(n+2)c_nt^n,\\
4y &= 4y_1\ln t + t^2\sum_{n=0}^{\infty}4c_nt^n.
\end{aligned}
$$

The sum of the terms on the left is zero since we are assuming a $y$ is a solution. The sum of the terms with $\ln t$ as a factor is also zero since $y_1$ is a solution. Observe also that the $n = 0$ terms occur in the first, third, and fourth series. In the first series the coefficient is $2c_0$, in the third series the coefficient is $-6c_0$, and in the fourth series the coefficient is $4c_0$. We can thus start all the series at $n = 1$ since the $n = 0$ terms cancel. Adding these terms together and simplifying gives

$$
0 = 2ty_1' - (t+4)y_1 + t^2\sum_{n=1}^{\infty}\left(n^2c_n - (n+1)c_{n-1}\right)t^n. \tag{25}
$$

Now let us calculate the power series for $2ty_1' - (t+4)y_1$ and factor $t^2$ out of the sum. A short calculation and some index shifting gives

$$
2ty_1' - (t+4)y_1 = t^2\sum_{n=1}^{\infty}\frac{n+2}{(n-1)!}t^n. \tag{26}
$$

We now substitute this calculation into (25), cancel out the common factor $t^2$, and equate coefficients to get

$$
\frac{n+2}{(n-1)!} + n^2c_n - (n+1)c_{n-1} = 0.
$$

Since $n \geq 1$, we can solve for $c_n$ to get the following recurrence relation:

$$
c_n = \frac{n+1}{n^2}c_{n-1} - \frac{n+2}{n(n!)}, \qquad n \geq 1.
$$

As in the previous example, such recurrence relations are difficult to solve in a closed form. There is no restriction on $c_0$ so we may assume it is zero. The first few terms thereafter are as follows:

$$
\begin{array}{llll}
n=0 & c_0=0, & n=4 & c_4=\dfrac{-173}{288}, \\
n=1 & c_1=-3, & n=5 & c_5=\dfrac{-187}{1200}, \\
n=2 & c_2=\dfrac{-13}{4}, & n=6 & c_6=\dfrac{-463}{14400}, \\
n=3 & c_3=\dfrac{-31}{18}, & n=7 & c_7=\dfrac{-971}{176400}.
\end{array}
$$

We substitute these values into (24) to obtain (an approximation to) a second linearly independent solution

$$y_2(t)=y_1(t)\ln t-\left(3t^3+\frac{13}{4}t^4+\frac{31}{18}t^5+\frac{173}{288}t^6+\frac{187}{1200}t^7+\frac{463}{14400}t^8+\cdots\right).$$

◀

Since the roots in this example are coincident, their difference is 0. The juncture mentioned in the example that proceeded this one thus occurs at $n=0$ and so we can make the choice $c_0=0$. If $c_0$ is chosen to be nonzero, then $y_2$ will include an extra term $c_0y_1$. Thus nothing is gained.

**Example 8 (Complex Roots).** Use Theorem 2 to solve the following differential equation:

$$t^2(t+1)y''+ty'+(t+1)^3y=0. \tag{27}$$

► **Solution.** It is easy to see that $t_0=0$ is a regular singular point. The indicial polynomial is $q(s)=s(s-1)+s+1=s^2+1$ and has roots $r=\pm\mathrm{i}$. Thus, there is a complex-valued Frobenius solution, and its real and imaginary parts will be two linear independent solutions to (27). A straightforward substitution gives

$$
\begin{aligned}
t^3y''&=t^r\sum_{n=1}^{\infty}(n+r-1)(n+r-2)c_{n-1}t^n,\\
t^2y''&=t^r\sum_{n=0}^{\infty}(n+r)(n+r-1)c_nt^n,\\
ty'&=t^r\sum_{n=0}^{\infty}(n+r)c_nt^n,\\
y&=t^r\sum_{n=0}^{\infty}c_nt^n,
\end{aligned}
$$

$$3ty = t^r \sum_{n=1}^{\infty} 3c_{n-1}t^n,$$

$$3t^2y = t^r \sum_{n=2}^{\infty} 3c_{n-2}t^n,$$

$$t^3y = t^r \sum_{n=3}^{\infty} c_{n-3}t^n.$$

As usual, the sum of these series is zero. Since the index of the sums have different starting points, we separate the cases $n = 0$, $n = 1$, $n = 2$, and $n \geq 3$ to get, after some simplification, the following:

$$\begin{aligned}
n &= 0 \qquad (r^2+1)c_0 = 0, \\
n &= 1 \qquad ((r+1)^2+1)c_1 + (r(r-1)+3)c_0 = 0, \\
n &= 2 \qquad ((r+2)^2+1)c_2 + ((r+1)r+3)c_1 + 3c_0 = 0, \\
n &\geq 3 \qquad ((r+n)^2+1)c_n + ((r+n-1)(r+n-2)+3)c_{n-1} \\
&\qquad\qquad + 3c_{n-2} + c_{n-3} = 0.
\end{aligned}$$

The $n = 0$ case implies that $r = \pm\text{i}$. We will let $r = \text{i}$ (the $r = -\text{i}$ case will give equivalent results). As usual, $c_0$ is arbitrary but nonzero. For simplicity, let us fix $c_0 = 1$. Substituting these values into the cases, $n = 1$ and $n = 2$ above, gives $c_1 = i$ and $c_2 = \frac{-1}{2}$. The general recursion relation is

$$((i+n)^2+1)c_n + ((i+n-1)(i+n-2)+3)c_{n-1} + 3c_{n-2} + c_{n-3} = 0 \quad (28)$$

from which we see that $c_n$ is determined as long as we know the previous three terms. Since $c_0$, $c_1$, and $c_2$ are known, it follows that we can determine all $c_n$'s. Although (28) is somewhat tedious to work with, straightforward calculations give the following values:

$$\begin{aligned}
n &= 0 \qquad c_0 = 1, & n &= 3 \qquad c_3 = \tfrac{-i}{6} = \tfrac{i^3}{3!}, \\
n &= 1 \qquad c_1 = i, & n &= 4 \qquad c_4 = \tfrac{1}{24} = \tfrac{i^4}{4!}, \\
n &= 2 \qquad c_2 = \tfrac{-1}{2} = \tfrac{i^2}{2!}, & n &= 5 \qquad c_5 = \tfrac{i}{120} = \tfrac{i^5}{5!}.
\end{aligned}$$

We will leave it as an exercise to verify by mathematical induction that $c_n = \frac{i^n}{n!}$. It follows now that

$$y(t) = t^i \sum_{n=0}^{\infty} \frac{i^n}{n!}t^n = t^i \text{e}^{it}.$$

We assume $t > 0$. Therefore, we can write $t^i = e^{i \ln t}$ and

$$y(t) = e^{i(t+\ln t)}.$$

By Euler's formula, the real and imaginary parts are

$$y_1(t) = \cos(t + \ln t) \quad \text{and} \quad y_2(t) = \sin(t + \ln t).$$

It is easy to see that these functions are linearly independent solutions. We remark that the $r = -i$ case gives the solution $y(t) = e^{-i(t+\ln t)}$. Its real and imaginary parts are, up to sign, the same as $y_1$ and $y_2$ given above. ◄

## Exercises

1–5. For each problem, determine the singular points. Classify them as regular or irregular.

1. $y'' + \frac{t}{1-t^2}y' + \frac{1}{1+t}y = 0$
2. $y'' + \frac{1-t}{t}y' + \frac{1-\cos t}{t^3}y = 0$
3. $y'' + 3t(1-t)y' + \frac{1-e^t}{t}y = 0$
4. $y'' + \frac{1}{t}y' + \frac{1-t}{t^3}y = 0$
5. $ty'' + (1-t)y' + 4ty = 0$

6–10. Each differential equation has a regular singular point at $t = 0$. Determine the indicial polynomial and the exponents of singularity. How many Frobenius solutions are guaranteed by Theorem 2? How many could there be?

6. $2ty'' + y' + ty = 0$
7. $t^2y'' + 2ty' + t^2y = 0$
8. $t^2y'' + te^t y' + 4(1-4t)y = 0$
9. $ty'' + (1-t)y' + \lambda y = 0$
10. $t^2y'' + 3t(1+3t)y' + (1-t^2)y = 0$

11–14. Verify the following claims that were made in the text.

11. In Example 5, verify the claims that $y_1(t) = \frac{\sin t}{t}$ and $y_2(t) = \frac{\cos t}{t}$.
12. In Example 8, we claimed that the solution to the recursion relation

$$((i+n)^2 + 1)c_n + ((i+n-1)(i+n-2) + 3)c_{n-1} + 3c_{n-2} + c_{n-3} = 0$$

was $c_n = \frac{i^n}{n!}$. Use mathematical induction to verify this claim.

13. In Remark 3, we stated that the logarithmic case could be obtained by a reduction of order argument. Consider the Cauchy–Euler equation

$$t^2y'' + 5ty' + 4y = 0.$$

One solution is $y_1(t) = t^{-2}$. Use reduction of order to show that a second independent solution is $y_2(t) = t^{-2}\ln t$, in harmony with the statement for the appropriate case of the theorem.

14. Verify the claim made in Example 6 that $y_1(t) = (t^3 + 4t^2)e^t$

15–26. Use the Frobenius method to solve each of the following differential equations. For those problems marked with a (*), one of the independent solutions can easily be written in closed form. For those problems marked with a (**), both independent solutions can easily be written in closed form. In each case below we let $y = t^r \sum_{n=0}^{\infty} c_n t^n$ where we assume $c_0 \neq 0$ and $r$ is the exponent of singularity.

15. $ty'' - 2y' + ty = 0$ (**) (real roots, differ by integer, two Frobenius solutions)
16. $2t^2 y'' - ty' + (1+t)y = 0$ (**) (real roots, do not differ by an integer, two Frobenius solutions)
17. $t^2 y'' - t(1+t)y' + y = 0$, (*) (real roots, coincident, logarithmic case)
18. $2t^2 y'' - ty' + (1-t)y = 0$ (**) (real roots, do not differ by an integer, two Frobenius solutions)
19. $t^2 y'' + t^2 y' - 2y = 0$ (**) (real roots, differ by integer, two Frobenius solutions)
20. $t^2 y'' + 2ty' - a^2 t^2 y = 0$ (**) (real roots, differ by integer, two Frobenius Solutions)
21. $ty'' + (t-1)y' - 2y = 0$, (*) (real roots, differ by integer, logarithmic case)
22. $ty'' - 4y = 0$ (real roots, differ by an integer, logarithmic case)
23. $t^2(-t+1)y'' + (t+t^2)y' + (-2t+1)y = 0$ (**) (complex)
24. $t^2 y'' + t(1+t)y' - y = 0$, (**) (real roots, differ by an integer, two Frobenius solutions)
25. $t^2 y'' + t(1-2t)y' + (t^2 - t + 1)y = 0$ (**) (complex)
26. $t^2(1+t)y'' - t(1+2t)y' + (1+2t)y = 0$ (**) (real roots, equal, logarithmic case)

## 7.4 Application of the Frobenius Method: Laplace Inversion Involving Irreducible Quadratics

In this section, we return to the question of determining formulas for the Laplace inversion of

$$\frac{b}{(s^2+b^2)^{k+1}} \quad \text{and} \quad \frac{s}{(s^2+b^2)^{k+1}}, \tag{1}$$

for $k$ a nonnegative integer. Recall that in Sect. 2.5, we developed reduction of order formulas so that each inversion could be recursively computed. In this section, we will derive a closed formula for the inversion by solving a distinguished second order differential equation, with a regular singular point at $t = 0$, associated to each simple rational function given above. Table 7.1 of Sect. 7.5 summarizes the Laplace transform formulas we obtain in this section.

To begin with, we use the dilatation principle to reduce the simple quadratic rational functions given in (1) to the case $b = 1$. Recall the dilation principle, Theorem 23 of Sect. 2.2. For an input function $f(t)$ and $b$ a positive number, we have

$$\mathcal{L}\{f(bt)\}(s) = \frac{1}{b}\mathcal{L}\{f(t)\}(s/b).$$

The corresponding inversion formula gives

$$\mathcal{L}^{-1}\{F(s/b\} = bf(bt), \tag{2}$$

where as usual $\mathcal{L}\{f\} = F$.

**Proposition 1.** *Suppose* $b > 0$. *Then*

$$\mathcal{L}^{-1}\left\{\frac{b}{(s^2+b^2)^{k+1}}\right\}(t) = \frac{1}{b^{2k}}\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^{k+1}}\right\}(bt),$$
$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+b^2)^{k+1}}\right\}(t) = \frac{1}{b^{2k}}\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^{k+1}}\right\}(bt).$$

*Proof.* We simply apply (2) to get

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{b}{(s^2+b^2)^{k+1}}\right\}(t) &= \frac{b}{b^{2(k+1)}}\mathcal{L}^{-1}\left\{\frac{1}{((s/b)^2+1)^{k+1}}\right\}(t)\\ &= \frac{1}{b^{2k}}\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^{k+1}}\right\}(bt).\end{aligned}$$

A similar calculation holds for the second simple rational function. □

It follows from Proposition 1 that we only need to consider the cases

$$\frac{s}{(s^2+1)^{k+1}} \quad \text{and} \quad \frac{1}{(s^2+1)^{k+1}}. \tag{3}$$

We now define $A_k(t)$ and $B_k(t)$ as follows:

$$A_k(t) = 2^k k! \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^{k+1}}\right\}$$
$$B_k(t) = 2^k k! \mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^{k+1}}\right\}. \quad (4)$$

The factor $2^k k!$ will make the resulting formulas a little simpler.

**Lemma 2.** *For $k \geq 1$, we have*

$$A_k(t) = -t^2 A_{k-2}(t) + (2k-1)A_{k-1}(t), \quad k \geq 2, \quad (5)$$
$$B_k(t) = tA_{k-1}(t), \quad (6)$$
$$A_k(0) = 0, \quad (7)$$
$$B_k(0) = 0. \quad (8)$$

*Proof.* Let $b = 1$ in Proposition 8 of Sect. 2.5, use the definition of $A_k$ and $B_k$, and simplify to get

$$A_k(t) = -tB_{k-1}(t) + (2k-1)A_{k-1}(t),$$
$$B_k(t) = tA_{k-1}(t).$$

Equation (6) is the second of the two above. Now replace $k$ by $k-1$ in (6) and substitute into the first equation above to get (5). By the initial value theorem, Theorem 1 of Sect. 5.4, we have

$$A_k(0) = 2^k k! \lim_{s\to\infty} \frac{s}{(s^2+1)^{k+1}} = 0,$$
$$B_k(0) = 2^k k! \lim_{s\to\infty} \frac{s^2}{(s^2+1)^{k+1}} = 0.$$
□

**Proposition 3.** *Suppose $k \geq 1$. Then*

$$A_k'(t) = B_k(t), \quad (9)$$
$$B_k'(t) = 2kA_{k-1}(t) - A_k(t). \quad (10)$$

*Proof.* By the input derivative principle and the Lemma above, we have

$$\begin{aligned}\frac{1}{2^k k!}\mathcal{L}\left\{A_k'(t)\right\} &= \frac{1}{2^k k!}(s\mathcal{L}\{A_k(t)\} - A_k(0)) \\ &= \frac{s}{(s^2+1)^{k+1}} \\ &= \frac{1}{2^k k!}\mathcal{L}\{B_k(t)\}.\end{aligned}$$

Equation (9) now follows. In a similar way, we have

$$\begin{aligned}\frac{1}{2^k k!}\mathcal{L}\left\{B_k'(t)\right\} &= \frac{1}{2^k k!}(s\mathcal{L}\{B_k(t)\} - B_k(0)) \\ &= \frac{s^2}{(s^2+1)^{k+1}} \\ &= \frac{1}{(s^2+1)^k} - \frac{1}{(s^2+1)^{k+1}} \\ &= \frac{2k}{2^k k!}\mathcal{L}\{A_{k-1}(t)\} - \frac{1}{2^k k!}\mathcal{L}\{A_k(t)\}.\end{aligned}$$

Equation (10) now follows. □

**Proposition 4.** *With notation as above, we have*

$$tA_k'' - 2kA_k' + tA_k = 0, \tag{11}$$

$$t^2B_k'' - 2ktB_k' + (t^2+2k)B_k = 0. \tag{12}$$

*Proof.* We first differentiate equation (9) and then substitute in (10) to get

$$A_k'' = B_k' = 2kA_{k-1} - A_k.$$

Now multiply this equation by $t$ and simplify using (6) and (9) to get

$$tA_k'' = 2ktA_{k-1} - tA_k = 2kB_k - tA_k = 2kA_k' - tA_k.$$

Equation (11) now follows. To derive equation (12), first differentiate equation (11) and then multiply by $t$ to get

$$t^2A_k''' + (1-2k)tA_k'' + t^2A_k' + tA_k = 0.$$

From (11), we get $tA_k = -tA_k'' + 2kA_k'$ which we substitute into the equation above to get

$$t^2A_k''' - 2ktA_k'' + (t^2+2k)A_k' = 0.$$

Equation (12) follows now from (9). □

Proposition 4 tells us that $A_k$ and $B_k$ both satisfy differential equations with a regular singular point at $t_0 = 0$. We know from Corollary 11 of Sect. 2.5 that $A_k$ and $B_k$ are sums of products of polynomials with $\sin t$ and $\cos t$ (see Table 2.5 for the cases $k = 0, 1, 2, 3$). Specifically, we can write

$$A_k(t) = p_1(t)\cos t + p_2(t)\sin t, \tag{13}$$

$$B_k(t) = q_1(t)\cos t + q_2(t)\sin t, \tag{14}$$

where $p_1$, $p_2$, $q_1$, and $q_2$ are polynomials of degree at most $k$. Because of the presence of the sin and cos functions, the Frobenius method will produce rather complicated power series and it will be very difficult to recognize these polynomial factors. Let us introduce a simplifying feature that gets to the heart of the polynomial coefficients. We will again assume some familiarity of complex number arithmetic. By Euler's formula, $\mathrm{e}^{it} = \cos t + i \sin t$ and $\mathrm{e}^{-\mathrm{i}t} = \cos t - i \sin t$. Adding these formulas together and dividing by 2 gives a formula for $\cos t$. Similarly, subtracting these formula and dividing by $2i$ gives a formula for $\sin t$. Specifically, we get

$$\cos t = \frac{\mathrm{e}^{it} + \mathrm{e}^{-\mathrm{i}t}}{2},$$
$$\sin t = \frac{\mathrm{e}^{it} - \mathrm{e}^{-\mathrm{i}t}}{2i}.$$

Substituting these formulas into (13) and simplifying gives

$$\begin{aligned}
A_k(t) &= p_1(t)\cos t + p_2(t)\sin t \\
&= p_1(t)\frac{\mathrm{e}^{it} + \mathrm{e}^{-\mathrm{i}t}}{2} + p_2(t)\frac{\mathrm{e}^{it} - \mathrm{e}^{-\mathrm{i}t}}{2i} \\
&= \frac{p_1(t) - i p_2(t)}{2}\mathrm{e}^{it} + \frac{p_1(t) + i p_2(t)}{2}\mathrm{e}^{-\mathrm{i}t} \\
&= \frac{a_k(t)}{2}\mathrm{e}^{it} + \overline{\frac{a_k(t)}{2}\mathrm{e}^{it}} \\
&= \mathrm{Re}(a_k(t)\mathrm{e}^{it}),
\end{aligned}$$

where $a_k(t) = p_1(t) - i p_2(t)$ is a complex-valued polynomial, which we determine below. Observe that since $p_1$ and $p_2$ have degrees at most $k$, it follows that $a_k(t)$ is a polynomial of degree at most $k$. In a similar way, we can write

$$B_k(t) = \mathrm{Re}(b_k(t)\mathrm{e}^{it}),$$

for some complex-valued polynomial $b_k(t)$ whose degree is at most $k$. We summarize the discussion above for easy reference.

**Proposition 5.** *There are complex-valued polynomials $a_k(t)$ and $b_k(t)$ so that*

$$A_k(t) = \mathrm{Re}(a_k(t)\mathrm{e}^{it}),$$
$$B_k(t) = \mathrm{Re}(b_k(t)\mathrm{e}^{it}).$$

We now proceed to show that $a_k(t)$ and $b_k(t)$ satisfy second order differential equations with a regular singular point at $t_0 = 0$. The Frobenius method will give only one polynomial solution in each case, which we identify with $a_k(t)$ and $b_k(t)$. From there, it is an easy matter to use Proposition 5 to find $A_k(t)$ and $B_k(t)$.

First a little Lemma.

**Lemma 6.** *Suppose $p(t)$ is a complex-valued polynomial and* $\mathrm{Re}(p(t)\mathrm{e}^{it}) = 0$ *for all $t \in \mathbb{R}$. Then $p(t) = 0$ for all $t$.*

*Proof.* Write $p(t) = \alpha(t)+i\beta(t)$, where $\alpha(t)$ and $\beta(t)$ are real-valued polynomials. Then the assumption that $\mathrm{Re}(p(t)\mathrm{e}^{it}) = 0$ becomes

$$\alpha(t)\cos t - \beta(t)\sin t = 0. \tag{15}$$

Let $t = 2\pi n$ in (15). Then we get $\alpha(2\pi n) = 0$ for each integer $n$. This means $\alpha(t)$ has infinitely many roots, and this can only happen when a polynomial is zero. Similarly, if $t = \frac{\pi}{2} + 2\pi n$ is substituted into 15, then we get $\beta\left(\frac{\pi}{2} + 2\pi n\right) = 0$, for all $n$. We similarly get $\beta(t) = 0$. It now follows that $p(t) = 0$. $\square$

**Proposition 7.** *The polynomials $a_k(t)$ and $b_k(t)$ satisfy*

$$\begin{aligned} ta_k'' + 2(it-k)a_k' - 2kia_k &= 0, \\ t^2 b_k'' + 2t(it-k)b_k' - 2k(it-1)b_k &= 0. \end{aligned}$$

*Proof.* Let us start with $A_k$. Since differentiation respects the real and imaginary parts of complex-valued functions, we have

$$\begin{aligned} A_k(t) &= \mathrm{Re}(a_k(t)\mathrm{e}^{it}), \\ A_k'(t) &= \mathrm{Re}((a_k(t)\mathrm{e}^{it})') = \mathrm{Re}((a_k'(t) + ia_k(t))\mathrm{e}^{it}), \\ A_k''(t) &= \mathrm{Re}((a_k''(t) + 2ia_k'(t) - a_k(t))\mathrm{e}^{it}). \end{aligned}$$

It follows now from Proposition 4 that

$$\begin{aligned} 0 &= tA_k''(t) - 2kA_k'(t) + tA_k(t) \\ &= t\,\mathrm{Re}(a_k(t)\mathrm{e}^{it})'' - 2k\,\mathrm{Re}(a_k(t)\mathrm{e}^{it})' + t\,\mathrm{Re}(a_k(t)\mathrm{e}^{it}) \\ &= \mathrm{Re}((t(a_k''(t) + 2ia_k'(t) - a_k(t)) - 2k(a_k'(t) + ia_k(t)) + ta_k(t))\mathrm{e}^{it}) \\ &= \mathrm{Re}\left((ta_k''(t) + 2(it-k)a_k'(t) - 2kia_k(t))\mathrm{e}^{it}\right). \end{aligned}$$

Now, Lemma 6 implies

$$ta_k''(t) + 2(it-k)a_k'(t) - 2kia_k(t) = 0.$$

The differential equation in $b_k$ is done similarly and left as an exercise (see Exercise 3). $\square$

Each differential equation involving $a_k$ and $b_k$ in Proposition 7 is second order and has $t_0 = 0$ as a regular singular point. Do not be troubled by the presence of complex coefficients; the Frobenius method applies over the complex numbers as well.

The following lemma will be useful in determining the coefficient needed in the Frobenius solutions given below for $a_k(t)$ and $b_k(t)$.

**Lemma 8.** *The constant coefficient of $a_k(t)$ is given by*

$$a_k(0) = -i\,\frac{(2k)!}{k!2^k}.$$

*The coefficient of $t$ in $b_k$ is given by*

$$b_k'(0) = -i\,\frac{(2(k-1))!}{(k-1)!2^{(k-1)}}.$$

*Proof.* Replace $k$ by $k+2$ in (5) to get

$$A_{k+2} = (2k+3)A_{k+1} - t^2 A_k,$$

for all $t$ and $k \geq 1$. Lemma 6 gives that $a_k$ satisfies the same recursion relation:

$$a_{k+2}(t) = (2k+3)a_{k+1}(t) - t^2 a_k(t).$$

If $t = 0$, we get $a_{k+2}(0) = (2k+3)a_{k+1}(0)$. Replace $k$ by $k-1$ to get

$$a_{k+1}(0) = (2k+1)a_k(0).$$

By Table 2.5, we have

$$\begin{aligned}
A_1(t) &= 2^1(1!)\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\} \\
&= \sin t - t\cos t \\
&= \mathrm{Re}((-t-i)e^{it}).
\end{aligned}$$

Thus, $a_1(t) = -t - i$ and $a_1(0) = -i$. The above recursion relation gives the following first four terms:

$$\begin{aligned}
a_1(0) &= -i, & a_3(0) &= 5a_2(0) = -5\cdot 3\cdot i, \\
a_2(0) &= 3a_1(0) = -3i, & a_4(0) &= 7a_3(0) = -7\cdot 5\cdot 3\cdot i.
\end{aligned}$$

Inductively, we get

$$\begin{aligned} a_k(0) &= -(2k-1)\cdot(2k-3)\cdots 5\cdot 3\cdot i \\ &= -i\,\frac{(2k)!}{k!2^k}. \end{aligned}$$

For any polynomial $p(t)$, the coefficient of $t$ is given by $p'(0)$. Thus, $b_k'(0)$ is the coefficient of $t$ in $b_k(t)$. On the other hand, (6) implies $b_{k+1}(t) = ta_k(t)$, and hence, the coefficient of $t$ in $b_{k+1}(t)$ is the same as the constant coefficient, $a_k(0)$ of $a_k(t)$. Replacing $k$ by $k-1$ and using the formula for $a_k(0)$ derived above, we get

$$b_k'(0) = a_{k-1}(0) = -i\,\frac{(2(k-1))!}{(k-1)!2^{(k-1)}}. \qquad \square$$

**Proposition 9.** *With the notation as above, we have*

$$a_k(t) = \frac{-i}{2^k}\sum_{n=0}^{k}\frac{(2k-n)!}{n!(k-n)!}(-2it)^n$$

*and* $$b_k(t) = \frac{-it}{2^{k-1}}\sum_{n=0}^{k-1}\frac{(2(k-1)-n)!}{n!(k-1-n)!}(-2it)^n.$$

*Proof.* By Proposition 7, $a_k(t)$ is the polynomial solution to the differential equation

$$ty'' + 2(it-k)y' - 2kiy = 0 \tag{16}$$

with constant coefficient $a_k(0)$ as given in Lemma 8. Multiplying equation (16) by $t$ gives $t^2y'' + 2t(it-k)y' - 2kity = 0$ which is easily seen to have a regular singular point at $t=0$. The indicial polynomial is given by $q(s) = s(s-1)-2ks = s(s-(2k+1))$. It follows that the exponents of singularity are 0 and $2k+1$. We will show below that the $r=0$ case gives a polynomial solution. We leave it to the exercises (see Exercise 1) to verify that the $r=2k+1$ case gives a nonpolynomial Frobenius solution. We thus let

$$y(t) = \sum_{n=0}^{\infty} c_n t^n.$$

Then

$$ty''(t) = \sum_{n=1}^{\infty}(n)(n+1)c_{n+1}t^n,$$

$$2ity'(t) = \sum_{n=1}^{\infty} 2inc_n t^n,$$

$$-2ky'(t) = \sum_{n=0}^{\infty} -2k(n+1)c_{n+1}t^n,$$

$$-2iky(t) = \sum_{n=0}^{\infty} -2ikc_n t^n.$$

By assumption, the sum of the series is zero. We separate the $n = 0$ and $n \geq 1$ cases and simplify to get

$$\begin{aligned} n = 0 \qquad & -2kc_1 - 2ikc_0 = 0, \\ n \geq 1 \qquad & (n-2k)(n+1)c_{n+1} + 2i(n-k)c_n = 0. \end{aligned} \tag{17}$$

The $n = 0$ case tells us that $c_1 = -ic_0$. For $1 \leq n \leq 2k-1$, we have

$$c_{n+1} = \frac{-2i(k-n)}{(2k-n)(n+1)}c_n.$$

Observe that $c_{k+1} = 0$, and hence, $c_n = 0$ for all $k+1 \leq n \leq 2k-1$. For $n = 2k$ we get from the recursion relation $0c_{2k+1} = 2ikc_k = 0$. This implies that $c_{2k+1}$ can be arbitrary. We will choose $c_{2k+1} = 0$. Then $c_n = 0$ for all $n \geq k+1$, and hence, the solution $y$ is a polynomial. We make the usual comment that if $c_{k+1}$ is chosen to be nonzero, then those terms with $c_{k+1}$ as a factor will make up the Frobenius solution for exponent of singularity $r = 2k+1$. Let us now determine the coefficients $c_n$ for $0 \leq n \leq k$. From the recursion relation, (17), we get

$$\begin{aligned} n = 0 \qquad & c_1 = -ic_0, \\ n = 1 \qquad & c_2 = \frac{-2i(k-1)}{(2k-1)2}c_1 = \frac{2(-i)^2(k-1)}{(2k-1)2}c_0 = \frac{(-2i)^2k(k-1)}{(2k)(2k-1)2}c_0, \\ n = 2 \qquad & c_3 = \frac{-2i(k-2)}{(2k-2)3}c_2 = \frac{(-2i)^3k(k-1)(k-2)}{2k(2k-1)(2k-2)3!}c_0, \\ n = 3 \qquad & c_4 = \frac{-2i(k-3)}{(2k-3)4}c_3 = \frac{(-2i)^4k(k-1)(k-2)(k-3)}{2k(2k-1)(2k-2)(2k-3)4!}c_0. \end{aligned}$$

and generally,

$$c_n = \frac{(-2i)^n k(k-1)\cdots(k-n+1)}{2k(2k-1)\cdots(2k-n+1)n!}c_0 \quad n = 1, \ldots, k.$$

We can write this more compactly in terms of binomial coefficients as

$$c_n = \frac{(-2i)^n \binom{k}{n}}{\binom{2k}{n} n!}c_0.$$

It follows now that

$$y(t) = c_0 \sum_{n=0}^{k} \frac{(-2i)^n \binom{k}{n}}{\binom{2k}{n} n!} t^n.$$

The constant coefficient is $c_0$. Thus, we choose $c_0 = a_k(0)$ as given in Lemma 8. Then $y(t) = a_k(t)$ is the polynomial solution we seek. We get

$$\begin{aligned} a_k(t) &= a_k(0) \sum_{n=0}^{k} \frac{(-2i)^n \binom{k}{n}}{\binom{2k}{n} n!} t^n \\ &= \sum_{n=0}^{k} -i \frac{(2k)!}{k! 2^k} \frac{(-2i)^n \binom{k}{n}}{\binom{2k}{n} n!} t^n \\ &= \frac{-i}{2^k} \sum_{n=0}^{k} \frac{(2k-n)!}{(k-n)! n!} (-2it)^n. \end{aligned}$$

It is easy to check that (6) has as an analogue the equation $b_{k+1}(t) = t a_k(t)$. Replacing $k$ by $k-1$ in the formula for $a_k(t)$ and multiplying by $t$ thus establishes the formula for $b_k(t)$. □

For $x \in \mathbb{R}$, we let $\lfloor x \rfloor$ denote the greatest integer function of $x$. It is defined to be the greatest integer less than or equal to $x$. We are now in a position to give a closed formula for the inverse Laplace transforms of the simple rational functions given in (4).

**Theorem 10.** *For the simple rational functions, we have*

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^{k+1}}\right\}(t) &= \frac{\sin t}{2^{2k}} \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} (-1)^m \binom{2k-2m}{k} \frac{(2t)^{2m}}{(2m)!} \\ &\quad - \frac{\cos t}{2^{2k}} \sum_{m=0}^{\lfloor \frac{k-1}{2} \rfloor} (-1)^m \binom{2k-2m-1}{k} \frac{(2t)^{2m+1}}{(2m+1)!}, \\ \mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^{k+1}}\right\}(t) &= \frac{2t \sin t}{k \cdot 2^{2k}} \sum_{m=0}^{\lfloor \frac{k-1}{2} \rfloor} (-1)^m \binom{2k-2m-2}{k-1} \frac{(2t)^{2m}}{(2m)!} \\ &\quad - \frac{2t \cos t}{k \cdot 2^{2k}} \sum_{m=0}^{\lfloor \frac{k-2}{2} \rfloor} (-1)^m \binom{2k-2m-3}{k-1} \frac{(2t)^{2m+1}}{(2m+1)!}. \end{aligned}$$

*The first formula is valid for $k \geq 0$, and the second formula is valid for $k \geq 1$. Sums where the upper limit is less than $0$ (which occur in the cases $k = 0$ and $1$) should be understood to be $0$.*

*Proof.* From Proposition 9, we can write

$$a_k(t) = \frac{1}{2^k}\sum_{n=0}^{k}\frac{(2k-n)!}{n!(k-n)!}(-1)^{n+1}(i)^{n+1}(2t)^n.$$

It is easy to see that the real part of $a_k$ consists of those terms where $n$ is odd. The imaginary part consists of those terms where $n$ is even. The odd integers from $n = 0, \ldots, k$ can be written $n = 2m + 1$ where $m = 0, \ldots, \lfloor \frac{k-1}{2} \rfloor$. Similarly, the even integers can be written $n = 2m$, where $m = 0, \ldots, \lfloor \frac{k}{2} \rfloor$. We thus have

$$\begin{aligned}\mathrm{Re}(a_k(t)) &= \frac{1}{2^k}\sum_{m=0}^{\lfloor \frac{k-1}{2} \rfloor}\frac{(2k-2m-1)!}{(2m+1)!(k-2m-1)!}(-1)^{2m+2}(i)^{2m+2}(2t)^{2m+1}\\ &= \frac{-1}{2^k}\sum_{m=0}^{\lfloor \frac{k-1}{2} \rfloor}\frac{(2k-2m-1)!}{(2m+1)!(k-2m-1)!}(-1)^m(2t)^{2m+1}\end{aligned}$$

and

$$\begin{aligned}\mathrm{Im}(a_k(t)) &= \frac{1}{i}\frac{1}{2^k}\sum_{m=0}^{\lfloor \frac{k}{2} \rfloor}\frac{(2k-2m)!}{(2m)!(k-2m)!}(-1)^{2m+1}(i)^{2m+1}(2t)^{2m}\\ &= \frac{-1}{2^k}\sum_{m=0}^{\lfloor \frac{k}{2} \rfloor}\frac{(2k-2m)!}{(2m)!(k-2m)!}(-1)^m(2t)^{2m}.\end{aligned}$$

Now $\mathrm{Re}(a_k(t)e^{it}) = \mathrm{Re}(a_k(t))\cos t - \mathrm{Im}(a_k(t))\sin t$. It follows from Propositions 5 that

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^{k+1}}\right\} &= \frac{1}{2^k k!}\mathrm{Re}(a_k(t)e^{it})\\ &= \frac{1}{2^k k!}\mathrm{Re}(a_k(t))\cos t - \frac{1}{2^k k!}\mathrm{Im}(a_k(t))\sin t\\ &= \frac{-\cos t}{2^{2k}}\sum_{m=0}^{\lfloor \frac{k-1}{2} \rfloor}\frac{(2k-2m-1)!}{k!(2m+1)!(k-2m-1)!}(-1)^m(2t)^{2m+1}\\ &\quad + \frac{\sin t}{2^{2k}}\sum_{m=0}^{\lfloor \frac{k}{2} \rfloor}\frac{(2k-2m)!}{k!(2m)!(k-2m)!}(-1)^m(2t)^{2m}\end{aligned}$$

$$= \frac{-\cos t}{2^{2k}} \sum_{m=0}^{\lfloor \frac{k-1}{2} \rfloor} \binom{2k-2m-1}{k} (-1)^m \frac{(2t)^{2m+1}}{(2m+1)!}$$

$$+ \frac{\sin t}{2^{2k}} \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} \binom{2k-2m}{k} (-1)^m \frac{(2t)^{2m}}{(2m)!}.$$

A similar calculation gives the formula for $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^{k+1}}\right\}$. $\square$

We conclude with the following corollary which is immediate from Proposition 1.

**Corollary 11.** *Let $b > 0$, then*

$$\mathcal{L}^{-1}\left\{\frac{b}{(s^2+b^2)^{k+1}}\right\}(t) = \frac{\sin bt}{(2b)^{2k}} \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} (-1)^m \binom{2k-2m}{k} \frac{(2bt)^{2m}}{(2m)!}$$

$$-\frac{\cos bt}{(2b)^{2k}} \sum_{m=0}^{\lfloor \frac{k-1}{2} \rfloor} (-1)^m \binom{2k-2m-1}{k} \frac{(2bt)^{2m+1}}{(2m+1)!},$$

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+b^2)^{k+1}}\right\}(t) = \frac{2bt \sin bt}{k \cdot (2b)^{2k}} \sum_{m=0}^{\lfloor \frac{k-1}{2} \rfloor} (-1)^m \binom{2k-2m-2}{k-1} \frac{(2bt)^{2m}}{(2m)!}$$

$$-\frac{2bt \cos bt}{k \cdot (2b)^{2k}} \sum_{m=0}^{\lfloor \frac{k-2}{2} \rfloor} (-1)^m \binom{2k-2m-3}{k-1} \frac{(2bt)^{2m+1}}{(2m+1)!}.$$

## Exercises

1–3. Verify the following unproven statements made in this section.

1. Verify the statement made that the Frobenius solution to

$$ty'' + 2(it - k)y' - 2kiy = 0$$

with exponent of singularity $r = 2k + 1$ is not a polynomial.

2. Verify the second inversion formula

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+b^2)^{k+1}}\right\}(t) = \frac{1}{b^{2k}}\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^{k+1}}\right\}(bt)$$

given in Proposition 1.

3. Verify the second differential equation formula

$$t^2 b_k'' + 2t(it - k)b_k' - 2k(it - 1)b_k = 0$$

given in Proposition 7.

4–15. *This series of exercises leads to closed formulas for the inverse Laplace transform of*

$$\frac{1}{(s^2-1)^{k+1}} \quad \textit{and} \quad \frac{s}{(s^2-1)^{k+1}}.$$

*Define $C_k(t)$ and $D_k(t)$ by the formulas*

$$\frac{1}{2^k k!}\mathcal{L}\{C_k(t)\}(s) = \frac{1}{(s^2-1)^{k+1}}$$

*and*

$$\frac{1}{2^k k!}\mathcal{L}\{D_k(t)\}(s) = \frac{s}{(s^2-1)^{k+1}}.$$

4. Show that $C_k$ and $D_k$ are related by

$$D_{k+1}(t) = tC_k(t).$$

5. Show that $C_k$ and $D_k$ satisfy the recursion formula

$$C_k(t) = tD_{k-1}(t) - (2k-1)C_{k-1}(t).$$

6. Show that $C_k$ satisfies the recursion formula

$$C_{k+2}(t) = t^2 C_k(t) - (2k+3)C_{k+1}(t).$$

7. Show that $D_k$ satisfies the recursion formula

$$D_{k+2}(t) = t^2 D_k(t) - (2k+1) D_{k+1}(t).$$

8. For $k \geq 1$, show that

$$C_k(0) = 0,$$
$$D_k(0) = 0.$$

9. Show that for $k \geq 1$,

   1. $C_k'(t) = D_k(t)$.
   2. $D_k'(t) = 2kC_{k-1}(t) + C_k(t)$.

10. Show the following:

   1. $tC_k''(t) - 2kC_k'(t) - tC_k(t) = 0$.
   2. $t^2 D_k''(t) - 2ktD_k'(t) + (2k - t^2)D_k(t) = 0$.

11. Show that there are polynomials $c_k(t)$ and $d_k(t)$, each of degree at most $k$, such that

   1. $C_k(t) = c_k(t)\mathrm{e}^t - c_k(-t)\mathrm{e}^{-t}$.
   2. $D_k(t) = d_k(t)\mathrm{e}^t + d_k(-t)\mathrm{e}^{-t}$.

12. Show the following:

   1. $tc_k''(t) + (2t - 2k)c_k'(t) - 2kc_k(t) = 0$.
   2. $td_k''(t) + 2t(t-k)d_k'(t) - 2k(t-1)d_k(t) = 0$.

13. Show the following:

   1. $c_k(0) = \dfrac{(-1)^k (2k)!}{2^{k+1} k!}$.
   2. $d_k'(0) = \dfrac{(-1)^{k-1}(2(k-1))!}{2^k (k-1)!}$.

14. Show the following:

   1. $c_k(t) = \dfrac{(-1)^k}{2^{k+1}} \displaystyle\sum_{n=0}^{k} \dfrac{(2k-n)!}{n!(k-n)!}(-2t)^n$.
   2. $d_k(t) = \dfrac{(-1)^k}{2^{k+1}} \displaystyle\sum_{n=1}^{k} \dfrac{(2k-n-1)!}{(n-1)!(k-n)!}(-2t)^n$.

15. Show the following:

1. $\mathcal{L}^{-1}\left\{\dfrac{1}{(s^2-1)^{k+1}}\right\}(t) = \dfrac{(-1)^k}{2^{2k+1}k!}\displaystyle\sum_{n=0}^{k}\dfrac{(2k-n)!}{n!(k-n)!}\left((-2t)^n\mathrm{e}^t - (2t)^n\mathrm{e}^{-t}\right).$

2. $\mathcal{L}^{-1}\left\{\dfrac{s}{(s^2-1)^{k+1}}\right\}(t) = \dfrac{(-1)^k}{2^{2k+1}k!}\displaystyle\sum_{n=1}^{k}\dfrac{(2k-n-1)!}{(n-1)!(k-n)!}\left((-2t)^n\mathrm{e}^t + (2t)^n\mathrm{e}^{-t}\right).$

## 7.5 Summary of Laplace Transforms

**Table 7.1** Laplace transforms

| $f(t)$ | $F(s)$ | Page |
|---|---|---|
| Laplace transforms involving the quadratic $s^2 + b^2$ | | |
| 1. $\dfrac{\sin bt}{(2b)^{2k}} \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} (-1)^m \binom{2k-2m}{k} \dfrac{(2bt)^{2m}}{(2m)!}$ | $\dfrac{b}{(s^2+b^2)^{k+1}}$ | 549 |
| $-\dfrac{\cos bt}{(2b)^{2k}} \sum_{m=0}^{\lfloor \frac{k-1}{2} \rfloor} (-1)^m \binom{2k-2m-1}{k} \dfrac{(2bt)^{2m+1}}{(2m+1)!}$ | | |
| 2. $\dfrac{2bt \sin bt}{k \cdot (2b)^{2k}} \sum_{m=0}^{\lfloor \frac{k-1}{2} \rfloor} (-1)^m \binom{2k-2m-2}{k-1} \dfrac{(2bt)^{2m}}{(2m)!}$ | $\dfrac{s}{(s^2+b^2)^{k+1}}$ | 549 |
| $-\dfrac{2bt \cos bt}{k \cdot (2b)^{2k}} \sum_{m=0}^{\lfloor \frac{k-2}{2} \rfloor} (-1)^m \binom{2k-2m-3}{k-1} \frac{(2bt)^{2m+1}}{(2m+1)!}$ | | |
| Laplace transforms involving the quadratic $s^2 - b^2$ | | |
| 3. $\dfrac{(-1)^k}{2^{2k+1}k!} \sum_{n=0}^{k} \dfrac{(2k-n)!}{n!(k-n)!} ((-2t)^n \mathrm{e}^t - (2t)^n \mathrm{e}^{-t})$ | $\dfrac{1}{(s^2-1)^{k+1}}$ | 553 |
| 4. $\dfrac{(-1)^k}{2^{2k+1}k!} \sum_{n=1}^{k} \dfrac{(2k-n-1)!}{(n-1)!(k-n)!} ((-2t)^n \mathrm{e}^t + (2t)^n \mathrm{e}^{-t})$ | $\dfrac{s}{(s^2-1)^{k+1}}$ | 553 |